

Computational Finance PDE

Copenhagen University
December 2022

Jesper Andreasen
Saxo Bank, Copenhagen
kwant.daddy@saxobank.com

Outline

- PDEs in finance.
- Backward solution.
- Forward solution.
- Let's get practical.

Material

- Andreassen, J (2011): “Finite Difference Methods for Financial Problems.”
PhD Course Copenhagen University.
- Andreassen, J (2022): “Catch-Up.” Forthcoming Wilmott.
- Andreassen, J, B Huge and F Kryger-Baggesen (2022):
<https://github.com/brnohu>.

PDE's in Finance

- Let

$$ds = \mu(t,s)dt + \sigma(t,s)dW \tag{1}$$

$$db/b = r(t,s)dt$$

- then the expectation

$$f(t,s(t)) = E_t[e^{-\int_t^T r(u)du} f(T,s(T))] \tag{2}$$

- ... is the solution to the *backward* PDE

$$0 = f_t + Af, A = -r + \mu \partial_s + \frac{1}{2} \sigma^2 \partial_{ss} \quad (3)$$

- ... and

$$f(0, s(0)) = \int f(T, s) p(T, s) ds \quad (4)$$

- ... where p solves the *forward* PDE

$$0 = -p_t + A^* p, A^* = -r - \partial_s r + \frac{1}{2} \partial_{ss} \sigma^2, p(0, s) = \delta(s - s(0)) \quad (5)$$

- A^* is denoted the *adjoint* operator.
- The backward PDE is solved backwards in time: from terminal value to current value.

- The forward PDE is solved forwards in time: from current density of current state to density of future state.
- One can possibly somewhat understand this intellectually but the way to really grasp it is to actually do it!

Backward Finite Difference Solution

- Theta scheme:

$$\begin{aligned} f(t_{h+1/2}) &= [1 + (1 - \theta)\Delta t \bar{A}] f(t_{h+1}) \\ [1 - \theta\Delta t \bar{A}] f(t_h) &= f(t_{h+1/2}) \end{aligned} \tag{6}$$

- where

$$f(t_h) = (f(t_h, s_0), \dots, f(t_h, s_{n-1}))' \tag{7}$$

- ... is a vector of solution values.
- ... and \bar{A} is a *finite difference* approximation to A

$$\bar{A} = -r + \mu \delta_s + \frac{1}{2} \sigma^2 \delta_{ss} \quad (8)$$

- ... can be represented as a *tridiagonal* matrix.
- We solve the system (6) backward in time:

$$f(t_h) \xleftarrow[\text{tridiagonal matrix inversion}]{\quad} f(t_{h+1/2}) \xleftarrow[\text{tridiagonal matrix multiplication}]{\quad} f(t_{h+1}) \quad (9)$$

Forward Finite Difference Solution

- Multiply the (6a-b) from the left by vectors $p(t_h), p(t_{h+1/2})$:

$$p(t_h)' f(t_h) = p(t_h)' [1 - \theta \Delta t \bar{A}]^{-1} f(t_{h+1/2}) \quad (10)$$

$$p(t_{h+1/2})' f(t_{h+1/2}) = p(t_{h+1/2})' [1 + (1 - \theta) \Delta t \bar{A}] f(t_{h+1})$$

- ... and thereby

$$[1 - \theta \Delta t \bar{A}'] p(t_{h+1/2}) = p(t_h) \quad (11)$$

$$p(t_{h+1}) = [1 + (1 - \theta) \Delta t \bar{A}'] p(t_{h+1/2})$$

- So the adjoint operator is simply the transpose: $\bar{A}^* = \bar{A}'$.

Practical Steps

- For this to work we need (at least) 6 things:
 1. A representation of a tridiagonal matrix: `kMatrix(n,3)`.
 2. A way of multiplying tridiagonal matrix with vector: `banmul()`.
 3. A way solving tridiagonal matrix system: `tridag()`.
 4. Routines for constructing operators: `dx()`, `dxx()`.
 5. Routine for constructing the matrix $[1+\Delta t \bar{A}]$ and its transpose: `calcAx()`.
 6. Put it all together: `rollBwd()`, `rollFwd()`.

- Today we will focus on the first three steps.

Notes

- In this course vectors (and matrixes) are always in increasing order of the state: $x[0] < x[1] < x[2] < \dots$