Consider the PDE

$$(1) \quad \frac{\partial f}{\partial t} = a(x,t)\frac{\partial^2 f}{\partial x^2} + b(x,t)\frac{\partial f}{\partial x} + c(x,t)f + d(x,t)$$

This can be solved using the Crank-Nicholson scheme:

$$(2) \quad \frac{f_i^{n+1} - f_i^n}{dt} = \frac{1}{2}L_i^n + \frac{1}{2}L_i^{n+1} \quad \text{or} \quad f_i^n := f(x_i, t_n), \quad i \in [\![1, N-1]\!].$$

and $L_i^n := a(x_i,t_n)\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{dx^2} + b(x_i,t_n)\frac{f_{i+1}^n - f_{i-1}^n}{2dx} + c(x_i,t_n)f_i^n + d(x_i,t_n)$

$$(2) \iff f_i^n + \frac{1}{2}L_i^n dt = f_i^{n+1} - \frac{1}{2}L_i^{n+1}dt$$

Denote $\partial_1 = \frac{dt}{dx^2}$ $\quad \partial_2 = \frac{dt}{dx}$.

$$\text{Then } L_i^n dt = a_i^n \partial_1 (f_{i+1}^n - 2f_i^n + f_{i-1}^n) + b_i^n \frac{\partial_2}{2}(f_{i+1}^n - f_{i-1}^n) + c_i^n f_i^n dt + d_i^n dt.$$

$$= f_{i+1}^n\left(a_i^n\partial_1 + \frac{1}{2}b_i^n\partial_2\right) + f_i^n\left(c_i^n dt - 2a_i^n\partial_1\right)$$

$$+ f_{i-1}^n\left(a_i^n\partial_1 - \frac{1}{2}b_i^n\partial_2\right) + d_i^n dt$$

$$\Rightarrow \frac{1}{2}L_i^n dt = C_i^n f_{i+1}^n + B_i^n f_i^n + A_i^n f_{i-1}^n + D_i^n$$

or $C_i^n := \frac{1}{2}a_i^n\partial_1 + \frac{1}{4}b_i^n\partial_2$

$B_i^n := \frac{1}{2}c_i^n dt - a_i^n\partial_1$

$A_i^n := \frac{1}{2}a_i^n\partial_1 - \frac{1}{4}b_i^n\partial_2$

$D_i^n := d_i^n dt \frac{1}{2}$.

So $f_i^n + \frac{1}{2} L_i^n \, dt = f_i^{n+1} - \frac{1}{2} L_i^{n+1} \, dt$

$$\Leftrightarrow C_i^n f_{i+1}^n + (B_i^n + 1) f_i^n + A_i^n f_{i-1}^n + D_i^n = -C_i^{n+1} f_{i+1}^{n+1} + (1 - B_i^{n+1}) f_i^{n+1} - A_i^{n+1} f_{i-1}^{n+1} - D_i^{n+1}.$$

$$\Rightarrow \begin{pmatrix} B_1^n+1 & C_1^n & 0 & \cdots & 0 \\ A_2^n & B_2^n+1 & C_2^n & 0 \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & C_{N-2}^n \\ 0 & \cdots & 0 & A_{N-1}^n & B_{N-1}^n+1 \end{pmatrix} \begin{pmatrix} f_1^n \\ \vdots \\ f_{N-1}^n \end{pmatrix} + \begin{pmatrix} f_0^n A_1^n + D_1^n \\ D_2^n \\ \vdots \\ D_{N-2}^n \\ f_N^n C_{N-1}^n + D_{N-1}^n \end{pmatrix}$$

$$= \begin{pmatrix} 1-B_1^{n+1} & -C_1^{n+1} & 0 & \cdots & 0 \\ -A_2^{n+1} & 1-B_2^{n+1} & -C_2^{n+1} & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & -C_{N-2}^{n+1} \\ & & 0 & -A_{N-1}^{n+1} & 1-B_{N-1}^{n+1} \end{pmatrix} \begin{pmatrix} f_1^{n+1} \\ \vdots \\ f_{N-1}^{n+1} \end{pmatrix} + \begin{pmatrix} -A_1^{n+1} f_0^{n+1} - D_1^{n+1} \\ -D_2^n \\ \vdots \\ -D_{N-2}^n \\ -C_{N-1}^{n+1} f_N^{n+1} - D_{N-1}^{n+1} \end{pmatrix}$$

$$\Rightarrow M_1^n f^n + k_1^n = M_2^{n+1} f^{n+1} + k_2^{n+1}$$

where $M_1^n, M_2^{n+1} \in \mathbb{R}^{(N-1)\times(N-1)}$, $f^n, f^{n+1} \in \mathbb{R}^{N-1}$, $k_1^n, k_1^{n+1} \in \mathbb{R}^{N-1}$

$$\Rightarrow f^n = M_1^{-1} \left[ M_2^{n+1} f^{n+1} + k_2^{n+1} - k_1^n \right]$$

We have $N-1$ equations to resolve given that $f_0$ and $f_N \in \mathbb{R}^T$ are fully determined by the boundary conditions.