Determining the Largest Volume of a Paralelepiped Constrained Inside Ellipsoid by Lagrange Multipliers Method

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1 Introduction

Consider ellipsoid described by the following equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

What is the largest of volume of paralelepiped that can be inscribed inside this ellipsoid? This is an optimization problem that can be solved by Lagrange multiplier method.

2 Lagrange Multipliers

Let $f(x, y, z)$ be our function that we want to optimize and $\phi(x, y, z) = \text{const}$ be our constraint. We then set the total differential of $f(x, y, z)$ and $\phi(x, y, z)$ equal to zero

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial y}dz = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial y}dz = 0 \tag{3}$$

Next, we construct the function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) \tag{4}$$

and set its total derivative to zero

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0 \tag{5}$$

It follows that, for any value of $dx,\ dy,\ dz$, we choose λ such that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \tag{6}$$

Putting it all together, to optimize $f(x, y, z)$ with constraint $\phi(x, y, z)$, we need to optimize $F(x, y, z)$, which obtained by solving three partial derivative equations and constraint equation $\phi(x, y, z) = \text{const}$. The equations in question are

$$\frac{\partial F}{\partial x} = 0 \tag{7}$$

$$\frac{\partial F}{\partial y} = 0 \tag{8}$$

$$\frac{\partial F}{\partial z} = 0 \tag{9}$$

$$\phi(x, y, z) = \text{const.} \tag{10}$$

3 Application

The ellipsoid function acts as constraints, that left is to determine the function that we want to optimize. This requires some clever thinking. We begin by defining point (x, y, z) be the corner of our parallelepiped. Now, this point is located in the first octant of our parallelepiped. The volume of this octant is

$$v = xyz \tag{11}$$

Since the parallelepiped's sides are parallel the axis, its total volume is

$$V = 8v \tag{12}$$

Hence, the volume of our parallelepiped is

$$V = 8xyz \tag{13}$$

This is the function that we want to maximize. We then construct the function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \tag{14}$$

The partial derivatives of F read as

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda}{a^2}x, \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda}{b^2}y, \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda}{c^2}z \tag{15}$$

To find the maximum of F, we then must solve the partial derivative equations and constraint equation

$$8yz + \frac{2\lambda}{a^2}x = 0 \tag{16}$$

$$8xz + \frac{2\lambda}{b^2}y = 0 \tag{17}$$

$$8xy + \frac{2\lambda}{c^2}z = 0 \tag{18}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{19}$$

Multiplying the first equation by x, the second by y, the third by z and adding them all together, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 24xyz + 2\lambda = 0 \tag{20}$$

Hence

$$\lambda = -12xyz \tag{21}$$

Substituting this into the partial derivative equation to obtain

$$8yz - \frac{24yz}{a^2}x^2 = 0 \implies x = \frac{\sqrt{3}}{3}a \tag{22}$$

$$8xz - \frac{24xz}{b^2}y^2 = 0 \implies y = \frac{\sqrt{3}}{3}b \tag{23}$$

$$8xy - \frac{24xy}{c^2}z^2 = 0 \implies z = \frac{\sqrt{3}}{3}c \tag{24}$$

Therefore, the maximum volume of said parallelepiped is

$$V = \frac{24\sqrt{3}}{27}abc \tag{25}$$