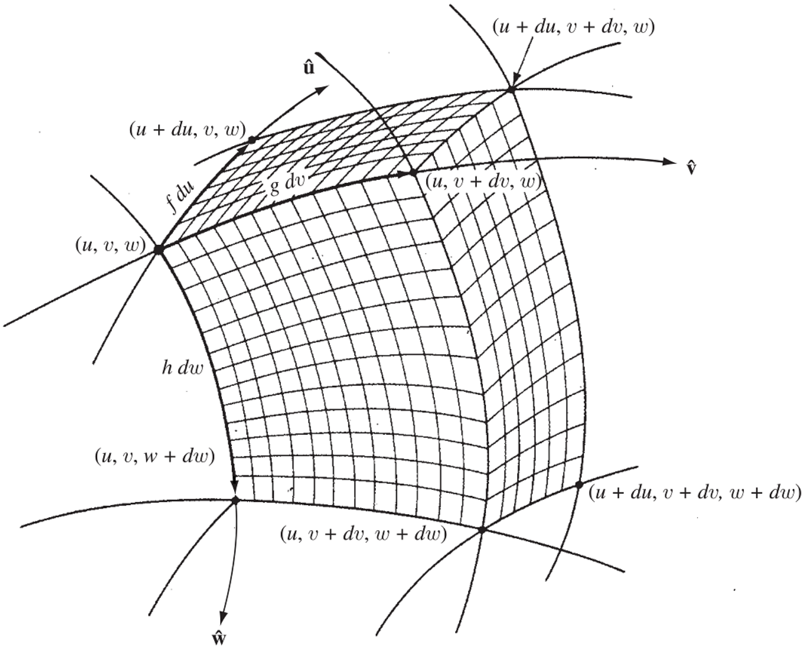
Physical Mathematics

E. F. Faust

with LaTeX



December 28, 2024

FUNDAMENTAL

Mainly consist of precalculus, and basic Calculus.

Algebra

Laws of Exponents.

$$\begin{aligned} x^{\frac{m}{n}} &= \sqrt[n]{m} \\ (x^m)^n &= x^{mn} \\ x^m x^n &= x^{m+n} \\ x^a y^a &= (xy)^a \end{aligned}$$

Special Factorization.

$$\begin{aligned} x^2 - y^2 &= (x+y)(x-y) \\ x^3 - y^3 &= (x-y)(x^2+xy+y^2) \\ x^3 + y^3 &= (x+y)(x^2-xy+y^2) \end{aligned}$$

Quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \begin{cases} D > 0 & \mathrm{re}(2) \\ D = 0 & \mathrm{re}(1) \\ D < 0 & \mathrm{im}(2) \end{cases}$$

Binomial theorem.

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k$$

with

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$

Trigonometry

Trigonometry Definition.

$$\begin{aligned} \sin\theta &= \frac{1}{\csc\theta} \\ \cos\theta &= \frac{1}{\sec\theta} \\ \tan\theta &= \frac{1}{\cot\theta} \end{aligned}$$

Pythagorean Identity.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1\\
\sec^2\theta - \tan^2\theta &= 1\\
\csc^2\theta - \cot^2\theta &= 1
\end{aligned}$$

Law of Sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Trigonometry Double Angle Identity.

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta\\
\cos 2\theta &= 1 - 2\sin^2\theta\\
&= 2\cos^2\theta - 1\\
&= \cos^2\theta - \sin^2\theta\\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

Trigonometry Addition and Difference Identity.

$$\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y\\
\sin(x-y) &= \sin x\cos y - \cos x\sin y\\
\cos(x+y) &= \cos x\cos y - \sin x\sin y\\
\cos(x-y) &= \cos x\cos y - \sin x\sin y\\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y}\\
\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}
\end{aligned}$$

Trigonometry Product Rule.

