Electrodynamics

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with $\LaTeX$

March 23, 2025

Electromagnetism

Vector Analysis

Vector Operation

There are four vector operation: Addition, Multiplication by a scalar, Dot product, and Cross Product. (i) Addition of two vectors. Addition is commutative and associative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

(ii) Multiplication by a scalar. Scalar multiplication is distributive.

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$$

(iii) Dot product of two vectors. The dot product of two vectors is defined

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos\theta$$

where θ is the angle they form. Note that dot product is commutative and distributive.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$
$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta \, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The cross product is distributive, but not commutative.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$
$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$$

Few rule for manipulating vector. (i): To add vectors, add like components

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

(ii): To multiply by a scalar, multiply each component.

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$$

Rule (iii): To calculate the dot product, multiply like components, and add.

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Rule (iv): To calculate the cross product, form the determinant whose first row is unit vector, whose second row is A (in component form), and whose third row is B.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Triple Product

(i) Scalar triple product.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = B\cdot(\mathbf{C}\times A) = C\cdot(\mathbf{A}\times B)$$

They are cyclic and in component form

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector triple product.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$$

The product is linear combination of vector in parentheses.

Separation Vector

Separation vector defined as vector from the source point $\bar{r'}$ to the field point $\bar{r}$

$$\boldsymbol{\mathscr{r}} \equiv \bar{\mathbf{r}} - \bar{\mathbf{r}'}.$$

Del Operator

Vector operator defined as follows.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Operation Involving Del Operator

There are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);

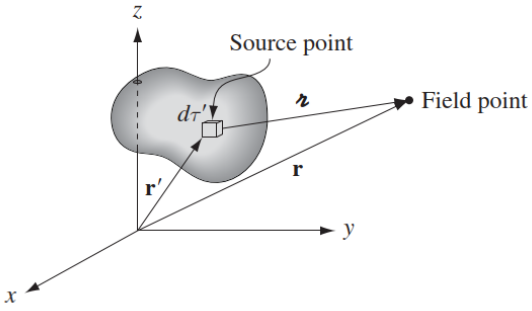

Figure: Separation vector

2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the divergence);

3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the curl).

Gradient of scalar function $T(x, y, z)$

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

can be used to define partial derivative of T

$$\begin{aligned} dT &= \left(\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= \nabla T \cdot \hat{\mathbf{u}} \end{aligned}$$

Note that ∇T is a vector quantity, with three components. The gradient ∇T points in the direction of maximum increase of the function T. Moreover, The magnitude ∇T gives the slope (rate of increase) along this maximal direction.

Divergence of vector function $\mathbf{V}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is a scalar. Divergence is a measure of how much the vector V spreads out (diverges) from the point in question.

Curl of vector function $\mathbf{V}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

The name curl is also well-chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.

Product Rule

There are two ways to construct a scalar as the product of two functions

$$\begin{aligned} fg &\quad \text{(product of two scalar functions)} \\ \mathbf{A} \cdot \mathbf{B} &\quad \text{(dot product of two vector functions)} \end{aligned}$$

and two ways to make a vector

$$\begin{aligned} f\mathbf{A} &\quad \text{(scalar times vector)} \\ \mathbf{A} \times \mathbf{B} &\quad \text{(cross product of two vectors)} \end{aligned}$$

Accordingly, there are six product rule, two for gradients

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

two for divergences

$$\nabla(f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

and two for curls

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivative

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(1) Divergence of gradient: $\nabla \cdot (\nabla T)$. Called Laplacian of T. Notice that the Laplacian of a scalar T is a scalar.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y} + \frac{\partial^2 T}{\partial z}$$

Occasionally, we shall speak of the Laplacian of a vector, $\nabla^2 \mathbf{v}$. By this we mean a vector quantity whose x-component is the Laplacian of V_x, and so on:

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 V_x)\hat{\mathbf{x}} + (\nabla^2 V_y)\hat{y} + (\nabla^2 V_z)\hat{\mathbf{z}}$$

(2) The curl of a gradient: $\nabla \times (\nabla T)$. Always zero.

$$\nabla \cdot (\nabla T) = 0$$

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$. $\nabla(\nabla \cdot \mathbf{v})$ is not the same as the Laplacian of a vector.

$$\nabla(\nabla \cdot \mathbf{v}) \neq \nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v}$$

(4) The divergence of a curl: $\nabla \cdot (\nabla \times \mathbf{v})$. Always zero.

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

(5) Curl of curl:$\nabla \times (\nabla \times \mathbf{v})$. From the definition of ∇,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Fundamental Theorem of Calculus

The fundamental theorem of calculus says the integral of a derivative over some region is given by the value of the function at the end points (boundaries).

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Gradient. The fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (line integral) of a derivative (gradient) is given by the value of the function at the boundaries (a and b).

$$\int_a^b (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Corollary 1: $\int_a^b (\nabla T) \cdot d\mathbf{l}$ is independent of the path.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$ since the beginning and end points are identical.

Divergences . Like the other "fundamental theorems," it says that the integral of a derivative (divergence) over a region (volume V) is equal to thevalue of the function at the boundary (surface S).

$$\int_V (\nabla \cdot \mathbf{v}) \, d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

If v represents the flow of an incompressible fluid, then the flux of **v** is the total amount of fluid passing out through the surface, per unit time. There are two ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. Alternatively,

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Curl. As always, the integral of a derivative (curl) over a region (patch of surface, S) is equal to the value of the function at the boundary (perimeter of the patch, P). Now, the integral of the curl over some surface (flux of the curl) represents the "total amount of swirl," and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Corollary 1. $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line. It doesn't matter which way you go as long as you are consistent. For a closed surface (divergence theorem), $d\mathbf{a}$ points in the direction of the outward normal; but for an open surface is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$.

Corollary 2. $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point.

Integration by Parts

It applies to the situation in which you are called upon to integrate the product of one function (f) and the derivative of another (g); it says you can transfer the derivative from g to f, at the cost of a minus sign and a boundary term.

$$\int_a^b f\left(\frac{dg}{dx}\right) dx = -\int_a^b g\left(\frac{df}{dx}\right) dx + fg\Big|_a^b$$

Curvilinear Coordinates

I shall use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. Infinitesimal displacement vector can be written

$$d\mathbf{l} = f\ du\ \hat{\mathbf{u}} + g\ dv\ \hat{\mathbf{v}} + h\ dw\ \hat{\mathbf{w}}$$

where f, g, and h are functions of position characteristic of the particular coordinate system. While infinitesimal volume is

$$d\tau = fgh\ du\ dv\ dw$$

Use table 1 for references.

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	r $\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Table 1

Gradient. The gradient of t is

$$\nabla \mathbf{t} \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$

Divergence. The divergence of $\mathbf{A}$ in curvilinear coordinates:

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$

Curl.

$$\nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}$$

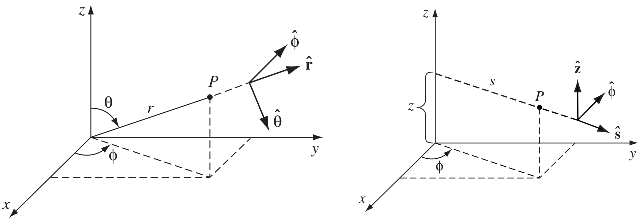
Laplacian.

$$\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]$$

Spherical.

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\ \hat{\mathbf{r}} + \cos\theta\cos\phi\ \hat{\boldsymbol{\theta}} - \sin\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\ \hat{\mathbf{r}} + \cos\theta\sin\phi\ \hat{\boldsymbol{\theta}} + \cos\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\sqrt{x^2+y^2}/z \\ \phi &= \arctan y/z \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\boldsymbol{y}} + \cos\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\ \hat{\mathbf{x}} + \cos\theta\sin\phi\ \hat{\boldsymbol{y}} - \sin\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\boldsymbol{y}} \end{cases}$$



Spherical Coordinates and Cylindrical Coordinates

Cylindrical.

$$\begin{cases} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan y/z \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

Dirac Delta

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1. That is to say

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

It follows that

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Since the product is zero anyway except at $x = a$, we may as well replace $f(x)$ by the value it assumes at the origin. In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

It's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx$$

It is easy to generalize the delta function to three dimensions

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

with $\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, and it's integral

$$\int_{\text{all space}} \delta^3(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dxdydz = 1$$

Generalizing Delta function, we get

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r\text{-}a})\, d\tau = f(\mathbf{a})$$

Few Dirac delta function

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\boldsymbol{\mathscr{r}}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$

$$\nabla^2\frac{1}{\boldsymbol{\mathscr{r}}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Fourier Transform of a δ function. Using the definition of a Fourier transform, we write

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\alpha x}\, dx = \frac{1}{2\pi}e^{-i\alpha x}$$

and its inverse transform

$$\delta(x-a) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-a)}\, d\alpha$$

The integral however does not converge. If we replace the limits by $-n, n$, we obtain a set of functions which are increasingly peaked around $x = a$ as n increases, but all have area 1.

Derivative of a δ function. Using repeated integrations by parts gives

$$\int_{-\infty}^{\infty} \phi(x)\delta^{(n)}(x-a)\, dx = (-1)^n\phi^{(n)}(a)$$

Few formulas involving δ function. For step function

$$u(x-a) = \begin{cases} 1, x > a \\ 0, x < a \end{cases}$$

$$u'(x-a) = \delta(x-a)$$

It is easy to see how the derivative of step function is equal to delta function.

Helmholtz Theorem

Suppose we are told that the divergence of a vector function F(r) is a specified scalar function D(r):

$$\nabla \cdot \mathbf{F} = D$$

and the curl of F(r) is a specified vector function C(r):

$$\nabla \times \mathbf{F} = \mathbf{C}$$

For consistency, C must be divergenceless $\nabla \cdot \mathbf{C} = 0$. Helmholtz theorem state if the divergence D(r) and the curl C(r) of a vector function F(r) are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$ and if F(r) goes to zero as $r \to \infty$, then F is given uniquely by

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$$

Potential Theorem

Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, F satisfies one if and only if it satisfies all the others):

- $\nabla \times \mathbf{F} = 0$ everywhere.
- $\int_a^b \mathbf{F} \cdot \; d\mathbf{l}$ is independent of path, for any given end points.
- $\oint \mathbf{F} \cdot \; d\mathbf{l} = 0$ for any closed loop.
- $\mathbf{F}$ is the gradient of some scalar function: $F = -\nabla V$.

Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

- $\nabla \cdot \mathbf{F} = 0$ everywhere.
- $\int \mathbf{F} \cdot \; d\mathbf{a}$ is independent of surface, for any given boundary line.
- $\oint \mathbf{F} \cdot \; d\mathbf{a} = 0$ for any closed surface.
- $\mathbf{F}$ is the curl of some scalar function: $\mathbf{F} = -\nabla \mathbf{A}$.

Electrostatics

Electric Field

Coulomb's Law. The force on a test charge Q due to a single point charge q is given by Coulomb's law

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q\,Q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

The constant ϵ_0 is called (ludicrously) the permittivity of free space. In SI units, where force is in newtons (N), distance in meters (m), and charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{Nm}^2}$$

Electric Field. Total force on Q can be written as

$$\mathbf{F} = Q\mathbf{E}$$

where

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}$$

Continuous Charge Distributions. If, instead, the charge is distributed continuously over some region, the sum becomes an integral

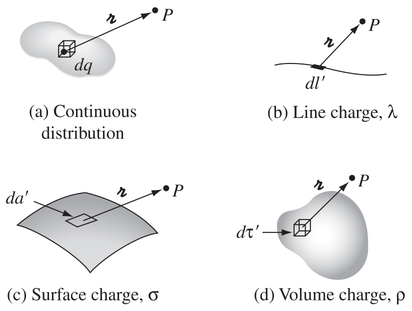
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{1}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}\,dq$$

If the charge is spread out along a line, then $dq = \lambda\, dl'$; if the charge is smeared out over a surface, then $dq = \sigma\, da'$; and if the charge fills a volume, then $dq = \rho\, d\tau'$:

$$dq \to \lambda\, dl' \sim \sigma\, da' \sim \rho\, d\tau'$$

Thus the electric field of a line charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}\,dl'$$



Charge Distribution

for a surface charge

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\boldsymbol{\mathscr{r}}_i^2}\hat{\boldsymbol{\mathscr{r}}}\, da'$$

and for a volume charge

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\boldsymbol{\mathscr{r}}_i^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

Gauss's law. The flux of **E** through a surface S

$$\Phi_E \equiv \int_S \mathbf{E}\cdot d\mathbf{a}$$

is a measure of the "number of field lines" passing through S. The flux through any closed surface is a measure of the total charge inside. This is the essence of Gauss's law.

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{enc}$$

applying the divergence theorem

$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{E})\, d\tau = \int_{\mathcal{V}} \frac{\rho}{\epsilon_0}\, d\tau$$

And since this holds for any volume, the integrands must be equal

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$$

Divergence of E. Gauss's law state the divergence of **E** in differential form

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$$

Curl of E. The integral of **E** around a closed path is zero

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

and hence, applying Stokes' theorem

$$\nabla\times\mathbf{E} = 0$$

Potential

Introduction to Potential. Because the line integral of **E** is independent of path, we can define a function

$$V(r) \equiv -\int_{\mathcal{O}}^{r} \mathbf{E}(\mathbf{r}')\cdot d\mathbf{l}'$$

The potential difference between two points a and b is

$$V(b) - V(a) = -\int_a^b \mathbf{E}(\mathbf{r}')\cdot d\mathbf{l}'$$

Applying fundamental theorem for gradients, we get

$$\mathbf{E} = -\nabla V$$

Electric field **E** will point from high potential to low potential.

Poisson's Equation and Laplace's Equation Poisson's Equation state the divergence of E in terms of volume. Since $\mathbf{E} = -\nabla V$, divergence of E in terms of V

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This is known as Poisson's equation. In regions where there is no charge, so $\rho = 0$, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$

The Potential of a Localized Charge Distribution. In general, the potential of a collection of charges is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\boldsymbol{\mathscr{r}}_i}$$

or, for a continuous distribution

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\boldsymbol{\mathscr{r}}} dq$$

and, for a volume charge, it's

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\boldsymbol{\mathscr{r}}} d\tau'$$

Boundary Conditions. We have, in the course of our discussion, discussed the three fundamental quantities of electrostatics: ρ, $\mathbf{E}$, and V and derived all six formulas interrelating them.

Electric field always undergoes a discontinuity when you cross a surface charge σ The normal component of $\mathbf{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary. In particular, where there is no surface charge, perpendicular electric $E^{\perp}$ field is continuous, as for instance at the surface of a uniformly charged solid sphere.

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

The tangential component of $\mathbf{E}$ ($E^{||}$), by contrast, is always continuous.

$$E^{||}_{\text{above}} = E^{||}_{\text{below}}$$

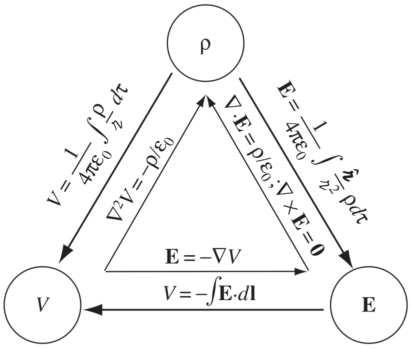


Figure: Electrostatics Holy Trinity

The equation then can be summarized by

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

In terms of potential,

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

or

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

denotes the normal derivative of V (that is, the rate of change in the direction perpendicular to the surface). Meanwhile, potential is continuous across any boundary $V_{\text{above}} = V_{\text{below}}$

Work

The Work It Takes to Move a Charge. The work you do to move a test charge Q from point a to point b is

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{l} = -Q\int_a^b \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

Work also defined as the difference in potential energy U of system

$$W = \Delta U$$

If you have set the reference point (point a) at infinity, therefore

$$W = QV(\mathbf{r})$$

In this sense, potential V is potential energy W (the work it takes to create the system) per unit charge Q, just as the field is the force per unit charge ($\mathbf{E} = \mathbf{F}/Q$).

The Energy of a Point Charge Distribution. The work it would take to assemble an entire collection of point charges is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

where $V(\mathbf{r}_i)$ is potential at point $\mathbf{r}_i$

The Energy of a Continuous Charge Distribution. For a volume charge density,

$$W = \frac{1}{2}\int \rho V \, d\tau$$

which can be written using Gauss's law and integration by parts as

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2 \, d\tau + \oint V\mathbf{E} \cdot d\mathbf{a}\right)$$

Integration can be done over whatever volume you use (as long as it encloses all the charge), but the contribution from the volume integral goes up, and that of the surface integral goes down, as you take larger and larger volumes. In particular, why not integrate over all space? Thus,

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau \quad \text{Over all Space}$$

Where is the energy stored? In the context of radiation theory it is useful (and in general relativity it is essential) to regard the energy as stored in the field, with a density

$$U = \frac{\epsilon_0}{2}E^2$$

But in electrostatics one could just as well say it is stored in the charge, with a density $\frac{1}{2}\rho V$.

Conductor

Basic Properties. A perfect conductor would contain an unlimited supply of free charges. From this definition, the basic electrostatic properties of ideal conductors immediately follow

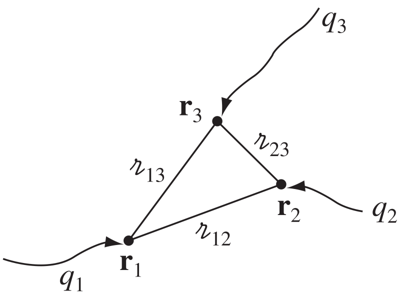
1. $\mathbf{E} = 0$ inside a conductor
2. $\rho = 0$ inside a conductor
3. Any net charge resides on the surface
4. A conductor is an equipotential
5. $\mathbf{E}$ is perpendicular to the surface, just outside a conductor

Surface Charge and the Force on a Conductor. Because the field inside a conductor is zero, the field immediately outside is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Applying boundary condition, the surface charge on a conductor in terms of potential is

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$



Point Charges

In the presence of an electric field, a surface charge will experience a force; the force per unit area, $\mathbf{f}$, is $\sigma\mathbf{E}$

$$\mathbf{f} = \sigma\mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}})$$

In the case of a conductor, the field is zero inside. Therefore,

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$$

This amounts to an outward electrostatic pressure on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field just outside the surface

$$P = \frac{\epsilon_0}{2}E^2$$

Capacitor. Since $\mathbf{E}$ is proportional to Q, so also is V. The constant of proportionality is called the capacitance of the arrangement:

$$C \equiv \frac{Q}{V}$$

Appendix: Electric Field

Discreet charges. Find the electric field a distance z above the midpoint between two equal charges q, a distance d apart. First we need to determine the separation vector. Let's say the right one is q_1 while the left one is q_2. Thus, for q_1

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}_1 &= \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}}_1^2 &= z^2 + (d/2)^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}|} = \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}}
\end{aligned}
$$

and for q_2

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}_1 &= z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}} \\
\boldsymbol{\mathscr{r}}_1^2 &= z^2 + (d/2)^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}}
\end{aligned}
$$

Applying superposition theorem,

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \frac{q_i}{\boldsymbol{\mathscr{r}}_i^2} \hat{\boldsymbol{\mathscr{r}}} \\
&= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{z^2 + (d/2)^2} \frac{z\hat{\mathbf{z}} - (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}} - \frac{q}{z^2 + (d/2)^2} \frac{z\hat{\mathbf{z}} + (d/2)\hat{\mathbf{x}}}{\sqrt{z^2 + (d/2)^2}} \right) \\
&= \frac{1}{4\pi\epsilon_0} \frac{2qz}{(z^2 + (d/2)^2)^{3/2}} \hat{\mathbf{z}}
\end{aligned}
$$

Continuous line charge. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ. As always, we will find the separation vector first

$$
\begin{aligned}
\boldsymbol{\mathscr{r}} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}} \\
|\boldsymbol{\mathscr{r}}|^2 &= z^2 + x^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{(z^2 + x^2)^{1/2}}
\end{aligned}
$$

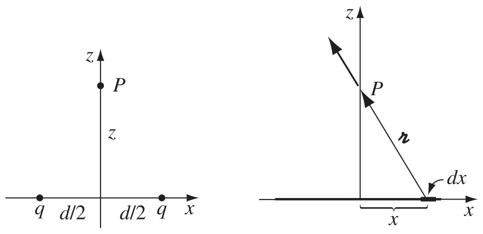


Figure: Discreet charges and continuous line charge

Thus, the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{\lambda}{(z^2+x^2)}\frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{(z^2+x^2)^{1/2}}\,dx \\
&= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{(z^2+x^2)^{3/2}}\,dx \\
&= \frac{\lambda}{4\pi\epsilon_0}\left[z\hat{\mathbf{z}}\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{3/2}}-\hat{\mathbf{x}}\int_{-L}^{L}\frac{x}{(z^2+x^2)^{3/2}}\,dx\right]
\end{aligned}$$

The first integral

$$I_1 = z\hat{\mathbf{z}}\int_{-L}^{L}\frac{dx}{(z^2+x^2)^{3/2}}$$

can be easily solved using trig substitution. Substituting

$$\tan\theta = \frac{x}{z}$$

solving for x and dx

$$x = z\tan\theta \qquad dx = z\sec^2\theta\;d\theta$$

Based on the substitution, we also get

$$\begin{aligned}
\sec\theta &= \frac{(z^2+x^2)^{1/2}}{z} \\
(z^2+x^2)^{3/2} &= z^3\sec^3\theta
\end{aligned}$$

and

$$\sin\theta = \frac{x}{(z^2+x^2)^{1/2}}$$

We finally get all the equation we need

$$\begin{aligned}
I_1 &= z\hat{\mathbf{z}}\int_{-L}^{L}\frac{z\sec^2\theta\;d\theta}{z^3\sec^3\theta} \\
&= \frac{\hat{\mathbf{z}}}{z}\int_{-L}^{L}\cos\theta\;d\theta \\
&= \frac{\hat{\mathbf{z}}}{z}\sin\theta\Big|_{-L}^{L} \\
&= \frac{\hat{\mathbf{z}}}{z}\frac{x}{(z^2+x^2)^{1/2}}\Bigg|_{-L}^{L} \\
I_1 &= 2\frac{L}{z(z^2+L^2)^{1/2}}\hat{\mathbf{z}}
\end{aligned}$$

For second integral, simple u-sub is enough

$$\begin{aligned}
u &= z^2+x^2 \\
du &= 2x\;dx
\end{aligned}$$

then

$$
\begin{aligned}
I_2 &= -\hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2+x^2)^{3/2}}\, dx \\
&= \int_{-L}^{L} u^{-3/2}\, du \\
&= \hat{\mathbf{x}} \frac{1}{(z^2+x^2)^{1/2}} \Bigg|_{-L}^{L} \\
I_2 &= 0
\end{aligned}
$$

Substituting back

$$
\mathbf{E} = 2\frac{\lambda}{4\pi\epsilon_0} \frac{L}{z(z^2+L^2)^{1/2}} \hat{\mathbf{z}}
$$

For $L \to \infty$ carries

$$
\mathbf{E}_{L\to\infty} = \frac{\lambda}{2\pi\epsilon_0} \hat{\mathbf{z}}
$$

Appendix: PR Listrik Magnet 27 Agustus 2024

Soal 1. Vector pemisahan $\boldsymbol{\mathscr{r}}$ (Gambar 1) dari titik sumber r' ke titik medan P adalah

$$
\boldsymbol{\mathscr{r}} = \mathbf{P} - \mathbf{r}' = z\hat{\boldsymbol{z}} - (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}) = z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}
$$

sehingga nilai kuadrat dan vektor satuan adalah

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}^2 &= \boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}} = x^2 + y^2 + z^2 \\
\hat{\boldsymbol{\mathscr{r}}} &= \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{1/2}}
\end{aligned}
$$

Selanjutnya, nilai $\mathbf{E}$ akibat lempengan persegi adalah

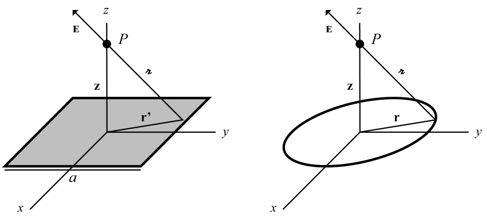
$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{A}} \frac{\sigma}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}}\, da \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{\mathcal{A}} \frac{1}{(x^2+y^2+z^2)} \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{1/2}}\, da \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}} - x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}}{(x^2+y^2+z^2)^{3/2}}\, dxdy
\end{aligned}
$$

Karena medan listrik komponen x dan y saling membatalkan, maka suku $-x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}}$ dapat dihilangkan. Sehingga, integral dapat disederhanakan menjadi

$$
\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}}\, dxdy
$$

Melihat tabel integral, integral pertama adalah

$$
\begin{aligned}
\mathbf{E} &= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \left(\frac{z\hat{\boldsymbol{z}}\, x}{(y^2+z^2)(x^2+y^2+z^2)^{1/2}} \Bigg|_{-a/2}^{a/2} \right) dy \\
&= \frac{\sigma}{4\pi\epsilon_0} \int_{-a/2}^{a/2} \frac{z\hat{\boldsymbol{z}}\, a}{(y^2+z^2)(a^2/4+y^2+z^2)^{1/2}}\, dy
\end{aligned}
$$



Gambar 1: Medan listrik akibat lempengan persegi (kiri) dan cincin lingkaran (kanan)

Menggunkan komputer, integral dapat dievaluasi menjadi

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \arctan\left(\frac{ay}{z(a^2+4y^2+z^2)^{1/2}} \right)\Bigg|_{-a/2}^{a/2}$$

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \left[\arctan\left(\frac{a^2}{2z(a^2+a^2+z^2)^{1/2}} \right) \right.$$
$$\left. - \arctan\left(-\frac{a^2}{2z(a^2+a^2+z^2)^{1/2}} \right) \right]$$

Karena arctan merupakan fungsi ganjil, maka $\arctan -x = -\arctan x$. Dengan demikian

$$\begin{aligned} \mathbf{E} &= \frac{\sigma}{4\pi\epsilon_0} 2\hat{\boldsymbol{z}} \cdot 2 \arctan\left(\frac{a^2}{2z(2a^2+z^2)^{1/2}} \right) \\ &= \frac{\sigma}{\pi\epsilon_0} \arctan\left(\frac{a^2}{2z(2a^2+z^2)^{1/2}} \right) \hat{\boldsymbol{z}} \end{aligned}$$

Sebagai cek, limit ketika $a \to \infty$ adalah bidang menjadi tak hingga dengan besar medan listrik $\mathbf{E} = \sigma/(2\epsilon_0)$. Persamaan tersebut menunjukan

$$\begin{aligned} \mathbf{E}_{a\to\infty} &= \lim_{a\to\infty} \frac{\sigma}{\pi\epsilon_0} \arctan\left(\frac{a^2}{2z(2a^2+z^2)^{1/2}} \right) \hat{\boldsymbol{z}} \\ &= \frac{\sigma}{\pi\epsilon_0} \frac{\pi}{2} \\ \mathbf{E}_{a\to\infty} &= \frac{\sigma}{2\epsilon_0} \end{aligned}$$

Sesuai dengan medan listik oleh bidang tak hingga.

