Mass on A Spring

For small displacements the force produced by the spring is described by Hooke's law:

$$F = -kx$$

Using Newton's second law of motion, we obtain the equation of motion of the mass

$$\ddot{x} = -\omega^2 x$$

where

$$\omega^2 = \frac{k}{m}$$

We can solve the equation using by rewritting in from

$$(D + \omega i)(D - \omega i)x = 0$$

the roots of auxilary equation are therefore $D = \pm\omega i$. Thus, the general solution is

$$x = a\cos\omega t + b\sin\omega t = A\cos\omega t + \phi$$

Energy of a mass on a spring. The work done on the spring, extending it from x' to $x' + dx'$, is $kx'dx'$. Hence the work done extending it from its unstretched length by an amount x

$$U = \int_0^x kx'dx' = \frac{1}{2}kx^2$$

Conservation of energy for the harmonic oscillator follows from Newton's second law

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{const.}$$

substituting the value of x and $v = dx/dt$, we get

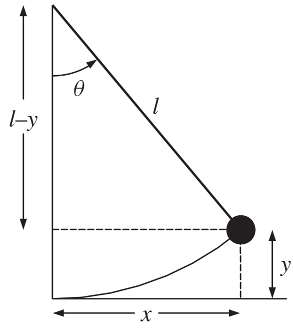
$$E = \frac{1}{2}kA^2$$

Pendulum

By Newton's second law

$$ml\ddot{\theta} = -mg\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$



The geometry of the simple pendulum

expanding $\sin\theta$ in power series

$$\ddot{\theta} = -\frac{g}{l}\theta$$

This is the equation of SHM with $\omega = \sqrt{g/l}$ and its general solution

$$\theta = \theta_0 \cos\omega t + \phi$$

Energy of pendulum. For small θ, we have

$$\begin{aligned} l^2 &= (l-y)^2 + x^2 \\ 2ly &= Y^2 + x^2 \end{aligned}$$

For small displacements of the pendulum, $x \ll l$, it follows that $y \ll x$, so that the term y^2 can be neglected and we can write

$$y = \frac{x^2}{2l}$$

The total energy of the system E is therefore

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

At the turning point of the motion, when x equals A, it follows that

$$\frac{1}{2}mg\frac{A^2}{2l} = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{2l}$$

We can use it to obtain expressions for velocity v

$$\frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}$$

and for displacements x

$$\begin{aligned} \int \frac{dx}{\sqrt{A^2 - x^2}} &= \int \sqrt{\frac{g}{l}}dt \\ \arcsin\frac{x}{A} &= \sqrt{\frac{g}{l}}t + \phi \\ x = A\sin\sqrt{\frac{g}{l}}t + \phi \end{aligned}$$

which describes SHM with $\omega = \sqrt{g/l}$and $T = 2\pi\sqrt{l/g}$ as before.

Notice that both equations have the form

$$E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$$

where α and β are constants. The constant α corresponds to the inertia of the system through which it can store kinetic energy. The constant β corresponds to the restoring force per unit displacement through which the system can store. When we differentiate the conservation of energy equation with respect to time

$$\frac{dE}{dt} = \alpha v\frac{dv}{dt} + \beta x\frac{dx}{dt} = 0$$

giving

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}v$$

it follows that the angular frequency of oscillation ω is equal to $\sqrt{\beta/\alpha}$.

Physical pendulum. In a physical pendulum the mass is not concentrated at a point as in the simple pendulum, but is distributed over the whole body. An example of a physical pendulum consists of a uniform rod of length l that pivots about a horizontal axis at its upper end.

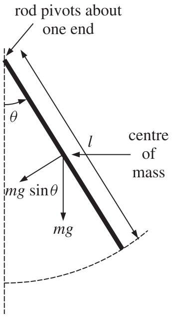
Noting that $\tau = I\ddot{\theta} = \mathbf{r} \times \mathbf{F}$

$$I\ddot{\theta} = \frac{l}{2}(-mg)\sin\pi - \theta$$
$$\frac{1}{3}ml^2\ddot{\theta} = -\frac{1}{2}mgl\sin\theta$$
$$\ddot{\theta} = \frac{3g}{2l}\sin\theta$$

Again we can use the small-angle approximation to obtain

$$\ddot{\theta} = \frac{3g}{2l}\theta$$

This is SHM with $\omega = \sqrt{3g/2l}$ and $T = 2\pi\sqrt{2l/3g}$.



