Mass on A Spring

For small displacements the force produced by the spring is described by Hooke's law:

$$F = -kx$$

Using Newton's second law of motion, we obtain the equation of motion of the mass

$$\ddot{x} = -\omega^2 x$$

where

$$\omega^2 = \frac{k}{m}$$

We can solve the equation using by rewritting in from

$$(D + \omega i)(D - \omega i)x = 0$$

the roots of auxilary equation are therefore $D = \pm\omega i$. Thus, the general solution is

$$x = a\cos\omega t + b\sin\omega t = A\cos\omega t + \phi$$

Energy of a mass on a spring. The work done on the spring, extending it from x' to $x' + dx'$, is $kx'dx'$. Hence, the work done extending it from its unstretched length by an amount x

$$U = \int_0^x kx'dx' = \frac{1}{2}kx^2$$

Conservation of energy for the harmonic oscillator follows from Newton's second law

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{const.}$$

Substituting the value of x and $v = dx/dt$, we get

$$E = \frac{1}{2}kA^2$$

Pendulum

By Newton's second law

$$ml\ddot{\theta} = -mg\sin\theta$$
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

expanding $\sin\theta$ in power series

$$\ddot{\theta} = -\frac{g}{l}\theta$$

This is the equation of SHM with $\omega = \sqrt{g/l}$ and its general solution

$$\theta = \theta_0\cos\omega t + \phi$$

Energy of pendulum. For small θ, we have

$$
\begin{aligned}
l^2 &= (l-y)^2 + x^2 \\
2ly &= Y^2 + x^2
\end{aligned}
$$

For small displacements of the pendulum, $x \ll l$, it follows that $y \ll x$, so that the term y^2 can be neglected, and we can write

$$y = \frac{x^2}{2l}$$

The total energy of the system E is therefore

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

At the turning point of the motion, when x equals A, it follows that

$$\frac{1}{2}mg\frac{A^2}{2l} = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

We can use it to obtain expressions for velocity v

$$\frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}$$

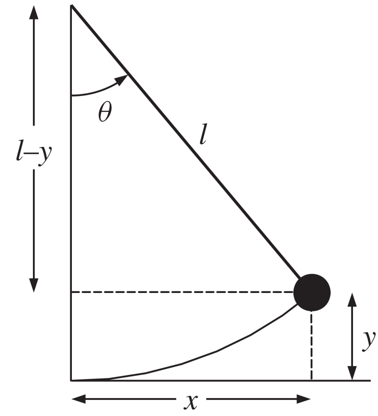
and for displacements x

$$
\begin{aligned}
\int \frac{dx}{\sqrt{A^2 - x^2}} &= \int \sqrt{\frac{g}{l}}dt \\
\arcsin \frac{x}{A} &= \sqrt{\frac{g}{l}}t + \phi \\
x = A \sin \sqrt{\frac{g}{l}}t + \phi
\end{aligned}
$$

which describes SHM with $\omega = \sqrt{g/l}$and $T = 2\pi\sqrt{l/g}$ as before.

Notice that both equations have the form

$$E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$$



The geometry of the simple pendulum

where α and β are constants. The constant α corresponds to the inertia of the system through which it can store kinetic energy. The constant β corresponds to the restoring force per unit displacement through which the system can store. When we differentiate the conservation of energy equation with respect to time

$$\frac{dE}{dt} = \alpha v \frac{dv}{dt} + \beta x \frac{dx}{dt} = 0$$

giving

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha} v$$

it follows that the angular frequency of oscillation ω is equal to $\sqrt{\beta/\alpha}$.

Physical pendulum. In a physical pendulum the mass is not concentrated at a point as in the simple pendulum, but is distributed over the whole body. An example of a physical pendulum consists of a uniform rod of length l that pivots about a horizontal axis at its upper end.

Noting that $\tau = I\ddot{\theta} = \mathbf{r} \times \mathbf{F}$

$$\begin{aligned}
I\ddot{\theta} &= \frac{l}{2}(-mg)\sin\pi - \theta \\
\frac{1}{3}ml^2\ddot{\theta} &= -\frac{1}{2}mgl\sin\theta \\
\ddot{\theta} &= \frac{3g}{2l}\sin\theta
\end{aligned}$$

Again we can use the small-angle approximation to obtain

$$\ddot{\theta} = \frac{3g}{2l}\theta$$

This is SHM with $\omega = \sqrt{3g/2l}$ and $T = 2\pi\sqrt{2l/3g}$.

Potential approach.

Suppose a system is oscillating inside potential $V(x)$. Using Taylor series, we rewite the potential at $x = x_0$ as

$$V(x) = V(x_0) + x\frac{dV}{dx}\bigg|_{x=x_0} + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \dots$$

The first term is a constant, while the second is zero due to dV/dx evaluated at $x = x_0$ is zero. Therefore,

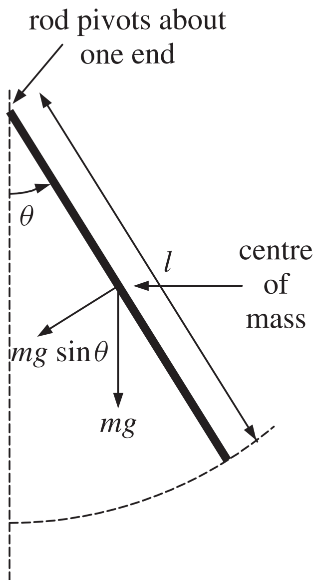
$$V(x) \approx V(x_0) + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

and

$$F = -\frac{dV(x)}{dx} \approx -x\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

Thus its frequency

$$\omega = \left(\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=x_0}\right)^{1/2}$$



Physical pendulum

Similarities in Physics

LC circuit. Initially, capacitor is charged to voltage $V_C = q/C$. Switch then closed and charge begins to flow through the inductor and a current $\dot{q}$ flows in the circuit. This is a time-varying current and produces a voltage across the inductor given $V_L = L\ddot{q}$. We can analyse the LC circuit using Kirchhoff's law, which states that the sum of the voltages around the circuit is zero

$$\begin{aligned} V_C + V_L &= \\ \frac{q}{C} + L\ddot{q} &= 0 \\ \ddot{q} &= -\frac{1}{LC}q \end{aligned}$$

It is of the same form as SHM equation and the frequency of the oscillation is given directly by, $\omega = \sqrt{1/LC}$. Since we have the initial condition that the charge on the capacitor has its maximum value at t = 0, then the solution is

$$q = q_0 \cos \omega t$$

The energy stored in a capacitor charged to voltage V_C is equal to $(1/2)CV_C^2$. This is electrostatic energy. The energy stored in an inductor is equal to $(1/2)LI^2$ and this is magnetic energy. Thus

$$\begin{aligned} E &= \frac{1}{2}CV_C^2 + \frac{1}{2}LI^2 \\ &= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2 \end{aligned}$$

Similarities in physics. We note the similarities in both cases

$$\ddot{Z} = -\frac{\beta}{\alpha} Z \qquad E = \frac{1}{2}\alpha \dot{Z}^2 + \frac{1}{2}\beta Z^2$$

where α and α are constants and $Z = Z(t)$ is the oscillating quantity. In the mechanical case Z stands for the displacement x, and in the electrical case for the charge q.