Soal 2. Vector pemisahan $\boldsymbol{\mathscr{r}}$ (Gambar 1) dari titik sumber r ke titik medan P adalah

$$\boldsymbol{\mathscr{r}} = \mathbf{P} - \mathbf{r} = z\hat{\boldsymbol{z}} - (r\hat{\boldsymbol{r}} + \theta\hat{\boldsymbol{\theta}}) = z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}$$

dalam koordinat tabung. Sehingga nilai kuadrat dan vektor satuan adalah

$$|\boldsymbol{\mathscr{r}}^2| = \boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}} = r^2 + z^2$$
$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\sqrt{\boldsymbol{\mathscr{r}}^2}} = \frac{z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}}{(r^2+z^2)^{1/2}}$$

Perpindahan dl dalam koordinat tabung adalah $dl = r\ dr + r\theta\ d\theta + z\ dz$. Karena integrasi akan dilakukan sepanjang keliling lingkaran, r dan z adalah konstan. Sehingga, $dl = r\theta\ d\theta$. Maka medan listrik akibat cincin lingkaran adalah:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_l \frac{\lambda}{\boldsymbol{\mathscr{r}}^2} \hat{\boldsymbol{\mathscr{r}}}\ dl \\
&= \frac{1}{4\pi\epsilon_0} \oint \frac{1}{(r^2+z^2)} \frac{z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}}{(r^2+z^2)^{1/2}} \lambda r\ d\theta \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \oint \frac{\lambda r(z\hat{\boldsymbol{z}} - r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}})}{(r^2+z^2)^{3/2}}\ d\theta
\end{aligned}$$

Karena medan listrik komponen r dan θ saling membatalkan, maka suku $-r\hat{\boldsymbol{r}} - \theta\hat{\boldsymbol{\theta}}$ dapat dihilangkan. Sehingga, integral dapat disederhanakan menjadi

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \oint \frac{\lambda r\ z\hat{\boldsymbol{z}}}{(r^2+z^2)^{3/2}}\ d\theta$$

Melihat tabel integral, integral adalah

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \lambda r\ z\hat{\boldsymbol{z}} \frac{\theta}{(r^2+z^2)^{3/2}} \Bigg|_0^{2\pi} \\
\mathbf{E} &= \frac{z\hat{\boldsymbol{z}}}{4\pi\epsilon_0} \frac{\lambda 2\pi r}{(r^2+z^2)^{3/2}}
\end{aligned}$$

Mengingat bahwa $\lambda 2\pi r = Q$, dimana Q adalah muatan total; maka

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Qz}{(r^2+z^2)^{3/2}} \hat{\boldsymbol{z}}$$

Potential

Laplace's Equation

One Dimension. Suppose V depends on only one variable, x. Then Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} = 0$$

The general solution is

$$V(x) = mx + b$$

the equation for a straight line. The result of this solution are as follows:

1. V (x) is the average of V $(x + a)$ and $V(x - a)$, for any a:

$$V(x) = \frac{1}{2}[V(x + a) + V(x - a)]$$

2. Laplace's equation tolerates no local maxima or minima; extreme values of V must occur at the end points.

Two Dimensions. If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Harmonic functions in two dimensions have the same properties we noted in one dimension

1. The value of V at a point (x, y) is the average of those around the point.

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl$$

2. V has no local maxima or minima; all extrema occur at the boundaries.

Three Dimensions. The same two properties remain true

1. The value of V at point $\mathbf{r}$ is the average value of V over a spherical surface of radius R centered at $\mathbf{r}$:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{sphere}} V \, da$$

2. As a consequence, V can have no local maxima or minima; the extreme values of V must occur at the boundaries.

Uniqueness Theorems

First Theorem. The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

> Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

Second Theorem. In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded.)

Image Method

Any stationary charge distribution near a grounded conducting plane can be treated introducing its mirror image–hence the name method of images.

Separation of Variable

Cartesian. The solution to partial differential equation can be obtained by assuming that the solution is in the form of products of two function. For Laplace equation in two dimension therefore,

$$V(x, y) = X(x)Y(y)$$

It follows that

$$\frac{d^2}{dx^2}X(x) = k^2 X(x) \quad \text{and} \quad \frac{d^2}{dy^2}Y(y) = -k^2 Y(y)$$

Thus

$$X(x) = Ae^{kx} + Be^{-kx} \quad \text{and} \quad Y(y) = C\sin ky + D\cos ky$$

We are left with

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

There are two extraordinary properties of the separable solutions: completeness and orthogonality. A set of functions $f_n(y)$ is said to be complete if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

A set of functions is orthogonal if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0$$

We will now discuss Laplace's Equation in three dimension. As always, we look for solutions that are products:

$$V(x, y) = X(x)Y(y)Z(z)$$

It follows that

$$\frac{d^2}{dx^2}X = (k^2 + l^2)X \quad \frac{d^2}{dy^2}Y = -k^2Y(y) \quad \frac{d^2}{dz^2}Z = -l^2Z(z)$$

The solutions are

$$\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x} \\ Y(y) &= C\sin ky + D\cos\cos ky \\ Z(z) &= E\sin lz + F\cos\cos lz \end{aligned}$$

Spherical Coordinates. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

I shall assume the problem has azimuthal symmetry, so that V is independent of ϕ, which reduces the equation introducing

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

As before, we look for solutions that are products

$$V(r, \theta) = R(r)\Theta(\theta)$$

Putting this into equation and dividing by V,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1), \quad \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -l(l+1)$$

As always, separation of variables has converted a partial differential equation into ordinary differential equations. The radial equation, second order ODE, has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

where A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. The solutions to the angular equation are Legendre polynomials in the variable $\cos\theta$.

$$\Theta(\theta) = P_l(\cos\theta)$$

$P_l(x)$ is most conveniently defined by the Rodrigues formula

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

The first few Legendre polynomials are listed as follows

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2 - 1)/2 \\
P_3(x) &= (5x^3 - 3x)/2 \\
P_4(x) &= (35x^4 - 30x^2 + 3)/8 \\
P_5(x) &= (63x^2 - 70x^3 + 15x)/8
\end{aligned}$$

Notice that $P_l(x)$ is (as the name suggests) an lth-order polynomial in x; it contains only even powers, if l is even, and odd powers, if l is odd. As before, separation of variables yields an infinite set of solutions, one for each l. The general solution is the linear combination of separable solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Multipole Expansion

I propose now to develop a systematic expansion for the potential of any localized charge distribution, in powers of $1/r$. The potential at $\mathbf{r}$ is given by

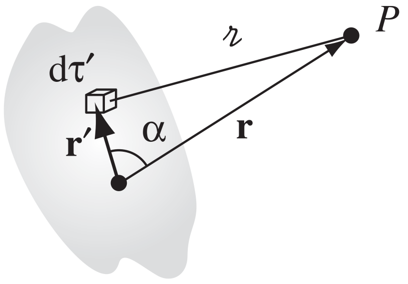
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\mathbf{r}') d\tau'$$

The separation vector can be written in terms of Legendre polynomials

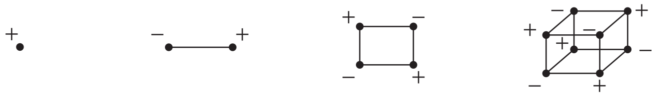
$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right) P_l(\cos\alpha)$$

Substituting this back, and noting that r is a constant, as far as the integration is concerned, I conclude that

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}')\, d\tau'$$



Potential at point P and relevant variables



Monopole, Dipole, Quadrupole, and Octopole

More explicitly

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\, d\tau' + \frac{1}{r^2}\int (r')\cos\alpha\rho(\mathbf{r}')\, d\tau' + \frac{1}{r^3}\int (r')^2(\frac{3}{2}\cos^2\alpha - \frac{1}{2})\rho(\mathbf{r}')\, d\tau' + \ldots\right]$$

This is the desired result–the multipole expansion of V in powers of $1/r$. The first term ($n = 0$) is the monopole contribution (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); the third is quadrupole; the fourth octopole; and so on. Remember that α is the angle between r and r', so the integrals depend on the direction to the field point. If we put together a pair of equal and opposite dipoles to make a quadrupole; back-to-back quadrupoles create an octopole; and so on.

The Monopole and Dipole Terms. Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r},$$

where $\int \rho\, d\tau = Q$. For a point charge at the origin, V_{mon} is the exact potential, not merely a first approximation at large r; in this case, all the higher multipoles vanish. If the total charge is zero, the dominant term in the potential will be the dipole:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\rho(\mathbf{r}')\, d\tau'.$$

Since α is the angle between $\mathbf{r}'$ and $\mathbf{r}$, then $r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$. Therefore, dipole potential can be written more succinctly:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'.$$

This integral (which does not depend on r) is defined the dipole moment of the distribution

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'.$$

Dipole contribution to the potential simplifies to

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Dipole moment translates in the usual way for point, line, and surface charges. Thus, the dipole moment of a collection of point charges is

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

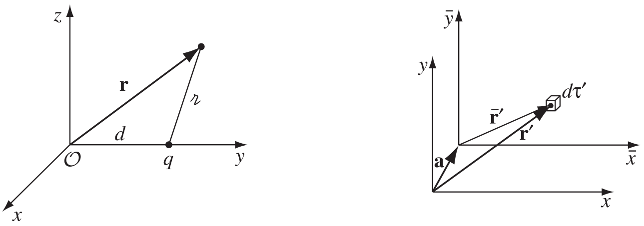


Figure 1: Point charge not at origin and point charge from different origin

For a physical dipole (equal and opposite charges, $\pm q$),

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q\mathbf{d}$$

where $\mathbf{d}$ is the vector from the negative charge to the positive one. However, that this is only the approximate potential of the physical dipole–evidently there are higher multipole contributions. A physical dipole becomes a pure dipole, then, in the rather artificial limit $d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed. Dipole moments are vectors, and they add accordingly.

Origin of Coordinates in Multipole Expansions. A point charge at the origin constitutes a "pure" monopole; if it is not at the origin, it's no longer a pure monopole. Point charge at 1 has a dipole moment $\mathbf{p} = qd\ \hat{\mathbf{y}}$, and a corresponding dipole term in its potential. The monopole potential $(1/4\pi\epsilon_0)q/r$ is not quite correct for this configuration; rather, the exact potential is $(1/4\pi\epsilon_0)q/\boldsymbol{\mathscr{r}}$. The multipole expansion is a series in inverse powers of r (the distance to the origin), and when we expand $1/\boldsymbol{\mathscr{r}}$, we get all powers, not just the first.

So moving the origin (or, what amounts to the same thing, moving the charge) can alter a multipole expansion. Ordinarily, the dipole moment does change when you shift the origin, but there is an important exception: If the total charge is zero, then the dipole moment is independent of the choice of origin.

$$\begin{aligned}
\bar{\mathbf{p}} &= \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' \\
&= \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau' \\
&= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' \\
&= \mathbf{p} - Q\mathbf{a}
\end{aligned}$$

Electric Field of a Dipole. If we choose coordinates so that p is at the origin and points in the z direction, then the field at r, θ is:

$$\mathbf{E}_{\text{dip}}(r,\theta) = -\nabla V_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

This formula makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for p (along z). Notice that the dipole field falls off as the inverse cube of r; the monopole field

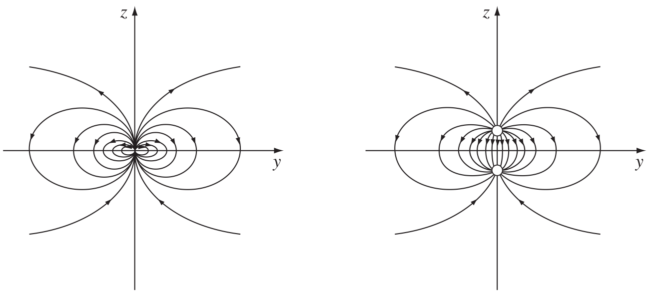


Figure: field of pure and real dipole

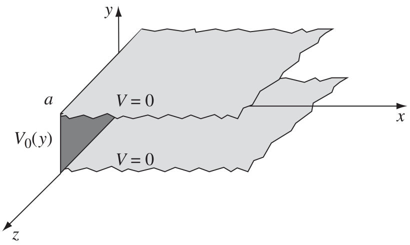
goes as the inverse square, of course. Quadrupole fields go like $1/r^4$, octopole like $1/r^5$, and so on.

Notice how similar the fields become if you blot out the central region; up close, however, they are entirely different. Only for points $r >> d$ does $\mathbf{E}_{\text{dip}}$ represent a valid approximation to the field of a physical dipole. This régime can be reached either by going to large r or by squeezing the charges very close together.

Appendix I: Laplace's Equation

Separation Variable in Spherical Domain. Next we will provide example of Laplace's equation in spherical coordinate, except I'm not gonna do that because it's too hard. We'll just skip to method of image.

Separation Variable in 2D Cartesian. We will first discuss example for two-dimensional Laplace's Equation. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot." The configuration is independent



of z, so this is really a two-dimensional problem. In mathematical terms, we must solve two-dimensional Laplace's equation subject to boundaries:

1. $V = 0$ when $y = 0$,
2. $V = 0$ when $y = a$,
3. $V = V_0(y)$ when $x = 0$,
4. $V \to 0$ as $x \to \inf$.

Using the general solution, condition (4) requires that A equal zero. Absorbing B into C and D, we are left with

$$V(x, y) = e^{-kx}(C \sin ky + D \cos \cos ky)$$

Condition (1) now demands that D equal zero

$$V(x, y) = e^{-kx} C \sin ky$$

Meanwhile (2) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a} \quad n = 1, 2, 3, \ldots$$

The "generaler" solution is therefore

$$V(x, y) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right) \tag{1}$$

With the final boundaries condition (3)

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Using Fourier's Trick, i.e. multiply by $\sin(n'\pi y/a)$ and integrate from 0 to a

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

Where the integrand on the left side is 0 if $n \neq n'$ and $a/2$ if $n = n'$. And the left side of equation reduces to $(a/2)Cn'$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy \tag{2}$$

That does it: 1 is the solution, with coefficients given by 2. As a concrete example, suppose the strip at x = 0 is a metal plate with constant potential V_0. Then

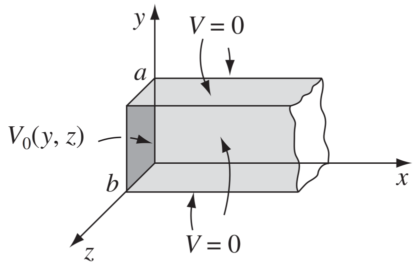
$$C_n = \begin{cases} 0, & \text{n is even} \\ \dfrac{4V_0}{n\pi}, & \text{n is odd} \end{cases}$$

Thus

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \exp\left(-\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right)$$

Separation Variable in 3D Cartesian. For the next question, we will discuss three-dimensional Laplace's Equation. For example, an infinitely long rectangular metal pipe (insides a and b) is grounded, but one end, at x = 0, is maintained at a specified potential $V_0(y,z)$. Find the potential inside the pipe. The boundaries conditions are therefore the following

1. $V = 0$ when $y = 0$,
2. $V = 0$ when $y = a$,
3. $V = 0$ when $z = 0$,
4. $V = 0$ when $z = b$,
5. $V \to 0$ as $x \to \infty$,
6. $V = V_0(y,z)$ when $x = 0$



Boundary condition (5) implies $A = 0$, (1) gives $D = 0$, and (3) yields $F = 0$, whereas (2) and (3) require that $k = n\pi/a$ and $l =$

$m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x,y,z) = C\exp\left(-\pi\sqrt{(n/a)^2+(m/b)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

The generaler solution is then

$$V = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\exp\left(-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{n}{b}\right)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) \tag{3}$$

We hope to fit the remaining boundary condition

$$V(0,y,z) = V_0(y,z)$$

or

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\exp\left(-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{n}{b}\right)^2}x\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) = V_0(y,z)$$

To determine these constants, we multiply by $\sin(n'\pi y/a)\sin(m'\pi z/b)$ and integrate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy \int_0^b \sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m'\pi z}{b}\right)dz = \int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\sin(m'\pi z/b)\,dydz$$

Using Fourier's Trick, the left side is $(ab/4)C_{n,m}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\sin(m'\pi z/b)\,dydz \tag{4}$$

Equation 3, with the coefficients given by Eq. 4, is the solution to our problem.

Appendix II: Method of Image

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. Question: What is the potential in the region above the plane? Trick: Forget about the actual problem; we're going to study a completely different situation. This new configuration consists of two point charges, $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$, and no conducting plane. For this configuration, I can easily write down the potential:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

It follows that:

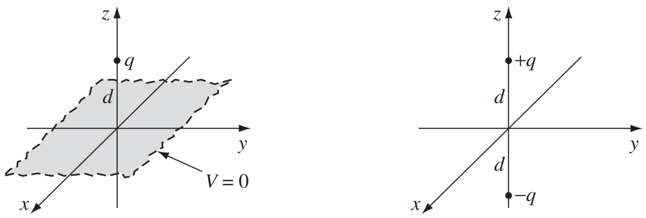
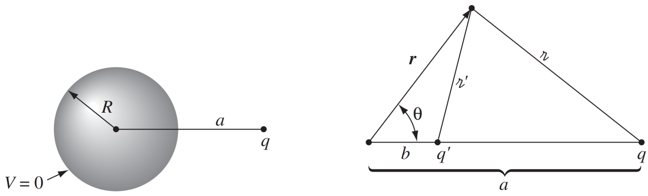


Figure: Charge q above grounded plane.

1. $V = 0$ when $z = 0$, and
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$.

Notice the crucial role played by the uniqueness theorem in this argument: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

Let us try another example. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere. As before, we examine the completely



different configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a} q$$

placed a distance

$$b = \frac{R^2}{a}$$

to the right of the center of the sphere. No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q}{\mathscr{r}'}\right)$$

where r and r' are the distances from q and q', respectively.

Electric Field in Matter

Dielectrics

We have already talked about conductors; these are substances that contain an "unlimited" supply of charges that are free to move about through the material. In dielectrics, by contrast, all charges are attached to specific atoms or molecules—they're on a tight leash, and all they can do is move a bit within the atom or molecule.

Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials. There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: stretching and rotating.

Stretching. When a neutral atom is placed in an electric field $\mathbf{E}$, the nucleus is pushed in the direction of the field, and the electrons the opposite way; stretching the atom. If the field is large enough, it can pull the atom apart completely, "ionizing" it. With less extreme fields, however, an equilibrium is soon established. The atom now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$:

$$\mathbf{p} = \alpha \mathbf{E}$$

the constant of proportionality α is called atomic polarizability.

Rotating. When molecules with built-in, permanent dipole moments are placed in a uniform electric field, the force on the positive end $\mathbf{F}_+$ exactly cancels the force on the negative end $\mathbf{F}_-$. However, there will be a torque:

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

about the center of the dipole.; about any other point $\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$. Notice that $\mathbf{N}$ is in such a direction as to line $\mathbf{p}$ up parallel to $\mathbf{E}$; a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is nonuniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net force on the dipole, in addition to the torque

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ + \mathbf{E}_-) = q(\Delta \mathbf{E})$$

Assuming the dipole is very short, we may write

$$\Delta \mathbf{E} = \nabla \mathbf{E} \cdot \mathbf{d} = (\mathbf{d} \cdot \nabla)\mathbf{E}$$

and therefore

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

Polarization

Notice that those two mechanisms produce the same basic result: a lot of little dipoles pointing along the direction of the field—the material becomes polarized. A convenient measure of this effect is

$$\mathbf{P} \equiv \text{ dipole moment per unit volume} = \frac{\mathbf{p}}{V}$$

which is called the polarization.

Bound Charges

Suppose we have a piece of polarized material–that is, an object containing a lot of microscopic dipoles lined up. The potential for a single dipole $\mathbf{p}$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$

we have a dipole moment $\mathbf{p} = \mathbf{P}\ d\tau'$ in each volume element $d\tau'$, so the total potential is

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\mathbf{P}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}\ d\tau'$$

That does it, in principle. However, observing that $\nabla'(1/\boldsymbol{\mathscr{r}}) = \hat{\boldsymbol{\mathscr{r}}}/\hat{\boldsymbol{\mathscr{r}}}$, where the differentiation is with respect to the source coordinates (r'), we have

$$V = \frac{1}{4\pi\epsilon_0}\left(\oint_S \frac{1}{\boldsymbol{\mathscr{r}}}\mathbf{P}\cdot d\mathbf{a}' + \int_V \frac{1}{\boldsymbol{\mathscr{r}}}(\nabla'\cdot\mathbf{P})d\tau'\right)$$

The first term looks like the potential of a surface charge

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$$

(where $\hat{\mathbf{n}}$ is the normal unit vector), while the second term looks like the potential of a volume charge

$$\rho_b \equiv -\nabla\cdot\mathbf{P}$$

With these definitions

$$V = \frac{1}{4\pi\epsilon_0}\left(\oint_S \frac{\sigma_b}{\boldsymbol{\mathscr{r}}}da' + \int_V \frac{\rho_b}{\boldsymbol{\mathscr{r}}}d\tau'\right)$$

What this means is that the potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density ρ_b plus a surface charge density σ_b. Instead of integrating the contributions of all the infinitesimal dipoles, we could first find those bound charges, and then calculate the fields they produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law).

Physical interpretation. Suppose we have a long string of dipoles. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over. We call the net charge at the ends a bound charge to remind ourselves that it cannot be removed.

To calculate the actual amount of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to $\mathbf{P}$. The dipole moment is given by

$$p = PAd$$

where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge

$$q = PA$$

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P$$

For an oblique cut, the charge is still the same, but $A = A_{end} \cos\theta$, so

$$\sigma_b = \frac{q}{A_{end} \cos\theta} = P$$

and thus

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$ over the surface of the material.

If the polarization is nonuniform, we get accumulations of bound charge within the material, as well as on the surface. Indeed, the net bound charge in a given volume is equal and opposite to the amount that has been pushed out through the surface:

$$\int_V \rho_b \, d\tau = -\oint_S \mathbf{P} \cdot d\mathbf{a} = -\int_V (\nabla \cdot \mathbf{P}) \, d\tau$$

Since this is true for any volume, we have

$$\rho_b = -\nabla \cdot \mathbf{P}$$

Electric Displacement

The effect of polarization is to produce accumulations of (bound) charge ρ_b within the dielectric and σ_b on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything else (which, for want of a better term, we call free charge, ρ). Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f$$

and Gauss's law reads

$$\begin{aligned} \epsilon_0 \nabla \cdot \mathbf{E} &= \rho \\ &= \rho_b + \rho_f \\ &= -\nabla \cdot \mathbf{P} + \rho_f \end{aligned}$$

It is convenient to combine the two divergence terms

$$\rho_f = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P})$$

The expression in parentheses is known as the electric displacement

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

In terms of **D**, Gauss's law reads

$$\nabla \cdot \mathbf{D} = \rho_f$$

or, in integral form

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

where $_{f_{enc}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because it makes reference only to free charges, and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, and this polarization produces the bound charge.

Deceptive parallel. You may be tempted to conclude that **D** is "just like" **E**, but the conclusion is false; in particular, there is no "Coulomb's law" for **D**:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \rho_f(\mathbf{r})\, d\tau'$$

The parallel between **E** and **D** is more subtle than that. For the divergence alone is insufficient to determine a vector field; you need to know the curl as well. One tends to forget this in the case of electrostatic fields because the curl of **E** is always zero. But the curl of **D** is not always zero.

$$\nabla \times \mathbf{D} = \epsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P} = \nabla \times \mathbf{P}$$

and there is no reason, in general, to suppose that the curl of **P** vanishes.

Linear Dielectrics

I shall call materials that obey

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

as linear dielectrics. For linear dielectric, the polarization is proportional to the field, provided **E** is not too strong. The constant of proportionality, χ_e, is called the electric susceptibility of the medium (a factor of ϵ_0 has been extracted to make χ_e dimensionless). **E** is the total field; it may be due in part to free charges and in part to the polarization itself.

If, for instance, we put a piece of dielectric into an external field E_0, we cannot compute **P** directly; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, which...

In linear media we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1+\chi_e)\mathbf{E}$$

So $\mathbf{D}$ is also proportional to $\mathbf{E}$

$$\mathbf{D} = \epsilon\mathbf{E}$$

where

$$\epsilon = \epsilon_0(1+\chi_e)$$

This new constant ϵ is called the permittivity of the material. In vacuum, where there is no matter to polarize, the susceptibility χ_e is zero, and the permittivity is ϵ_0. That's why ϵ_0 is called the permittivity of free space.

If you remove a factor of ϵ_0, the remaining dimensionless quantity

$$\epsilon_r \equiv 1+\chi_e = \frac{\epsilon}{\epsilon_0}$$

is called the relative permittivity, or dielectric constant, of the material.

Deceptive (?) parallel. You might suppose that linear dielectrics escape the defect in the parallel between $\mathbf{E}$ and $\mathbf{D}$; since $\mathbf{P}$ and $\mathbf{D}$ are now proportional to $\mathbf{E}$, does it not follow that their curls must vanish? Unfortunately, it does not, for the line integral of $\mathbf{P}$ around a closed path that straddles the boundary between one type of material and another need not be zero, even though the integral of $\mathbf{E}$ around the same loop must be.

Of course, if the space is entirely filled with a homogeneous, that is medium is one whose properties (in this case the susceptibility) do not vary with position, linear dielectric, then this objection is void; in this rather special circumstance

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}$$

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant.

For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{1\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}}$$

A common way to beef up a capacitor is to fill parallel-plate capacitor with insulating material of dielectric constant ϵ_r. Since the field is confined to the space between the plates, the dielectric will reduce $\mathbf{E}$, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is increased by a factor of the dielectric constant

$$C = \epsilon_r C_{\text{vac}}$$

Energy in Dielectric Systems It takes work to charge up a capacitor

$$W = \frac{1}{2}CV^2$$

If the capacitor is filled with linear dielectric

$$C = \epsilon_r C_{\text{vac}}$$

I have also derived a general formula for the energy stored in any electrostatic system

$$W = \frac{\epsilon_0}{2}\int E^2 \, d\tau$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2 \, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E} \, d\tau$$

Magnetostatics

Lorentz Force Law

Magnetic forces. The magnetic force on a charge Q, moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, is

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

This is known as the Lorentz force law. In the presence of both electric and magnetic fields, the net force on Q would be

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

One implication of the Lorentz force law deserves special attention

$$\boxed{\textbf{Magnetic forces do no work.}}$$

For if Q moves an amount $d\mathbf{l} = \mathbf{v}\ dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\ dt = 0$$

This follows because $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.

Current. The current in a wire is the charge per unit time passing a given point. A line charge λ traveling down a wire at speed $\mathbf{v}$ constitutes a current

$$\mathbf{I} = \lambda \mathbf{v}$$

The magnetic force on a segment of current-carrying wire is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda\ dl$$

thus

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\ dl$$

Inasmuch as $\mathbf{I}$ and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})$$

where the current is constant (in magnitude) along the wire.

Surface current density. Consider a "ribbon" of infinitesimal width $dl\perp$, running parallel to the flow. If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl\perp}$$

In words, $\mathbf{K}$ is the current per unit width. In particular, if the (mobile) surface charge density is σ and its velocity is $\mathbf{v}$, then

$$\mathbf{K} \equiv \sigma \mathbf{v}$$

The magnetic force on the surface current is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \sigma\ da = \int (\mathbf{K} \times \mathbf{B})\ da$$

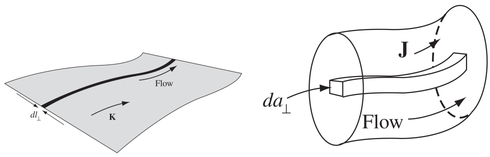


Figure: surface and volume current.

Volume current density. Consider a tube of infinitesimal cross-section $da\perp$, running parallel to the flow. If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da\perp}$$

In words, $\mathbf{J}$ is the current per unit area. If the (mobile) volume charge density is ρ and the velocity is $\mathbf{v}$, then

$$\mathbf{J} \equiv \rho\mathbf{v}$$

The magnetic force on a volume current is therefore

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau$$

Continuity equation. From the definition of volume current, the total current crossing a surface $\mathcal{S}$ can be written as

$$I = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$$

In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau$$

Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau = -\int_{\mathcal{V}} \frac{d\rho}{dt} \, d\tau$$

where minus sign reflects the fact that an outward flow decreases the charge left in $\mathcal{V}$. Since this applies to any volume, we conclude that

$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

This is the precise mathematical statement of local charge conservation; it is called the continuity equation.

Dictionary. For future reference, let me summarize the "dictionary" we have implicitly developed for translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{n=1}^{n} q_i \mathbf{v}_i \sim \int_{\mathcal{L}} \mathbf{I} \, dl \sim \int_{\mathcal{S}} \mathbf{K} \, da \sim \int_{\mathcal{V}} \mathbf{J} \, d\tau$$

Biot-Savart Law

Steady currents produce magnetic fields that are constant in time; the theory of steady currents is called magnetostatics. By steady current I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere. Formally, electro/magnetostatics is the régime

$$\frac{\partial \rho}{\partial t} = 0 \quad \frac{\partial \mathbf{J}}{\partial t} = 0$$

At all places and all times. Notice that a moving point charge cannot possibly constitute a steady current.