Physical pendulum

Similarities in Physics

LC circuit. Initially, capacitor is charged to voltage $V_C = q/C$. Switch then closed and charge begins to flow through the inductor and a current $\dot{q}$ flows in the circuit. This is a time-varying current and produces a voltage across the inductor given $V_L = L\ddot{q}$. We can analyse the LC circuit using Kirchhoff's law, which states that the sum of the voltages around the circuit is zero

$$V_C + V_L =$$
$$\frac{q}{C} + L\ddot{q} = 0$$
$$\ddot{q} = -\frac{1}{LC}q$$

It is of the same form as SHM equation and the frequency of the oscillation is given directly by, $\omega = \sqrt{1/LC}$. Since we have the initial condition that the charge on the capacitor has its maximum value at t = 0, then the solution is

$$q = q_0 \cos\omega t$$

The energy stored in a capacitor charged to voltage V_C is equal to $(1/2)CV_C^2$. This is electrostatic energy. The energy stored in an inductor is equal to $(1/2)LI^2$ and this is magnetic energy. Thus

$$\begin{aligned} E &= \frac{1}{2}CV_C^2 + \frac{1}{2}LI^2 \\ &= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2 \end{aligned}$$

Potential approach.

Suppose a system is oscillating inside potential $V(x)$. Using Taylor series, we rewite the potential at $x = x_0$ as

$$V(x) = V(x_0) + x\frac{dV}{dx}\bigg|_{x=x_0} + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \dots$$

The first term is a constant, while the second is zero due to dV/dx evaluated at $x = x_0$ is zero. Therefore

$$V(x) \approx V(x_0) + \frac{x^2}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

and

$$F = -\frac{dV(x)}{dx} \approx -x\frac{d^2V}{dx^2}\bigg|_{x=x_0}$$

Thus its frequency

$$\omega = \left(\frac{1}{m}\frac{d^2V}{dx^2}\bigg|_{x=x_0}\right)^{1/2}$$

Similarities in physics. We note the similarities in both cases

$$\ddot{Z} = -\frac{\beta}{\alpha}Z \qquad E = \frac{1}{2}\alpha\dot{Z}^2 + \frac{1}{2}\beta Z^2$$

where α and α are constants and $Z = Z(t)$ is the oscillating quantity. In the mechanical case Z stands for the displacement x, and in the electrical case for the charge q.

Damped harmonic oscillator

The damping force F_d acting on system is proportional to its velocity v so long as v is not too large. In another word

$$F_d = -bv$$

The resulting equation of motion is

$$m\ddot{x} = -kx - b\dot{x}$$

We introduce the parameters

$$\begin{aligned} \omega_0 &= \frac{k}{m} \\ \gamma &= \frac{b}{m} \end{aligned}$$

Using these parameter, the equation become

$$\ddot{x} + \gamma\dot{x} + \omega_0 x = 0$$

Now we designate the angular frequency ω_0 and describe it as the natural frequency of oscillation, or the oscillation frequency if there were no damping. We can write the equation as

$$D^2 x + D\gamma x + \omega_0^2 x = 0$$
$$(D^2 + D\gamma + \omega_0^2)x = 0$$

Using the quadratic equation, we find the value of D

$$D = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

The solution is therefore depend on the value of the square root term; which can either be real, imaginary or simply zero. The value of the square root also determine the cases of damping that occur on the system.

