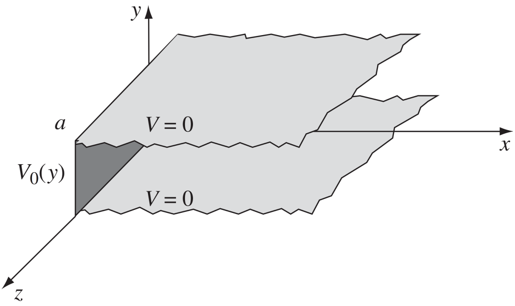
Appendix I: Laplace's Equation

Separation Variable in Spherical Domain. Next we will provide example of Laplace's equation in spherical coordinate, except I'm not gonna do that because it's too hard. We'll just skip to method of image.

Separation Variable in 2D Cartesian. We will first discuss example for two-dimensional Laplace's Equation. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot." The configuration is independent



of z, so this is really a two-dimensional problem. In mathematical terms, we must solve two-dimensional Laplace's equation subject to boundaries:

1. $V = 0$ when $y = 0$,
2. $V = 0$ when $y = a$,
3. $V = V_0(y)$ when $x = 0$,
4. $V \to 0$ as $x \to \text{inf}$.

Using the general solution, condition (4) requires that A equal zero. Absorbing B into C and D, we are left with

$$V(x, y) = e^{-kx}(C \sin ky + D \cos \cos ky)$$

Condition (1) now demands that D equal zero

$$V(x, y) = e^{-kx} C \sin ky$$

Meanwhile (2) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a} \quad n = 1, 2, 3, \ldots$$

The "generaler" solution is therefore

$$V(x, y) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right) \tag{1}$$

With the final boundaries condition (3)

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Using Fourier's Trick, i.e. multiply by $\sin(n'\pi y/a)$ and integrate from 0 to a

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

Where the integrand on the left side is 0 if $n \neq n'$ and $a/2$ if $n = n'$. And the left side of equation reduces to $(a/2)Cn'$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy \tag{2}$$

That does it: 1 is the solution, with coefficients given by 2. As a concrete example, suppose the strip at x = 0 is a metal plate with constant potential V_0. Then

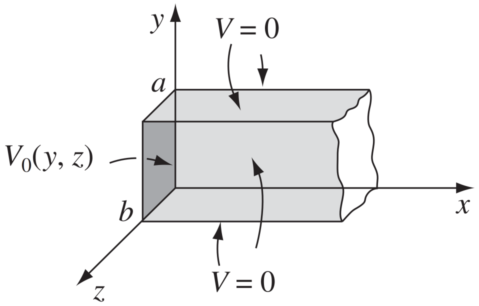
$$C_n = \begin{cases} 0, & \text{n is even} \\ \dfrac{4V_0}{n\pi}, & \text{n is odd} \end{cases}$$

Thus

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \exp\left(-\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)$$

Separation Variable in 3D Cartesian. For the next question, we will discuss three-dimensional Laplace's Equation. For example, an infinitely long rectangular metal pipe (insides a and b) is grounded, but one end, at x = 0, is maintained at a specified potential $V_0(y, z)$. Find the potential inside the pipe. The boundaries conditions are therefore the following

1. $V = 0$ when $y = 0$,
2. $V = 0$ when $y = a$,
3. $V = 0$ when $z = 0$,
4. $V = 0$ when $z = b$,
5. $V \to 0$ as $x \to \infty$,
6. $V = V_0(y, z)$ when $x = 0$



Boundary condition (5) implies $A = 0$, (1) gives $D = 0$, and (3) yields $F = 0$, whereas (2) and (3) require that $k = n\pi/a$ and $l =$

$m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x,y,z) = C\exp\left(-\pi\sqrt{(n/a)^2+(m/b)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

The generaler solution is then

$$V = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\exp\left(-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{n}{b}\right)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) \tag{3}$$

We hope to fit the remaining boundary condition

$$V(0,y,z) = V_0(y,z)$$

or

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\exp\left(-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{n}{b}\right)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) = V_0(y,z)$$

To determine these constants, we multiply by $\sin(n'\pi y/a)\sin(m'\pi z/b)$ and integrate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy \int_0^b \sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m'\pi z}{b}\right)dz = \int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\sin(m'\pi z/b)\;dydz$$

Using Fourier's Trick, the left side is $(ab/4)C_{n,m}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\sin(m'\pi z/b)\;dydz \tag{4}$$

Equation 3, with the coefficients given by Eq. 4, is the solution to our problem.

Appendix II: Method of Image

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. Question: What is the potential in the region above the plane? Trick: Forget about the actual problem; we're going to study a completely different situation. This new configuration consists of two point charges, $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$, and no conducting plane. For this configuration, I can easily write down the potential:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

It follows that:

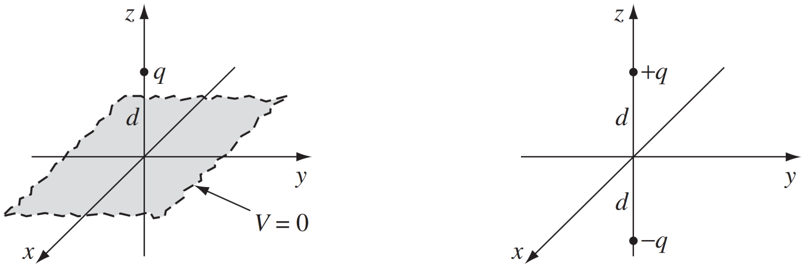
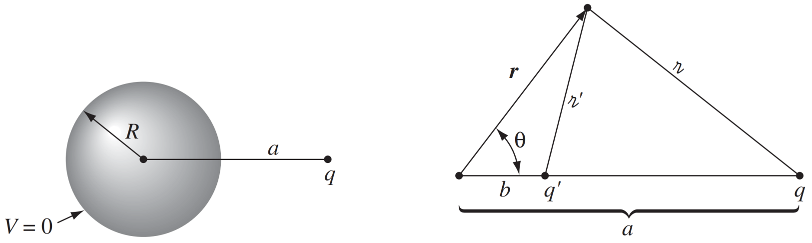


Figure: Charge q above grounded plane.

1. $V = 0$ when $z = 0$, and
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$.

Notice the crucial role played by the uniqueness theorem in this argument: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

Let us try another example. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere. As before, we examine the completely



different configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q$$

placed a distance

$$b = \frac{R^2}{a}$$

to the right of the center of the sphere. No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

where r and r' are the distances from q and q', respectively.