When a steady current flows in a wire, its magnitude I must be the same all along the line. More generally, since $\partial\rho/\partial t = 0$ in magnetostatics, the continuity equation becomes

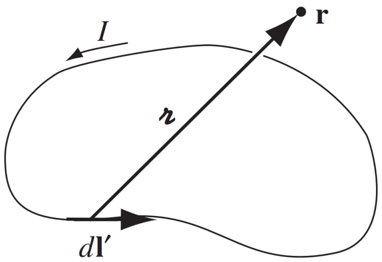
$$\nabla \cdot \mathbf{J} = 0$$

Biot-Savart law. The magnetic field of a steady line current is given by the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

The integration is along the current path, in the direction of the flow; $d\mathbf{l}'$ is an element of length along the wire, and $\boldsymbol{\mathscr{r}}$, as always, is the vector from the source to the point $\mathbf{r}$. The constant μ_0 is called the permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7} \quad \mathrm{N/A^2}$$



The Hall Effect

The phenomenon known as the Hall effect occur when a current-carrying conductor is placed in a magnetic field, a potential difference is generated in a direction perpendicular to both the current and the magnetic field. The arrangement itself consists of a flat conductor carrying a current I in the x direction, with a uniform magnetic field $\mathbf{B}$ is applied in the y direction. If the charge carriers are electrons moving in the negative x direction with a drift velocity $\mathbf{v}_d$, they experience an

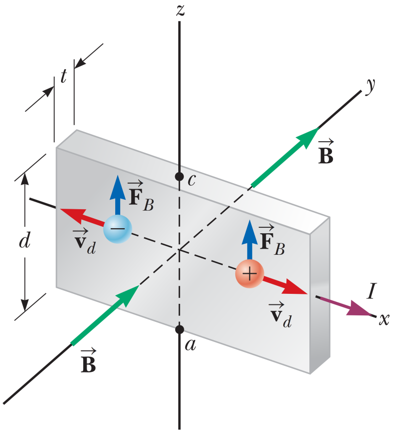


Figure: arrangement of the Hall effect

upward magnetic force $\mathbf{F} = -q\mathbf{v}_d \times \mathbf{B}$, and are deflected upward. This accumulation of charge at the edges establishes an electric field in the conductor and increases until the electric force on carriers remaining in the bulk of the conductor balances the magnetic force acting on the carriers. In equilibrium, the magnetic is balanced by the electric force, therefore

$$-qv_dB = -qE_H$$
$$E_H = v_dB$$

If d is the width of the conductor, the Hall voltage is

$$V_H = E_Hd = v_dBd$$

Since infinitesimal charge dq inside infinitesimal length dx inside conductors can be expressed as $dq = qnAdx = qnAv_ddt$

$$V_H = \frac{IBd}{nqA} = \frac{IB}{nqt} = \frac{IB}{t}R_H$$

where $R_H = 1/nq$ is called the Hall coefficient. This relationship shows that a properly calibrated conductor can be used to measure the magnitude of an unknown magnetic field.

Divergence and Curl of Magnetic Field

Curl. Magnetic field of an infinite straight wire is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

The integral of $\mathbf{B}$ around a circular path of radius s, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

Notice that the answer is independent of s; that's because $\mathbf{B}$ decreases at the same rate as the circumference increases. With more work, you can generalize for any arbitrary wire and get the same result.

Now suppose we have a bundle of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing. The line integral will then be

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

Using Stokes' theorem, we can write

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Most current configurations cannot be constructed out of infinite straight wires, so the next section is devoted to the formal derivation of the divergence and curl of B, starting from the Biot-Savart law itself.

Divergence. Evidently the divergence of the magnetic field is

$$\nabla \cdot \mathbf{B} = 0$$

Ampere's Law

The equation for the curl of **B**

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

is called Ampère's law (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l}$$

Thus

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

Since $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface which we call I_{enc}

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Like Gauss's law, Ampère's law is always true (for steady currents), but it is not always useful. The current configurations that can be handled by Ampère's law are Infinite straight lines, Infinite planes, Infinite solenoids, Toroids.

Magnetic Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics $\mathbf{E} = -\nabla V$, so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a vector potential $\mathbf{A}$ in magnetostatics

$$B = \nabla \times \mathbf{A}$$

The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is always zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$$
$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

With $\mathbf{A}$, Ampère's law becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This again is nothing but Poisson's equation—or rather, it is three Poisson's equations, one for each Cartesian component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

For line current

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} d\mathbf{l}'$$

and surface currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} da'$$

Multipole Expansion

If you want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. As we found

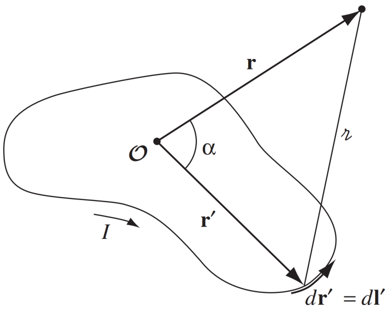
$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

where α is the angle between **r** and **r'**. Accordingly, the vector potential of a current loop can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint r'^n P_n(\cos\alpha)\, d\mathbf{l}'$$

more explicitly

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\mathbf{l}' + \frac{1}{r^2} \oint r' \cos\alpha\, d\mathbf{l}' + \right.$$
$$\left. \frac{1}{r^3} \oint r'^2 \left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right) \alpha\, d\mathbf{l}' + \dots \right]$$



As in the multipole expansion of V, we call the first term (which goes like $1/r$) the monopole term, the second dipole, the third quadrupole, and so on. Now, the magnetic monopole term is always zero, for the integral is just the total vector displacement around a closed loop. This reflects the fact that there are no magnetic monopoles in nature.

In the absence of any monopole contribution, the dominant term is the dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2}\oint \hat{\mathbf{r}}\cdot\mathbf{r}' \, d\mathbf{l}'$$

This integral can be rewritten in a more illuminating way if we invoke

$$\oint (\mathbf{c}\cdot\mathbf{r}) \, d\mathbf{l} = \left(\oint \mathbf{r}\times d\mathbf{l}\right)\times\mathbf{c}$$

for any constant vector $\mathbf{c}$. Then

$$\oint (\hat{\mathbf{r}}\cdot\mathbf{r}) \, d\mathbf{l} = \int d\mathbf{a}'\times\hat{\mathbf{r}}$$

and

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

where $\mathbf{m}$ is the magnetic dipole moment

$$\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$$

Field of Dipole. In practice, the dipole potential is a suitable approximation whenever the distance r greatly exceeds the size of the loop. You must take an infinitesimally small loop at the origin, but then, in order to keep the dipole moment finite, you have to crank the current up to infinity, with the product $m = Ia$ held fixed. The magnetic field of a (perfect) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z-direction. Accordingly, the potential at point (r, θ, ϕ) is

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

and hence

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

Surprisingly, this is identical in structure to the field of an electric dipole. Up close, however, the field of a physical magnetic dipole–a small current loop–looks quite different from the field of a physical electric dipole–plus and minus charges a short distance apart.

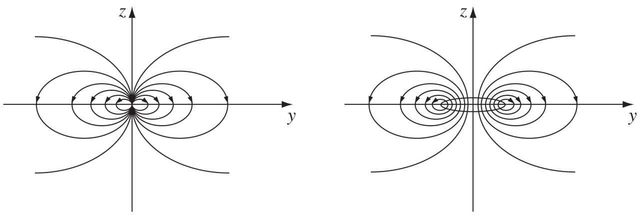


Figure: Field of a "pure" dipole and "physical" dipole

Boundary Condition

In ELECTROSTATICS, I drew a triangular diagram to summarize the relations among the three fundamental quantities of electrostatics. A similar figure can be constructed for magnetostatics, relating the current density $\mathbf{J}$, the field $\mathbf{B}$, and the potential $\mathbf{A}$.

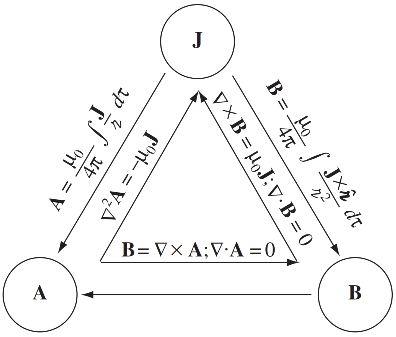


Figure: Magnetostatic Holy Trinity

Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current. Only this time it is the tangential component that changes. For perpendicular component

$$\mathbf{B}^{\perp}_{\text{above}} = \mathbf{B}^{\perp}_{\text{below}}$$

and tangential component

$$\mathbf{B}^{||}_{\text{above}} - \mathbf{B}^{||}_{\text{below}} = \mu_0 K$$

Thus, the component of $\mathbf{B}$ that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$. These results can be summarized in a single formula:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$$

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

But the derivative of $\mathbf{A}$ inherits the discontinuity of $\mathbf{B}$:

$$\frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

Appendix I: Magnetic Forces

Cyclotron motion. A uniform magnetic field points into the page; if the charge Q moves counterclockwise, with speed v, around a circle of radius R, the magnetic force points inward, and has a fixed magnitude QvB—just right to sustain uniform circular motion:

$$QvB = m\frac{v^2}{R}$$
$$p = QBR$$

where m is the particle's mass and $p = mv$ is its momentum.

Cycloid Motion. Suppose, for instance, that B points in the x-direction, and E in the z-direction. A positive charge is released from the origin; what path will it follow?

There being no force in the x-direction, the position of the particle at any time t can be described by the vector $(0, y(t), z(t))$; the velocity is therefore

$$\mathbf{v} = \dot{y}\,\hat{\mathbf{y}} + \dot{z}\,\hat{\mathbf{z}}$$

Thus

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{\mathbf{y}} - B\dot{y}\hat{\mathbf{z}}$$

applying Newton's second law

$$Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = m\mathbf{a}$$
$$Q(B\dot{z}\hat{\mathbf{y}} + (E - B\dot{y})\hat{\mathbf{z}}) = m(\ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}})$$

Or, treating the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components separately

$$\ddot{y} = \omega\dot{z} \quad \text{and} \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

where

$$\omega \equiv \frac{QB}{m}$$

To solve these differential equation, first I solve for $\ddot{z}$

$$\ddot{z} = \frac{1}{\omega}\dddot{y} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

or

$$\frac{d}{dt}\ddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right)$$

Then I will do change of variable $\psi = \dot{y}$, thus

$$\ddot{\psi} + \omega^2\psi = \omega^2\frac{E}{B}$$

Reminder that the solution to differential equation is the superposition of its homogenous equation solution–called complementary– and its particular solution. Its homogenous form is

$$\ddot{\psi} + \omega^2\psi = 0$$

which has the solution

$$\psi_c = \mathcal{A}\sin\omega t + \mathcal{B}\cos\omega t$$

For its particular solution, I shall rewrite the equation

$$\ddot{\psi} + \omega^2\psi = \omega^2\frac{E}{B}$$
$$(D+\omega i)(D-\omega i)\psi = e^{0\cdot t}\omega^2\frac{E}{B}$$

which has the form of

$$(D+a)(D-b)y = e^{cx}P_n(x)$$

Since $c \neq a, c \neq b$ and $\omega^2 E/B$ is polynomial of the zeroth degree, I have the particular solution

$$\psi_p = A$$

Substituting into my equation, I have

$$A = \frac{E}{B}$$

Then, my particular solution

$$\psi_p = \frac{E}{B}$$

And my total solution

$$\psi = \mathcal{A}\sin\omega t + \mathcal{B}\cos\omega t + \frac{E}{B}$$

Changing into my original variable, I have the solution of y

$$y = \mathcal{A}\cos\omega t + \mathcal{B}\sin\omega t + \frac{E}{B}t + \mathcal{D}$$

Substituting y into

$$\ddot{y} = \omega\dot{z}$$

I have

$$\mathcal{B}\cos\omega t - \mathcal{A}\sin\omega t = \omega\dot{z}$$

Thus, I have the solution for z

$$z = \mathcal{B}\cos\omega t - \mathcal{A}\sin\omega t + \mathcal{D}$$

Now, we apply the boundaries condition. We know that the particle started from rest $\dot{y}_0 = \dot{z}_0 = 0$ at the origin $y_0 = z_0 = 0$. We have

$$\begin{cases} y &= \mathcal{A}\cos\omega t + \mathcal{B}\sin\omega t + \frac{E}{B}t + \mathcal{D} \\ \dot{y} &= \mathcal{B}\omega\sin\omega t - \mathcal{A}\omega\cos\omega t + \frac{E}{B} \\ z &= \mathcal{B}\cos\omega t - \mathcal{A}\sin\omega t + \mathcal{D} \\ \dot{z} &= -\mathcal{B}\omega\cos\omega t - \mathcal{A}\omega\sin\omega t \end{cases}$$

applying the boundaries condition

$$\begin{cases} \mathcal{A}+\mathcal{C} & = 0 \\ \mathcal{B} & = -\frac{E}{B\omega} \\ \mathcal{D} & = \frac{E}{B\omega} \\ -\mathcal{A}\omega & = 0 \end{cases}$$

Finally, I have

$$y = \frac{E}{B\omega}(\omega t - \sin\omega t) \quad \text{and} \quad z = \frac{E}{B\omega}(1-\cos\omega t)$$

In this form, the answer is not terribly enlightening, but if we let

$$R \equiv \frac{E}{B\omega}$$

and eliminate the sines and cosines by exploiting the trigonometric identity, we find that

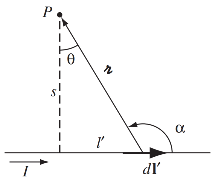
$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

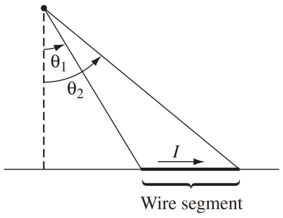
This is the formula for a circle, of radius R, whose center $(0, R\omega t, R)$ travels in the y-direction at a constant speed

$$u = \omega R = \frac{E}{B}$$

Appendix II: Biot-Savart's Law

Straight Wire. Find the magnetic field a distance s from a long straight wire carrying a steady current I.





I shall evaluate the magnetic field in cylindrical coordinate. I define the direction of current as the z axis. Also, let's say that the length of the wire is L. First, we need to determine the separation vector

$$\boldsymbol{\mathscr{r}} = s\hat{\mathbf{s}} + z\hat{\mathbf{z}}$$
$$|\boldsymbol{\mathscr{r}}| = (s^2+z^2)^{1/2}$$

Then, the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}\times\boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}^3|}$$
$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{(ds\ \hat{\mathbf{s}} + s\ d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}})\times(s\hat{\mathbf{s}}+z\hat{\mathbf{z}})}{(s^2+z^2)^{3/2}}$$

since I'm only integrating along z, $ds = d\phi = 0$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{(dz\ \hat{\mathbf{z}}) \times (s\hat{\mathbf{s}} + z\hat{\mathbf{z}})}{(s^2 + z^2)^{3/2}}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{s\ dz}{(s^2 + z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

Evaluating the integral using trigonometric substitution

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} s \frac{z}{s^2(s^2 + z^2)^{1/2}} \Bigg|_{-L/2}^{L/2} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi s} \frac{L}{(s^2 + L^2/4)^{1/2}} \hat{\boldsymbol{\phi}}$$

Or, in terms of the initial and final angles θ_1 and θ_2

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r} (\sin\theta_2 - \sin\theta_1) \hat{\boldsymbol{\phi}}$$

In the case of an infinite wire, we integrate from $z = -\infty$ to $z = \infty$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} s \frac{z}{s^2(s^2 + z^2)^{1/2}} \Bigg|_{-\infty}^{\infty} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}$$

Two Straight Wire. Let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2. The field at (2) due to (1) is

$$\mathbf{B} = \frac{\mu_0 I_1}{2\pi d} \hat{\boldsymbol{\phi}}$$

and it points into the page. The Lorentz force law predicts a force directed towards (1)

$$F = I_2 \frac{\mu_0 I_1}{2\pi d} \int dl$$

The total force, not surprisingly, is infinite, but the force per unit length is

$$f = \frac{\mu_0 I_1 I_2}{2\pi d}$$

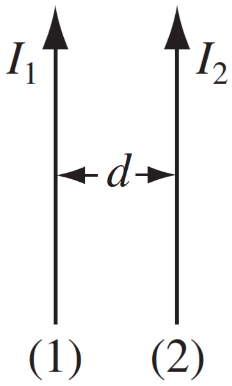
If the currents are antiparallel (one up, one down), the force is repulsive.

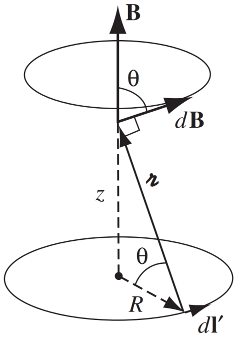
Loop. Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I. We'll just jump straight to solving $\mathbf{B}$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}^3|}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{(ds\ \hat{\mathbf{s}} + s\ d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}}) \times (z\hat{\mathbf{z}} - s\hat{\mathbf{s}} - \phi\hat{\boldsymbol{\phi}})}{(R^2 + z^2)^{3/2}}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{(z\ d\phi\ \hat{\mathbf{s}} + R^2 d\phi\ \hat{\mathbf{z}})}{(R^2 + z^2)^{3/2}}$$





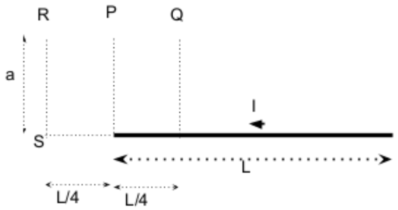
Since the $\hat{\mathbf{s}}$ cancel itself, we only need to evaluate along z axis

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{R^2 d\phi \; \hat{\mathbf{z}}}{R^2 + z^{2^{3/2}}}$$

$$\mathbf{B} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

Appendix III: Tugas 3 Listrik Magnet

Soal 1. Kawat lurus (cetak tebal) yang panjangnya L dialiri arus I. Dengan menggunakan hukum Biot-Savart, tentukanlah medan magnet yang terjadi di titik P, Q, R, dan S.



Medan magnet akan dievaluasi dalam koordinat tabung. Vektor pemisahan pada titik P, Q dan R adalah

$$\boldsymbol{\mathscr{r}}(s, \phi, z) = s\hat{\mathbf{s}} + z\hat{\mathbf{z}}$$

dengan magnitude

$$|\boldsymbol{\mathscr{r}}| = (s^2 + r^2)^{1/2}$$

Maka, medan magnet adalah

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}^3|}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{(ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}) \times (s\hat{\mathbf{s}} + z\hat{\mathbf{z}})}{(s^2 + r^2)^{3/2}}$$

Karena integrasi dilakukan sepanjang sumbu z, maka s, ϕ adalah konstan dan $ds = d\phi = 0$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{(dz\,\hat{\mathbf{z}}) \times (s\hat{\mathbf{s}} + z\hat{\mathbf{z}})}{(s^2 + z^2)^{3/2}}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{s\,dz}{(s^2 + z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

dimana $\hat{\mathbf{z}} \times \hat{\mathbf{s}} = \hat{\boldsymbol{\phi}}$ dan $\hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$. Kemudian diketahui titik P berjarak a dari kawat, sehingga $s = a$ dan integral dihitung dari $z = 0$ hingga $z = L$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int_0^L \frac{a\,dz}{(a^2 + z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

Integral dapat dievaluasi menggunakan subsitusi trigonometri atau hanya dengan melihat tabel, yang menghasilkan

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \frac{az}{a^2(a^2 + z^2)^{1/2}} \bigg|_0^L \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi a} \frac{L}{(a^2 + L^2)^{1/2}} \hat{\boldsymbol{\phi}}$$

Karena sumbu z positif didefinisikan sesuai arah arus (ke kiri) maka, arah medah (positif $\hat{\boldsymbol{\phi}}$) adalah ke dalam bidang.

Pada titik Q, integral dihitung dari $z = -L/4$ hingga $z = L/3$. Maka

$$\mathbf{B}_Q = \frac{\mu_0 I}{4\pi} \int_{-L/4}^{L/3} \frac{a\,dz}{(a^2 + z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

Seperti sebelumnya, integral dapat dievaluasi dengan subsitusi trigonometri atau tabel

$$\mathbf{B}_Q = \frac{\mu_0 I}{4\pi} \frac{az}{a^2(a^2 + z^2)^{1/2}} \bigg|_{-l/4}^{L/3} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_Q = \frac{\mu_0 I}{4\pi a} \left(\frac{L/3}{(a^2 + L^2/9)^{1/2}} + \frac{L/4}{(a^2 + L^2/16)^{1/2}} \right) \hat{\boldsymbol{\phi}}$$

Seperti sebelumnya, arah medan adalah ke dalam bidang.

Pada titik R, integral dihitung dari $z = L/4$ hingga $z = 5L/4$. Maka

$$\mathbf{B}_R = \frac{\mu_0 I}{4\pi} \int_{L/4}^{5L/4} \frac{a\,dz}{(a^2 + z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_R = \frac{\mu_0 I}{4\pi} \frac{az}{a^2(a^2 + z^2)^{1/2}} \bigg|_{L/4}^{5L/4} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_R = \frac{\mu_0 I}{4\pi a} \left(\frac{5L/4}{(a^2 + 25L^2/16)^{1/2}} - \frac{L/4}{(a^2 + L^2/16)^{1/2}} \right) \hat{\boldsymbol{\phi}}$$

Medan pada titik R berarah ke dalam bidang.

Pada titik S, medan adalah nol karena cross product dari vektor arah integrasi $d\mathbf{l}$ dengan vector pemisahan $\boldsymbol{\mathscr{r}}$ adalah nol. Diketahui vector pemisahan pada titik S

$$\boldsymbol{\mathscr{r}}(s, \phi, z) = z\hat{\mathbf{z}}$$

dan vektor arah integrasi

$$d\mathbf{l} = ds\ \hat{\mathbf{s}} + s\ d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}} = dz\ \hat{\mathbf{z}}$$

karena $ds = d\phi = 0$. Sehingga cross product kedua vector

$$d\mathbf{l} \times \boldsymbol{\mathscr{r}} = z\hat{\mathbf{z}} \times dz\ \hat{\mathbf{z}} = 0$$

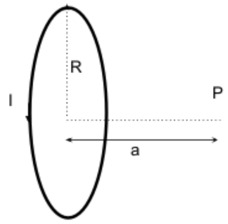
mengingat $\hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$. Hal ini menyebabkan medan pada S

$$\mathbf{B}_S = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}^3|} = 0$$

menjadi nol.

Soal 2. Sebuah loop berbentuk lingkaran berjari-jari R dialiri arus listrik I. Dengan menggunakan hukum BiotSavart, tentukanlah :

- Medan magnet di titik P.
- Medan magnet di pusat lingkaran loop.



Medan akan dievaluasi dalam koordinat tabung. Selanjutnya sumbu z didefinisikan sebagai arah dari pusat lingkaran ke titik P. Diketahui vektor pemisahan pada titik P sebagai

$$\boldsymbol{\mathscr{r}} = a\hat{\mathbf{z}} - R\hat{\mathbf{s}} - \phi\hat{\boldsymbol{\phi}}$$

dengan magnitude

$$|\boldsymbol{\mathscr{r}}| = (R^2 + z^2)^{1/2}$$

Maka, medan pada titik P

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \boldsymbol{\mathscr{r}}}{|\boldsymbol{\mathscr{r}}^3|}$$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{(ds\ \hat{\mathbf{s}} + R\ d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}}) \times (a\hat{\mathbf{z}} - R\hat{\mathbf{s}} - \phi\hat{\boldsymbol{\phi}})}{(R^2 + z^2)^{3/2}}$$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{R\ d\phi\ \hat{\boldsymbol{\phi}} \times (a\hat{\mathbf{z}} - R\hat{\mathbf{s}} - \phi\hat{\boldsymbol{\phi}})}{(R^2 + z^2)^{3/2}}$$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int \frac{(z\ d\phi\ \hat{\mathbf{s}} + R^2 d\phi\ \hat{\mathbf{z}})}{(R^2 + z^2)^{3/2}}$$

Karena komponen radial $\hat{\mathbf{s}}$ saling membatalkan, maka suku $z\ d\phi\ \hat{\mathbf{s}}$ dibuang. Kemudian integral dievaluasi dari $\phi = 0$ hingga $\phi = 2\pi$

$$\mathbf{B}_P = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R^2 d\phi\ \hat{\mathbf{z}}}{(R^2 + z^2)^{3/2}}$$
$$\mathbf{B}_P = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

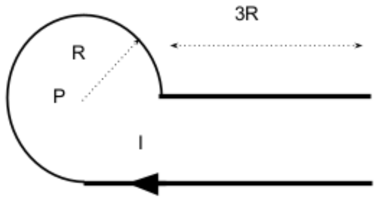
Arah medan adalah ke kanan bidang (positif z).

Pada pusat lingkaran, $a = 0$, sehingga

$$\mathbf{B}_{\text{pusat}} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2)^{3/2}} \hat{\mathbf{z}} = \frac{\mu_0 I}{2R} \hat{\mathbf{z}}$$

Arah medan adalah ke arah titik P (positif z).

Soal 3. Suatu sistem terdiri atas kawat $\frac{3}{4}$ lingkaran dihubungkan dengan dua kawat lurus sejajar seperti gambar. Jika pada sistem mengalir arus I seperti gambar, tentukanlah medan magnet di titik P (pusat lingkaran).



Medan pada pusat lingkaran P merupakan superposisi dari 3 medan yang diakibatkan oleh tiga kawat: kawat lurus 3R, kawat melingkar $\frac{3}{2}\pi R$ dan kawat lurus 4R.

Medan akibat kawat lurus 3R adalah nol karena titik P paralel dengan arah arus, seperti pada soal pertama bagian titik S

$$\mathbf{B}_1 = 0$$

Medan akibat kawat melingkar adalah

$$\mathbf{B}_2 = \frac{\mu_0 I}{4\pi} \int \frac{R^2 d\phi\ \hat{\mathbf{z}}}{R^2 + z^{2^{3/2}}}$$

Karena lingkaran melingkar sepanjang $\frac{3}{2}\pi R$, integrasi dilakukan sepanjang $\phi = 0$ hingga $\phi = \frac{3}{2}\pi$

$$\mathbf{B}_2 = \frac{\mu_0 I}{4\pi} \int_0^{\frac{3}{2}\pi} \frac{R^2 d\phi\ \hat{\mathbf{z}}}{R^2 + z^{2^{3/2}}}$$
$$\mathbf{B}_2 = \frac{3\mu_0 I}{8} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

dimana positif z didefinisikan ke arah dalam bidang. Karena titik P berada pada pusat lingkaran maka $z = 0$

$$\mathbf{B}_2 = \frac{\mu_0 I}{2} \frac{R^2}{(R^2)^{3/2}} \hat{\mathbf{z}} = \frac{\mu_0 I}{2R} \hat{\mathbf{z}}$$

Medan akibat kawat lurus adalah

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{s\ dz}{(s^2+z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

Karena kawat membentang sepanjang $4R$, maka integrasi dilakukan dari $z = 0$ hingga $z = 4R$. Perlu dicatat bahwa sumbu z yang didefinisikan pada kawat lurus tidak sama dengan sumbu z pada kawat melingkar. Namun, arah medan kedua medan adalah sama: ke dalam bidang. Kembali ke pengevaluasian medan

$$\mathbf{B}_3 = \frac{\mu_0 I}{4\pi} \int_0^{4R} \frac{R\ dz}{(R^2+z^2)^{3/2}} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_3 = \frac{\mu_0 I}{4\pi} \frac{Rz}{R^2(R^2+z^2)^{1/2}} \Bigg|_0^{4R} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_3 = \frac{\mu_0 I}{4\pi R} \frac{4R}{(R^2+16R^2)^{1/2}} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_3 = \frac{\mu_0 I \sqrt{17}}{17\pi R} \hat{\boldsymbol{\phi}}$$

dengan arah ke dalam bidang. Karena arah ke dalam bidang sebelumnya didefinisikan sebagai sumbu z maka medan dapat ditulis

$$\mathbf{B}_3 = \frac{\mu_0 I \sqrt{17}}{17\pi R} \hat{\mathbf{z}}$$

Selanjutnya, medan pada titik P merupakan resultan dari ketiga medan yang telah dievaluasi

$$\sum \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$$

$$\sum \mathbf{B} = \frac{\mu_0 I}{2R} \hat{\mathbf{z}} + \frac{\mu_0 I \sqrt{17}}{17\pi R} \hat{\mathbf{z}}$$

$$\sum \mathbf{B} = \frac{\mu_0 I}{R} \left(\frac{1}{2} + \frac{\sqrt{17}}{17\pi} \right) \hat{\mathbf{z}}$$

dengan arah ke dalam bidang.

Appendix IV: Ampere's Law

Infinite straight lines. Find the magnetic field a distance s from a long straight wire carrying a steady current I, This is the same problem we solved using the Biot-Savart's Law.

We know the direction of $\mathbf{B}$ is "circumferential," circling around the wire as indicated by the right-hand rule. By symmetry, the magnitude of $\mathbf{B}$ is constant around an Amperian loop of radius s, centered on the wire. So Ampère's law gives

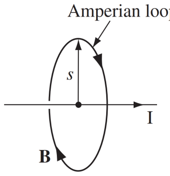
$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

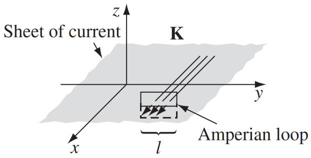
$$B 2\pi s = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

Since we know the direction of $\mathbf{B}$ is "circumferential," we can write the expression as

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$





Infinite planes. Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\hat{\mathbf{x}}$, flowing over the xy plane.

