Project - Maths part

* Theta Scheme outside of the boundaries:

We consider the following PDE:

$$\frac{\partial f}{\partial t}(x,t) = a(x,t)\frac{\partial^2 f}{\partial x^2}(x,t) + b(x,t)\frac{\partial f}{\partial x}(x,t) + c(x,t) f(x,t) + d(x,t),$$

for a, b, c, d some functions $\mathbb{R}\times[0,T]\to\mathbb{R}$. We consider the sets $\{x_i ; i=0,\dots,N\}$ of log-spot values in the mesh, and $\{t_i ; i=0,\dots,T\}$ of time-steps. We have, reusing the notation of the subject: $\forall n\in\{0,\dots,T-1\}$, $\forall i\in\{1,\dots,N-1\}$,

$$L_i^n = a_i^n \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{dx^2} + b_i^n \frac{f_{i+1}^n - f_{i-1}^n}{2dx} + c_i^n f_i^n + d_i^n$$

$$= \xi_i^n f_{i-1}^n + \alpha_i^n f_i^n + \beta_i^n f_{i+1}^n + d_i^n,$$

where:

* $\xi_i^n = \frac{a_i^n}{dx^2} - \frac{b_i^n}{2dx}$, * $\alpha_i^n = c_i^n - \frac{2a_i^n}{dx^2}$, * $\beta_i^n = \frac{a_i^n}{dx^2} + \frac{b_i^n}{2dx}$

The Theta scheme is thus,

$$\frac{f_i^{n+1} - f_i^n}{dt} = \theta L_i^n + (1-\theta) L_i^{n+1}$$

$$\Leftrightarrow f_i^{n+1} - dt(1-\theta) L_i^{n+1} = f_i^n + dt\,\theta L_i^n$$

$$\Leftrightarrow f_i^{n+1}\left[1 - dt(1-\theta)\alpha_i^{n+1}\right] + f_{i-1}^{n+1}\left[-dt(1-\theta)\xi_i^{n+1}\right] + f_{i+1}^{n+1}\left[-dt(1-\theta)\beta_i^{n+1}\right] - dt(1$$

$$= f_i^n\left[1 + dt\,\theta\alpha_i^n\right] + f_{i-1}^n\left[dt\,\theta\xi_i^n\right] + f_{i+1}^n\left[dt\,\theta\beta_i^n\right] + dt\,\theta d_i^n$$

$$\Leftrightarrow A_i^n f_i^{n+1} + B_i^n f_{i+1}^{n+1} + C_i^n f_{i-1}^{n+1} - dt\left(\theta d_i^n + (1-\theta) d_i^{n+1}\right) = \tilde{A}_i^n f_i^n + \tilde{B}_i^n f_{i+1}^n + \tilde{C}_i^n f_{i-1}^n$$

with:

- $A_i^n = 1 - dt(1-\theta)\alpha_i^{n+1}$
- $B_i^n = -dt(1-\theta)\beta_i^{n+1}$
- $C_i^n = -dt(1-\theta)\xi_i^{n+1}$

- $\tilde{A}_i^n = 1 + dt\,\theta\alpha_i^n$
- $\tilde{B}_i^n = dt\,\theta\beta_i^n$
- $C_i^n = dt\,\theta\xi_i^n$

- Theta scheme at the boundaries:

Using the finite differences presented in p. 2 of the subject, we have:

- **At x_0:** $\forall n \in \{0, \dots, T-1\}$,

$$L_0^n = \gamma_0^n f_0^n + \upsilon_0^n f_1^n + \mu_0^n f_2^n + d_0^n,$$

where:

- $\gamma_0^n = \dfrac{a_0^n}{dx^2} - \dfrac{b_0^n}{dx} + c_0^n$
- $\upsilon_0^n = -\dfrac{2a_0^n}{dx^2} + \dfrac{b_0^n}{dx}$
- $\mu_0^n = \dfrac{a_0^n}{dx^2}$