$$\begin{aligned}
\cos x\cos y &= \frac{1}{2}[\cos(x-y) + \cos(x+y)]\\
\sin x\sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)]\\
\sin x\cos y &= \frac{1}{2}[\sin(x+y) + \sin(x-y)]\\
\cos x\sin y &= \frac{1}{2}[\sin(x+y) - \sin(x-y)]
\end{aligned}$$

Neat Mnemonics.

$$\begin{vmatrix}\mathrm{S}^+\\ \mathrm{S}^-\\ \mathrm{C}^+\\ \mathrm{C}^-\end{vmatrix} = \begin{vmatrix}\mathrm{SC}+\mathrm{CS}\\ \mathrm{SC}-\mathrm{CS}\\ \mathrm{CC}-\mathrm{SS}\\ \mathrm{CC}+\mathrm{SS}\end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix}\mathrm{CC}\\ \mathrm{SS}\\ \mathrm{SC}\\ \mathrm{CS}\end{vmatrix} = \begin{vmatrix}\mathrm{C}^- + \mathrm{C}^+\\ \mathrm{C}^- - \mathrm{C}^+\\ \mathrm{S}^+ + \mathrm{S}^-\\ \mathrm{S}^+ - \mathrm{S}^-\end{vmatrix}$$

Logarithm

Definition (informal). $\log_a b$ means a to the power of what equal b.

Few important log rule.

$$\begin{aligned}
\log_c(ab) &= \log_c(a) + \log_c(b) \\
\log_c(\frac{a}{b}) &= \log_c(a) - \log_c(b) \\
\log_a b &= \frac{\log_c(b)}{\log_c(a)} \\
a^{\log_a b} &= b
\end{aligned}$$

Limit

Few Important Limits.

$$\begin{aligned}
\lim_{x \to a} c &= c \\
\lim_{x \to 0^+} \frac{1}{x} &= \infty \to \frac{1}{0^+} = \infty \\
\lim_{x \to 0^-} \frac{1}{x} &= -\infty \to \frac{1}{0^-} = -\infty
\end{aligned}$$

Limit as Definition of Derivative.

$$\frac{d}{dx} y = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative

How to determine the order of derivation: last computation is the first thing to do.

General Formula.

$$\begin{aligned}
\mathrm{D}\ x^n &= nx^{n-1} \\
\mathrm{D}\ (uv) &= \mathrm{D}\ u \cdot v + u \cdot \mathrm{D}\ v \\
\mathrm{D}\ \left(\frac{u}{v}\right) &= \frac{\mathrm{D}\ u \cdot v - u \cdot \mathrm{D}\ v}{v^2}
\end{aligned}$$

Trigonometry Formula.

$$\begin{aligned}
\mathrm{D}\ \sin x &= \cos x \\
\mathrm{D}\ \cos x &= -\sin x \\
\mathrm{D}\ \tan x &= \sec^2 x \\
\mathrm{D}\ \cot x &= -\csc^2 x \\
\mathrm{D}\ \sec x &= \sec x \tan x \\
\mathrm{D}\ \csc x &= -\cot x \csc x
\end{aligned}$$

Neat Mnemonics.

$$\begin{array}{ccc} \sec & \sec & \tan \\ \csc & -\csc & \cot \\ & \leftrightarrow \text{multiply} & \end{array} \quad \downarrow \text{cofunction}$$

Exponential and Logarithmic Functions.

$$\begin{aligned} \mathrm{D}\ \ln x &= \frac{1}{x} \\ \mathrm{D}\ a^n &= a^n \ln x \\ \mathrm{D}\ \log_a b &= \frac{1}{b \ln a} \end{aligned}$$

Minima and Maxima test. First derivative test:

- Determine critical points ($\mathrm{D}y = 0$), then divine into region;
- Pick value from each region and plug into *derivative*;
- Do the sign-graph.
- Determine local minima and maxima, the plug into *original function*

Second derivative test:

- determine critical points;
- plug critical into second derivative; and
- positive $\mathrm{D}^2 y$ means concave up ($\smile$), negative means concave down ($\frown$), and 0 means inconclusive.