$\mathbf{B}$ can only have a y component, and a quick check with your right hand should convince you that it points to the left above the plane and to the right below it. With this in mind, we draw a rectangular Amperian loop parallel to the yz plane and extending an equal distance above and below the surface. Applying Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

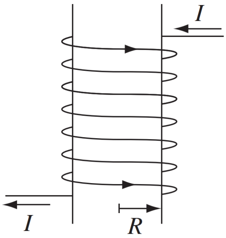
$$2Bl = \mu_0 K l$$

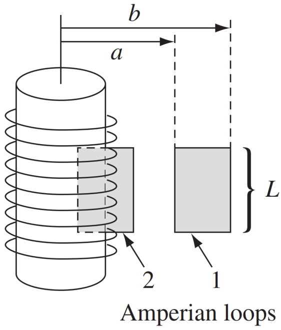
$$B = \frac{\mu_0}{2} K$$

or, more precisely

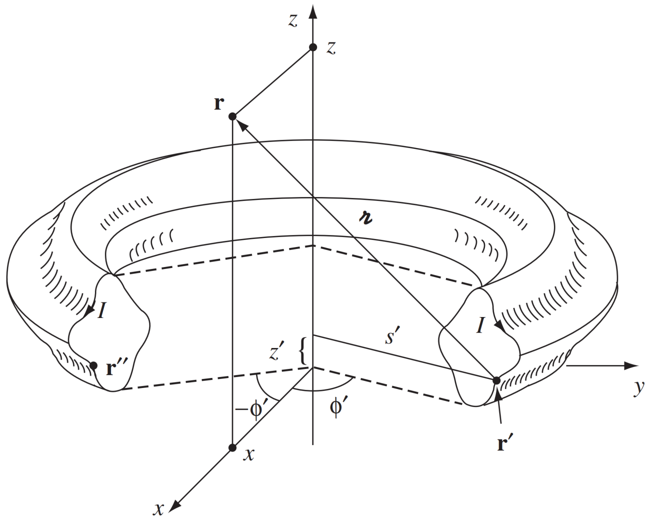
$$\mathbf{B} = \begin{cases} \frac{\mu_0}{2} K\,\hat{\mathbf{y}}, & z < 0 \\ -\frac{\mu_0}{2} K\,\hat{\mathbf{y}}, & z > 0 \end{cases}$$

Infinite solenoids. Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R, each carrying a steady current I.





Magnetic field of an infinite, closely wound solenoid runs parallel to the axis. From the right-hand rule, we expect that it points upward inside the solenoid and downward outside. Moreover, it certainly approaches zero as you go very far away. With this in mind, let's apply



Ampère's law to the two rectangular loops. Loop 1 lies entirely outside the solenoid, with its sides at distances a and b from the axis

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$[B(a) - B(b)]l = 0$$

so

$$B(a) = B(b)$$

Evidently the field outside does not depend on the distance from the axis. But we agreed that it goes to zero for large s. It must therefore be zero everywhere! As for loop 2, which is half inside and half outside, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$BL = \mu_0 nIL$$

where B is the field inside the solenoid. The right side of the loop contributes nothing, since B = 0 out there. Conclusion

$$\mathbf{B} = \begin{cases} \mu_0 nI\,\hat{\mathbf{z}} & \text{inside solenoid} \\ 0 & \text{outside solenoid} \end{cases}$$

Toroid. A toroidal coil consists of a circular ring, or "donut," around which a long wire is wrapped. The winding is uniform and tight enough so that each turn can be considered a plane closed loop. The cross-sectional shape of the coil is immaterial. In that case, it follows that the magnetic field of the toroid is circumferential at all points, both inside and outside the coil. I will first determine the direction of the field using Biot-Savart, then I will determine the magnitude using Ampere's Law.

First the direction. According to the Biot-Savart law, the field at **r** due to the current at **r'** is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} dl'$$

The separation vector is

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= (s\hat{\mathbf{s}} + \phi\hat{\boldsymbol{\phi}} + z\hat{\mathbf{z}}) - (s'\hat{\mathbf{s}} + \phi'\hat{\boldsymbol{\phi}} + z'\hat{\mathbf{z}}) \\
&= (s-s')\hat{\mathbf{s}} + (\phi-\phi')\hat{\boldsymbol{\phi}} + (z-z')\hat{\mathbf{z}}
\end{aligned}$$

Then, since the current has no ϕ component

$$\mathbf{I} = I = I_s\hat{\mathbf{s}} + I_z\hat{\mathbf{z}}$$

Accordingly

$$\begin{aligned}
\mathbf{I}\times\boldsymbol{\mathscr{r}} &= \begin{vmatrix} \hat{\mathbf{s}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ I_s & 0 & I_z \\ s-s' & \phi-\phi' & z-z' \end{vmatrix} \\
&= I_z(\phi'-\phi)\hat{\mathbf{s}} + \left[I_z(s-s') - I_s(\phi-\phi')\right]\hat{\boldsymbol{\phi}} + I_s(\phi-\phi')\hat{\mathbf{z}} \\
&= I_z(s-s')\hat{\boldsymbol{\phi}}
\end{aligned}$$

where the last line is evaluated due to symmetry. The symmetry in question situated at r'', with the same s', the same r, the same dl'', the same I_s, and the same I_z, but negative ϕ'. Because ψ' changes sign, the $\hat{\mathbf{s}}$ and $\hat{\mathbf{z}}$ leaving only a $\hat{\boldsymbol{\phi}}$ term. Thus, the field is circumferential. ■

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$\begin{aligned}
\oint \mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} \\
B 2\pi s &= \mu_0 N I
\end{aligned}$$

where N is the total number of turns. And hence

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 N I}{2\pi s} & \text{for points inside the coil} \\ 0 & \text{for points outside the coil} \end{cases}$$

Magnetic Field in Matter

Magnetized Object

All magnetic phenomena are due to electric charges in motion, and in fact, if you could examine a piece of magnetic material on an atomic scale you would find tiny currents: electrons orbiting around nuclei and spinning about their axes–in other words, dipole. Ordinarily, they cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.

Unlike electric polarization, which is almost always in the same direction as **E**, some materials acquire a magnetization parallel to **B** (paramagnets) and some opposite to **B** (diamagnets). A few substances, called ferromagnets, retain their magnetization even after the external field has been removed—for these, the magnetization is not determined by the present field but by the whole magnetic "history" of the object.

Paramagnetism and Torques

Center the loop at the origin, and tilt it an angle θ from the z axis towards the y-axis. Let **B** point in the z direction. The forces on the two sloping sides cancel–they tend to stretch the loop, but they don't rotate it.

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}$$

The magnitude of the force on each of these segments is

$$F = IbB$$

and therefore

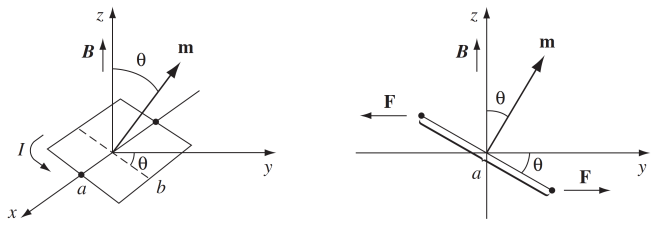
$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

or

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}$$

where $m = Iab$ is the magnetic dipole moment of the loop.

Notice that magnetic torque $\mathbf{N} = \mathbf{m}\times\mathbf{B}$ is identical in form to the electrical analog, $\mathbf{N} = \mathbf{p}\times\mathbf{E}$. In particular, the torque is again in such a direction as to line the dipole up parallel to the field. It is this torque that accounts for paramagnetism.



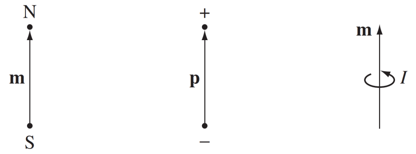


Figure: Gilbert model, electric dipole and Ampere model

Actually, quantum mechanics tends to lock the electrons within a given atom together in pairs with opposing spins, and this effectively neutralizes the torque on the combination. As a result, paramagnetism most often occurs in atoms or molecules with an odd number of electrons. Even here, the alignment is far from complete, since random thermal collisions tend to destroy the order.

In a uniform field, the net force on a current loop is zero:

$$\mathbf{F} = I\oint(d\mathbf{l}\times\mathbf{B}) = I\left(\oint d\mathbf{l}\right)\times\mathbf{B} = 0$$

the constant $\mathbf{B}$ comes outside the integral, and the net displacement. Around a closed loop vanishes. In a nonuniform field this is no longer the case. In general, for an infinitesimal loop, with dipole moment $\mathbf{m}$, in a field $\mathbf{B}$, the force is

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

Early physicists thought magnetic dipoles consisted of positive and negative magnetic "charges" (north and south "poles," they called them), separated by a small distance, just like electric dipoles. It's not a bad model, but it's bad physics, because there's no such thing as a single magnetic north pole or south pole. Magnetism is not due to magnetic monopoles, but rather to moving electric charges; magnetic dipoles are tiny current loops.

Diamagnetism and Electrons Orbit

Let's assume the orbit of electrons is a circle of radius R. Although technically this orbital motion does not constitute a steady current, in practice it's going to look like a steady current with period the period $T = 2\pi R/v$

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R}$$

Accordingly, the orbital dipole moment $(I\pi R^2)$ is

$$\mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}$$

The centripetal acceleration v^2/R is ordinarily sustained by electrical forces alone

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R^2}$$

However, in the presence of a magnetic field there is an additional force, Lorentz's force. For the sake of argument, let's say that $\mathbf{B}$ is perpendicular to the plane of the orbit

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} - e\bar{v}B = m_e\frac{\bar{v}^2}{R}$$

Under these conditions, the new speed $\bar{v}$ is greater than v

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$$

or, assuming the change $\Delta v = \bar{v} - v$ is small

$$\Delta v = \frac{eRB}{2m_e}$$

Thus, when **B** is turned on, then, the electron speeds up. A change in orbital speed means a change in the dipole moment

$$\Delta\mathbf{m} = -\frac{1}{2}e\Delta vR\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

Notice that the change in m is opposite to the direction of **B**. Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But in the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and these increments are all antiparallel to the field. This is the mechanism responsible for diamagnetism. It is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with even numbers of electrons, where paramagnetism is usually absent.

Magnetization

In the presence of a magnetic field, matter becomes magnetized. We have discussed two mechanisms that account for this magnetic polarization: (1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field) and (2) diamagnetism (the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field). Whatever the cause, we describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \text{magnetic moment per unit volume} = \frac{\mathbf{m}}{\mathcal{V}}$$

called the magnetization.

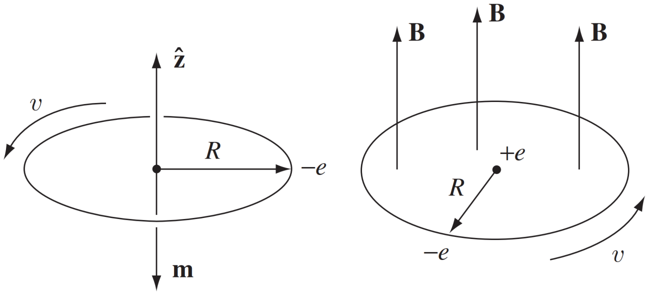


Figure: Electron orbit before and after external field

Bound Current

The vector potential of a single dipole $\mathbf{m}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^2}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}\,d\tau'$, so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

That does it, in principle. But, as in the electrical case, the integral can be cast in a more illuminating form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int\frac{1}{\mathscr{r}}(\nabla'\times\mathbf{M}(\mathbf{r}'))d\tau' + \oint\frac{1}{\mathscr{r}}(\mathbf{M}(\mathbf{r}')\times\ )d\mathbf{a}'\right]$$

The first term looks just like the potential of a volume current

$$\mathbf{J}_b = \nabla\times\mathbf{M}$$

while the second looks like the potential of a surface current

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the normal unit vector. With these definitions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{\mathcal{V}}\frac{\mathbf{J}_b}{\mathscr{r}}d\tau' + \oint_{\mathcal{S}}\frac{\mathbf{K}_b}{\mathscr{r}}d\mathbf{a}'\right]$$

What this means is that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current throughout the material, plus a surface current on the boundary. Instead of integrating the contributions of all the infinitesimal dipoles, we first determine the bound currents, and then find the field they produce, in the same way we would calculate the field of any other volume and surface currents.

Physical interpretation. Consider thin slab of uniformly magnetized material, with the dipoles represented by tiny current loops–notice that all the "internal" currents cancel. Say that each of the tiny loops has area a and thickness t. In terms of the magnetization M, its dipole moment is $m = Mat$. In terms of the circulating current I, however, $m = Ia$. Therefore, $I = Mt$, so the surface current is $K_b = I/t = M$, more precisely

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

When the magnetization is nonuniform, the internal currents no longer cancel. Consider two adjacent chunks of magnetized material. On the surface where they join, there is a net current in the x direction, given by

$$I_x = [M_z(y+dy) - M_z(y)]dz = \frac{\partial M_z}{\partial y}dy\,dz$$

The corresponding volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}$$

By the same token, a nonuniform magnetization in the y direction would contribute an amount $-\partial M_y/\partial z$, so

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \partial M_y/\partial z$$

In general, then

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

Auxiliary Field H

---

the effect of magnetization is to establish bound currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ within the material and $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$ on the surface. We are now ready to put everything together: the field attributable to bound currents, plus the field due to everything else–which I shall call the free current

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$$

The free current is there because somebody hooked up a wire to a battery—it involves actual transport of charge; the bound current is there because of magnetization—it results from the conspiracy of many aligned atomic dipoles.

Ampère's law then can be written

$$\begin{aligned} \frac{1}{\mu_0}\nabla \times \mathbf{B} &= \mathbf{J} \\ \frac{1}{\mu_0}\nabla \times \mathbf{B} &= \nabla \times \mathbf{M} + \mathbf{J}_f \\ \mathbf{J}_f &= \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) \end{aligned}$$

The quantity in parentheses is designated by the letter $\mathbf{H}$

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In terms of $\mathbf{H}$, then, Ampère's law reads

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

or

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\mathrm{f_{enc}}}$$

where $I_{\mathrm{f_{enc}}}$ is the total free current passing through the Amperian loop.

Deceptive parallel. Whereas $\nabla \cdot \mathbf{B} = 0$, the divergence of $\mathbf{H}$ is not, in general, zero. In fact

$$\nabla \cdot \mathbf{H} = -\nabla \mathbf{M}$$

Only when the divergence of $\mathbf{M}$ vanishes is the parallel between $\mathbf{B}$ and $\mu_0\mathbf{M}$ faithful. If the problem to find $\mathbf{H}$ exhibits cylindrical, plane, solenoidal, or toroidal symmetry, then you can get H by the usual Ampère's law methods. If the requisite symmetry is absent, you'll have to think of another approach, and in particular you must not assume that $\mathbf{H}$ is zero just because there is no free current in sight.

Linear Medium

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when $\mathbf{B}$ is removed, $\mathbf{M}$ disappears. In fact, for most substances the magnetization is proportional to the field. Thus, I express the proportionality in terms of $\mathbf{H}$, instead of $\mathbf{B}$

$$\mathbf{M} = \chi_m \mathbf{H}$$

The constant of proportionality χ_m is called the magnetic susceptibility; it is a dimensionless quantity that varies from one substance to another–positive for paramagnets and negative for diamagnets.

Materials that this equation are called linear media. In terms of $\mathbf{B}$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$$

Thus $\mathbf{B}$ is also proportional to $\mathbf{H}$

$$\mathbf{B} = \mu\mathbf{H}$$

where

$$\mu \equiv \mu_0(1 + \chi_m)$$

is called the permeability of the material. In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the permeability of free space. If you factor out μ_0, what's left is called the relative permeability

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

Incidentally, the volume bound current density in a homogeneous linear material is proportional to the free current density

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times \chi_m \mathbf{H} = \chi_m \mathbf{J}_f$$

Defective parallel. You might suppose that linear media escape the defect in the parallel between $\mathbf{B}$ and $\mathbf{H}$: since $\mathbf{M}$ and $\mathbf{H}$ are now proportional to $\mathbf{B}$, does it not follow that their divergence, must always vanish? Unfortunately, it does not. Formally,

$$\nabla \cdot \mathbf{H} = \nabla \cdot \frac{\mathbf{B}}{\mu}$$

$$\nabla \cdot \mathbf{H} = \frac{1}{\mu}\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \cdot \frac{1}{\mu}$$

$$\nabla \cdot \mathbf{H} = \mathbf{B} \cdot \nabla \cdot \frac{1}{\mu}$$

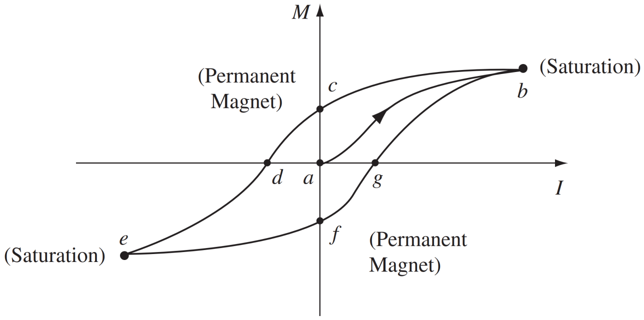


Figure: Hysteresis Loop

so **H** is not divergenceless (in general) at points where μ is changing. For instance, at the end of a cylinder of linear paramagnetic material, **M** is zero on one side but not on the other. Then

$$\oint \mathbf{M} \neq 0$$

and hence, by the divergence theorem, $\nabla \cdot \mathbf{M}$ cannot vanish everywhere within it.

Ferromagnets

In a linear medium, the alignment of atomic dipoles is maintained by a magnetic field imposed from the outside. Ferromagnets–which are emphatically not linear–require no external fields to sustain the magnetization; the alignment is "frozen in." In a ferromagnet, each dipole "likes" to point in the same direction as its neighbors. The reason for this preference is essentially quantum mechanical.

But if that is true, why isn't every wrench and nail a powerful magnet? The answer is that the alignment occurs in relatively small patches, called domains. Each domain contains billions of dipoles, all lined up, but the domains themselves are randomly oriented and their magnetic fields cancel, so as a whole the object is not magnetized.

Hysteresis loop. How, then, would you produce a permanent magnet? If you put a piece of iron into a strong magnetic field, the torque $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ tends to align the dipoles parallel to the field. Since they like to stay parallel to their neighbors, most of the dipoles will resist this torque. However, at the boundary between two domains, there are competing neighbors, and the torque will throw its weight on the side of the domain most nearly parallel to the field. The net effect of the magnetic field, then, is to move the domain boundaries and domains parallel to the field grow. If the field is strong enough, one domain takes over entirely, and the iron is said to be saturated.

This process is not entirely reversible. When the field is switched off, there will be some return to randomly oriented domains, but it is far from complete. You now have a permanent magnet.

A simple way to accomplish this, in practice, is to wrap a coil of wire around the object to be magnetized and run a current I through

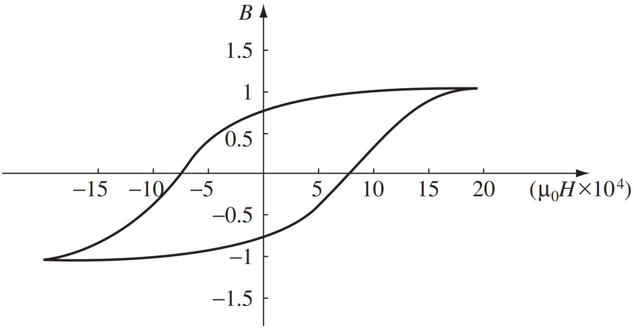


Figure: Hysteresis Loop

the coil. As you increase the current, the field increases, the domain boundaries move, and the magnetization grows. Eventually, you reach the saturation point, with all the dipoles aligned, and a further increase in current has no effect on $\mathbf{M}$ (point b).

Now suppose you reduce the current. Instead of retracing the path back to $M = 0$, there is only a partial return to randomly oriented domains. M decreases, but even with the current off there is some residual magnetization (point c).

The object is now a permanent magnet. If you want to eliminate the remaining magnetization, you'll have to run a current backwards through the coil (a negative I). As you increase I (negatively), M drops down to zero (point d).

If you turn I still higher, you soon reach saturation in the other direction (point e). At this stage, switching off the current will leave the object with a permanent magnetization to the other direction (point f). To complete the story, turn I on again in the positive sense: M returns to zero (point g), and eventually to the forward saturation point (point b).

Notice that the magnetization of the object depends not only on the applied field (that is, on I), but also on its previous magnetic "history." It is customary to draw hysteresis loops as plots of B against H, rather than M against I. If our coil is approximated by a long solenoid, with n turns per unit length, then $H = nI$, so H and I are proportional. Meanwhile, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, but in practice M is huge compared to H, so to all intents and purposes B is proportional to M.

To make the units consistent (Tesla), I have plotted $(\mu_0 H)$ horizontally; notice, however, that the vertical scale is 10^4 times greater than the horizontal one. Roughly speaking, $\mu\mathbf{H}$ is the field our coil would have produced in the absence of any iron; B is what we actually got. That's why anyone who wants to make a powerful electromagnet will wrap the coil around an iron core. It doesn't take much of an external field to move the domain boundaries, and when you do that, you have all the dipoles in the iron working with you.

One final point about ferromagnetism: It all follows, remember, from the fact that the dipoles within a given domain line up parallel to one another. Random thermal motions compete with this ordering, but as long as the temperature doesn't get too high, they cannot budge the dipoles out of line. It's not surprising, though, that very high temperatures do destroy the alignment. What is surprising is that this occurs at a precise temperature, called the Curie point.

Electrodynamics

Ohm's Law

To make a current flow, you have to push on the charges. For most substances, the current density $\mathbf{J}$ is proportional to the force per unit charge, $\mathbf{f}$:

$$\mathbf{J} = \sigma \mathbf{f}$$

The proportionality factor σ (not to be confused with surface charge) is an empirical constant that varies from one material to another; it's called the conductivity of the medium. Notice that even insulators conduct slightly, though the conductivity of a metal is astronomically greater; in fact, for most purposes metals can be regarded as perfect conductors, with $\sigma = \infty$, while for insulators we can pretend $\sigma = 0$.

In principle, the force that drives the charges to produce the current could be anything. For our purposes, though, it's usually an electromagnetic force that does the job. In this

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Ordinarily, the velocity of the charges is sufficiently small that the second term can be ignored

$$\mathbf{J} = \sigma \mathbf{E}$$

I know: you're confused because I said $\mathbf{E} = 0$ inside a conductor. But that's for stationary charges ($\mathbf{J} = 0$). Moreover, for perfect conductors $\mathbf{E} = \mathbf{J}/\sigma = 0$ even if current is flowing. In practice, metals are such good conductors that the electric field required to drive current in them is negligible. Thus, we routinely treat the connecting wires in electric circuits (for example) as equipotential.

Proof. Within the cylinder V obeys Laplace's equation. What are the boundary conditions? At the left end the potential is constant—we may as well set it equal to zero. At the right end the potential is likewise constant—call it V_0. In other words, I am going to solve one dimensional Laplace's equation

$$\frac{d^2V(z)}{dz^2} = 0$$

with boundary conditions

$$\begin{cases} V(0) & = 0 \\ V(z) & = V_0 \end{cases}$$

Laplace's equation in one dimensional has the solution

$$V(z) = Az + B$$

Applying the boundary conditions

$$V(z) = \frac{V_0}{L} z$$

The uniqueness theorem guarantees that this is the solution. The corresponding field is

$$\mathbf{E} = -\nabla V = -\frac{V_0}{L}\hat{\mathbf{z}}$$

which is indeed uniform. ■

The more familiar version of Ohm's law is

$$V = IR$$

Notice that the proportionality between V and I is a direct consequence of $\mathbf{J} = \sigma\mathbf{E}$ if you want to double V, you simply double the charge on the electrodes—that doubles $\mathbf{E}$, which (for an ohmic material) doubles $\mathbf{J}$, which doubles I. The work done by the electrical force is converted into heat in the resistor. Since the work done per unit charge is V and the charge flowing per unit time is I, the power delivered is

$$P = VI = I^2R$$

This is the Joule heating law.

Electromotive Force

There are really two forces involved in driving current around a circuit: the source, $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say), and an electrostatic force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

The physical agency responsible for $\mathbf{f}_s$ can be many things. Whatever the mechanism, its net effect is determined by the line integral of $\mathbf{f}$ around the circuit:

$$\varepsilon \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

where ε is called the electromotive force, or emf, of the circuit. Because closed line integral of $\mathbf{E}$ is zero, it doesn't matter whether you use $\mathbf{f}$ or $\mathbf{f}_s$.

Within an ideal source of emf, the net force on the charges is zero, so $\mathbf{E} = \mathbf{f}_s$. The potential difference between the terminals is therefore

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint_a^b \mathbf{f}_s \cdot d\mathbf{l} = \varepsilon$$

where we extend the integral to the entire loop because $\mathbf{f}_s = 0$ outside the source.

Motional emf

The flux rule for motional emf is

$$\varepsilon = -\frac{d\Phi}{dt}$$

Apart from its delightful simplicity, the flux rule has the virtue of applying to non-rectangular loops moving in arbitrary directions through nonuniform magnetic fields; in fact, the loop need not even maintain a fixed shape.

Proof. Suppose we compute the flux of arbitrary loop at time t, using surface $\mathcal{S}$, and the flux at time $t+dt$, using the surface consisting of $\mathcal{S}$ plus the "ribbon" that connects the new position of the loop to the old. The change in flux, then, is

$$d\Phi = \Phi_{\text{rib}} = \int_{\text{rib}} \mathbf{B} \cdot d\mathbf{a}$$

In time dt, P moves to P'. Let $\mathbf{v}$ be the velocity of the wire, and $\mathbf{u}$ the velocity of a charge down the wire; then $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the resultant. Velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})dt$$

Therefore

$$d\Phi = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l})dt$$

Since $\mathbf{v} = \mathbf{w} - \mathbf{u}$ and $\mathbf{u}$ is parallel to d$\mathbf{l}$, we can just as well write this as

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l})$$

Now, the scalar triple-product can be rewritten

$$\mathbf{B} \cdot \mathbf{w} \times d\mathbf{l} = \mathbf{B} \times \mathbf{w} \cdot d\mathbf{l} = -\mathbf{w} \times \mathbf{B} \cdot d\mathbf{l}$$

so

$$\frac{d\Phi}{dt} = -\oint \mathbf{w} \times \mathbf{B} \cdot d\mathbf{l}$$

But $(\mathbf{w} \times \mathbf{B})$ is the magnetic force per unit charge, $\mathbf{f}_{\text{mag}}$, so

$$\frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l}$$

and the integral of $\mathbf{f}_{\text{mag}}$ is the emf

$$\varepsilon = -\frac{d\phi}{dt}$$

Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, which resulted in an ingenious idea: A changing magnetic field induces an electric field. Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

then $\mathbf{E}$ is related to the change in $\mathbf{B}$ by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

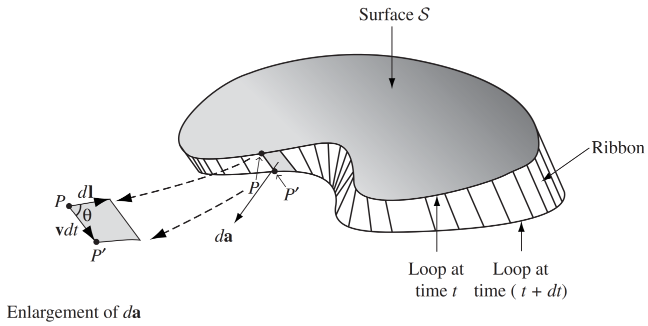


Figure: arbitrary loop of wire at time t, and also a short time dt later

This is Faraday's law, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

For the first experiments, he pulled a loop of wire to the right through a magnetic field. A current flowed in the loop. This is, of course, is a straightforward case of motional emf $\varepsilon = -d\Phi/dt$. It's Lorentz force law at work; the emf is magnetic. For the second experiments, He moved the magnet to the left, holding the loop still. Again, a current flowed in the loop. This is what causes Faraday to think that changing magnetic field induces an electric field. The third, With both the loop and the magnet at rest, he changed the strength of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop. Because the magnetic field changes, it induces electric field, giving rise to an emf $-d\Phi/dt$.

Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for another time.

Lenz's Law. Keeping track of the signs in Faraday's law can be a real headache. But there's a handy rule, called Lenz's law, whose sole purpose is to help you get the directions right

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux it produces tends to cancel the change.

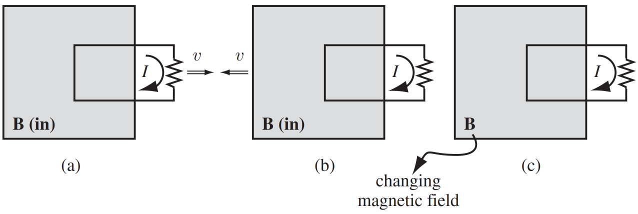


Figure: Faraday's experiments

Inductance

Suppose you have two loops of wire, at rest. If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some field lines pass through loop 2; let Φ_2 be the flux of B_1 through 2. You might have a tough time actually calculating $\mathbf{B}_1$, but a glance at the Biot-Savart law

$$\mathbf{B}_1 = \frac{\mu_0 I_1}{4\pi} \int \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}$$

reveals one significant fact about this field: It is proportional to the current I_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}$$

Thus

$$\Phi_2 = M_{21} I_1$$

where $M_2 1$ is the constant of proportionality; it is known as the mutual inductance of the two loops. Expressing the flux in terms of the vector potential, and invoking Stokes' theorem

$$\begin{aligned}
\Phi_2 &= \int \nabla \times \mathbf{A}_1 \cdot d\mathbf{a}_2 \\
&= \oint \mathbf{A}_1 \cdot d\mathbf{l}_2 \\
&= \oint \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\boldsymbol{\imath}} \cdot d\mathbf{l}_2 \\
\Phi_2 &= \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\boldsymbol{\imath}} I_1
\end{aligned}$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\boldsymbol{\imath}}$$

This is the Neumann formula; which is not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. $M_2 1$ is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral is unchanged if we switch the roles of loops 1 and 2; it follows that

   $$M_{21} = M_{12}$$

   We may as well drop the subscripts and call them both M.

Suppose, now, that you vary the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2:

$$\varepsilon = -\frac{\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

Meaning, very time you change the current in loop 1, an induced current flows in loop 2–even though there are no wires connecting them!

A changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop itself. Once again, the field (and therefore also the flux) is proportional to the current:

$$\Phi = LI$$

The constant of proportionality L is called the self inductance. If the current changes, the emf induced in the loop is

$$\varepsilon = -L\frac{dI}{dt}$$

Inductance is measured in henries (H); a Henry is a volt-second per ampere

Energy in Magnetic Fields

The work done on a unit charge by you against the back emf, in one trip around the circuit is $-\varepsilon$. The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt}$$

If we start with zero current and build it up to a final value I, the work done

$$W = \frac{1}{2}LI^2$$

There is a nicer way to write W, which has the advantage that it is readily generalized to surface and volume currents

$$W = \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\, d\tau - \oint \mathbf{A} \times \mathbf{B} \cdot d\mathbf{a}\right]$$

Now, the integration is to be taken over the entire volume occupied by the current, but $\mathbf{J}$ is zero out there anyway. In particular, if we agree to integrate over all space, then the surface integral goes to zero, and we are left with

$$W = \frac{1}{2\mu_0}\int B^2\, d\tau \qquad \text{Over all Space}$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount

$$U_m = \frac{B^2}{\mu_0}$$

per unit volume. Although some might prefer to say that the energy is stored in the current distribution, in the amount $\frac{1}{2}\mathbf{A} \cdot \mathbf{J}$ per unit volume.

Maxwell's Equation

Maxwell's equations, or Maxwell-Heaviside equations, are as follows

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or in integral form

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$
$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$
$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} + \mu_0 \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}$$

How Maxwell Fixed Ampère's Law. Ampère's law is bound to fail for non-steady currents. Suppose we're in the process of charging up a capacitor. In integral form, Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

The total current passing through the loop, or, more precisely, the current piercing a surface that has the loop for its boundary. In this case, the simplest surface lies in the plane of the loop–the wire punctures this surface, so $I_{\text{enc}} = I$. Fine–but what if I draw instead the balloon-shaped surface? No current passes through this surface, and I conclude that $I_{\text{enc}} = 0$!

We will try to fix it by considering the continuity equation and invoking Gauss's law

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \epsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

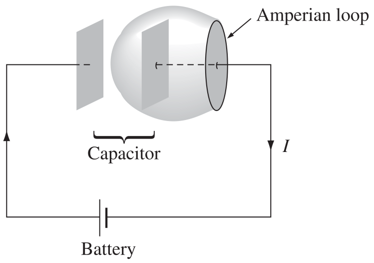


Figure: Weird Amperian loop

If we were to combine $\epsilon_0 \partial \mathbf{E}/\partial t$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} + \mu_0 \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}$$

Apart from curing the defect in Ampère's law, Maxwell's term has a certain aesthetic appeal: Just as a changing magnetic field induces an electric field (Faraday's law), a changing electric field induces a magnetic field.

Appendix I: Ohm's Law

Example 1. Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ. If they are maintained at a potential difference V, what current flows from one to the other, in a length L?

First we need to determine the Field. Using Gauss' Theorem,

$$\oint \mathbf{E} = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$E 2\pi s L = \frac{\lambda L}{\epsilon_0}$$

thus

$$\mathbf{E} = \frac{\lambda}{2\pi s \epsilon_0}\hat{\mathbf{s}}$$

While current is

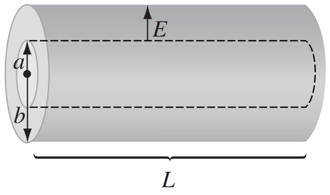
$$I = \int \mathbf{J} \cdot d\mathbf{a} = \int \sigma \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L$$

and the potential difference between the cylinders is

$$V(a) - V(b) = V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

so

$$I = \frac{2\pi\sigma L}{\ln b/a}V$$



Appendix II: Circuits

Impedance. For resistor, impedance, or rather resistance, is defined by Ohm's Law (derivated)

$$R = \frac{V}{I}$$

We define capacitive reactance as:

$$X_C = -i\frac{1}{2\pi f C}$$

The inductive reactance can be found using:

$$X_L = i2\pi f L$$

Resonance. This phenomenon occurs when our circuit has maximum value of current, which resulted from having minimum impedance. Since minimum impedance occur when $X_L = X_C$,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

RC Circuits. Applying Kirchhoff's loop rule to the circuit after the switch is thrown to position a, we get

$$iR + \frac{1}{C}q = \varepsilon$$
$$\dot{q} + \frac{1}{RC}q = \frac{\varepsilon}{R}$$

this is first order ODE, which can be easily solved using integral factor

$$I = \int \frac{1}{RC}dt = \frac{t}{RC}$$

Then

$$q = e^{-t/RC} \int \frac{\varepsilon}{R}e^{t/RC}\ dt + Ae^{-t/RC}$$
$$q = C\varepsilon + Ae^{-t/RC}$$

Applying boundary condition $q(0) = 0$, we get $A = -C\varepsilon$, thus

$$q = C\varepsilon(1 - e^{-t/RC})$$

Since $i = dq/dt$

$$i = \frac{\varepsilon}{R}e^{-t/RC}$$

Time constant $\tau \equiv RC$ of the circuit represents the time interval during which the current decreases to $1/e$ of its initial value. Now, imagine that the capacitor is completely charged. If the switch is now thrown to position b, the capacitor begins to discharge through the resistor. The differential equation becomes

$$\dot{q} + \frac{1}{RC}q = 0$$

with general solution

$$q = Ae^{-t/RC}$$

Applying boundary condition $q(0) = Q_i$, we get $A = Q_i$, thus

$$q = Q_i e^{-t/RC}$$

As for the instantaneous current

$$i = -\frac{Q_i}{RC}e^{-t/RC} = -I_i e^{-t/RC}$$

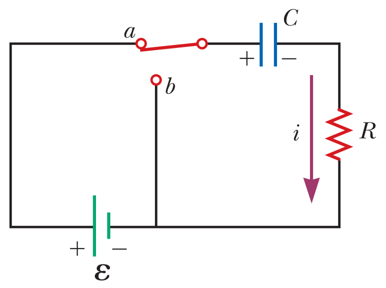


Figure: RC Circuit

RL Circuits. Let's apply Kirchhoff's loop rule to the circuit when S_2 is set to a and switch S_1 is closed

$$L\dot{i} + iR = \varepsilon$$
$$\dot{i} + i\frac{R}{L} = \frac{\varepsilon}{L}$$

As before, I use integral factor

$$I = \int \frac{R}{L}dt = \frac{R}{L}t$$

Then

$$i = e^{-Rt/L}\int \frac{\varepsilon}{L}e^{Rt/L}dt + Ae^{-Rt/L}$$
$$i = \frac{\varepsilon}{L} + Ae^{-Rt/L}$$

Applying boundary condition $i(0) = 0$, we get $A = -\varepsilon/R$, thus

$$i = \frac{\varepsilon}{R}(1 - e^{-Rt/L})$$

Since $q = \int i\ dt$

$$q = \frac{\varepsilon}{R}(1\frac{L}{R}e^{-Rt/L})$$

Now, suppose S_2 is thrown from a to b. The differential equation becomes

$$\dot{i} + i\frac{R}{L} = 0$$

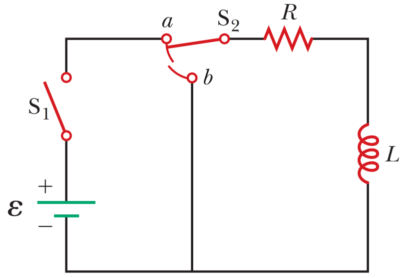


Figure: RL Circuit

with general solution

$$i = Ae^{-Rt/L}$$

Applying boundary condition $i(0) = I_i$, we get $A = I_i$, thus

$$i = I_i e^{-Rt/L}$$

RLC Circuits. If the applied voltage varies sinusoidally with time

$$v = V_m \sin \omega t$$

then current in the circuit is given by

$$i = I_m \sin \omega t - \phi$$

where

$$\phi = \arctan^{-1} \frac{Z_{im}}{Z_{Re}}$$

is some phase angle between the current and the applied voltage.

Wave

Simple Harmonic Motion

Mass on A Spring

For small displacements the force produced by the spring is described by Hooke's law:

$$F = -kx$$

Using Newton's second law of motion, we obtain the equation of motion of the mass

$$\ddot{x} = -\omega^2 x$$

where

$$\omega^2 = \frac{k}{m}$$

We can solve the equation using by rewritting in from

$$(D + \omega i)(D - \omega i)x = 0$$

the roots of auxilary equation are therefore $D = \pm\omega i$. Thus, the general solution is

$$x = a\cos\omega t + b\sin\omega t = A\cos\omega t + \phi$$

Energy of a mass on a spring. The work done on the spring, extending it from x' to $x' + dx'$, is $kx'dx'$. Hence, the work done extending it from its unstretched length by an amount x

$$U = \int_0^x kx'dx' = \frac{1}{2}kx^2$$

Conservation of energy for the harmonic oscillator follows from Newton's second law

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{const.}$$

Substituting the value of x and $v = dx/dt$, we get

$$E = \frac{1}{2}kA^2$$

Pendulum

By Newton's second law

$$ml\ddot{\theta} = -mg\sin\theta$$
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

expanding $\sin\theta$ in power series

$$\ddot{\theta} = -\frac{g}{l}\theta$$

This is the equation of SHM with $\omega = \sqrt{g/l}$ and its general solution

$$\theta = \theta_0\cos\omega t + \phi$$

Energy of pendulum. For small θ, we have

$$\begin{aligned} l^2 &= (l-y)^2 + x^2 \\ 2ly &= Y^2 + x^2 \end{aligned}$$

For small displacements of the pendulum, $x \ll l$, it follows that $y \ll x$, so that the term y^2 can be neglected, and we can write

$$y = \frac{x^2}{2l}$$

The total energy of the system E is therefore

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

At the turning point of the motion, when x equals A, it follows that

$$\frac{1}{2}mg\frac{A^2}{2l} = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

We can use it to obtain expressions for velocity v

$$\frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}$$

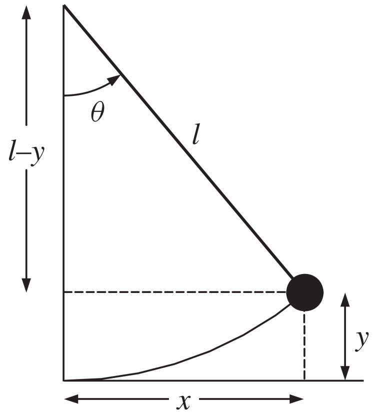
and for displacements x

$$\begin{aligned} \int \frac{dx}{\sqrt{A^2 - x^2}} &= \int \sqrt{\frac{g}{l}} dt \\ \arcsin\frac{x}{A} &= \sqrt{\frac{g}{l}}t + \phi \\ x = A\sin\sqrt{\frac{g}{l}}t + \phi \end{aligned}$$

which describes SHM with $\omega = \sqrt{g/l}$and $T = 2\pi\sqrt{l/g}$ as before.

Notice that both equations have the form

$$E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$$



The geometry of the simple pendulum

where α and β are constants. The constant α corresponds to the inertia of the system through which it can store kinetic energy. The constant β corresponds to the restoring force per unit displacement through which the system can store. When we differentiate the conservation of energy equation with respect to time

$$\frac{dE}{dt} = \alpha v \frac{dv}{dt} + \beta x \frac{dx}{dt} = 0$$

giving

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha} v$$

it follows that the angular frequency of oscillation ω is equal to $\sqrt{\beta/\alpha}$.

Physical pendulum. In a physical pendulum the mass is not concentrated at a point as in the simple pendulum, but is distributed over the whole body. An example of a physical pendulum consists of a uniform rod of length l that pivots about a horizontal axis at its upper end.

Noting that $\tau = I\ddot{\theta} = \mathbf{r} \times \mathbf{F}$

$$\begin{aligned} I\ddot{\theta} &= \frac{l}{2}(-mg)\sin\pi - \theta \\ \frac{1}{3}ml^2\ddot{\theta} &= -\frac{1}{2}mgl\sin\theta \\ \ddot{\theta} &= \frac{3g}{2l}\sin\theta \end{aligned}$$

Again we can use the small-angle approximation to obtain

$$\ddot{\theta} = \frac{3g}{2l}\theta$$

This is SHM with $\omega = \sqrt{3g/2l}$ and $T = 2\pi\sqrt{2l/3g}$.

Potential approach.

Suppose a system is oscillating inside potential $V(x)$. Using Taylor series, we rewite the potential at $x = x_0$ as

$$V(x) = V(x_0) + x\frac{dV}{dx}\bigg|_{x=x_0} + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \dots$$

The first term is a constant, while the second is zero due to dV/dx evaluated at $x = x_0$ is zero. Therefore,

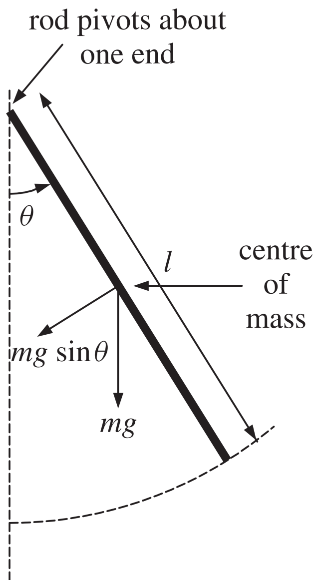
$$V(x) \approx V(x_0) + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

and

$$F = -\frac{dV(x)}{dx} \approx -x\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

Thus its frequency

$$\omega = \left(\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=x_0}\right)^{1/2}$$



Physical pendulum

Similarities in Physics

LC circuit. Initially, capacitor is charged to voltage $V_C = q/C$. Switch then closed and charge begins to flow through the inductor and a current $\dot{q}$ flows in the circuit. This is a time-varying current and produces a voltage across the inductor given $V_L = L\ddot{q}$. We can analyse the LC circuit using Kirchhoff's law, which states that the sum of the voltages around the circuit is zero

$$\begin{aligned} V_C + V_L &= \\ \frac{q}{C} + L\ddot{q} &= 0 \\ \ddot{q} &= -\frac{1}{LC}q \end{aligned}$$

It is of the same form as SHM equation and the frequency of the oscillation is given directly by, $\omega = \sqrt{1/LC}$. Since we have the initial condition that the charge on the capacitor has its maximum value at t = 0, then the solution is

$$q = q_0 \cos \omega t$$

The energy stored in a capacitor charged to voltage V_C is equal to $(1/2)CV_C^2$. This is electrostatic energy. The energy stored in an inductor is equal to $(1/2)LI^2$ and this is magnetic energy. Thus

$$\begin{aligned} E &= \frac{1}{2}CV_C^2 + \frac{1}{2}LI^2 \\ &= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2 \end{aligned}$$

Similarities in physics. We note the similarities in both cases

$$\ddot{Z} = -\frac{\beta}{\alpha} Z \qquad E = \frac{1}{2}\alpha \dot{Z}^2 + \frac{1}{2}\beta Z^2$$

where α and α are constants and $Z = Z(t)$ is the oscillating quantity. In the mechanical case Z stands for the displacement x, and in the electrical case for the charge q.

Damped Harmonic Oscillation

Equation of Motion

The damping force F_d acting on system is proportional to its velocity v so long as v is not too large. In another word

$$F_d = -bv$$

The resulting equation of motion is

$$m\ddot{x} = -kx - b\dot{x}$$

We introduce the parameters

$$\omega_0^2 = \frac{k}{m}$$
$$\gamma = \frac{b}{m}$$

Using these parameters, the equation become

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Now we designate the angular frequency ω_0 and describe it as the natural frequency of oscillation, or the oscillation frequency if there were no damping. We can write the equation as

$$D^2 x + D\gamma x + \omega_0^2 x = 0$$
$$(D^2 + D\gamma + \omega_0^2)x = 0$$

Using the quadratic equation, we find the value of D

$$D = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

The solution is therefore depend on the value of the square root term; which can either be real, imaginary or simply zero. The value of the square root also determine the cases of damping that occur on the system.

Light damping. This case occur if $\gamma^2/4 < \omega_0^2$, which causes the square root term to be imaginary. Let us introduce yet another constant

$$\omega^2 = \omega_0^2 - \gamma^2/4$$

Substituting back into D

$$D = -\frac{\gamma}{2} \pm \sqrt{-\omega^2} = -\frac{\gamma}{2} \pm \omega i$$

Thus, we can say that the equation is second order differential equation with imaginary auxiliary equation roots. The solution is

$$x = A\exp\left(-\frac{\gamma t}{2}\right)\cos\omega t + \phi$$

Now consider the graph of x. The term $\exp -\gamma t/2$ represent an envelope for the oscillations. $x = 0$ occur when $\cos \omega t$ is zero and so are separated by π/ω with period $T = 2\pi/\omega$. Successive maxima are also separated by T. If A_n occurs at time t_0 and A_{n+1} at $t_0 + T$, then

$$x(t_0) = A \exp\left(-\frac{\gamma t_0}{2}\right) \cos \omega t_0$$

$$x(t_0 + T) = A \exp\left(-\frac{\gamma (t_0 + T)}{2}\right) \cos \omega (t_0 + T)$$

Since $\cos \omega t_0 = \cos \omega (t_0 + T) = \cos \omega t_0 + 2\pi$

$$\frac{A_n}{A_{n+1}} = \exp \frac{\gamma T}{2}$$

or the natural logarithm version

$$\ln \frac{A_n}{A_{n+1}} = \frac{\gamma T}{2}$$

which is called the logarithmic decrement and is a measure of this decrease.

Heavy damping. Heavy damping occurs when the degree of damping is sufficiently large that the system returns sluggishly to its equilibrium position without making any oscillations at all. In another words, $\gamma^2/4 > \omega_0^2$ and the square root term is real. Thus, we can say that the equation is second order differential equation with two real auxilary equation roots. The solution is

$$x = A \exp\left[\left(-\frac{\gamma}{2} + \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}\right)t\right] + B \exp\left[\left(-\frac{\gamma}{2} - \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}\right)t\right]$$

Critical damping. Occurs when $\gamma^2/4 = \omega_0^2$, which makes the suqre roots zero. Thus the equation is second order differential equation with one real auxilary equation roots. The solution is

$$x = (At + B) \exp\left(-\frac{\gamma t}{2}\right)$$

Here the mass, or whatever oscillating, returns to its equilibrium position in the shortest possible time without oscillating.

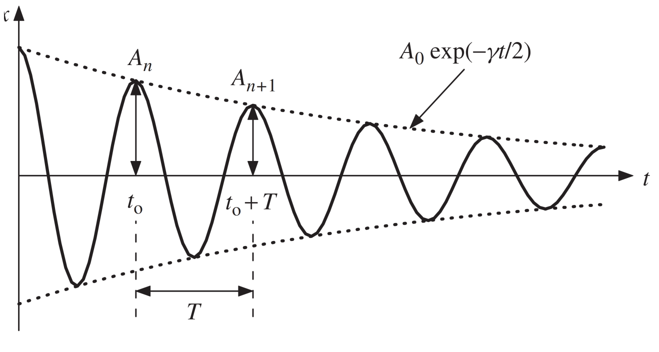


Figure: Graph of $x = A_0 \exp(-\gamma^2 t/4) \cos \omega t$

Putting all together. In summary we find three types of damped motion:

1. $\gamma^2/4 < \omega_0^2$ Light damping, Imaginary square root, Damped oscillations;

2. $\gamma^2/4 > \omega_0^2$ Heavy damping, Real Square root, Exponential decay of displacement;

3. $\gamma^2/4 = \omega_0^2$ Critical damping, Zero square root, Quickest return to equilibrium position without oscillation.

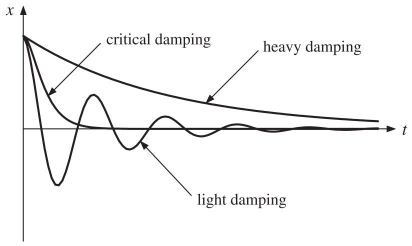


Figure: Motion of a damped oscillator for various cases

RLC circuit.

In the case of an electrical oscillator it is the resistance in the circuit that impedes the flow of current. Kirchoff's law gives

$$
\begin{aligned}
L\ddot{q} + R\dot{q} + \frac{1}{C}q &= 0 \\
\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q &= 0 \\
\ddot{q} + \gamma\dot{q} + \omega_0^2 q &= 0
\end{aligned}
$$

This is the equation of DHO with q as x, L as m, k as 1/C and R as b; so R/L is the equivalent of $\gamma = b/m = R/L$ and $\omega_0^2 = 1/LC$. Now assuming that this this the case of light damping, in another words $R^2/4L^2 < 1/LC$, the solution is

$$q = q_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega t$$

with

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$$

Since the voltage V_C across the capacitor is equal to q/C, dividing the solution by C

$$V_C = V_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega t$$

We find that the quality factor Q of the circuit is given by

$$Q = \frac{\omega_0}{\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Energy of DHO

In the case of very lightly damped oscillator $\gamma^2/4 \ll \omega_0^2$ we have

$$x = A_0 \exp\left(-\frac{\gamma t}{2}\right)\cos\omega_0 t$$
$$v = -A_0\omega_0 \exp\left(-\frac{\gamma t}{2}\right)\left[\sin\omega_0 t + \frac{\gamma}{2\omega_0}\cos\omega_0 t\right]$$

where we approximate $\omega = \omega_0$. Since $\gamma \ll \omega_0$, we can ignore the second term at velocity equation

$$v = -A_0\omega_0 \exp\left(-\frac{\gamma t}{2}\right)\sin\omega_0 t$$

Then

$$E = \frac{1}{2}A_0^2\exp(-\gamma t)(m\omega_0^2\sin^2\omega_0 t + k\cos\omega_0 t)$$

considering $\omega_0^2 = k/m$

$$E(t) = \frac{1}{2}kA_0^2\exp(-\gamma t) = E_0\exp(-\gamma t)$$

The reciprocal of γ is the time taken $\tau = 1/\gamma$ for the energy of the oscillator to reduce by a factor of e^{-1}, thus

$$E(t) = E_0\exp\left(-\frac{t}{\tau}\right)$$

Rate of dissipation. The energy of an oscillator is dissipated because it does work against the damping force at the rate (damping force $\times$ velocity). We can see this by differentiating energy with respect to time

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

since the damping force $F_d = m\ddot{x} + kx = -b\dot{x}$, we can write

$$\frac{dE}{dt} = -b\dot{x}^2$$

Q factor

The quality factor Q of the oscillator describe how good an oscillator is, where we imply that the smaller the degree of damping the higher the quality of the oscillator. Oscillator with a high Q-value would make an appreciable number of oscillations before its energy is reduced substantially. The quality factor Q is defined as

$$Q = \frac{\omega}{\gamma} \approx \frac{\omega_0}{\gamma}$$

Another way to define Q factor is

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$$

Now, consider energy of a very lightly damped oscillator one period apart

$$E_1 = E_0 \exp(-\gamma t)$$
$$E_2 = E_0 \exp[-\gamma(t+T)]$$

giving

$$\frac{E_2}{E_1} = \exp(-\gamma T)$$

Using series expansion

$$\frac{E_2}{E_1} \approx 1 - \gamma T$$

therefore

$$\frac{E_1 - E_2}{E_1} \approx \gamma T \approx \frac{2\pi\gamma}{\omega_0} \approx \frac{2\pi}{Q}$$

where we have $\gamma T \ll 1$ and $\omega \approx \omega_0$. The fractional change in energy per cycle is equal to $2\pi/Q$ and so the fractional change in energy per radian is equal to 1/Q. Thus our definition is proved.

We can also recast DHO equation using Q factor

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$$

and the angular frequency ω

$$\omega = \omega_0 \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

This confirm confirms our assumption that ω is equal to ω_0 to a good approximation under most circumstances. Even when Q is as low as 5, ω is different from ω_0 by just 0.5%.

Forced Oscillation

Undamped forced oscillations.

We begin with a mass m on a horizontal spring with a periodic driving force $F = F_0 \cos \omega t$ is applied to it. We obtain

$$m\ddot{x} + kx = F_0 \cos \omega t$$

Another form of this equation is

$$m\ddot{x} = -k(x - \xi)$$
$$m\ddot{x} + kx = ka \cos \omega t$$

with $\xi = a \cos \omega t$ as displacement due to driving force. Furthermore, we can rewrite the equation as

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \; a \cos \omega t$$
$$(D + \omega_0^2 i)(D - \omega_0^2 i)x = \omega_0^2 \; a \cos \omega t$$

To solve this, first we solve

$$(D + \omega_0^2 i)(D - \omega_0^2 i)X = \omega_0^2 \; a \exp i\omega t$$

This has a particular solution

$$X_p = C \exp i\omega t$$

Thus

$$\ddot{X}_p = -C\omega^2 \exp i\omega t$$

Substituting back, we get

$$(-\omega^2 + \omega_0^2)C \exp i\omega t = \omega_0^2 a \exp i\omega t$$

Solving for C

$$C = \frac{a}{1 - \omega^2/\omega_0^2}$$

The solution to the exponential equation is

$$X = \frac{a}{1 - \omega^2/\omega_0^2}(\cos \omega t + i \sin \omega t)$$

To solve our original, we take the real part

$$x = \frac{a}{1 - \omega^2/\omega_0^2} \cos \omega t$$

The fractional term is the amplitude of our oscillator as function of ω or $A(\omega)$.

Damped forced oscillations.

