Appendix: Electric Field

Discreet charges. Find the electric field a distance z above the midpoint between two equal charges q, a distance d apart. First we need to determine the separation vector. Let's say the right one is q_1 while the left one is q_2. Thus, for q_1

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}_1 &= \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}}_1^2 &= z^2 + (d/2)^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|} = \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}}
\end{aligned}
$$

and for q_2

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}_1 &= z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}}_1^2 &= z^2 + (d/2)^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}}
\end{aligned}
$$

Applying superposition theorem,

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\boldsymbol{\mathscr{r}}_i^2} \hat{\boldsymbol{\mathscr{r}}} \\
&= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{z^2 + (d/2)^2} \frac{z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}} - \frac{q}{z^2 + (d/2)^2} \frac{z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{2qz}{(z^2 + (d/2)^2)^{3/2}} \hat{\mathbf{z}}
\end{aligned}
$$

Continuous line charge. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ. As always, we will find the separation vector first

$$
\begin{aligned}
\boldsymbol{\mathscr{r}} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
|\boldsymbol{\mathscr{r}}|^2 &= z^2 + x^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{(z^2 + x^2)^{1/2}}
\end{aligned}
$$

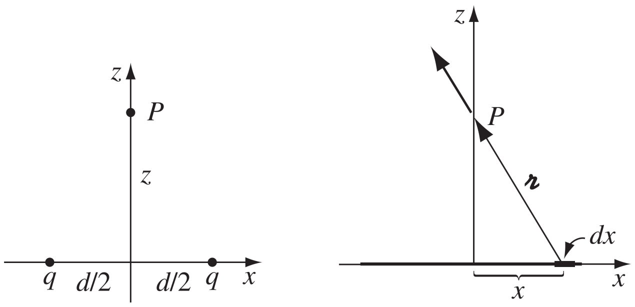


Figure: Discreet charges and continuous line charge

Thus, the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda}{(z^2+x^2)}\frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{(z^2+x^2)^{1/2}}\,dx \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{(z^2+x^2)^{3/2}}\,dx \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[z\hat{\mathbf{z}}\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{3/2}}-\hat{\mathbf{x}}\int_{-L}^{L}\frac{x}{(z^2+x^2)^{3/2}}\,dx\right]
\end{aligned}$$

The first integral

$$I_1 = z\hat{\mathbf{z}}\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{3/2}}$$

can be easily solved using trig substitution. Substituting

$$\tan\theta = \frac{x}{z}$$

solving for x and dx

$$x = z\tan\theta \qquad dx = z\sec^2\theta\;d\theta$$

Based on the substitution, we also get

$$\begin{aligned}
\sec\theta &= \frac{(z^2+x^2)^{1/2}}{z} \\
(z^2+x^2)^{3/2} &= z^3\sec^3\theta
\end{aligned}$$

and

$$\sin\theta = \frac{x}{(z^2+x^2)^{1/2}}$$

We finally get all the equation we need

$$\begin{aligned}
I_1 &= z\hat{\mathbf{z}}\int_{-L}^{L}\frac{z\sec^2\theta\;d\theta}{z^3\sec^3\theta} \\
&= \frac{\hat{\mathbf{z}}}{z}\int_{-L}^{L}\cos\theta\;d\theta \\
&= \frac{\hat{\mathbf{z}}}{z}\sin\theta\bigg|_{-L}^{L} \\
&= \frac{\hat{\mathbf{z}}}{z}\frac{x}{(z^2+x^2)^{1/2}}\bigg|_{-L}^{L} \\
I_1 &= 2\frac{L}{z(z^2+L^2)^{1/2}}\hat{\mathbf{z}}
\end{aligned}$$

For second integral, simple u-sub is enough

$$\begin{aligned}
u &= z^2+x^2 \\
du &= 2x\;dx
\end{aligned}$$

then

$$\begin{aligned}
I_2 &= -\hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2+x^2)^{3/2}}\, dx \\
&= \int_{-L}^{L} u^{-3/2}\, du \\
&= \hat{\mathbf{x}} \frac{1}{(z^2+x^2)^{1/2}} \Bigg|_{-L}^{L} \\
I_2 &= 0
\end{aligned}$$

Substituting back

$$\mathbf{E} = 2\frac{\lambda}{4\pi\epsilon_0} \frac{L}{z(z^2+L^2)^{1/2}} \hat{\mathbf{z}}$$

For $L \to \infty$ carries

$$\mathbf{E}_{L\to\infty} = \frac{\lambda}{2\pi\epsilon_0} \hat{\mathbf{z}}$$

Appendix: PR Listrik Magnet 27 Agustus 2024

Soal 1. Vector pemisahan $\boldsymbol{\mathscr{r}}$ (Gambar 1) dari titik sumber r' ke titik medan P adalah

$$\boldsymbol{\mathscr{r}} = \mathbf{P} - \mathbf{r}' = z\hat{\boldsymbol{z}} - (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}) = z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}$$

sehingga nilai kuadrat dan vektor satuan adalah

$$\begin{aligned}
\boldsymbol{\mathscr{r}}^2 &= \boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}} = x^2 + y^2 + z^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{1/2}}
\end{aligned}$$

