To: **Computational Finance Course Participants**
Fr: **Jesper Andreasen**
St: **Week 3**
Dt: **5 Dec 2022**

Agenda

Class 9 Dec 2022 13:15-16 in room 4-0-05 at the Bio Center.

Bring your Windows laptop. We recommend that you try starting on the list below before class on Friday.

We summarize a few results about differential operators and their representation as tridiagonal matrixes. We use this to represent the backward equation as a tridiagonal matrix equation.

To do list

Fill the interior of the methods kFiniteDifference::dx() and kFiniteDifference::dxx() that produce the operator matrixes $\delta_x^+, \delta_x^-, \delta_x$ and δ_{xx}, respectively.

Fill the interior of the method kFd1d::calcAx() that produces the operator $1 + \Delta t A$, $A = -r + \mu\delta_x + \frac{1}{2}\sigma^2\delta_{xx}$, as a tridiagonal matrix.

Test the function xFd1d() from excel.

Difference Operators

First order operators

$$\delta_x^+ f(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\delta_x^- f(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$\delta_x f(x_i) = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} \delta_x^- f(x_i) + \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \delta_x^+ f(x_i)$$

Second order operators

$$\delta_{xx} f(x_i) = 2\frac{\delta_x^+ f(x_i) - \delta_x^- f(x_i)}{x_{i+1} - x_{i-1}}$$

Matrix Representation

First order difference operators as tridiagonal matrixes in compact form. For the rows we have

$$(\delta_x^+)_{i\cdot} = \begin{bmatrix} 0 & \dfrac{-1}{x_{i+1}-x_i} & \dfrac{1}{x_{i+1}-x_i} \end{bmatrix} \qquad ,0 \le i < n-1$$

$$(\delta_x^-)_{i\cdot} = \begin{bmatrix} \dfrac{-1}{x_i-x_{i-1}} & \dfrac{1}{x_i-x_{i-1}} & 0 \end{bmatrix} \qquad ,0 < i \le n-1$$

and

$$(\delta_x)_{i\cdot} = \frac{1}{x_{i+1}-x_{i-1}} \begin{bmatrix} -\dfrac{x_{i+1}-x_i}{x_i-x_{i-1}} & \dfrac{x_{i+1}-x_i}{x_i-x_{i-1}} - \dfrac{x_i-x_{i-1}}{x_{i+1}-x_i} & \dfrac{x_i-x_{i-1}}{x_{i+1}-x_i} \end{bmatrix} \qquad ,0 < i < n-1$$

Similar for the second order operator

$$(\delta_{xx})_{i\cdot} = \frac{2}{x_{i+1}-x_{i-1}} \cdot \begin{bmatrix} \dfrac{1}{x_i-x_{i-1}} & \left(\dfrac{-1}{x_i-x_{i-1}} + \dfrac{-1}{x_{i+1}-x_i}\right) & \dfrac{1}{x_{i+1}-x_i} \end{bmatrix} \qquad ,0 < i < n-1$$

Accuracy

Accuracy of the first order operators

$$\delta_x^- f(x_i) = f'(x_i) + O(\Delta x)$$
$$\delta_x^+ f(x_i) = f'(x_i) + O(\Delta x)$$
$$\delta_x f(x_i) = f'(x_i) + O(\Delta x^2)$$

For the second order operator we have

$$\delta_{xx} f(x_i) = f''(x_i) + \frac{1}{3} f'''(x_i) \frac{(x_{i+1}-x_i)^2 - (x_i - x_{i-1})^2}{(x_{i+1}-x_{i-1})} + O(\Delta x^2)$$

Hence, for equidistant spacing, the accuracy of the second order operator is $O(\Delta x^2)$. Otherwise it is $O(\Delta x)$.

Proof

From Taylor expansion we get

$$f(x_{i+1})-f(x_i)=f'(x_i)(x_{i+1}-x_i)+\frac{1}{2}f''(x_i)(x_{i+1}-x_i)^2+\frac{1}{6}f'''(x_i)(x_{i+1}-x_i)^3+O(\Delta x^4)$$
$$f(x_{i-1})-f(x_i)=-f'(x_i)(x_i-x_{i-1})+\frac{1}{2}f''(x_i)(x_i-x_{i-1})^2-\frac{1}{6}f'''(x_i)(x_i-x_{i-1})^3+O(\Delta x^4)$$
(1)

Hence,

$$\delta_x^+ f(x_i)=f'(x_i)+\frac{1}{2}f''(x_i)(x_{i+1}-x_i)+\frac{1}{6}f'''(x_i)(x_{i+1}-x_i)^2+O(\Delta x^3)$$
$$\delta_x^- f(x_i)=f'(x_i)-\frac{1}{2}f''(x_i)(x_{i-1}-x_i)+\frac{1}{6}f'''(x_i)(x_i-x_{i-1})^2+O(\Delta x^3) \qquad (2)$$

From this we get from (2) that for any λ:

$$(1-\lambda)\delta_x^- f(x_i)+\lambda\delta_x^+ f(x_i)=f'(x_i)+\frac{1}{2}f''(x_i)[\lambda(x_{i+1}-x_i)-(1-\lambda)(x_i-x_{i-1})]+O(\Delta x^2)$$
(3)

Specifically, for $\lambda=(x_i-x_{i-1})/(x_{i+1}-x_{i-1})$ we get

$$\delta_x f(x_i)=(1-\lambda)\delta_x^- f(x_i)+\lambda\delta_x^+ f(x_i)=f'(x_i)+O(\Delta x^2) \qquad (4)$$

Subtracting (2a-b) yields

$$\delta_{xx} f(x_i)=f''(x_i)+\frac{1}{6}f'''(x_i)\frac{(x_{i+1}-x_i)^2-(x_i-x_{i-1})^2}{(x_{i+1}-x_{i-1})}+O(\Delta x^2) \qquad (5)$$

QED.