We add the damping term into our equation

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

or

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 \; a \cos \omega t$$

As before, we write the equation as

$$\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)x = \omega_0^2 \; a \cos \omega t$$

By the method of complex exponentials, we solve first

$$\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)X = \omega_0^2 \; a \exp i\omega t$$

This has a particular solution

$$X_p = C \exp i\omega t$$

thus

$$\begin{aligned} \dot{X} &= C\omega i \exp i\omega t \\ \ddot{X} &= -C\omega \exp i\omega t \end{aligned}$$

Substituting back, we get

$$(-\omega^2 + \omega\gamma i + \omega_0^2)C \exp i\omega t = \omega_0^2 a \exp i\omega t$$

Solving fo C

$$C = \frac{a\omega_0^2}{(\omega_0^2 - \omega^2) + \omega\gamma i} = \frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) - \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

It is convenient to write the complex number C in the polar $|C| \exp i\delta$ form. We have

$$\begin{aligned} |C| &= \left(\frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) - \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) + \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right)^{1/2} \\ &= \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \end{aligned}$$

Angle of $C = -\delta$ is formed by the real term $(\omega_0^2 - \omega^2)$ and imaginary term $-\omega\gamma i$. Thus,

$$C = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \exp(-i\delta)$$

and

$$X_p = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \exp i(\omega t - \delta)$$

To find x_p we take the real part of X_p:

$$x = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \cos(\omega t - \delta)$$

As before, the fractional term is the amplitude of our oscillator. Here δ is phase angle between the driving force and the resultant displacement. The minus sign of δ in Equation implies that the displacement lags behind the driving force and this is indeed the case in forced oscillations. Finally, in order to make our equation more general, we make use of the substitution $F_0 = ka$

$$x = \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \cos(\omega t - \delta)$$

Mechanical impedance. I will now try to solve the damped forced oscillations' equation by introducing mechanical impedance

$$\mathbf{Z}_m = \Re\, b + \Im\, (m\omega - \frac{k}{\omega})$$

where it has complex value. I have also written it in bold to imply that it is a vector quantity, this will be important later. First, I write the constant C not in polar form, but instead I write

$$C = \frac{F_0}{-m\omega^2 + b\omega i + k}$$

where I have substituted the constant. Factoring ωi

$$C = \frac{F_0}{\omega i\left[b + (m\omega - k\omega)i\right]} = \frac{F_0}{\omega i \mathbf{Z}_m}$$

Multiplying with its conjugate

$$C = -\frac{F_0 i}{\omega i Z_m \exp i\delta}$$

where I have written the mechanical impedance in its polar form. Therefore, the solution is

$$x = -\frac{F_0 i}{\omega Z_m \exp i\delta} \exp i\omega t$$

by Euler's formula

$$x = \frac{F_0}{\omega Z_m} \sin \omega t - \delta$$

Maximum amplitude. The amplitude

$$A(\omega) = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}}$$

is maximum when the denominator is minimum

$$\frac{d}{d\omega}\left[\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}\right] = 0$$

from which

$$\omega = \omega_0 \left(1 - \frac{\gamma^2}{2\omega_0^2}\right)^{1/2} \equiv \omega_{\max}$$

follows. We can find the maximum value of the amplitude $A_{\max}$ by substituting $\omega_{\max}$

$$A_{\max} = \frac{a\omega/\gamma}{\left(1 - \gamma^2/4\omega_0^2\right)^{1/2}}$$

In the meantime we use the substitution $Q = \omega_0/\gamma$ in the equations for $\omega_{\max}$ and $A_{\max}$

$$\omega_{\max} = \omega_0 \left(1 - \frac{1}{2Q^2}\right)^{1/2}$$
$$A_{\max} = \frac{aQ}{\left(1 - 1/4Q^2\right)^{1/2}}$$

Power Absorbed

The rate of energy loss due to damping is equal to the damping force times the velocity of the mass. Since the damping force and the velocity are time-dependent, we must define the instantaneous power absorbed in time t by

$$P(t) = b\big[v(t)\big]^2$$

with

$$v = -\frac{a\omega\omega_0^2}{\big[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\big]^{1/2}} \sin\omega t - \delta = -v_0(\omega)\sin\omega t - \delta$$

It is more convenient to talk about average power $\bar{P}(\omega)$ absorbed over a complete cycle of oscillation between times t_o and $t_o + T$

$$\bar{P}(\omega) = \frac{1}{T}\int_{t_0}^{t_0+T} P(t)\, dt = \frac{b\big[v_0(\omega)\big]^2}{2}$$

substituting for b, ω_0^2, and a

$$\bar{P}(\omega) = \frac{\omega^2 F_0^2 \gamma}{2m\big[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\big]}$$

Another method. I will now try tofind the rate of energy loss by using mechanical impedance. The solution of damped forced oscillation has the solution

$$x = \frac{F_0}{\omega Z_m}\sin\omega t - \delta$$

with first derivative

$$\dot{x} = \frac{F_0}{Z_m}\cos\omega t - \delta$$

The power of the oscillator is

$$P(t) = b\dot{x}^2 = \frac{bF_0^2}{Z_m^2}\cos\omega t - \delta$$

with average power of

$$\bar{P} = \frac{1}{T}\int_0^T \frac{bF_0^2}{Z_m^2}\cos\omega t - \delta\; dt = \frac{1}{T}\frac{F_0^2}{Z_m}\frac{b}{Z_m}\frac{T}{2}$$

thus

$$\bar{P} = \frac{F_0^2}{2Z_m}\cos\delta$$

Full width half height. An important parameter of a power resonance curve is its full width at half height ω_{fwhh}; which characterises the sharpness of the response of the oscillator to an applied force. When the driving frequency is close to the frequency ω_0, we can replace

$$\omega^2 - \omega_0^2 = (\omega_0 + \omega) \approx 2\omega_0\Delta\omega$$

where

$$\Delta\omega \equiv \omega - \omega_0$$

With these approximations,

$$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma(4\Delta\omega^2/\gamma + 1)}$$

with maximum value

$$\bar{P}_{\text{max}} = \frac{F_0^2}{2m\gamma}$$

which occur at $\Delta\omega = 0$. The half heights of the curve, equal to $\bar{P}_{\text{max}}/2$. Thus,

$$\omega_{\text{fwhh}} = 2\Delta\omega = \gamma = \frac{\omega_0}{Q}$$

or

$$Q = \frac{\omega_0}{\omega_{\text{fwhh}}} = \frac{\text{resonance frequency}}{\text{full width at half height of power curve}}$$

Electrical Circuit

Applying Kirchoff's law to circuit driven by an alternative (AC) voltage $V(t) = V_0\cos\omega t$ gives the equation

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t$$

Corresponding replacements are

$$\omega_0^2 = \frac{1}{LC} \quad \gamma = \frac{R}{L} \quad Q = R\sqrt{\frac{L}{C}}$$

The solution is

$$q = q(\omega)\cos(\omega t - \delta)$$

where

$$\begin{aligned} q(\omega) &= \frac{V_0/L}{\left[(\omega_0^2 - \omega^2)^2 + (\omega R/L)^2\right]^{1/2}} \\ &= \frac{V_0}{\omega\left[(1/\omega C - \omega L)^2 + R^2\right]^{1/2}} \end{aligned}$$

The current I flowing in the circuit is given by

$$I = -\frac{V_0 \sin \omega t - \delta}{\left[(1/\omega C - \omega L)^2 + R^2\right]^{1/2}}$$

The resultant alternating voltage V_C across the capacitor is equal to q/C, hence

$$V_C = V_C(\omega)\cos(\omega t - \delta)$$

where

$$V_(\omega) = \frac{V_0/LC}{\left[(\omega_0^2 - \omega^2)^2 + (\omega R/L)^2\right]^{1/2}}$$

At resonance when $\omega = \omega_0$, we have

$$V_C(\omega_0) = \frac{V_0}{RC\omega_0} = QV_0$$

We see that the resonance circuit has amplified the AC voltage applied to the circuit by the Q-value of the circuit.

Traveling Wave

Travelling waves may be either transverse waves or longitudinal waves. In transverse waves the change in the corresponding physical quantity, e.g. displacement, occurs in the direction at right angles to the direction of travel of the wave, as for the outgoing ripples on a pond. For longitudinal waves, the change occurs along the direction of travel.

The Wave Equation

One-dimensional wave equation is written as

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

The general solution of it is

$$y = f(x - vt) + g(x + vt)$$

We can write the wave equation more generally as

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

and its general solution as

$$\psi = f(x - vt) + g(x + vt)$$

Traveling Sinusoidal Wave

We represent the travelling sinusoidal wave by

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt) = A \sin(kx - \omega t)$$

where A is the amplitude and λ is the wavelength. This function repeats itself each time x increases by the distance λ. The frequency ν is equal to the velocity v of the wave divided by the wavelength λ

$$\lambda \nu = v$$

The time or period T that a wave crest takes to travel a distance λ is equal to λ / v

$$\nu = \frac{1}{T}$$

Displacement varies sinusoidally with time t with an angular frequency ω where

$$\omega = \frac{2\pi v}{\lambda} = 2\pi\nu$$

We define the quantity $2\pi/\lambda$ as the wavenumber

$$k = \frac{2\pi}{\lambda}$$

Using the relationships $\nu\lambda = v$ and $2\pi\nu = \omega$, we have

$$v = \frac{\omega}{k}$$

Finally, we can write the following alternative mathematical expressions for travelling sinusoidal waves

$$\begin{aligned} y(x,t) &= A \exp \frac{2\pi}{\lambda} i(x - vt) \\ &= A \exp 2\pi i(\frac{x}{\lambda} - \nu t) \\ &= A \exp i(kx - \omega t) \end{aligned}$$

Direction of propagation.

Traveling wave will propagate to positive x direction if the wave number k and angular frequency ω have different sign

$$y = A \exp i(kx - \omega t) \longleftrightarrow y = A \exp i(\omega t - ky)$$

In another hand, wave will propagate to negative x if the wave number k and angular frequency ω have same sign

$$y = A \exp i(kx + \omega t) \longleftrightarrow y = A \exp i(\omega t + ky)$$

Velocity. Three types of velocity. First, particle velocity

$$v_p = \frac{\partial y}{\partial t}$$

which describes oscillation velocity of particle in respect to equilibrium. Second, wave velocity

$$v = \frac{\partial x}{\partial t}$$

which describes velocity of point wave with respect to propagation direction. Third, group velocity

$$v_g = \frac{d\omega}{dk}$$

which describes displacement velocity of group wave, which formed from superposition of multiple waves with different frequency.

Vibrating String

The segment of the string will be subject to a restoring force due to the tension T in the string. We can resolve this force into its components in the x and y direction. At x the y component of the force F_y is equal to $T \sin\theta$. For small values of θ we have

$$\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}$$

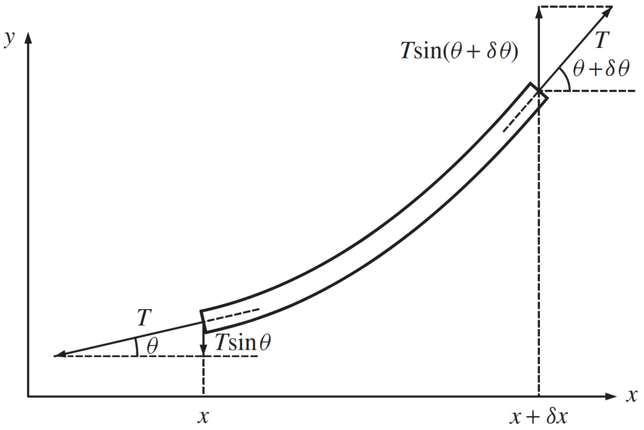


Figure: Vibrating string

and thus

$$F_y(x) = T\frac{\partial y}{\partial x}$$

Similarly, the transverse force at $x + dx$ is equal to the tension T times the slope at that point, which equal slope at x plus rate of change of slope times dx

$$\left.\frac{\partial y}{\partial x}\right|_{x+dx} = \left.\frac{\partial y}{\partial x}\right|_{x} + \frac{\partial}{\partial x}\frac{\partial y}{\partial x}dx = \left.\frac{\partial y}{\partial x}\right|_{x} + \frac{\partial^2 y}{\partial^2 x}dx$$

and thus

$$F_y(x + dx) = T\left[\left.\frac{\partial y}{\partial x}\right|_{x} + \frac{\partial^2 y}{\partial^2 x}dx\right]$$

This acts in the opposite direction to the transverse force at x, therefore

$$\Sigma F_y = T\frac{\partial^2 y}{\partial^2 x}dx$$

We now use Newton's second law

$$\mu dx\frac{\partial^2 y}{\partial^2 t} = T\frac{\partial^2 y}{\partial^2 x}dx$$
$$\frac{\partial^2 y}{\partial^2 t} = \frac{T}{\mu}\frac{\partial^2 y}{\partial^2 x}$$

This is the equation that describes wave motion on a taut string, with

$$v = \sqrt{\frac{T}{\mu}}$$

Kenergy and Venergy and Eenergy

As the wave moves along the string, short segments of width dx will oscillate in the transverse direction and so will have kinetic energy K given by

$$K = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 dx$$

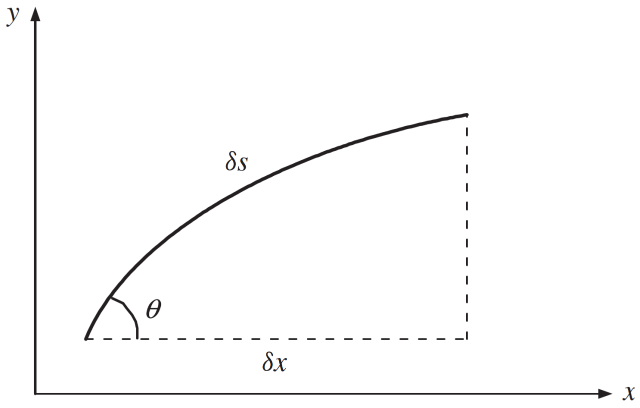


Figure: String under tension

In addition, the segments will be slightly stretched, therefore also have potential energy V. This potential energy is equal to the extension times the tension V in the string, which we assume to be constant. To a good approximation, the extended length of a segment ds is related to the unstretched length dx by

$$ds = \frac{dx}{\cos\theta} = \frac{dx}{(1-\sin^2\theta)^{1/2}} \approx \frac{dx}{(1-\theta^2)^{1/2}} \approx dx\left(1+\frac{1}{2}\theta^2\right)$$
$$\approx dx\left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]$$

To a good approximation the potential energy is therefore given by

$$V = T(ds-dx) = \frac{1}{2}Tdx\left(\frac{\partial y}{\partial x}\right)^2$$

The energy in a portion $a \leq x \leq b$ of a string at time t is given by

$$E = \frac{1}{2}\int_a^b \left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]dx$$
$$E = \frac{1}{2}\mu\int_a^b \left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]dx$$

Example. As an example of the above discussion, we consider the sinusoidal wave

$$y = A\sin kx - \omega t$$

In particular we consider a length of the string equal to one wavelength λ. The kinetic energy of a segment dx is

$$K = \frac{1}{2}\mu\omega^2 A^2\cos^2(kx-\omega t)dx$$

with resultant Kenergy of

$$K = \frac{1}{2}\mu\omega^2 A^2\int_0^\lambda \cos^2(kx-\omega t)dx = \frac{1}{4}\mu\lambda\omega^2 A^2$$

Similarly, we find potential energy U of a string

$$U = \frac{1}{4}\mu\lambda\omega^2 A^2$$

Therefore

$$E = \frac{1}{2}\mu\lambda\omega^2 A^2$$

Power

Energy distribution is carried along with the wave at the velocity v. The distance travelled by the wave in unit time is equal to v. The energy contained within this length is therefore

$$E \times \frac{v}{\lambda} = \frac{1}{2}\mu v\omega^2 A^2$$

In another word

$$P = \frac{1}{2}\mu v\omega^2 A^2$$

or by using impedance

$$P = \frac{1}{2}Z\omega^2 A^2$$

Impedance

Impedance of travelling wave defined as

$$Z \equiv \frac{\text{Transversal force}}{\text{Transversal velocity}} = \frac{F}{v_p}$$

In vibrating wave, the transversal force is the tension, while its transversal velocity is $v = \sqrt{T/\mu}$. For vibrating wave, therefore

$$Z = \frac{T}{\sqrt{T/\mu}} = \mu v$$

We also define transmission coefficient t

$$t \equiv \frac{A_T}{A_I}$$

and reflection coefficient r

$$r \equiv \frac{A_R}{A_I}$$

Not only that, we define, again, reflectance R

$$R \equiv \frac{\text{Reflected wave power}}{\text{Incident Wave power}} = \frac{Z_1 A_R^2}{Z_1 A_I^2} = r^2$$

and transmittance T

$$T \equiv \frac{\text{Transmitted wave power}}{\text{Incident Wave power}} = \frac{Z_2 A_T^2}{Z_1 A_I^2} = \frac{Z_2}{Z_1}t^2$$

It can also be proven that

$$R + T = 1$$

Wave at Discontinuites

The following conditions exist at the boundary between the two strings.

$\mathbf{y_1} = \mathbf{y_2}$ at boundary. Since the two ends of the strings are joined they must move up and down together, the displacements of the strings at the boundary must be the same at $x = 0$, if we define it as the position of the discontinuites, for all times.

$\sum \mathbf{F} = \mathbf{0}$. Otherwise a finite difference in the force would act on an infinitesimally small mass of the string giving an infinite acceleration, which is unphysical. The transverse force is equal to $T(\partial y/\partial x)$; since the tension T is constant, the slopes$(\partial y/\partial x)$ therefore must be the same at $x = 0$ for all times.

We now use these boundary conditions to determine the relative amplitudes and phases of the incident, transmitted and reflected waves. We let the incident wave be

$$y_I = A_I \cos(\omega t - k_1 x)$$

while the reflected wave

$$y_R = A_R \cos(\omega t + k_1 x)$$

and the transmitted wave

$$y_T = A_T \cos(\omega t - k_2 x)$$

Thus, applying condition 1, we get the resultant wave at boundary

$$A_I \cos(\omega t - k_1 x) + A_R \cos(\omega t + k_1 x) = A_T \cos(\omega t - k_2 x)$$

Since this equation must be true for all times we can take $t = 0$ to obtain

$$A_I + A_R = A_T$$

Condition 2 gives

$$k_1 A_I \sin(\omega t - k_1 x) + k_1 A_R \sin(\omega t + k_1 x) = k_2 A_T \sin(\omega t - k_2 x)$$

This time we choose $t = \pi/2\omega$, which gives

$$k_1 A_I + k_1 A_R = k_2 A_T$$

Using those two equation, we get the transmission coefficient of amplitude t:

$$t = \frac{A_T}{A_I} = \frac{2k_1}{k_1 + k_2}$$

and the reflection coefficient of amplitude

$$r = \frac{A_R}{A_I} = \frac{k_1 - k_2}{k_1 + k_2}$$

The transmission coefficient t is always a positive quantity and can have a value within the range 0 to 2. The reflection coefficient r can have both positive and negative values within the range +1 to -1. It also readily follows that

$$t = 1 + r$$

Another method. I will now solve the boundaries condition using impedance. For the first condition, I get the same result

$$A_I + A_R = A_T$$

For the second condition however, I get

$$T_1 \frac{\partial y_I}{\partial t} + T_1 \frac{\partial y_R}{\partial t} = T_2 \frac{\partial y_T}{\partial t}$$

Before, we assume that $T_1 = T_2$; now, however, do not use that assumption and proceed as it is. Then

$$T_1 k_1 A_I \sin(\omega t - k_1 x) + T_2 k_1 A_R \sin(\omega t + k_1 x) = T_2 k_2 A_T \sin(\omega t - k_2 x)$$

As before, we're also evaluating at $x = 0, t = 0$

$$T_1 k_1 A_I + T_2 k_1 A_R = T_2 k_2 A_T$$

This is where I insert concept impedance. Since $Z = \mu v$ for vibrating string,

$$Tk = \mu v^2 \frac{\omega}{v} = Z\omega$$

and since ω is the same for all wave,

$$Z_1 A_I + Z_1 A_R = Z_2 k_2 A_T$$

Solving for t

$$t = \frac{2Z_1}{Z_1 + Z_2}$$

while for r

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

And

$$t = 1 + r$$

also applies.

Impedance Compatibility

When a wave encounters the discontinuity at the boundary between two different strings, there will be a reflected wave. However, by inserting a third string between them, there will be two discontinuities each of which produces a reflection.

If we assume that the wavenumbers in the three strings are $k_3 > k_2 > k_1$, then the reflected waves y_4 and y_5 suffer a phase change of π upon reflection. However, wave y_5 has to travel the additional distance $2L$ before it reaches $x = 0$. Hence, there will be a phase difference $\Delta\phi$

$$\Delta\phi = 2\pi \times \frac{2L}{2\lambda_2}$$

Since maximum destructive interference will occur when $\Delta\phi$ is equal to π

$$L = \frac{\lambda_2}{4}$$

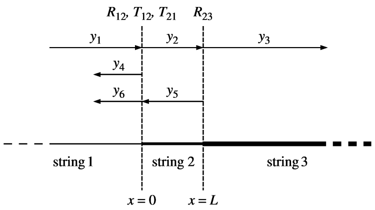


Figure: Two long strings of different mass per unit length connected by an intermediate piece of string.

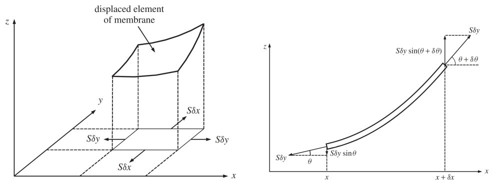


Figure: Taut membrane in the xyz plane and its projection in the xz

Then, we consider reflected the amplitude A_4 of reflected wave y_4

$$A_4 = r_1 A_1$$

for amplitude A_6

$$A_6 = t_1 A_5 = t_1 r_2 A_2 = t_1 r_2 t_2 A_1$$

Hence

$$\frac{A_6}{A_4} = \frac{t_1 r_2 t_2 A_1}{r_1 A_1} = \frac{r_2}{r_1}$$

where we have assumed that t_1 and t_2 are equal to unity. Putting $A_6 = A_4$ as required and subsituting the values of reflection coefficient, we get

$$k_2 = \sqrt{k_1 + k_3}$$

or

$$Z_2 = \sqrt{Z_1 + Z_3}$$

Waves in Two Dimension

We start by considering waves on a taut membrane which is the two-dimensional analogue of the taut string. The membrane has a mass per unit area σ and is stretched uniformly under surface tension S, which has units of force per unit length. From comparison with the one-dimensional result, we see that the resultant force acting on the element in the $x - z$ plane is given by

$$\sum F_{xz} = S\ dy \left[\frac{\partial z}{\partial x}\bigg|_x + \frac{\partial^2 z}{\partial x^2} dx - \frac{\partial z}{\partial x}\bigg|_x \right] = S\ dy \frac{\partial^2 z}{\partial x^2} dx$$

Similarly, the result force in the $y - z$ plane is given by

$$\sum F_{xz} = S\ dx \left[\frac{\partial z}{\partial y}\bigg|_y + \frac{\partial^2 z}{\partial y^2} dx - \frac{\partial z}{\partial y}\bigg|_y \right] = S\ dx \frac{\partial^2 z}{\partial y^2} dy$$

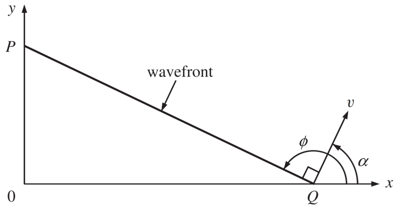


Figure: Wavefront

Thus the total force acting on the element in the z-direction is equal to

$$\sum F_z = S\ dy\ dx \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

Since the mass of the element is $\sigma\ dx\ dy$, we have as the equation of motion

$$\sigma dx dy \frac{\partial^2 z}{\partial t^2} = S\ dy\ dx \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

giving

$$\frac{\partial^2 z}{\partial t^2} = \frac{S}{\sigma}\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

This is the two-dimensional wave equation with

$$v = \sqrt{\frac{S}{\sigma}}$$

For a sinusoidal wave travelling in two dimensions, the corresponding solution of is

$$z(x, y, t) = A \cos k_1 x + k_2 y - \omega t$$

Substituting this solution into our equation gives

$$\omega^2 = v^2(k_1^2 + k_2^2)$$

and hence

$$v = \frac{\omega}{k}$$

with $k = \sqrt{k_1 + k_2}$.

k_1 and k_2 determine the direction of travel as well as the velocity v.

$$\tan\alpha = \frac{k_2}{k_1}$$
$$\tan\phi = -\frac{k_1}{k_2}$$

Waves of Circular Symmetry

In some situations the wavefronts are circular as in outgoing ripples on a pond. Then it is more appropriate to use the polar coordinate.

In this coordinate system a point is specified in terms of r, θ, and z. Again, we specify the displacement of the element by $z = z(r,t)$. Then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x}$$

and

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r}\frac{\partial^2 z}{\partial x^2}$$

We evaluate

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(\sqrt{x^2+y^2}) = \frac{x}{r}$$

and

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$$

Thus

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial r^2}\left(\frac{x}{r}\right)^2 + \frac{y^2}{r^3}\frac{\partial z}{\partial r}$$

Similarly

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2}\left(\frac{y}{r}\right)^2 + \frac{x^2}{r^3}\frac{\partial z}{\partial r}$$

Substituting our result to the equation for two-dimensional wave, we get

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

This is the wave equation for two-dimensional waves of circular symmetry. Its solutions are special functions called Bessel functions. However, at sufficiently large values of r the second term on the left-hand side of becomes negligible compared with the first. The equation then approximates to

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

This equation has the same form as the one-dimensional wave equation and has analogous solutions such as

$$z(r,t) = A\cos kr - \omega t$$

where v now corresponds to the radial velocity dr/dt. Hence, circular waves emanating from a point source become plane waves at large distances from the source.

Waves in Three Dimension

For the case of a wave propagating in a three-dimensional medium the wave equation becomes

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2}$$

with solution

$$\psi(x,y,z,t) = A\sin k_1 x + k_2 y + k_3 z - \omega t$$

and the velocity v is given by

$$v = \frac{\omega}{\sqrt{k_1^2 + k_2^2 + k_3^2}}$$

Spherical waves ψ depends only on the radial distance $r = (x^2 + y^2 + z^2)^{1/2}$ and the time t. Hence, we can write $\psi = \psi(r, t)$ for which it can be shown that the wave equation is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

To find the solutions, we consider the quantity

$$u(r, t) = r\psi(r, t)$$

instead of $\psi(r, t)$. Then

$$\frac{\partial u}{\partial r} = r\frac{\partial \psi}{\partial r} + \psi$$
$$\frac{\partial u^2}{\partial r^2} = r\frac{\partial^2 \psi}{\partial r^2} + 2\psi$$

giving

$$\frac{\partial \psi}{\partial r} = \frac{1}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)$$
$$\frac{\partial \psi^2}{\partial r^2} = \frac{1}{r}\left[\frac{\partial^2 u}{\partial r^2} - \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right]$$

and

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{r}\frac{\partial^2 u}{\partial t^2}$$

Substituting our result to the wave equation, we get

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

one-dimensional wave equation in the variable u, which satisfied by $u = A\cos\omega t - kr$. Thus

$$\psi = \frac{A}{r}\cos\omega t - kr$$

Problem 5.5 from Pain. A point mass M is concentrated at a point on a string of characteristic impedance ρc. A transverse wave of frequency ω moves in the positive x direction and is partially reflected and transmitted at the mass. The boundary conditions are that the string displacements just to the left and right of the mass are equal $y_i + y_r = y_t$ and that the difference between the transverse forces just to the left and right of the mass equal the mass times its acceleration. If A_1, B_1 and A_2 are respectively the incident, reflected and transmitted wave amplitudes show that

$$\frac{B_1}{A_1} = \frac{-iq}{1+iq} \qquad \text{and} \qquad \frac{A_2}{A_1} = \frac{1}{1+iq}$$

where $q = M\omega/2\rho c$.

Let us assume that the force acting on the point mass is described by

$$F = M\frac{\partial^2 y}{\partial t^2} = M\frac{\partial^2}{\partial t^2} A \exp i(kx - \omega t) = -M\omega^2 y$$

Where I have assumed $y = A\exp i(kx - \omega t)$. Its velocity thus

$$v_p = \frac{\partial}{\partial t} A \exp i(kx - \omega t) = \omega i y$$

The Impedance of the point mass M is therefore

$$Z_M = \frac{F}{v_p} = \frac{-M\omega^2 y}{\omega i y} = M\omega i$$

We now can evaluate the impedance Z_2 of string with point mass

$$Z_2 = Z_{\text{string}} + Z_M = \rho c + M\omega i$$

The transmission coefficient is

$$t = \frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2} = \frac{2\rho c}{2\rho c + M\omega i} = \frac{1}{1 + \frac{\omega M}{2\rho c} i} = \frac{1}{1+iq} \qquad \blacksquare$$

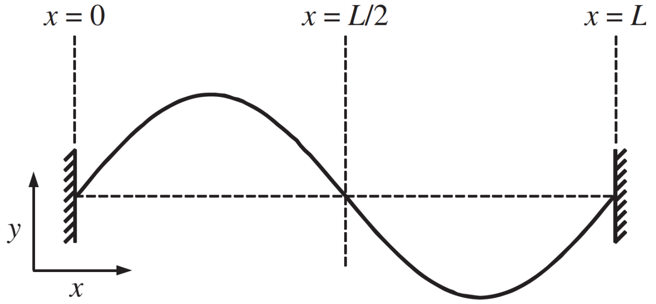
while the reflected coefficient

$$r = \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{-M\omega i}{2\rho c + M\omega i} = \frac{-i\frac{\omega M}{2\rho c}}{1 + \frac{\omega M}{2\rho c} i} = \frac{-iq}{1+iq} \qquad \blacksquare$$

Standing Wave

Standing Wave on a String

We shall explore the physical characteristics of standing waves by considering transverse waves on a taut string. The string is stretched between two fixed points, which we take to be at $x = 0$ and $x = L$, respectively. However, midway between the fixed ends we can see that the displacement of the string is also zero at all times. This point is called a node. Midway between this node and each end point the wave reaches its maximum displacement. These points are called antinodes. The positions of these maxima and minima do not move along the x-axis with time and hence the name standing or stationary waves. The number of antinodes in each standing wave is equal to the respective value of n.