Differentiation under integral sign. Differentiation under integral sign stated by Leibniz' rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x,t)\ dt = \int_u^v \frac{\partial f}{\partial x}\ dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

Proof. Suppose we want dI/dx where

$$I = \int_u^v f(t)\ dt$$

By the fundamental theorem of calculus

$$I = F(v) - F(u) = \mathcal{F}(v,u)$$

or I is a function of v and u. Finding dI/dx is then a partial differentiation problem. We can write

$$\frac{dI}{dx} = \frac{\partial I}{\partial v}\frac{dv}{dx} + \frac{\partial I}{\partial u}\frac{du}{dx}$$

By the fundamental theorem of calculus, we have

$$\frac{d}{dv} \int_a^v f(x)\ dt = \frac{d}{dv}\big[F(v) - F(a)\big] = f(v)$$

$$\frac{d}{dv}\int_u^b f(x)\,dt = \frac{d}{dv}\big[F(b) - F(u)\big] = -f(u)$$

where u and v are a function of x, while a and b are a constant. This is the case when we consider $\partial I/\partial v$ or $\partial I/\partial v$; the other variable is constant. Then

$$\frac{d}{dx}\int_u^v f(t)\,dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Under not too restrictive conditions,

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt$$

where, as before, a and b are constant. In other words, we can differentiate under the integral sign. It is convenient to collect these formulas into one formula known as Leibniz' rule:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = \int_u^v \frac{\partial f}{\partial x}\,dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} \qquad \blacksquare$$

Leibniz' rule for differentiating a product.

$$\left(\frac{d}{dx}\right)^n fg = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{d}{dx}\right)^{n-k} f\left(\frac{d}{dx}\right)^k g$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Integral

Basic Formula (integration constant omitted).

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

$$\int \frac{1}{x}\,dx = \ln|x|$$

$$\int u\,dv = uv - \int v\,du$$

$$\int a^x\,dx = \frac{a^x}{\ln a}$$

Trigonometry.

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \csc^2 x\,dx = -\cot x$$

$$\int \sec x\tan x\,dx = \sec x$$

$$\int \csc x\tan x\,dx = -\csc x$$

Root.

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin \frac{x}{a}$$
$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \ln x + \sqrt{x^2 \pm a^2}$$
$$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \frac{1}{a} \arctan \frac{x}{a}$$

Integration by part.

1. Splits the integrand. Choose u using LIATEN and let the rest be dv. (LIATEN: Log, Inverse trigonometry, Algebra, Trigonometry, ExponeN)

2. Do the box:

u	v	$\downarrow$ diff.
du	dv	$\uparrow$ int.

3. $\int u\, dv = uv - \int v\, du$

Tabular Method. Refer to the table

	D	I
+	$a \searrow$	b
-	$a' \searrow$	b
+	$a'' \searrow$	b
$\vdots$	$\vdots$	$\vdots$

1. 0 in D collumn or use LIATEN

2. integrate a row

3. a row repeats

Trigonometry Integral. Phytagorian Identity.

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

note that argument inside quadratic trigonometry is half of trigonometry, which means $\cos^2 2x = (1 + \cos 4x)/2$. There are few cases of tricky trigonometry integral. First, if power of sin is odd and positive.

- lop one power off
- convert remaining (even power) using Phytagorian Identity in term of cosine
- integrate using subs method

If the power of sine is odd and positive.

- same as before

If the power of sine and cosine is even and nonegative, then:

- convert using Phytagorian Identity and solve

Trigonometry substitution. Trigonometry function and its radical pair

$$\tan\theta = \sqrt{u^2 + a^2}$$
$$\sin\theta = \sqrt{a^2 - u^2}$$
$$\sec\theta = \sqrt{u^2 - a^2}$$

where u is the variable we are differentiating with respect to. Mnemonics: + looks like tangent; - for sin and sec; and it is a $s$in. Trigonometry substitution step is then.

