Equation of Motion

The damping force F_d acting on system is proportional to its velocity v so long as v is not too large. In another word

$$F_d = -bv$$

The resulting equation of motion is

$$m\ddot{x} = -kx - b\dot{x}$$

We introduce the parameters

$$\omega_0^2 = \frac{k}{m}$$
$$\gamma = \frac{b}{m}$$

Using these parameters, the equation become

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Now we designate the angular frequency ω_0 and describe it as the natural frequency of oscillation, or the oscillation frequency if there were no damping. We can write the equation as

$$D^2 x + D\gamma x + \omega_0^2 x = 0$$
$$(D^2 + D\gamma + \omega_0^2)x = 0$$

Using the quadratic equation, we find the value of D

$$D = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

The solution is therefore depend on the value of the square root term; which can either be real, imaginary or simply zero. The value of the square root also determine the cases of damping that occur on the system.

Light damping. This case occur if $\gamma^2/4 < \omega_0^2$, which causes the square root term to be imaginary. Let us introduce yet another constant

$$\omega^2 = \omega_0^2 - \gamma^2/4$$

Substituting back into D

$$D = -\frac{\gamma}{2} \pm \sqrt{-\omega^2} = -\frac{\gamma}{2} \pm \omega i$$

Thus, we can say that the equation is second order differential equation with imaginary auxiliary equation roots. The solution is

$$x = A\exp\left(-\frac{\gamma t}{2}\right)\cos\omega t + \phi$$

Now consider the graph of x. The term $\exp -\gamma t/2$ represent an envelope for the oscillations. $x = 0$ occur when $\cos\omega t$ is zero and so are separated by π/ω with period $T = 2\pi/\omega$. Successive maxima are also separated by T. If A_n occurs at time t_0 and A_{n+1} at $t_0 + T$, then

$$x(t_0) = A \exp\left(-\frac{\gamma t_0}{2}\right) \cos \omega t_0$$
$$x(t_0 + T) = A \exp\left(-\frac{\gamma(t_0 + T)}{2}\right) \cos \omega(t_0 + T)$$

Since $\cos \omega t_0 = \cos \omega(t_0 + T) = \cos \omega t_0 + 2\pi$

$$\frac{A_n}{A_{n+1}} = \exp \frac{\gamma T}{2}$$

or the natural logarithm version

$$\ln \frac{A_n}{A_{n+1}} = \frac{\gamma T}{2}$$

which is called the logarithmic decrement and is a measure of this decrease.

Heavy damping. Heavy damping occurs when the degree of damping is sufficiently large that the system returns sluggishly to its equilibrium position without making any oscillations at all. In another words, $\gamma^2/4 > \omega_0^2$ and the square root term is real. Thus, we can say that the equation is second order differential equation with two real auxilary equation roots. The solution is

$$x = A \exp\left[\left(-\frac{\gamma}{2} + (\frac{\gamma^2}{4} - \omega_0^2)^{1/2}\right)t\right] + B \exp\left[\left(-\frac{\gamma}{2} - (\frac{\gamma^2}{4} - \omega_0^2)^{1/2}\right)t\right]$$

Critical damping. Occurs when $\gamma^2/4 = \omega_0^2$, which makes the suqre roots zero. Thus the equation is second order differential equation with one real auxilary equation roots. The solution is

$$x = (At + B) \exp\left(-\frac{\gamma t}{2}\right)$$

Here the mass, or whatever oscillating, returns to its equilibrium position in the shortest possible time without oscillating.

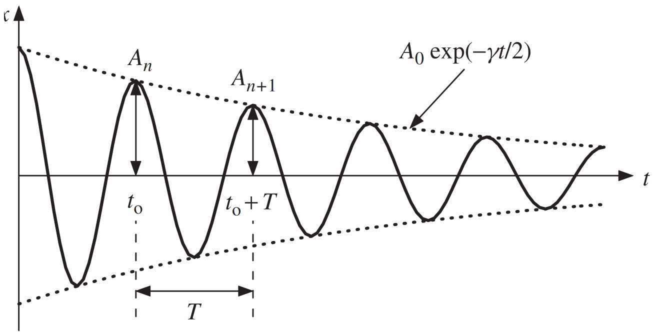

Figure: Graph of $x = A_0 \exp(-\gamma^2 t/4) \cos \omega t$

Putting all together. In summary we find three types of damped motion:

1. $\gamma^2/4 < \omega_0^2$ Light damping, Imaginary square root, Damped oscillations;

2. $\gamma^2/4 > \omega_0^2$ Heavy damping, Real Square root, Exponential decay of displacement;

3. $\gamma^2/4 = \omega_0^2$ Critical damping, Zero square root, Quickest return to equilibrium position without oscillation.