The theta scheme then yields:

$$\frac{f_0^{n+1} - f_0^n}{dt} = \theta L_0^n + (1-\theta) L_0^{n+1}$$

$$-dt(\theta d_0^n + (1-\theta) d_0^{n+1}) + A_0^n f_0^{n+1} + B_0^n f_1^{n+1} + C_0^n f_2^{n+1} = \tilde{A}_0^n f_0^n + \tilde{B}_0^n f_1^n + \tilde{C}_0^n f_2^n$$

where:

- $A_0^n = 1 - dt(1-\theta)\gamma_0^{n+1}$
- $B_0^n = -dt(1-\theta)\upsilon_0^{n+1}$
- $C_0^n = -dt(1-\theta)\mu_0^{n+1}$
- $\tilde{A}_0^n = 1 + dt\,\theta\,\gamma_0^n$
- $\tilde{B}_0^n = dt\,\theta\,\upsilon_0^n$
- $\tilde{C}_0^n = dt\,\theta\,\mu_0^n$

- **At x_N:** $\forall n \in \{0, \dots, T-1\}$:

$$L_N^n = \gamma_N^n f_N^n + \upsilon_N^n f_{N-1}^n + \mu_N^n f_{N-2}^n + d_N^n$$

- $\gamma_N^n = \dfrac{a_N^n}{dx^2} + \dfrac{b_N^n}{dx} + c_N^n$
- $\upsilon_N^n = \dfrac{-2a_N^n}{dx^2} - \dfrac{b_N^n}{dx}$
- $\mu_N^n = \dfrac{a_N^n}{dx^2}$

The Theta Schema then yields:

$$-dt(\theta d_N^n + (1-\theta) d_N^{n+1}) + A_N^n \rho_N^{n+1} + B_N^n \rho_{N-1}^{n+1} + C_N^n \rho_{N-2}^{n+1} = \tilde{A}_N^n \rho_N^n + \tilde{B}_N^n \rho_{N-1}^n + \tilde{C}_N^n \rho_{N-2}^n$$

- $A_N^n = 1 - dt(1-\theta)\gamma_N^{n+1}$
- $B_N^n = -dt(1-\theta)\theta_N^{n+1}$
- $C_N^n = -dt(1-\theta)\mu_N^{n+1}$
- $\tilde{A}_N^n = 1 + dt\,\theta\gamma_N^n$
- $\tilde{B}_N^n = dt\,\theta\,\theta_N^n$
- $\tilde{C}_N^n = dt\,\theta\mu_N^n$

- Final system:

Finally, we have the system $\forall n \in \{0, \dots, T-1\}$,

$$K^n \rho^{n+1} - dt(\theta d^n + (1-\theta) d^{n+1}) = \tilde{K}^n \rho^n \iff \rho^n = \tilde{K}^{n-1} K^n \rho^{n+1} - dt\,\tilde{K}^{n-1}(\theta d^n + (1-\theta) d^{n+1})$$

where, $\rho^n = (\rho_0^n, \dots, \rho_N^n)'$,

$$K^n = \begin{pmatrix} A_0^n & B_0^n & C_0^n & 0 & \cdots & & 0 \\ C_1^n & A_1^n & B_1^n & 0 & & & 0 \\ 0 & C_2^n & A_2^n & B_2^n & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & 0 & C_{N-1}^n & A_{N-1}^n & B_{N-1}^n \\ 0 & \cdots & & 0 & C_N^n & B_N^n & A_N^n \end{pmatrix} \qquad \tilde{K}^n = \begin{pmatrix} \tilde{A}_0^n & \tilde{B}_0^n & \tilde{C}_0^n & 0 & 0 & \cdots & 0 \\ \tilde{C}_1^n & \tilde{A}_1^n & \tilde{B}_1^n & 0 & 0 & \cdots & 0 \\ 0 & \tilde{C}_2^n & A_2^n & B_2^n & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & 0 & \tilde{C}_{N-1}^n & \tilde{A}_{N-1}^n & \tilde{B}_{N-1}^n \\ 0 & & & 0 & \tilde{C}_N^n & \tilde{B}_N^n & \tilde{A}_N^n \end{pmatrix}$$