C++ Project - Preliminaries

Case $i \neq 0$, $i \neq N$

$$\frac{f_i^{t+1} - f_i^t}{dt} = \theta L_i^t + (1-\theta) L_i^{t+1}$$

where $L_i^t = \alpha \frac{f_{i+1}^t - 2f_i^t + f_{i-1}^t}{dx^2} + \beta \frac{f_{i+1}^t - f_{i-1}^t}{2dx} + \gamma f_i^t + \Delta$

$\alpha, \beta, \gamma, \Delta$ functions (e.g. of r, σ...)

ie

$$\Omega f_i^t = a_i f_i^{t+1} + b_i f_{i-1}^t + c_i f_{i+1}^t + d_i f_{i+1}^{t+1} + e_i f_{i-1}^{t+1} + \Delta dt$$

where:

$$\Omega = \theta dt \left(\frac{2\alpha}{dx^2} - \gamma \right) - 1$$

$$a_i = (1-\theta) dt \left(-\frac{2\alpha}{dx^2} + \gamma \right) - 1$$

$$b_i = \theta dt \left(\frac{\alpha}{dx^2} - \frac{\beta}{2dx} \right)$$

$$c_i = \theta dt \left(\frac{\alpha}{dx^2} + \frac{\beta}{2dx} \right)$$

$$d_i = (1-\theta) dt \left(\frac{\alpha}{dx^2} + \frac{\beta}{2dx} \right)$$

$$e_i = (1-\theta) dt \left(\frac{\alpha}{dx^2} - \frac{\beta}{2dx} \right)$$

Case $i = 0$ (boundary)

$$\frac{f_0^{t+1} - f_0^t}{dt} = \theta L_0^t + (1-\theta) L_0^{t+1}$$

where $L_0^t = \alpha \frac{f_2^t - 2f_1^t + f_0^t}{dx^2} + \beta \frac{f_1^t - f_0^t}{dx} + \gamma f_0^t + \Delta$

ie

$$\Omega_0 f_0^t = a_0 f_0^{t+1} + b_0 f_1^t + c_0 f_1^{t+1} + d_0 f_2^t + e_0 f_2^{t+1} + \Delta dt$$

where

$$\Omega_0 = -\theta dt \left(\frac{\alpha}{dx^2} - \frac{\beta}{dx} + \gamma \right) - 1$$

$$a_0 = (1-\theta) dt \left(\frac{\alpha}{dx^2} - \frac{\beta}{dx} + \gamma \right) - 1$$

$$b_0 = \theta dt \left(\frac{\beta}{dx} - \frac{2\alpha}{dx^2} \right)$$

$$c_0 = (1-\theta) dt \left(\frac{\beta}{dx} - \frac{2\alpha}{dx^2} \right)$$

$$d_0 = \theta dt \frac{\alpha}{dx^2}$$

$$e_0 = (1-\theta) dt \frac{\alpha}{dx^2}$$

Case $i=N$ (boundary)

$$\frac{\rho_N^{t+1} - \rho_N^t}{dt} = \theta L_N^t + (1-\theta) L_N^{t+1}$$

where $L_N^t = \alpha \frac{\rho_N^t - 2\rho_{N-1}^t + \rho_{N-2}^t}{dx^2} + \beta \frac{\rho_N^t - \rho_{N-1}^t}{dx} + \gamma \rho_N^t + \Delta$

ie

$$\Omega_N \rho_N^t = a_N \rho_N^{t+1} + b_N \rho_{N-1}^t + c_N \rho_{N-1}^{t+1} + d_N \rho_{N-2}^t + e_N \rho_{N-2}^{t+1} + \Delta dt$$

where

$$\Omega_N = -\theta dt \left(\frac{\alpha}{dx^2} + \frac{\beta}{dx} + \gamma \right) - 1$$

$$a_N = (1-\theta) dt \left(\frac{\alpha}{dx^2} + \frac{\beta}{dx} + \gamma \right) - 1$$

$$b_N = -\theta dt \left(\frac{2\alpha}{dx^2} + \frac{\beta}{dx} \right)$$

$$c_N = -(1-\theta) dt \left(\frac{2\alpha}{dx^2} + \frac{\beta}{dx} \right)$$

$$d_N = \theta dt \frac{\alpha}{dx^2}$$

$$e_N = (1-\theta) dt \frac{\alpha}{dx^2}$$

Matrix format

$$\Omega X_t = A' X_{t+1} + A'' X_t + b'$$

ie $(\Omega - A'') X_t = A' X_{t+1} + b'$

ie $X_t = \underbrace{(\Omega - A'')^{-1} A'}_{A} X_{t+1} + \underbrace{(\Omega - A'')^{-1} b'}_{b} \Rightarrow X_t = A X_{t+1} + b$ (1,02)

where

$$X_t = \begin{pmatrix} f_N^t \\ \vdots \\ f_i^t \\ \vdots \\ f_0^t \end{pmatrix}, \quad X_{t+1} = \begin{pmatrix} f_N^{t+1} \\ \vdots \\ f_i^t \\ \vdots \\ f_0^{t+1} \end{pmatrix}, \quad b' = \begin{pmatrix} \Delta d_t \\ \vdots \\ \vdots \\ \Delta d_t \end{pmatrix}$$

$$A' = \begin{pmatrix} e_N & c_N & a_N & 0 & \cdots & 0 \\ e_{N-1} & a_{N-1} & d_{N-1} & & & \\ & \ddots & & \ddots & & \\ & & e_1 & a_1 & d_1 & \\ 0 & \cdots & & 0 & a_0 & c_0 & e_0 \end{pmatrix}$$

$$A'' = \begin{pmatrix} d_N & b_N & 0 & 0 & \cdots & 0 \\ b_{N-1} & 0 & c_{N-1} & & & \\ & \ddots & & \ddots & & \\ 0 & \cdots & & 0 & b_1 & 0 & c_1 \\ & & & & 0 & b_0 & d_0 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} \Omega_N & & & \\ & \Omega & & \\ & & \ddots & \\ & & & \Omega_0 \end{pmatrix}$$