1. Draw a right triangle where trigonometry pair equal $\frac{u}{a}$
2. using the trigonometry pair equation*, solve for x and dx
3. find trigonometry where $\frac{\surd}{a}$
4. subs again if equation* still contain θ and solve

Partial Fraction.

1. Factor out denominator
2. Breakup the function and put unknown (Capital Letter) into numerator. Put numerator normally if factor is linear, put $Px+Q$ Irreducible quadratic factor IQF. In general,

$$\frac{Ax^{n-1} + Bx^{n-2} + \cdots}{x^n + x^{n-1} + \cdots}$$

3. Multiply both side by left side's denominator
4. Take the roots of the linear factors and plug them into x, and solve for the unknowns
5. Put unknowns into step 2
6. Splits Integral, then solve
7. For equating coefficients like terms, after step 3, expand equation*. Then, collect like terms and equate coefficient of like terms from both side

Appendix

Integration Technique Example. 1. Trigonometry substitution. Find $\int \frac{dx}{\sqrt{9x^2+4}}$. Refer to the Mnemonics, the trigonometry pair is tangent.

$$I = \int \frac{dx}{\sqrt{(3x)^2 + 2^2}}$$

$$\tan\theta = \frac{3x}{2}$$

solving for x and dx

$$x = \frac{2}{3}\tan\theta$$

$$dx = \frac{2}{3}\sec^2\theta \; d\theta$$

trigonometry where $\frac{\surd}{a}$ holds is secant, solving for radical

$$\sec\theta = \frac{\sqrt{9x^2+4}}{2}$$

$$\sqrt{9x^2+4} = 2\sec\theta$$

the integral is then

$$I = \frac{1}{3}\int \sec\theta \; d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

substituting the θ function

$$I = \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2} + \frac{3x}{2}\right| + C$$

$$= \frac{1}{3}\ln\left|\sqrt{9x^2+4} + 3\right| + C$$

Basic. Most common integrals.

$$\int \frac{1}{x}\, dx = \ln|x|$$

$$\int u \; dv = uv - \int v \; du$$

$$\int u \; dv = uv - \int v \; du$$

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b|$$

Rational Functions. Integrals of rational function

$$\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a}$$

$$\int (x+a)^n\,dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$

$$\int x(x+a)^n\,dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln|a^2+x^2|$$

$$\int \frac{x^2}{a^2+x^2}\,dx = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3}{a^2+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, \quad a \neq b$$

$$\int \frac{x}{(x+a)^2}\,dx = \frac{a}{a+x} + \ln|a+x|$$

$$\int \frac{x}{ax^2+bx+c}\,dx = \frac{1}{2a}\ln|ax^2+bx+c| -$$
$$\frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

Roots. Integrals of roots.

$$\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2}$$

$$\int \frac{1}{\sqrt{x\pm a}}\,dx = 2\sqrt{x\pm a}$$

$$\int \frac{1}{\sqrt{a-x}}\,dx = -2\sqrt{a-x}$$

$$\int x\sqrt{x-a}\,dx = \begin{cases} \frac{2a}{3}(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}, & \text{or} \\ \frac{2}{3}x(x-a)^{3/2} - \frac{4}{15}(x-a)^{5/2}, & \text{or} \\ \frac{2}{15}(2a+3x)(x-a)^{3/2} \end{cases}$$

$$\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$$

$$\int (ax+b)^{3/2}\,dx = \frac{2}{5a}(ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x\pm a}}\,dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$$

$$\int \sqrt{\frac{x}{a-x}}\,dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$$

$$\int \sqrt{\frac{x}{a+x}}\, dx = \sqrt{x(a+x)} - a\ln[\sqrt{x}+\sqrt{x+a}]$$

$$\int x\sqrt{ax+b}\, dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b}$$

$$\int \sqrt{x^3(ax+b)}\, dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)}$$

$$+ \frac{b^3}{8a^{5/2}} \ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\int x\sqrt{x^2\pm a^2}\, dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2}$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\, dx = \ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}\, dx = \sqrt{x^2\pm a^2}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\, dx = -\sqrt{a^2-x^2}$$

$$\int \frac{x^2}{\sqrt{x^2\pm a^2}}\, dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\, dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