Light damping. This case occur if $\gamma^2/4 < \omega_0^2$, which causes the square root term to be imaginary. Let's us introduce yet another constant

$$\omega^2 = \omega_0^2 - \gamma^2/4$$

Substituting back into D

$$D = -\frac{\gamma}{2} \pm \sqrt{-\omega^2} = -\frac{\gamma}{2} \pm \omega i$$

Thus, we can say that the equation is second order differential equation with imaginary auxilary equation roots. The solution is

$$x = A \exp\left(-\frac{\gamma t}{2}\right) \cos \omega t + \phi$$

Now consider the graph of x. The term $\exp -\gamma t/2$ represent an envelope for the oscillations. $x = 0$ occur when $\cos \omega t$ is zero and so are separated by π/ω with period $T = 2\pi/\omega$. Successive maxima are also separated by T. If A_n occurs at time t_0 and A_{n+1} at $t_0 + T$, then

$$x(t_0) = A \exp\left(-\frac{\gamma t_0}{2}\right) \cos \omega t_0$$
$$x(t_0 + T) = A \exp\left(-\frac{\gamma(t_0 + T)}{2}\right) \cos \omega(t_0 + T)$$

Since $\cos \omega t_0 = \cos \omega(t_0 + T) = \cos \omega t_0 + 2\pi$

$$\frac{A_n}{A_{n+1}} = \exp \frac{\gamma T}{2}$$

or the natural logarithm version

$$\ln \frac{A_n}{A_{n+1}} = \frac{\gamma T}{2}$$

which is called the logarithmic decrement and is a measure of this decrease.

Heavy damping. Heavy damping occurs when the degree of damping is sufficiently large that the system returns sluggishly to its equilibrium position without making any oscillations at all. In another words, $\gamma^2/4 > \omega_0^2$ and the square root term is real. Thus, we can say that the equation is second order differential equation with two real auxilary equation roots. The solution is

$$x = A\exp\left[\left(-\frac{\gamma}{2} + (\frac{\gamma^2}{4} - \omega_0^2)^{1/2}\right)t\right] + B\exp\left[\left(-\frac{\gamma}{2} - (\frac{\gamma^2}{4} - \omega_0^2)^{1/2}\right)t\right]$$

Critical damping. Occurs when $\gamma^2/4 = \omega_0^2$, which makes the suqre roots zero. Thus the equation is second order differential equation with one real auxilary equation roots. The solution is

$$x = (At + B)\exp\left(-\frac{\gamma t}{2}\right)$$

Here the mass, or whatever oscillating, returns to its equilibrium position in the shortest possible time without oscillating.

Putting all together. In summary we find three types of damped motion:

1. $\gamma^2/4 < \omega_0^2$ Light damping, Imaginary square root, Damped oscillations;
2. $\gamma^2/4 > \omega_0^2$ Heavy damping, Real Square root, Exponential decay of displacement;
3. $\gamma^2/4 = \omega_0^2$ Critical damping, Zero square root, Quickest return to equilibrium position without oscillation.

RLC circuit. In the case of an electrical oscillator it is the resistance in the circuit that impedes the flow of current. Kirchoff's law gives

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$
$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$
$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0$$

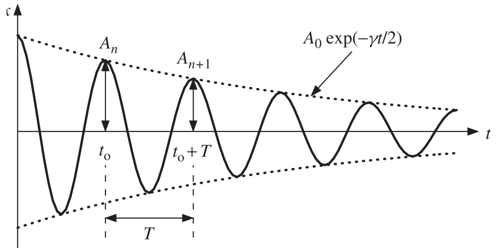


Figure: Graph of $x = A_0\exp(-\gamma^2 t/4)\cos\omega t$

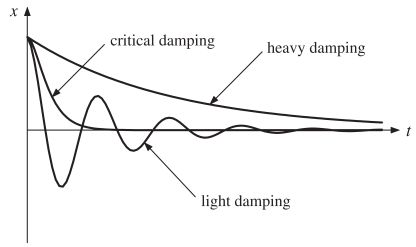


Figure: Motion of a damped oscillator for various cases

This is the equation of DHO with q as x, L as m, k as 1/C and R as b; so R/L is the equivalent of $\gamma = b/m = R/L$ and $\omega_0^2 = 1/LC$. Now assuming that this this the case of light damping, in another words $R^2/4L^2 < 1/LC$, the solution is

$$q = q_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega t$$

with

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$$

Since the voltage V_C across the capacitor is equal to q/C, dividing the solution by C

$$V_C = V_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega t$$

We find that the quality factor Q of the circuit is given by

$$Q = \frac{\omega_0}{\gamma} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Energy of DHO

