Problem 5.5 from Pain. A point mass M is concentrated at a point on a string of characteristic impedance ρc. A transverse wave of frequency ω moves in the positive x direction and is partially reflected and transmitted at the mass. The boundary conditions are that the string displacements just to the left and right of the mass are equal $y_i + y_r = y_t$ and that the difference between the transverse forces just to the left and right of the mass equal the mass times its acceleration. If A_1, B_1 and A_2 are respectively the incident, reflected and transmitted wave amplitudes show that

$$\frac{B_1}{A_1} = \frac{-iq}{1+iq} \qquad \text{and} \qquad \frac{A_2}{A_1} = \frac{1}{1+iq}$$

where $q = M\omega/2\rho c$.

Let us assume that the force acting on the point mass is described by

$$F = M\frac{\partial^2 y}{\partial t^2} = M\frac{\partial^2}{\partial t^2} A \exp i(kx - \omega t) = -M\omega^2 y$$

Where I have assumed $y = A \exp i(kx - \omega t)$. Its velocity thus

$$v_p = \frac{\partial}{\partial t} A \exp i(kx - \omega t) = \omega i y$$

The Impedance of the point mass M is therefore

$$Z_M = \frac{F}{v_p} = \frac{-M\omega^2 y}{\omega i y} = M\omega i$$

We now can evaluate the impedance Z_2 of string with point mass

$$Z_2 = Z_{\text{string}} + Z_M = \rho c + M\omega i$$

The transmission coefficient is

$$t = \frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2} = \frac{2\rho c}{2\rho c + M\omega i} = \frac{1}{1 + \frac{\omega M}{2\rho c} i} = \frac{1}{1+iq} \qquad \blacksquare$$

while the reflected coefficient

$$r = \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{-M\omega i}{2\rho c + M\omega i} = \frac{-i\frac{\omega M}{2\rho c}}{1 + \frac{\omega M}{2\rho c} i} = \frac{-iq}{1+iq} \qquad \blacksquare$$