Integrals with Logarithms.

$$\int \ln ax\; dx = x\ln(ax) - x$$

$$\int \frac{\ln ax}{x}\, dx = \frac{1}{2}(\ln ax)^2$$

$$\int \ln(ax+b)dx = (x+\frac{b}{a})\ln(ax+b) - x, \quad a\neq 0$$

$$\int \ln(x^2+a^2)\; dx = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x$$

$$\int \ln(x^2-a^2)\; dx = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x$$

$$\int \ln(x^2-a^2)\; dx = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x$$

$$\int \ln(ax^2+bx+c)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x$$

$$+ (\frac{b}{2a}+x)\ln\;(ax^2+bx+c)$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}(x^2-\frac{b^2}{a^2})\ln(ax+b)$$

$$\frac{1}{2}(x^2-\frac{a^2}{b^2})\ln(a^2-b^2x^2)$$

Integrals with Exponential.

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \sqrt{x} e^{ax} dx = \frac{1}{a}\sqrt{x} e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}} \operatorname{erf}\left(i\sqrt{ax}\right)$$

$$\int x e^{x} dx = (x-1)e^{x}$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{x} dx = \left(x^2 - 2x + 2\right) e^{x}$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$$

$$\int x^3 e^{x} dx = \left(x^3 - 3x^2 + 6x - 6\right) e^{x}$$

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$$

$$\int x^n e^{ax}\, dx = \frac{(-1)^n}{a^{n+1}} \Gamma[1+n, -ax]$$

$$\int e^{ax^2}\, dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(ix\sqrt{a}\right)$$

$$\int e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erf}\left(x\sqrt{a}\right)$$

$$\int x e^{-ax^2}\, dx = -\frac{1}{2a} e^{-ax^2}$$

$$\int x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \operatorname{erf}(x\sqrt{a}) - \frac{x}{2a} e^{-ax^2}$$

Integrals with Trigonometry Functions.

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax$$

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^n ax\, dx = -\frac{1}{a}\cos ax \times {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right]$$

$$\int \sin^3 ax\, dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax$$

$$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^p ax\, dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right]$$

$$\int \cos^3 ax\, dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos ax \sin bx \; dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, \quad a \neq b$$

$$\int \sin^2 ax \cos bx \; dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$

$$\int \sin^2 x \cos x \; dx = \frac{1}{3} \sin^3 x$$

$$\int \cos^2 ax \sin bx \; dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$

$$\int \cos^2 ax \sin ax \; dx = -\frac{1}{3a} \cos^3 ax$$

$$\int \sin^2 ax \cos^2 bx \; dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b}$$

$$- \frac{\sin[2(a+b)x]}{16(a+b)}$$

$$\int \sin^2 ax \cos^2 ax \; dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \tan ax \; dx = -\frac{1}{a} \ln \cos ax$$

$$\int \tan^2 ax \; dx = -x + \frac{1}{a} \tan ax$$

$$\int \tan^n ax \; dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right)$$

$$\int \tan^3 ax \; dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| = 2 \tanh^{-1}\left(\tan \frac{x}{2}\right)$$

$$int \sec^2 ax \; dx = \frac{1}{a} \tan ax$$

$$\int \sec^3 x \; dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x|$$

$$\int \sec x \tan x \; dx = \sec x$$

$$\int \sec^2 x \tan x \; dx = \frac{1}{2} \sec^2 x$$

$$\int \sec^n x \tan x \; dx = \frac{1}{n} \sec^n x, \quad n \neq 0$$

$$\int \csc x \; dx = \ln\left|\tan \frac{x}{2}\right| = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax \; dx = -\frac{1}{a} \cot ax$$

$$\int \csc^3 x \; dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln|\csc x - \cot x|$$

$$\int \csc^n x \cot x \; dx = -\frac{1}{n} \csc^n x, n \neq 0$$

$$\int \sec x \csc x \; dx = \ln|\tan x|$$

Products of Trigonometry Functions and Monomials.