In the case of very lightly damped oscillator $\gamma^2/4 \ll \omega_0^2$ we have

$$x = A_0 \exp\left(-\frac{\gamma t}{2}\right) \cos \omega_0 t$$
$$v = -A_0 \omega_0 \exp\left(-\frac{\gamma t}{2}\right) \left[\sin \omega_0 t + \frac{\gamma}{2\omega_0} \cos \omega_0 t\right]$$

where we approximate $\omega = \omega_0$. Since $\gamma \ll \omega_0$, we can ignore the second term at velocity equation

$$v = -A_0 \omega_0 \exp\left(-\frac{\gamma t}{2}\right) \sin \omega_0 t$$

Then

$$E = \frac{1}{2} A_0^2 \exp(-\gamma t)(m\omega_0^2 \sin^2 \omega_0 t + k \cos \omega_0 t)$$

considering $\omega_0^2 = k/m$

$$E(t) = \frac{1}{2} k A_0^2 \exp(-\gamma t) = E_0 \exp(-\gamma t)$$

The reciprocal of γ is the time taken $\tau = 1/\gamma$ for the energy of the oscillator to reduce by a factor of e, thus

$$E(t) = E_0 \exp\left(-\frac{t}{\tau}\right)$$

Rate of dissipation. The energy of an oscillator is dissipated because it does work against the damping force at the rate (damping force $\times$ velocity). We can see this by differentiating energy with respectto time

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x}$$

since the damping force $F_d = m\ddot{x} + kx = -b\dot{x}$, we can write

$$\frac{dE}{dt} = -b\dot{x}^2$$

Q factor

The quality factor Q of the oscillator describe how good an oscillator is, where we imply that the smaller the degree of damping the higher the quality of the oscillator. Oscillator with a high Q-value would make an appreciable number of oscillations before its energy is reduced substantially. The quality factor Q is defined as

$$Q = \frac{\omega_0}{\gamma}$$

Another way to define Q factor is

$$Q = \frac{\text{energy stored in the oscillator}}{\text{energy dissipated per radian}}$$

Now, consider energy of a very lightly damped oscillator one period apart

$$E_1 = E_0 \exp(-\gamma t)$$
$$E_2 = E_0 \exp[-\gamma(t+T)]$$

giving

$$\frac{E_2}{E_1} = \exp(-\gamma T)$$

Using series expansion

$$\frac{E_2}{E_1} = 1 - \gamma T$$

therefore

$$\frac{E - 2 - E_1}{E_1} \approx \gamma T \approx \frac{2\pi\gamma}{\omega_0} \approx \frac{2\pi}{Q}$$

where we have $\gamma T \ll 1$ and $\omega \approx \omega_0$. The fractional change in energy per cycle is equal to $2\pi/Q$ and so the fractional change in energy per radian is equal to 1/Q. Thus our definition is proved.

We can also recast DHO equation using Q factor

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x}\omega_0^2 x = 0$$

and the angular frequency ω

$$\omega = \omega_0 \left(1 - \frac{1}{4Q^2}\right)^{1/2}$$

This confirm confirms our assumption that ω is equal to ω_0 to a good approximation under most circumstances. Even when Q is as low as 5, ω is different from ω_0 by just 0.5%.