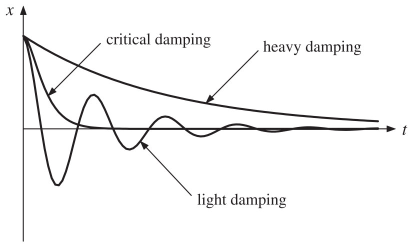

Figure: Motion of a damped oscillator for various cases

RLC circuit.

In the case of an electrical oscillator it is the resistance in the circuit that impedes the flow of current. Kirchoff's law gives

$$
\begin{aligned}
L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0 \\
\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0 \\
\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0
\end{aligned}
$$

This is the equation of DHO with q as x, L as m, k as 1/C and R as b; so R/L is the equivalent of $\gamma = b/m = R/L$ and $\omega_0^2 = 1/LC$. Now assuming that this this the case of light damping, in another words $R^2/4L^2 < 1/LC$, the solution is

$$q = q_0 \exp\left(-\frac{\gamma t}{2}\right) \cos\omega t$$

with

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$$

Since the voltage V_C across the capacitor is equal to q/C, dividing the solution by C

$$V_C = V_0 \exp\left(-\frac{\gamma t}{2}\right) \cos\omega t$$

We find that the quality factor Q of the circuit is given by

$$Q = \frac{\omega_0}{\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Energy of DHO

In the case of very lightly damped oscillator $\gamma^2/4 \ll \omega_0^2$ we have

$$x = A_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega_0 t$$
$$v = -A_0\omega_0 \exp\left(-\frac{\gamma t}{2}\right) \left[\sin \omega_0 t + \frac{\gamma}{2\omega_0} \cos \omega_0 t\right]$$

where we approximate $\omega = \omega_0$. Since $\gamma \ll \omega_0$, we can ignore the second term at velocity equation

$$v = -A_0\omega_0 \exp\left(-\frac{\gamma t}{2}\right) \sin \omega_0 t$$

Then

$$E = \frac{1}{2} A_0^2 \exp(-\gamma t)(m\omega_0^2 \sin^2 \omega_0 t + k \cos \omega_0 t)$$

considering $\omega_0^2 = k/m$

$$E(t) = \frac{1}{2} k A_0^2 \exp(-\gamma t) = E_0 \exp(-\gamma t)$$

The reciprocal of γ is the time taken $\tau = 1/\gamma$ for the energy of the oscillator to reduce by a factor of e^{-1}, thus

$$E(t) = E_0 \exp\left(-\frac{t}{\tau}\right)$$

Rate of dissipation. The energy of an oscillator is dissipated because it does work against the damping force at the rate (damping force $\times$ velocity). We can see this by differentiating energy with respect to time

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

since the damping force $F_d = m\ddot{x} + kx = -b\dot{x}$, we can write

$$\frac{dE}{dt} = -b\dot{x}^2$$

Q factor

The quality factor Q of the oscillator describe how good an oscillator is, where we imply that the smaller the degree of damping the higher the quality of the oscillator. Oscillator with a high Q-value would make an appreciable number of oscillations before its energy is reduced substantially. The quality factor Q is defined as

$$Q = \frac{\omega}{\gamma} \approx \frac{\omega_0}{\gamma}$$

Another way to define Q factor is

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$$

Now, consider energy of a very lightly damped oscillator one period apart

$$E_1 = E_0 \exp(-\gamma t)$$
$$E_2 = E_0 \exp[-\gamma(t+T)]$$

giving

$$\frac{E_2}{E_1} = \exp(-\gamma T)$$

Using series expansion

$$\frac{E_2}{E_1} \approx 1 - \gamma T$$

therefore

$$\frac{E_1 - E_2}{E_1} \approx \gamma T \approx \frac{2\pi\gamma}{\omega_0} \approx \frac{2\pi}{Q}$$

where we have $\gamma T \ll 1$ and $\omega \approx \omega_0$. The fractional change in energy per cycle is equal to $2\pi/Q$ and so the fractional change in energy per radian is equal to 1/Q. Thus our definition is proved.

We can also recast DHO equation using Q factor

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$$

and the angular frequency ω

$$\omega = \omega_0 \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

This confirm confirms our assumption that ω is equal to ω_0 to a good approximation under most circumstances. Even when Q is as low as 5, ω is different from ω_0 by just 0.5%.