$$\int x\cos x\;dx = \cos x + x\sin x$$

$$\int x\cos ax\;dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

$$\int x^2\cos x\;dx = 2x\cos x + \left(x^2-2\right)\sin x$$

$$\int x^2\cos ax\;dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$$

$$\int x^n\cos x\;dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix)+(-1)^n\Gamma(n+1,ix)\right]$$

$$\int x^n cosax\;dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax)-\Gamma(n+1,ixa)\right]$$

$$\int x\sin x\;dx = -x\cos x + \sin x$$

$$\int x\sin ax\;dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2\sin x\;dx = \left(2-x^2\right)\cos x + 2x\sin x$$

$$\int x^2\sin ax\;dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$$

$$\int x^n\sin x\;dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix)-(-1)^n\Gamma(n+1,-ix)\right]$$

Products of Trigonometry Functions and Exponential.

$$\int e^x\sin x\;dx = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\int e^{bx}\sin ax\;dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$$

$$\int e^x\cos x\;dx = \frac{1}{2}e^x(\sin x + \cos x)$$

$$\int e^{bx}\cos ax\;dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$$

$$\int xe^x\sin x\;dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$$

$$\int xe^x\cos x\;dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$$

Integrals of Hyperbolic Functions.

$$\int \cosh ax dx = \frac{1}{a} \sinh ax$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} \left[a \cosh bx - b \sinh bx \right], & a \neq b \\ \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2}, & a = b \end{cases}$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} \left[-b \cosh bx + a \sinh bx \right], & a \neq b \\ \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} a = b \end{cases}$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[1 + \dfrac{a}{2b}, 1, 2 + \dfrac{a}{2b}, -e^{2bx} \right] \\ \\ -\dfrac{1}{a} e^{ax} {}_2F_1 \left[\dfrac{a}{2b}, 1, 1E, -e^{2bx} \right], & a \neq b \\ \\ \dfrac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a}, & a = b \end{cases}$$

$$\int \tanh ax \, dx = \frac{1}{a} \ln \cosh ax$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[a \sin ax \cosh bx + b \cos ax \sinh bx \right]$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cos ax \cosh bx + a \sin ax \sinh bx \right]$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[-a \cos ax \cosh bx + b \sin ax \sinh bx \right]$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cosh bx \sin ax - a \cos ax \sinh bx \right]$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} \left[-2ax + \sinh 2ax \right]$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} \left[b \cosh bx \sinh ax - a \cosh ax \sinh bx \right]$$

Series

Test

Geometric Series (for r less than one):

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$$

Preliminary Test:

$$\text{if } \lim_{n\to\infty} a_n \neq 0 \text{ , then series Diverges}$$

Comparation Test:

$$A \le C \text{ , A Converges}$$
$$A \ge D \text{ , A Diverges}$$

Integral Test:

$$\int^{\infty} A \;\; dn \begin{cases} \text{A finite} \to \text{Converges} \\ \text{A infinite} \to \text{Diverges} \end{cases}$$

Ratio Test:

$$\rho_n = \left| \frac{a_{(n+1)}}{a_n} \right|$$
$$\rho = \lim_{n\to\infty} \rho_n \begin{cases} \rho > 1 \text{ Diverge} \\ \rho = 0 \text{ Inconclusive} \\ \rho < 1 \text{ Converges} \end{cases}$$

Special Comparation:

$$\lim_{n\to\infty} \frac{A}{C} \text{ Limit finite} \to \text{A Converges}$$
$$\lim_{n\to\infty} \frac{A}{D} \text{ Limit} > 0 \to \text{A Diverges}$$

Raabe Test:

$$\rho \equiv \lim_{n\to\infty} \left[n(\frac{a_n}{a_{n+1}} - 1) \right] \begin{cases} \rho > 1, \text{ Converge} \\ \rho = 1, \text{ Inconclusive} \\ \rho < 1, \text{ Diverge} \end{cases}$$

Root Test:

$$\rho \equiv \lim_{n\to\infty} \sqrt[n]{|a_n|}$$
$$\equiv \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \begin{cases} \rho > 1, \text{ Converge} \\ \rho = 1, \text{ Inconclusive} \\ \rho < 1, \text{ Diverge} \end{cases}$$

Alternating series Test

$$\text{if} \quad |a_{n+1}| \le |a_n| \quad \text{and} \quad \lim_{n\to\infty} a_n = 0$$

then series converges.