Forced Oscillations

Undamped forced oscillations. We begin with a mass m on a horizontal spring with a periodic driving force $F = F_0 \cos \omega t$ is applied to it. We obtain

$$m\ddot{x} + kx = F_0 \cos \omega t$$

Another form of this equation is

$$m\ddot{x} = -k(x - \xi)$$
$$m\ddot{x} + kx = ka \cos \omega t$$

with $\xi = a \cos \omega t$ as displacement due to driving force. We can rewrite the equation as

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \; a \cos \omega t$$
$$(D + \omega_0^2 i)(D - \omega_0^2 i)x = \omega_0^2 \; a \cos \omega t$$

To solve this, first we solve

$$(D + \omega_0^2 i)(D - \omega_0^2 i)X = \omega_0^2 \; a \exp i\omega t$$

This has a particular solution

$$X_p = C \exp i\omega t$$

Thus

$$\ddot{X}_p = -C\omega^2 \exp i\omega t$$

Substituting back, we get

$$(-\omega^2 + \omega_0^2)C \exp i\omega t = \omega_0^2 a \exp i\omega t$$

Solving for C

$$C = \frac{a}{1 - \omega^2/\omega_0^2}$$

The solution to the exponential equation is

$$X = \frac{a}{1 - \omega^2/\omega_0^2}(\cos \omega t + i \sin \omega t)$$

To solve our original, we take the real part

$$x = \frac{a}{1 - \omega^2/\omega_0^2} \cos \omega t$$

The fractional term is the amplitude of our oscillator as function of ω or $A(\omega)$.

Damped forced oscillations. We add the damping term into our equation

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

or

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2\, a \cos \omega t$$

As before, we write the equation as

$$\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right) x = \omega_0^2\, a \cos \omega t$$

By the method of complex exponentials, we solve first

$$\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right)\left(D - \frac{\gamma}{2} + \sqrt{\frac{\gamma}{4} - \omega_0^2}\right) X = \omega_0^2\, a \exp i\omega t$$

This has a particular solution

$$X_p = C \exp i\omega t$$

thus

$$\begin{aligned} \dot{X} &= C\omega i \exp i\omega t \\ \ddot{X} &= -C\omega \exp i\omega t \end{aligned}$$

Substituting back, we get

$$(-\omega^2 + \omega\gamma i + \omega_0^2) C \exp i\omega t = \omega_0^2 a \exp i\omega t$$

Solving fo C

$$C = \frac{a\omega_0^2}{(\omega_0^2 - \omega^2) + \omega\gamma i} = \frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) - \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

It is convenient to write the complex number C in the polar $|C| \exp i\delta$ form. We have

$$\begin{aligned} |C| &= \left(\frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) - \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \, \frac{a\omega_0^2\left[(\omega_0^2 - \omega^2) + \omega\gamma i\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \right)^{1/2} \\ &= \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \end{aligned}$$

Angle of $C = -\delta$ is formed by the real term $(\omega_0^2 - \omega^2)$ and imaginary term $-\omega\gamma i$. Thus

$$C = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \exp(-i\delta)$$

and

$$X_p = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \exp i(\omega t - \delta)$$

To find x_p we take the real part of X_p:

$$x = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \cos i(\omega t - \delta)$$

As before, the fractional term is the amplitude of our oscillator. Here δ is phase angle between the driving force and the resultant displacement. The minus sign of δ in Equation implies that the displacement lags behind the driving force and this is indeed the case in forced oscillations.

The amplitude

$$A(\omega) = \frac{a\omega_0^2}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}}$$

is maximum when the denominator is minimum

$$\frac{d}{d\omega}\left[\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}\right] = 0$$

from which

$$\omega = \omega_0\left(1 - \frac{\gamma^2}{2\omega_0^2}\right)^{1/2} \equiv \omega_{\mathrm{max}}$$

follows. We can find the maximum value of the amplitude A_{max} by substituting ω_{max}

$$A_{\mathrm{max}} = \frac{a\omega/\gamma}{\left(1 - \gamma^2/4\omega_0^2\right)^{1/2}}$$

Finally, in order to make our equation more general, we make use of the substitution $F_0 = ka$

$$x = \frac{F_0/m}{\left[(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right]^{1/2}} \cos i(\omega t - \delta)$$

In the meantime we use the substitution $Q = \omega_0/\gamma$ in the equations for ω_{max} and A_{max}

$$\omega_{\mathrm{max}} = \omega_0\left(1 - \frac{1}{2Q^2}\right)^{1/2}$$

$$A_{\mathrm{max}} = \frac{aQ}{\left(1 - 1/4Q^2\right)^{1/2}}$$