Selanjutnya, nilai $\mathbf{E}$ akibat lempengan persegi adalah

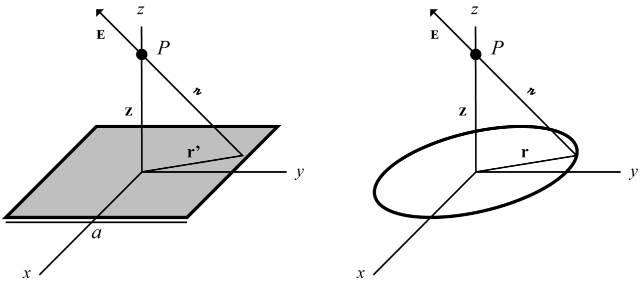
$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{A}} \frac{\sigma}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}}\, da \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{\mathcal{A}} \frac{1}{(x^2+y^2+z^2)} \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{1/2}}\, da \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{3/2}}\, dxdy
\end{aligned}$$

Karena medan listrik komponen x dan y saling membatalkan, maka suku $-x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}$ dapat dihilangkan. Sehingga, integral dapat disederhanakan menjadi

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}}\, dxdy$$

Melihat tabel integral, integral pertama adalah

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \left(\frac{z\hat{\boldsymbol{z}}\, x}{(y^2+z^2)(x^2+y^2+z^2)^{1/2}} \Bigg|_{-a/2}^{a/2} \right) dy \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}}\, a}{(y^2+z^2)(a^2/4+y^2+z^2)^{1/2}}\, dy
\end{aligned}$$



Gambar 1: Medan listrik akibat lempengan persegi (kiri) dan cincin lingkaran (kanan)

Menggunkan komputer, integral dapat dievaluasi menjadi

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \arctan\left(\frac{ay}{z(a^2 + 4y^2 + z^2)^{1/2}} \right) \Bigg|_{-a/2}^{a/2}$$

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \left[\arctan\left(\frac{a^2}{2z(a^2 + a^2 + z^2)^{1/2}} \right) - \arctan\left(- \frac{a^2}{2z(a^2 + a^2 + z^2)^{1/2}} \right) \right]$$

Karena arctan merupakan fungsi ganjil, maka $\arctan -x = -\arctan x$. Dengan demikian

$$\begin{aligned} \mathbf{E} &= \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \cdot 2\arctan\left(\frac{a^2}{2z(2a^2 + z^2)^{1/2}} \right) \\ &= \frac{\sigma}{\pi\epsilon_0} \arctan\left(\frac{a^2}{2z(2a^2 + z^2)^{1/2}} \right) \hat{\boldsymbol{z}} \end{aligned}$$

Sebagai cek, limit ketika $a \to \infty$ adalah bidang menjadi tak hingga dengan besar medan listrik $\mathbf{E} = \sigma/(2\epsilon_0)$. Persamaan tersebut menunjukan

$$\begin{aligned} \mathbf{E}_{a\to\infty} &= \lim_{a\to\infty} \frac{\sigma}{\pi\epsilon_0} \arctan\left(\frac{a^2}{2z(2a^2 + z^2)^{1/2}} \right) \hat{\boldsymbol{z}} \\ &= \frac{\sigma}{\pi\epsilon_0} \frac{\pi}{2} \\ \mathbf{E}_{a\to\infty} &= \frac{\sigma}{2\epsilon_0} \end{aligned}$$

Sesuai dengan medan listik oleh bidang tak hingga.

Soal 2. Vector pemisahan $\boldsymbol{\mathscr{r}}$ (Gambar 1) dari titik sumber r ke titik medan P adalah

$$\boldsymbol{\mathscr{r}} = \mathbf{P} - \mathbf{r} = z\hat{\boldsymbol{z}} - (r\hat{\boldsymbol{r}} + \theta\hat{\boldsymbol{\theta}}) = z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}$$

dalam koordinat tabung. Sehingga nilai kuadrat dan vektor satuan adalah

$$|\boldsymbol{\mathscr{r}}^2| = \boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}} = r^2 + z^2$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}}{(r^2 + z^2)^{1/2}}$$

Perpindahan dl dalam koordinat tabung adalah $dl = r\ dr + r\theta\ d\theta + z\ dz$. Karena integrasi akan dilakukan sepanjang keliling lingkaran, r dan z adalah konstan. Sehingga, $dl = r\theta\ d\theta$. Maka medan listrik akibat cincin lingkaran adalah:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_l \frac{\lambda}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}}\ dl \\
&= \frac{1}{4\pi\epsilon_0} \oint \frac{1}{(r^2 + z^2)} \frac{z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}}{(r^2 + z^2)^{1/2}} \lambda r\ d\theta \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \oint \frac{\lambda r (z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}})}{(r^2 + z^2)^{3/2}}\ d\theta
\end{aligned}$$

Karena medan listrik komponen r dan θ saling membatalkan, maka suku $-r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}$ dapat dihilangkan. Sehingga, integral dapat disederhanakan menjadi

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \oint \frac{\lambda r\ z\hat{\boldsymbol{z}}}{(r^2 + z^2)^{3/2}}\ d\theta$$

Melihat tabel integral, integral adalah

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \lambda r\ z\hat{\boldsymbol{z}} \frac{\theta}{(r^2 + z^2)^{3/2}} \Bigg|_0^{2\pi} \\
\mathbf{E} &= \frac{z\hat{\boldsymbol{z}}}{4\pi\epsilon_0} \frac{\lambda 2\pi r}{(r^2 + z^2)^{3/2}}
\end{aligned}$$

Mengingat bahwa $\lambda 2\pi r = Q$, dimana Q adalah muatan total; maka

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Qz}{(r^2 + z^2)^{3/2}} \hat{\boldsymbol{z}}$$