These characteristics suggest that the displacement y can be represented by

$$y(x,t) = f(x)\cos(\omega t + \phi)$$

If we choose the maximum displacements of the particles to occur at $t = 0$, then the phase angle ϕ is zero and

$$y(x,t) = f(x)\cos\omega t$$

We now substitute this solution into the one-dimensional wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

and we obtain

$$-\omega^2 f(x)\cos\omega t = v^2\frac{\partial^2 f(x)}{\partial t^2}\cos\omega t$$
$$\frac{\partial^2 f(x)}{\partial t^2}\cos\omega t = -\frac{\omega^2}{v^2}\cos\omega t$$

Compare this result with the equation of SHM

$$f(x) = A\sin\frac{\omega}{v}x + B\cos\frac{\omega}{v}x$$

The boundary conditions are $f(x) = 0$ at $x = 0$ and at $x = L$. The first condition gives B $= 0$. The second condition gives

$$A\sin\frac{\omega}{v}L = 0$$

which is satisfied if

$$\frac{\omega}{v} L = n\pi$$

Thus, ω must take one of the values given by n, and so we write it as

$$\omega_n = \frac{n\pi v}{L}$$

Substituting for $\omega = \omega_n$ and recalling that B = 0, we obtain

$$f_n(x) = A_n \sin k_n x$$

where

$$k_n = \frac{n\pi}{L}$$

Therefore, we finally obtain

$$y_n(x,t) = A_n \sin k_n x \ \cos \omega_n t$$

This equation describes the standing waves on the string, where each value of n corresponds to a different standing wave pattern. The standing wave patterns are alternatively called the modes of vibration of the string.

Parameter. Here's some parameter used in this section

$$\begin{aligned}
\omega_n &= \frac{n\pi v}{L} & k_n &= \frac{n\pi}{L} \\
T &= \frac{2\pi}{\omega_n} = \frac{2L}{nv} & \lambda_n &= \frac{2L}{n} \\
\nu_n &= \frac{vn}{2L} = \frac{v}{\lambda} & v &= \lambda\nu
\end{aligned}$$

For open-ended wave, we have these parameters

$$\begin{aligned}
\omega_n &= (n-1/2)\frac{\pi v}{L} & k_n &= (n-1/2)\frac{\pi}{L} \\
\nu_n &= (n-1/2)\frac{v}{2L} & \lambda_n &= \frac{2L}{n-1/2}
\end{aligned}$$

In general, sin and cos waves will reach their minima or maxima at

$$2n\pi \quad (2n-1)\pi \quad \frac{2n-1}{2}\pi$$

Standing Wave as Superposition of Two Travelling Waves

A standing wave is the superposition of two travelling waves of the same frequency and amplitude travelling in opposite directions. The general solution of the one-dimensional wave equation is

$$y = f(x - vt) + g(x + vt)$$

A specific example is

$$y = \frac{A}{2}\sin(kx - \omega t) + \frac{A}{2}\sin(kx + \omega t)$$

The first term in the right-hand side of this equation represents a sinusoidal wave of amplitude $A/2$ travelling in the positive x-direction and the second term represents a sinusoidal wave of amplitude $A/2$ travelling in the negative x-direction. Using the identity $\sin(x+y)+\sin(x-y) = 2\sin x \cos y$, we obtain

$$y = A \sin kx \cos \omega t$$

Energy

The general expression for the total energy E contained in a portion $a \leq x \leq b$ of a string vibrating in a single normal mode

$$y_n(x,t) = A_n \sin k_n x \cos \omega_n t$$

that carries a transverse wave

$$E = \frac{1}{2}\mu \int_a^b \left[\left(\frac{\partial y}{\partial t} \right)^2 + v^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

The first term in the integral relates to the kinetic energy of the string and the second term to its potential energy. We now use this expression to find the total energy associated with a standing wave, i.e. the energy of a string of length L vibrating in a single mode.

$$E_n = \frac{1}{2}\mu \int_0^L \left[A_n^2 \omega_n^2 \sin^2 \omega_n t \sin^2 k_n x + v^2 A_n^n k_n^2 \cos^2 \omega_n t \cos^2 k_n x \right] dx$$
$$E_n = \frac{1}{2}\mu \left[A_n^2 \omega_n^2 \sin^2 \omega_n t \frac{L}{2} + v^2 A_n^n k_n^2 \cos^2 \omega_n t \frac{L}{2} \right]$$
$$E_n = \frac{1}{2}\mu A_n^2 \left[\frac{v^2 \pi^2 n^2}{2L} \sin^2 \omega_n t + \frac{v^2 \pi^2 n^2}{2L} \cos^2 \omega_n t \right]$$
$$E_n = \frac{1}{2}\mu A_n^2 \frac{v^2 \pi^2 n^2}{2L}$$
$$E_n = \frac{1}{4}\mu L A_n^2 \omega_n^2$$

Multiple modes. The general superposition of normal modes is given by,

$$y(x,t) = \sum_n A_n \sin k_n x \cos \omega_n t$$

and we must use this expression, for calculating the energy E of the wave

$$E = \frac{1}{2}\mu \int_a^b \left[\left(\frac{\partial y}{\partial t} \right)^2 + v^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

The expressions for the derivatives are

$$\frac{\partial y}{\partial t} = -\sum_n A_n \omega_n \sin\left(\frac{n\pi}{L} x \right) \sin \omega_n t$$

and

$$\frac{\partial y}{\partial x} = \sum_n A_n k_n \cos\left(\frac{n\pi}{L} x \right) \cos \omega_n t$$

Squaring these derivatives

$$\left(\frac{\partial y}{\partial t}\right)^2 = \sum_m A_m \omega_m \sin\left(\frac{m\pi}{L}x\right) \sin \omega_m t \sum_n A_n \omega_n \sin\left(\frac{n\pi}{L}x\right) \sin \omega_n t$$

and

$$\left(\frac{\partial y}{\partial x}\right)^2 = \sum_m A_m k_m \cos\left(\frac{m\pi}{L}x\right) \cos \omega_m t \sum_n A_n k_n \cos\left(\frac{n\pi}{L}x\right) \cos \omega_n t$$

will lead to cross terms containing the products

$$\sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \quad \text{and} \quad \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right)$$

As a consequence, the expression for the energy E will contain integrals over these product terms. However, the integrals involving the cross terms ($m \neq n$) have the value 0. Hence, the cross terms with $m \neq n$ vanish in the integration and the total energy E is given by

$$E = \frac{1}{4}\mu L \sum_n A_n^2 \omega_n^2$$

Fourier Analysis

The idea that an essentially arbitrary function $f(x)$ can be expanded in a Fourier series of these sine functions with appropriate values for the coefficients

$$f(x) = \sum_n A_n \sin \frac{n\pi}{L} x$$

The expression for the Fourier amplitude is

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Proof. Multiplying the expression for Fourier expansion with sin $(m\pi x/L)$ and integrating the resulting equation with respect to x over the range $x = 0$ to $x = L$ gives

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L f(x) \sum_n A_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

Since only the term with $m = n$ is different from zero, and has the value $L/2$

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = A_n \frac{L}{2}$$

In this way we obtain the final expression for the Fourier amplitude.

Interference

We write the interference conditions as

$$s = n\lambda \qquad \text{constructive interference}$$
$$s = \left(n + \frac{1}{2}\right)\lambda \qquad \text{destructive interference}$$

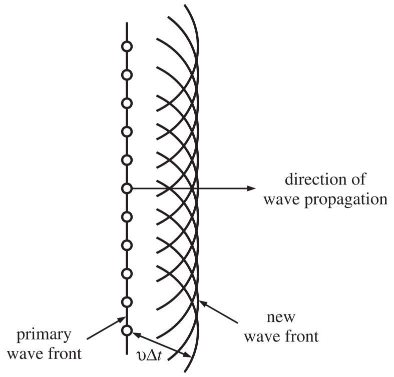
where s is the difference in their path lengths from the common source. For other values of path difference s the resulting amplitude will lie between these two extremes of total constructive and destructive interference. Since phase difference $\phi = 2\pi s/\lambda$, we can also write the interference conditions as:

$$\phi = 2n\pi \qquad \text{constructive interference}$$
$$\phi = (2n + 1)\pi \qquad \text{destructive interference}$$

These are the basic results for the interference of waves.

Huygen's Principle

Huygen postulated that each point on a primary wavefront acts as a source of secondary wavelets such that the wavefront at some later time is the envelope of these wavelets. Consider a plane wave. Each point on the primary wavefront acts as a source of secondary wavelets. These secondary wavelets combine and their envelope represents the new wavefront, which is also a plane wave.



Young's double-slit experiment

A monochromatic plane wave of wavelength λ is incident upon an opaque barrier that contains two very narrow slits S_1 and S_2. Since these secondary wavelets are driven by the same incident wave there is a well-defined phase relationship between them. This condition is called coherence and implies a systematic phase relationship between the secondary wavelets when they are superposed at some distant point P.

The separation of the slits is a, typically ≈ 0.5 mm while the distance L to the screen is typically of the order of a few metres. Hence, $L \gg a$ and $\lambda \approx da$. The point of this approximation is that the small slit only able to pass piece of wave, unlike the cases of diffraction.

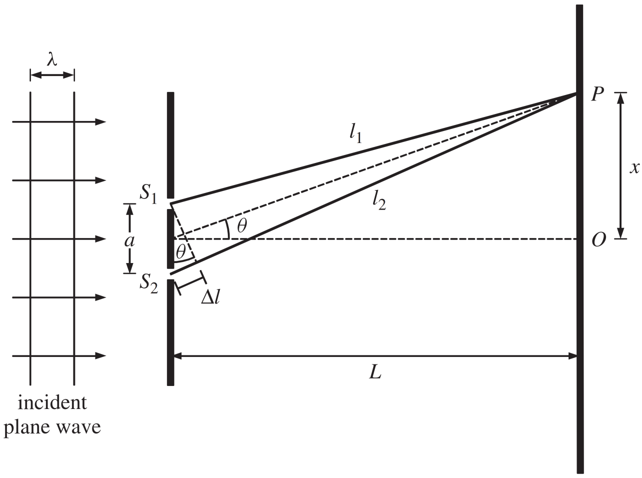


Figure: Schematic diagram of Young's double-slit experiment

We consider the secondary wavelets from S_1 and S_2 arriving at an arbitrary point P on the screen. The superposition of the wavelets at P gives the resultant amplitude.

$$\begin{aligned} R &= A[\cos(\omega t - kl_1) + \cos(\omega t - kl_2)] \\ &= 2A\cos\left[\omega t - \frac{k(l_1+l_2)}{2}\right]\cos\left[\frac{k(l_2-l_1)}{2}\right] \end{aligned}$$

Since $L \gg a$, the lines from S_1 and S_2 to P can be assumed to be parallel and also to make the same angle θ with respect to the horizontal axis. Hence,

$$l_1 \approx \frac{L}{\cos\theta} \approx l_2$$

and so

$$l_1 + l_2 = \frac{2L}{\cos\theta} \approx 2L$$

Hence, we can write the resultant amplitude as

$$R = 2A\cos(\omega t - kL)\cos(k\Delta l/2)$$

The intensity I at point P is equal to the square of the resultant amplitude R:

$$I = 4A^2\cos^2(\omega t - kL)\cos^2(k\Delta l/2)$$

This equation describes the instantaneous intensity at P. The time average of the intensity is given by

$$I = 2A^2\cos^2(k\Delta l/2) = I_0\cos^2(k\Delta l/2)$$

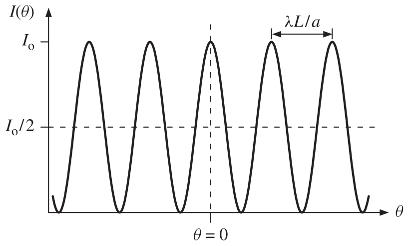


Figure: The interference pattern observed in Young's double-slit experiment

We see that $\Delta l \approx a \sin\theta$. Substituting for Δl and making the small angle approximation, we get

$$I(\theta) = I_0 \cos^2 \frac{\pi a \theta}{\lambda}$$

Intensity maxima occur when

$$\theta = \frac{n\lambda}{a}$$

and so the bright fringes occur at distances from the point O given by

$$x = L\theta = \frac{n\lambda}{a} L$$

Similarly, intensity minima occur when

$$\theta = (n + 1/2)\frac{\lambda}{a}$$

and so the dark fringe

$$x = \frac{(n + 1/2)}{a}\lambda L$$

Another Derivation. From the figure we have

$$\begin{aligned} l_1^2 &= l^2 + \frac{a^2}{4} - al\cos\frac{\pi}{2} + \theta = l^2 + \frac{a^2}{4} + al\sin\theta \\ l_2^2 &= l^2 + \frac{a^2}{4} - al\cos\frac{\pi}{2} - \theta = l^2 + \frac{a^2}{4} - al\sin\theta \end{aligned}$$

Then

$$l_2^2 - l_1^2 = 2al\sin\theta$$

Rewriting the right side as

$$l_2^2 - l_1^2 = (l_2 + l_1)(l_2 - l_1) \approx 2l\Delta l$$

Thus we have

$$\Delta l = a\sin\theta$$

Applying the interference conditions, we get the same results.

Critical angle. We emphasise that there would be no interference pattern if the two sources of secondary wavelets S_1 and S_2 were not coherent. Instead, the resultant intensity would be uniform across the screen with a value equal to $I_o/2$. We could ensure that the secondary wavelets from the two slits are coherent by illuminating them with a point source. However, we can still obtain an interference pattern with such a source if its spatial extent is smaller than a critical value.

Consider an extended source of width w that is used to illuminate the two slits S_1 and S_2. An extended source of width w behaves like a coherent light source so long

$$w \ll \frac{2l\lambda}{a}$$

is satisfied. The extended source subtends an angle θ at each slit where

$$\theta \ll \frac{2\lambda}{a} = \frac{w}{l}$$

which gives the maximum divergence angle that the source can have to produce clear interference fringes.

Finite Width. Consider each of the two slits to be composed of infinitely narrow strips that act as sources of secondary wavelets. Then the resultant amplitude R at a point P is the superposition of the secondary wavelets from both slits. This is given by

$$R = \int_{-a/2-d/2}^{-a2+d/2} \alpha \cos[\omega t - k(l - x\sin\theta)]\, dx + \int_{a/2-d/2}^{a2+d/2} \alpha \cos[\omega t - k(l - x\sin\theta)]\, dx$$

where d is the width of each slit and a is their separation. Evaluating these integrals gives

$$R = 2\alpha d \cos(2\omega t - kl)\frac{\sin\beta}{\beta}\cos\beta$$

The resultant intensity is

$$I(\theta) = I_0 \frac{\sin^2\beta}{\beta^2}\cos^2\beta$$

where I_o is the maximum intensity of the pattern. This result is the product of two functions: square of a sinc function–corresponding to diffraction at a single slit–and cosine-squared term–corresponding to double-slit interference pattern.

Michelson Spectral Interferometer

Michelson spectral interferometer is an important example of interference by division of amplitude, whereas Young's double-slit experiment is an example of interference by division of wavefront. Consider beam of light froma monochromatic source is split into two equal beams

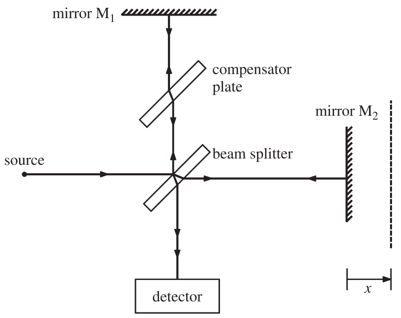


Figure: Schematic diagram of the Michelson spectral interferometer.

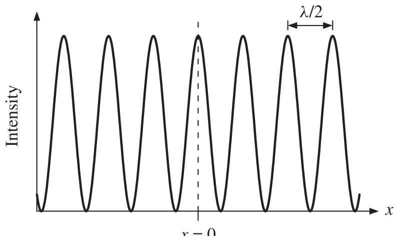


Figure: The measured light intensity is plotted as a function of the displacement x of the moveable mirrorM_2.

by the semi-reflecting front face of the beam splitter. The two separate beams travel to mirror M_1 and M_2, respectively, and then return to the beamsplitter from where they travel along the same path to the detector. Mirror M_1 is fixed in position. The position of mirror M_2 can be adjusted with a very fine micrometer screw.

If the path difference is an integral number of wavelengths, the beams will interfere constructively. However, if the path lengths are different by an odd number of half-wavelengths, there will be destructive interference and the detected light intensity will be zero. When the detected light intensity is plotted as a function of the displacement x of mirror M_2 an interference pattern is obtained.

Another example

Bragg's law. The angles for constructive interference are given by

$$2d \sin \theta = n\lambda$$

where d is the separation of the atomic planes and λ is the wavelength.

Diffraction at a Single Slit

Any obstacle in the path of the wave affects the way it spreads out; the wave appears to "bend" around the obstacle. Similarly, the wave spreads out beyond any aperture that it meets. Such bending

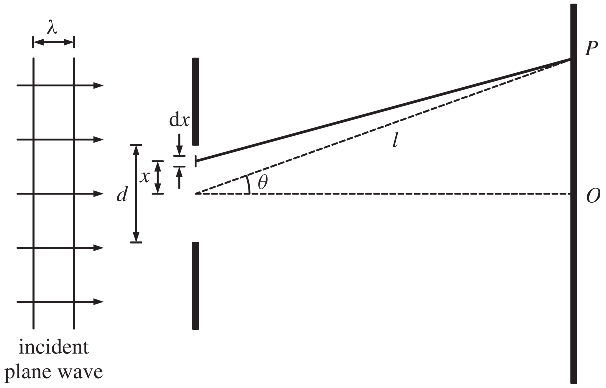


Figure: (Fraunhofer) Diffraction at a single slit.

or spreading of the wave is called diffraction. Diffraction can also be thought as interfere from many sources.

In our discussion of Young's double-slit experiment, we considered the width of each slit to be very narrow. In practice a real slit is not arbitrarily narrow but has a finite extent. Hence, the path lengths from different points across the slit to the point P will be different and consequently the secondary wavelets arriving at P will have a variation in phase.

Fraunhofer's diffraction. Consider a monochromatic plane wave of wavelength λ that is incident on a single slit in an opaque barrier. The point P was sufficiently far from the slit that the secondary wavelets had become plane waves by the time they reached P.

The amplitude dR of the wavelet arriving at P from the strip dx at x is proportional to the width dx of the strip, and its phase depends on the distance of P from the strip

$$dR = \alpha \cos[\omega t - k(l - x\sin\theta)]\, dx$$

where α is a constant. The resultant amplitude at P due to the contributions of the secondary wavelets from all the strips is

$$R = \int_{-d/2}^{d/2} \alpha \cos[\omega t - k(l - x\sin\theta)]\, dx$$

The integral can be evaluated to

$$R = -\frac{\lambda}{k\sin\theta} \sin[\omega t - kl - kx\sin\theta]$$

Inserting the limit and doing some algebraic manipulation, we get

$$R = \frac{\alpha d}{(kd/2)\sin\theta} \sin\left(\frac{kd}{2}\sin\theta\right) \cos(\omega t - kl)$$

Since, the instantaneous intensity I at P is equal to the square of the amplitude R and that the time average over many cycles of

$\cos^2(\omega t - kl)$ is equal to 1/2, the time average of the intensity is given by

$$I(\theta) = I_0 \frac{\sin^2 \beta}{\beta^2}$$

where

$$I_0 = \frac{\alpha^2 d^2}{2} \quad \text{and} \quad \beta = \frac{kd}{2} \sin\theta$$

The square of a sinc function has its maximum value of unity when $\beta = 0$. The maximum intensity I_o thus occurs when $\theta = 0$. The zeros in the intensity occur when

$$\frac{\sin^2 \beta}{\beta^2} = 0$$

$$\sin^2 \beta = 0$$

where I have assumed that β is not zero. Then

$$\frac{kd}{2} \sin\theta = n\pi$$

using $k = 2\pi/\lambda$

$$\sin\theta = n\frac{\lambda}{d}$$

In general, zeros in intensity occur when

$$\theta = n\frac{\lambda}{d}$$

where n is any interger except zero.

Fresnel diffraction. The case of Fresnel diffraction occurs when the source of the primary waves or P is so close to the slit that we have to take into account the curvature of the incoming or outgoing wavefronts. The path-length difference s at a distance x from the centre of the slit is not linearly proportional to x. It is easy to show that s is given by

$$s = \frac{x^2}{2R}$$

when $x^2/R^2 \ll 1$. Hence, the phase difference $\phi(x)$ for a point at x is

$$\phi(x) = \frac{2\pi x^2}{2\lambda R}$$

Circular apertures A circular aperture will also produce a diffraction pattern. For an aperture of diameter d, the first zeros on either side of the central maximum occur at angles $\pm\theta_R$, where

$$\theta_R = 1.22\frac{\lambda}{d}$$

The Rayleigh criterion states that the images of the two point objects can be just resolved when the maximum of one diffraction pattern overlaps the first minimum of the other. If two objects with a spatial separation b are at a large distance L from a lens or mirror, then we can write $\theta = b/L$. Hence, we can just resolve them if

$$b = 1.22\frac{\lambda L}{d}$$

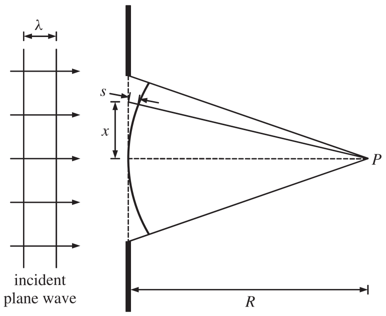


Figure: (Fresnel) Diffraction at a single slit.

Appendix I: Tugas Gelombang 20 November 2024

King 7.3: *"Plane waves of monochromatic light of wavelength 500 nm are incident upon a pair of very narrow slits producing an interference pattern on a screen. When one of the slits is covered by a thin film of transparent material of refractive index 1.60 the central (n = 0) bright fringe moves to the position previously occupied by the n =* 15 *bright fringe. What is the thickness of the film?"*

Jarak titik terang $n = 15$ setelah diberikan film tipis adalah

$$x_{15} = \frac{15\lambda_u L}{a},$$

dengan a sebagai jarak antar celah dan λ_u sebagai panjang gelombang di udara. Perbedaan lintasan Δl timbul akibat adanya film tipis. Panjang Δl adalah

$$\Delta l = a \sin\theta = a\frac{x_{15}}{L} = a\frac{15\lambda L}{aL} = 15\lambda_u.$$

Mengingat syarat terjadinya interferensi konstruktif

$$s = m\lambda.$$

Maka, panjang Δl adalah

$$\Delta l = m\Delta\lambda = m(\lambda_u - \lambda_r).$$

dimana λ_r sebagai panjang gelombang setelah diberikan film tipis. Nilai m meupakan interger yang dipengaruhi oleh panjang film tipis t:

$$m = \frac{t}{\lambda_r}.$$

Sehingga

$$\begin{aligned}
\frac{t}{\lambda_r}(\lambda_u - \lambda_r) &= 15\lambda_u \\
t &= 15\frac{\lambda_u\lambda_r}{\lambda_u - \lambda_r} \\
t &= 15\frac{\lambda_u\lambda_u/n}{\lambda_u - \lambda_u/n} \\
t &= 15\frac{\lambda_u^2}{n\lambda_u - \lambda_u} \\
t &= 15\frac{\lambda_u}{n-1}
\end{aligned}$$

dan diperoleh

$$t = 15\frac{500 \cdot 10^{-9}\ \text{m}}{1.6 - 1} = \boxed{1.25 \cdot 10^{-5}\ \text{m}}$$

King 7.4: *"(a) Estimate the divergence angle of the sunlight we receive on Earth given that the diameter of the Sun is* $1.4 \cdot 10^6$ *km and its distance from the Earth is* $1.5 \cdot 10^8$ *km. (b) In a Young's double-slit experiment, the slit spacing is 0.75 mm and the wavelength of the* *incident light is 550 nm. What should be the maximum divergence angle of the source for the interference fringes to be clearly visible? Compare this value with your answer from (a)."*

(a) Sudut maksimum divergensi diberikan oleh

$$\theta \approx \frac{w}{l}.$$

dengan w dengan panjang sumber cahaya dan l sebagai jarak dari sumber. Maka,

$$\theta \approx \frac{1.4 \cdot 10^6 \text{ km}}{1.5 \cdot 10^8 \text{ km}} = \boxed{9.3 \cdot 10^{-3} \text{ rad.}}$$

(b) Sudut maksimum divergensi juga diberikan oleh

$$\theta \approx \frac{2\lambda}{a},$$

dengan a sebagai jarak antar celah. Maka,

$$\theta \approx \frac{2 \cdot 550 \cdot 10^{-9} \text{ m}}{0.75 \cdot 10^{-3} \text{ m}} = \boxed{1.47 \cdot 10^{-3} \text{ rad.}}$$

Nilai yang diperoleh pada bagian (b) lebih kecil dibandingkan nilai pada bagian (a).

King 7.5: *"The two slits in a Young's double-slit experiment each have a width of 0.06 mm and are separated by a distance a. If an* $n = 15$ *bright fringe of the double-slit interference pattern falls at the first minimum of the diffraction pattern due to each slit, what is the value of the separation of the slits a?"*

Titik terang pada interferensi celah ganda diberikan oleh

$$\theta_i = n\frac{\lambda}{a},$$

sedangkan titik gelap pada celah difraksi adalah

$$\theta_d = n\frac{\lambda}{d}$$

Diketahui bahwa titik terang interferensi ke 15 jatuh pada titik gelap difraksi pertama, maka

$$\begin{gathered}\theta_i(n = 15) = \theta_d(n = 1) \\ \frac{\lambda}{d} = 15\frac{\lambda}{a} \\ a = 15d\end{gathered}$$

dan diperoleh

$$a = 15 \cdot 0.06 \text{ mm} = \boxed{0.9 \text{ mm}}$$

King 7.6: *"Two loudspeakers are separated by a distance of 1.36 m. They are connected to the same amplifier and emit sound waves of frequency 1.0 kHz. How many maxima in sound intensity would you hear if you walked in a complete circle around the loudspeakers at a* *large distance from them? Assume that the sound waves are emitted isotropically."*

Panjang gelombang diberikan oleh

$$\lambda = \frac{v}{f} = \frac{340 \text{ m/s}}{10^3 \text{ kHz}} = 0.34 \text{ m}.$$

dimana kecepatan gelombang suara diasumsikan 340 m/s (perintah soal). Kemudian, syarat terjadinya interferensi maksima adalah

$$\sin\theta = n\frac{\lambda}{d} = n\frac{0.34 \text{ m}}{1.36 \text{ m}} = \frac{n}{4}.$$

Nilai $\sin\theta$ akan berubah seiring kita melingkari speaker. Selajutnya dicari nilai n yang memenuhi persamaan

$$\sin\theta = \frac{n}{4}$$

selama satu siklus fungsi sinus. Pada kuadran pertama ($0 \leq \sin\theta < 1$), nilai n yang memenuhi adalah

$$n = \{0, 1, 2, 3\}.$$

Pada kuadran kedua ($1 \leq \sin\theta < 0$),

$$n = \{4, 3, 2, 1\}.$$

Pada kuadran ketiga ($0 \leq \sin\theta < -1$),

$$n = \{0, 1-, 2, -3\}.$$

Terakhir, pada kuadran kedua ($-1 \leq \sin\theta < 0$),

$$n = \{-4, -3, -2, -1\}$$

Karena terdapat 16 nilai n yang memenuhi, maka maksima terjadi sebanyak $\boxed{\text{16 kali}}$.

King 7.7: *" (a) Monochromatic light is directed into a Michelson spectral interferometer. It is observed that 4001 maxima in the detected light intensity span exactly 1.0 mm of mirror movement. What is the wavelength of the light? (b) Light from a sodium discharge lamp is directed into a Michelson spectral interferometer. The light contains two wavelength components having wavelengths of 589.0 nm and 589.6 nm, respectively. The interferometer is initially set up with its two arms of equal length so that a maximum in the detected light is observed. How far must the moveable mirror be moved so that the 589.0 nm component produces one more maximum in the detected intensity than the 589.6 nm component?"*

(a) Panjang gelombang dapat diperoleh dengan

$$\lambda = 2 \cdot M,$$

dimana M adalah jarak antar maksima. Jarak tersebut diperoleh dengan

$$M = \frac{\text{Perpindahan cermin}}{\text{Maxima terdeteksi}}.$$

Sehingga diperoleh

$$\lambda = 2\frac{10^{-3}\ \text{m}}{4001} = \boxed{5 \cdot 10^{-7}\ \text{m}}$$

(b) Diketahui syarat terjadinya peristiwa tersebut adalah

$$\Delta l = n\lambda_1 = (n+1)\lambda_2,$$

dimana $\lambda_1 = 589.6$ nm dan $\lambda_2 = 589.0$ nm. Sedangkan,

$$\Delta l = 2x$$

dimana x adalah perpindahan cermin. Maka diperoleh

$$\lambda_1 = \frac{2x}{n}, \tag{5}$$

$$\lambda_2 = \frac{2x}{n+1}. \tag{6}$$

Mengalikan persamaan 5 dan 6, diperoleh

$$\begin{aligned} \lambda_1\lambda_2 &= \frac{4x^2}{n(n+1)} \\ n(n+1) &= \frac{4x^2}{\lambda_1\lambda_2}. \end{aligned}$$

Mengurangi persamaan 5 dan 6, diperoleh

$$\begin{aligned} \lambda_1 - \lambda_2 &= \frac{2x(n+1) - 2xn}{n(n+1)} \\ &= \frac{2x}{n(n+1)} \\ n(n+1) &= \frac{2x}{\lambda_1 - \lambda_2}. \end{aligned}$$

Dengan demikian,

$$\begin{aligned} \frac{4x^2}{\lambda_1\lambda_2} &= \frac{2x}{\lambda_1 - \lambda_2} \\ x &= \frac{\lambda_1\lambda_2}{2(\lambda_1 - \lambda_2)}, \end{aligned}$$

dan diperoleh

$$x = \frac{589.6 \cdot 589 \cdot 10^{-16}\ \text{m}^2}{2 \cdot 10^{-9}(589.6 - 589)\ \text{m}} = \boxed{2.89 \cdot 10^{-4}\ \text{m}}$$

Dispersion Dispersion is the dependence of the velocity of a wave on its frequency, leading to the separation of different frequency components in a medium.

Beats

Beats refer to phenomenon where waves sometimes add constructively and another times destructively because of their different frequencies. Consider superposition of two monochromatic waves

$$\psi_1 = A\cos(k_1 x - \omega_1 t), \quad \psi_2 = A\cos(k_2 x - \omega_2 t)$$

that have the same amplitude A but different frequencies ω_1 and ω_2, respectively. In a non-dispersive medium, the two waves travel at the same velocity. The superposition of the two waves gives

$$\psi(x,t) = A\big[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\big]$$
$$\psi(x,t) = 2A\cos\left[\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t\right]\cos\left[\frac{k_2 + k_1}{2}x - \frac{\omega_2 + \omega_1}{2}t\right]$$

We consider how ψ varies at a fixed value of position x

$$\psi(0,t) = 2A\cos\left[\frac{\omega_2 - \omega_1}{2}t\right]\cos\left[\frac{\omega_2 + \omega_1}{2}t\right]$$

The resultant wave is contained within an envelope $A(t)$ given by

$$A(t) = 2A\cos\left[\frac{\omega_2 - \omega_1}{2}t\right]$$

Thus, we can write

$$\psi(0,t) = A(t)\cos\omega_0 t$$

where $\omega_0 = (\omega_2 + \omega_1)/2$.

Amplitude Modulation

A carrier wave of frequency ω_c is modulated by a sinusoidal wave of frequency ω_m, where $\omega_c \ll \omega_m$, The resultant wave can be represented by

$$\begin{aligned}\psi &= (A + B\cos\omega_m t)\sin\omega_c t \\ &= A\sin\omega_c t + \frac{B}{2}\big[\sin(\omega_c + \omega_m)t - \sin(\omega_c + \omega_m)t\big]\end{aligned}$$

which shows that there are three frequency components present in the modulated wave.

Phase and Group Velocities

We again consider the superposition of two monochromatic waves that have the same amplitude but slightly different frequencies. The superposition of ψ_1 and ψ_2 is the same as before

$$\psi(x,t) = 2A\cos\left[\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t\right]\cos\left[\frac{k_2 + k_1}{2}x - \frac{\omega_2 + \omega_1}{2}t\right]$$

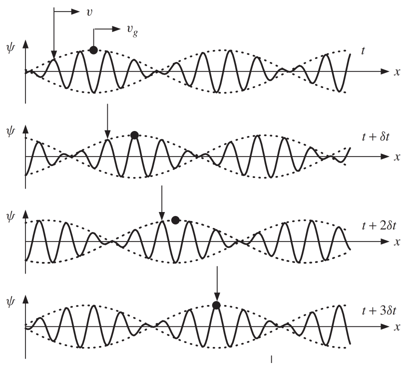


Figure: The propagation of the modulated wave ψ in a dispersive medium

We let

$$k_0 = \frac{k_2 + k_1}{2}, \quad \Delta\omega_o = \frac{\omega_2 + \omega_1}{2}$$

and

$$\Delta k = \frac{k_2 - k_1}{2}, \quad \Delta\omega = \frac{\omega_2 - \omega_1}{2}$$

In this case

$$\psi(x,t) = A(x,t)\cos(k_0 x - \omega_o t)$$

where

$$A(x,t) = 2Acos(\Delta k x - \omega t).$$

This equation represent a wave that has a frequency ω_o, a wavenumber k_0 phase velocity v given by

$$v = \frac{\omega_o}{k_0} = \frac{\omega}{k}\bigg|_{k=k_0}$$

and group velocity v_p

$$v_p = \frac{\Delta\omega}{\Delta\omega} = \frac{d\omega}{dk}\bigg|_{k=k_0}$$

The phase velocity v represent the velocity of modulated wave $\psi(x,t)$, while group velocity v_g represent the velocity of an envelope wave $A(x,t)$.

Proof. From the equation representing modulated wave ψ

$$\psi = A(x,t)\cos(k_0 x - \omega_o t)$$

we know that

$$k_0 x - \omega_o t = \text{Const.}$$

Rearangging this equation, we get the phase velocity at which the wave travels. As for the modulated term

$$A = 2Acos(\Delta k x - \omega t).$$

we have

$$\Delta k x - \omega t = \text{Const.}$$

Differentiating this equation with respect to t, we obtain the velocity at which the envelope travels.

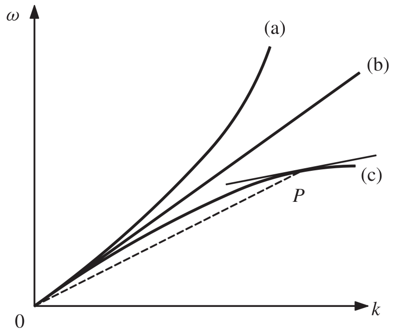


Figure Plots of frequency ω against wavenumber k for various dispersion relations

Dispersion Relation

The relationship between the frequency ω and the wavenumber k is called the dispersion relation of the medium. The dispersion relation is determined by the physical properties of the medium. In a non-dispersive medium, the velocity of a wave is independent of the wavenumber k

$$\omega = \text{Const.} \times k$$

The expression for the group velocity, may be rewritten in various different forms

$$v_g = \frac{d}{dk}kv = v + k\frac{dv}{dk} = v + k\frac{dv}{d\lambda}\frac{d}{dk}\frac{2\pi}{k}$$

and hence

$$v_g = v - \lambda\frac{dv}{d\lambda}$$

Usually $dv/d\lambda$ is positive and so $v_g < v$. This is called normal dispersion (c). Anomalous dispersion occurs when $dv/d\lambda$ is negative so that $v_g > v$ (a). If there is no dispersion, $dv/d\lambda = 0$ and the group and phase velocities are equal (b).

Electromagnetic Waves. We can apply these considerations to the propagation of electromagnetic waves. Electromagnetic waves travel with phase velocity

$$v = \frac{c}{\sqrt{1 - \frac{\omega_0^2}{\omega^2}}}$$

and group velocity

$$v_g = c\sqrt{1 - \frac{\omega_0^2}{\omega^2}}$$

Proof. In vacuum, electromagnetic waves propagate with a velocity

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

while in dielectric

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

where ϵ and μ are the permittivity and permeability of the material, respectively. Now conside the refractive index n

$$n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \mu_r\epsilon_r$$

where μ_r and ϵ_r are the relative permittivity and permeability of the material, respectively. For most materials μ_r is constant and approximately equal to 1, but ϵ_r does vary with frequency. Thus we can write

$$v_g = v - \lambda\frac{d}{d\epsilon_r}\left(\frac{c}{\sqrt{\mu_r\epsilon_r}}\right)\frac{d\epsilon_r}{d\lambda} = v + \lambda\frac{v}{2\epsilon_r}\frac{d\epsilon_r}{d\lambda}$$

The dispersion relation for Electromagnetic waves is

$$\omega^2 = \omega_o^2 + c^2k^2$$

for frequencies greater than ω_0 where ω_0 is a constant called the plasma oscillation frequency. Differentiating this equation gives

$$2\omega\ d\omega = 2kc^2\ dk$$
$$\frac{d\omega}{dk} = c^2\frac{k}{\omega}$$

The phase velocity is the given by

$$v = \frac{\sqrt{\omega_o^2 + c^2k^2}}{k}$$
$$v = \frac{c}{ck\left(\dfrac{1}{\omega_o^2 + c^2k^2}\right)^{1/2}}$$
$$v = \frac{c}{\left(\dfrac{c^2k^2 + \omega_0^2 - \omega_0^2}{\omega_o^2 + c^2k^2}\right)^{1/2}}$$
$$v = \frac{c}{\left(1 - \dfrac{\omega_0^2}{\omega^2}\right)^{1/2}}$$

Hence the group velocity is given by

$$v_g = \frac{c^2}{v} = c\sqrt{1 - \frac{\omega_0^2}{\omega^2}}$$

Wave Packets

Consider the superposition of a group of monochromatic waves having a set of discrete wavenumbers,

$$\psi = \sum_n a_n\cos(k_nx - \omega_nt)$$

We need some identities to manipulate the equation above. First we consider

$$\sum_{n=0}^{N} e^{inx}$$

Using complex analysis, we get the useful result

$$\sum_{n=0}^{N} \cos nx = \frac{\sin Nx/2}{\sin x/2} \cos \frac{(N-1)}{2} x$$

Then we can write

$$\psi = A(x,t)\cos(k_0 x - \omega_0 t)$$

where

$$A(x,t) = a\frac{\sin[n(x\delta k - t\delta\omega)/2]}{\sin[(x\delta k - t\delta\omega)/2]}$$

Suppose now that we have a group of waves that have a continuous distribution of wavenumbers, then the summation is replaced by an integral of the form

$$\psi = \int a(k)\cos(kx - \omega t)\; dk$$

Suppose also that the wave amplitude $a(k)$ is given by

$$a(k) = \begin{cases} \text{if } |k - k_0| \leq \Delta k/2 \\ \text{if } |k - k_0| > \Delta k/2 \end{cases}$$

The superposition of the corresponding group of waves is

$$\psi = a\int_{k_0-\Delta k/2}^{k_0+\Delta k/2} \cos(kx - \omega t)\; dk$$

Using Taylor's theorem and assuming that the range of wavenumbers is sufficiently small so that we need to retain only the linear term, we have

$$\omega = \omega_o + \alpha(k - k_0)$$

where $\omega_0 = \omega(k_0)$ and α is the derivative term evaluated at $k = k_0$. Hence, substituting for ω in $(kx - \omega t)$:

$$kx - \omega t = k(x - \alpha t) - \beta t$$

where $\beta \equiv \omega_0 - \alpha k_0$. We introduce new variable of integration

$$\xi = k(x - \alpha t) - \beta t$$
$$d\xi = (x - \alpha t)\; dk$$

Hence

$$\psi = a\int_{\xi_1}^{\xi_2} \frac{\cos\xi}{(x - \alpha t)}\; d\xi$$

with the range of

$$\xi_1 = (k_0 - \Delta k/2)(x - \omega t) - \beta t$$
$$\xi_2 = (k_0 + \Delta k/2)(x - \omega t) - \beta t$$

Therefore

$$\begin{aligned}\psi &= \frac{a}{x-\alpha t}\big(\sin\xi_1 - \sin\xi_2\big)\\ &= \frac{2a}{x-\alpha t}\sin\big(\frac{x_1-\xi_2}{2}\big)\cos\big(\frac{x_1+\xi_2}{2}\big)\\ &= A(x,t)\cos(k_0 x - \omega_0 t)\end{aligned}$$

where

$$A(x,t) = a\Delta k\frac{\sin[\Delta k(x-\alpha t)/2]}{\Delta k(x-\alpha t)/2}$$

A familiar result. The sinc first function becomes equal to zero when $x\Delta k/2 = \pm\pi$, giving

$$\Delta x\Delta k \approx 2\pi$$

This result is called the bandwidth theorem, which state that the shorter the length of the wave packet, the greater is the range of wavenumbers that is necessary to represent it. Using the relationship $\Delta k = \Delta\omega/v$ and $v = \Delta x/\Delta t$

$$\Delta\omega\Delta t \approx 2\pi$$

Using $\Delta k = \Delta p/\hbar$

$$\Delta p\Delta x \approx h$$

Using $\Delta\omega = \Delta E/\hbar$

$$\Delta E\Delta t \approx h$$

Optics

Introduction

Fermat's Principle

Optical path. Corresponds to the distance in vacuum equivalent to the distance traversed in the medium of index n. An optical path form point S to point P is defined as

$$OPL = \int_S^P n(s)\, ds$$

Fermat's principle. State that light will travel the route such that OPL in minimum

Law of Reflection

State that the angle-of-incidence equals the angle-of-reflection

$$\theta_i = \theta_r$$

Derivation. 404 not found.

Snell's Law

Also called law of refraction

$$n_i \sin\theta_i = n_r \sin\theta_r$$

Derivation. 404 not found.

Total internal reflection. In the case of $n_i > n_r$, when the incidence angel θ_i is equal or greater than the critical angle θ_c, total internal reflection occur. Snell's law state

$$\sin\theta_i = \frac{n_r}{n_i}\sin\theta_r$$

The critical angel occur when the reflected angle is perpendicular to normal, then said incidence angle is defined as critical angle

$$\sin\theta_i = \frac{n_r}{n_i} \implies \theta_c = \arcsin\frac{n_r}{n_i}$$

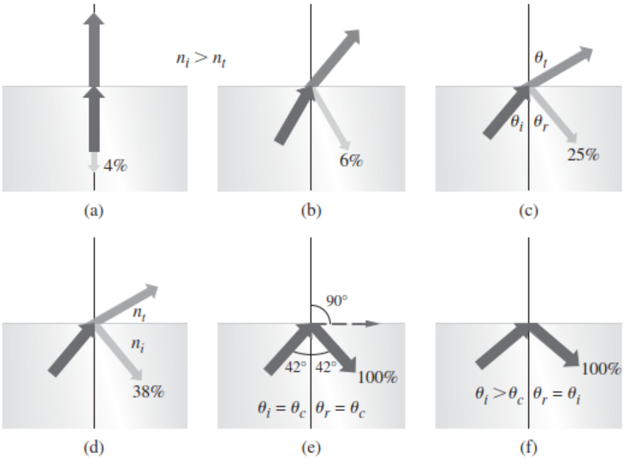


Figure: total internal reflection

Optical Geometry

Spherical Surfaces

I will discuss light behavior at spherical surfaces. Consider wave from the point source S impinging on a spherical interface of radius R centered at C. For the ray in question

$$OPL = n_1\ell_o + n_2\ell_i$$

Using the low of cosine in triangles SAC and ACP

$$\ell_o = [R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi]^{1/2}$$

$$\begin{aligned}\ell_i &= [R^2 + (s_i + R)^2 - 2R(s_i - R)\cos(\pi - \phi)]^{1/2} \\ &= [R^2 + (s_i + R)^2 + 2R(s_i - R)\cos\phi]^{1/2}\end{aligned}$$

Recall that the Fermat's Principle state that the optical path is stationary with respect to its position variable, which is ϕ in this case. Then setting the derivative of OPL to zero

$$\begin{aligned}\frac{d}{d\phi}OPL = n_1 &\frac{R(s_o + R)\sin\phi}{[R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi]^{1/2}} \\ &- n_2\frac{R(s_i - R)\sin\phi}{[R^2 + (s_i + R)^2 + 2R(s_i - R)\cos\phi]^{1/2}} = 0\end{aligned}$$

Or simply

$$\begin{aligned}n_1\frac{R(s_o + R)\sin\phi}{\ell_o} &= n_2\frac{R(s_i - R)\sin\phi}{\ell_i} \\ \frac{n_1 R s_o}{\ell_o} + \frac{n_1 R^2}{\ell_o} &= \frac{n_2 R s_i}{\ell_i} - \frac{n_2 R^2}{\ell_i} \\ \frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} &= \frac{1}{R}\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\end{aligned}$$

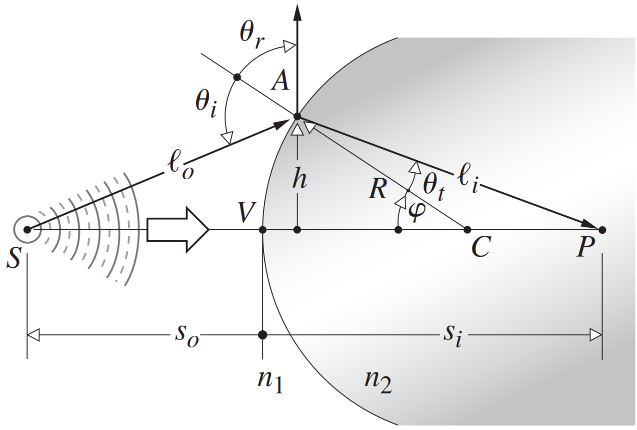


Figure: Refraction at spherical surface.

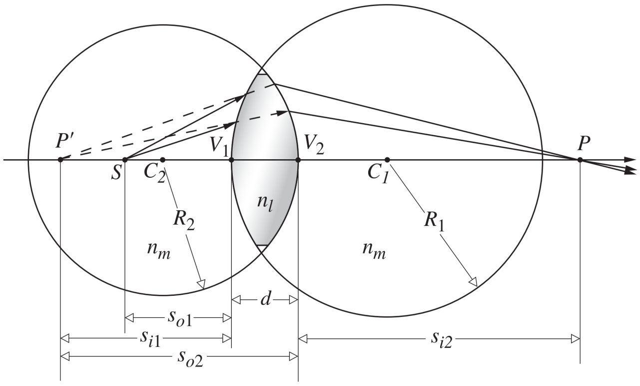


Figure: Light entering thin lens.

If we assume small ϕ, the value of ℓ_o and ℓ_i approach s_o and s_i respectively. Then, we write

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

The object focus is defined as the object distance when the image is at infinity

$$f_o = \lim_{s_i \to \infty} \left(\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \right) = \frac{n_1}{n_2 - n_1} R$$

The image focus is defined as the image distance when the object is at infinity

$$f_o = \lim_{s_o \to \infty} \left(\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} \right) = \frac{n_2}{n_2 - n_1} R$$

Thin Lenses

The equation for lenses equation, also called the lens-maker equation, is written as

$$\frac{1}{s_0} + \frac{1}{s_i} = (n_l - 1) \left(\frac{1}{R_1} \frac{1}{R_2} \right)$$

or simply

$$\frac{1}{s_0} + \frac{1}{s_i} = \frac{1}{f}$$

where

$$\frac{1}{f} = (n_l - 1) \left(\frac{1}{R_1} \frac{1}{R_2} \right)$$

The newtonian form of lenses equation is

$$x_o x_i = f^2$$

To determine the transversal magnification of lenses, we use

$$M_T \equiv \frac{y_i}{y_o} = -\frac{s_i}{s_o} = -\frac{x_i}{f} = -\frac{f}{x_o}$$

As for the longitudinal magnification

$$M_L \equiv \frac{dx_i}{dx_o} = -\frac{f^2}{x_o^2} = -M_T^2$$

Sign convention. All quantities used before are defined to be positive.

Table: Sign convention for lenses with respect to light

Quantities	Positive	Negative
$s_o,\ f_o$	front of V	behind of V
x_0	front of F_o	behind of F_o
$s_i,\ f_i$	behind of V	front of V
x_f	behind of F_i	front of $F_i i$
R	C is behind of V	C is front of V
$y_o,\ y_i$	above optical axis	below optical axis

Image formed by lenses. Summarized as follows. One for convex lenses.

Table: Images formed by convex lenses

Object	Image			
Location	Type	Location	Orientation	Size
$s_o > 2f$	Real	$f < s_i < 2f$	Inverted	Minified
$s_0 = 2f$	Real	$s_i = 2f$	Inverted	Same
$f < s_o < 2f$	Real	$s_i > 2f$	Inverted	Magnified
$s_o = f$		$\pm\infty$	Inverted	
$s_0 < f$	Virtual	$\lvert s_i\rvert > s_o$	Erect	Magnified

And other for concave lenses.

Table: Images formed by concave lenses

Object	Image			
Location	Type	Location	Orientation	Size
Anywhere	Virtual	$\lvert s_i\rvert < f$ $\lvert s_i\rvert < s_o$	Erect	Minified

Derivation. Thin lenses can be considered as two spherical surface, one is convex while other is convex. Now suppose we have lens of index n_l surrounded by medium of index n_m. The position of object s_{o1} and image s_{i1} for the first surface is given by

$$\frac{n_m}{s_{o1}} + \frac{n_l}{s_{i1}} = \frac{n_l - n_m}{R_1}$$

while for the second surface

$$\frac{n_l}{s_{o2}} + \frac{n_m}{s_{i2}} = \frac{n_m - n_l}{R_2}$$

From figure, we also have

$$|s_{o2}| = |s_{i1}| + d$$

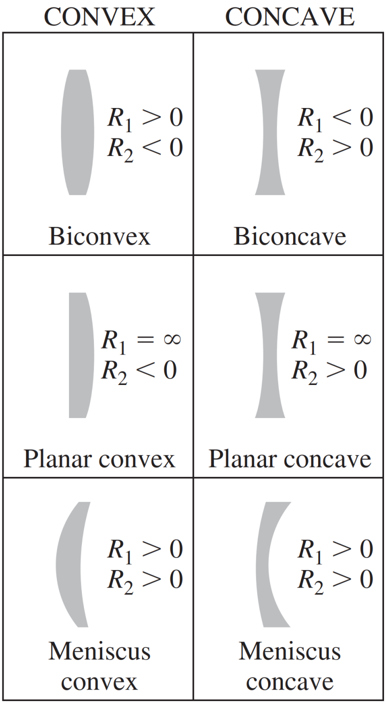


Figure: Crossection of various thin lenses.

Then by the definition, s_{o2} is positive and $s_{i1}$is negative

$$s_{o2} = -s_{i1} + d$$

On using this, the equation for second surface reads as

$$\frac{n_l}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_m - n_l}{R_2}$$

Now we add the equation for both surface

$$\frac{n_m}{s_{01}} + \frac{n_m}{s_{i2}} = (n_l - n_m)\left(\frac{1}{R_2} - \frac{1}{R_2}\right) - \frac{n_l}{s_{i1} - d}\frac{n_l}{s_{i1}}$$

$$\frac{n_m}{s_{01}} + \frac{n_m}{s_{i2}} = (n_l - n_m)\left(\frac{1}{R_2} - \frac{1}{R_2}\right) + \frac{n_l d}{s_{i1(s_{i1}-d)}}$$

For the case of thin lens, we then make the assumtion of small lens and that the surrounding medium is air

$$\frac{n_m}{s_{01}} + \frac{n_m}{s_{i2}} = (n_l - n_m)\left(\frac{1}{R_2} - \frac{1}{R_2}\right)$$

To obtain the Newton's expression for lenses, consider an image in front of convex lens. Using the definition of lateral magnification

$$\frac{y_o}{|y_i|} = \frac{f}{s_i - f} = \frac{f}{x_i}$$

This applies since the triangles AOF_i and $P_2P_1F_i$ are similiar. Appliying the same logic to triagle S_1S_2O and P_2P_1O

$$\frac{y_o}{|y_i|} = \frac{s_o}{s_i}$$

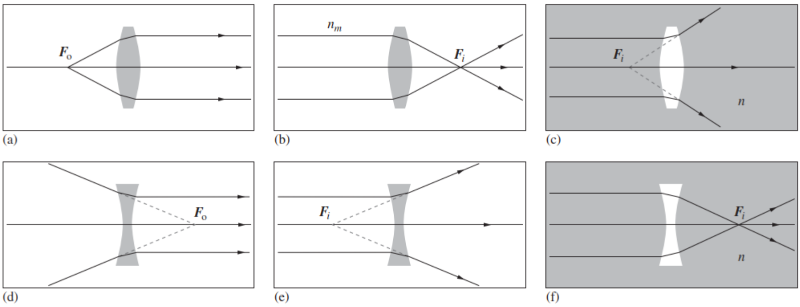


Figure: Light refracted by thin lenses.

On equating them we obtain

$$\frac{f}{s_o - f} = \frac{s_o}{f}$$
$$\frac{s_i}{f} - 1 = \frac{s_i}{s_o}$$
$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i}$$

the lens maker equation. To obtain Newton's expression, we consider the triangles BOF_o and $S_2S_1F_o$

$$\frac{y_o}{|y_i|} == \frac{s_o - f}{f} = \frac{x_o}{f}$$

then equating with the expression from AOF_i and $P_2P_1F_i$.

$$\frac{x_o}{f} = \frac{f}{x_i}$$
$$x_o x_i = f^2 \quad \blacksquare$$

According to the figure, the image is inverted, thus the value of y_o is negative. Hence, on using the value from triangle P_2P_1O to the definition of lateral magnification

$$M_T = -\frac{s_i}{s_o}$$

on using the value of triangle $P_2P_1F_i$

$$M_T = -\frac{x_i}{x_f}$$

and on using the value of triangle BOF_o

$$M_T = -\frac{f}{x_o}$$

To obtain the expression for lateral magnification, we write the Newtonian form as $x_i = f^2/x_o$ and using the definition of lateral magnification

$$m_L \equiv \frac{dx_i}{dx_o} = -\frac{f^2}{x_o^2} = -M_T^2$$

Types of thin lenses. See figure.

Light behavior. Also see figure.

Mirror

Sign convention. Listed as follows.

Table: Sign convention for lenses with respect to light

Quantities	Positive	Negative
s_o	front of V	behind of V
s_i	front of V	behind of V
f	concave	convex
R	C is behind of V	C is front of V
$y_o,\ y_i$	above optical axis	below optical axis

Planar mirrors. The equation for planar mirrors is simply

$$|s_o| = |s_i|$$

that is, the image and object are equidistant from the surface.

Spherical mirrors. The equation that governed spherical mirrors is the same with the one for lenses, albeit with different sign convention

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

where

$$\frac{1}{f} = -\frac{R}{2}$$

The definition for transversal and lateral magnification is the same as the case for lenses.

Derivation. I shall not derive the equation for planar mirrors, as it was trivial. now, consider a concave spherical mirror. From the triangle SAP, we observe

$$\frac{SC}{SA} = \frac{CP}{PA}$$

Furthermore

$$\begin{aligned} SC &= s_o - |R| = s_o + R \\ CP &= |R| - s_i = -(s_i - R) \end{aligned}$$

In the paraaxial region appoximation, $SA \approx s_o$ and $PA \approx s_i$

$$\begin{aligned} \frac{s_o + R}{s_o} &= -\frac{s_i + R}{s_i} \\ \frac{R}{s_o} + \frac{R}{s_i} &= -2 \\ \frac{1}{s_o} + \frac{1}{s_i} &= -\frac{2}{R} \end{aligned}$$

We define object focus and image focus respectively as

$$\lim_{s_i \to \infty} s_o = f_o \qquad \lim_{s_o \to \infty} s_i = f_i$$

Notice that $f_o = f_i$, so for convinience's sake we write

$$\frac{1}{f} = -\frac{R}{2}$$

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