Function Expansion

Taylor Series about $x = a$:

$$\frac{1}{n!}(x-a)^n f^n(a) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots$$

Maclaurin Series:

$$\frac{1}{n!}(x)^n f^n(0) = f(0) + (x)f'(0) + \frac{1}{2!}(x)^2 f''(0) + \frac{1}{3!}(x)^3 f'''(0) + \cdots$$

Maclaurin Expansion for basic function:

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} && = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} && = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad -1 < x \le 1 \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n)} && = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \\
(1+x)^p &= \sum_{n=0}^{\infty} \binom{p}{n} x^n && = +px + \frac{p(p-1)}{2!}x^2 \\
& && + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \quad |x| < 1
\end{aligned}$$

Complex Analysis

Introduction

Complex Number. Complex number may be written in the rectangular form or polar form

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

The quantity r is called the modulus or absolute value of z, and θ is called the angle of z (or the phase, or the argument, or the amplitude of z). In symbols

$$\begin{array}{ll} \text{Re } z = x & |z| = \text{mod } z = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} \\ \text{Im } z = y \text{ (not } iy) & \text{angle of } z = \theta \end{array}$$

The values of θ should be found from a diagram rather than a formula, although we do sometimes write $\theta = \arctan(y/x)$. Word of caution however, the domain of $\arctan x$ is restricted to $(-\pi/2, \pi/2)$. here's another useful (?) operator formula

$$\begin{aligned} \text{Re } z &= \frac{z+\bar{z}}{z} \\ \text{Im } z &= \frac{z-\bar{z}}{zi} \end{aligned}$$

Conjugate. Complex numbers come in conjugate pairs; for such pairs are mirror images of each other with the x axis as the mirror. If we write $z = r(\cos\theta + i\sin\theta)$, then

$$\bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

Euler's Formula.

Using series expansion, we write $e^{i\theta}$, where θ is real

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(+\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right) \end{aligned}$$

We then have the very useful result known as Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Remembering that any complex number can be written in the form $re^{i\theta}$, we get

$$z_1 \times z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \quad \text{and} \quad z_1 \div z_2 = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Using the rules for multiplication and division of complex numbers, we have

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

for any integral n. The case $r = 1$, the equation becomes DeMoivre's theorem:

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

Function of Complex Number

. We define e^z by the power series

$$e^z = \sum_0^{\infty} \frac{z}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

then we can write

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

We then write Euler's formula, in θ and $-\theta$

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

These two equations can be solved for $\sin\theta$ and $\cos\theta$

$$\begin{aligned} \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \end{aligned}$$

These formulas are useful in evaluating integrals since products of exponentials are easier to integrate than products of sines and cosines.

We could also define $\sin z$ and $\cos z$ for complex z by their power series as we did for e^z, however it is simpler to use the complex equations we just obtained to define $\sin z$ and $\cos z$

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} \end{aligned}$$

The rest of the trigonometric functions of z are defined in the usual way in terms of these; for example, $\tan z = \sin z / \cos z$

Hyperbolic Function

Hyperbolic sine (abbreviated sinh) and the hyperbolic cosine (abbreviated cosh) is defined from pure imaginary, that is, $z = iy$. Their definitions for all z are

$$\begin{aligned} \sinh z &= \frac{e^z - e^{-z}}{2} \\ \cosh z &= \frac{e^z + e^z}{2} \end{aligned}$$

As before, the other hyperbolic functions are named and defined in a similar way to parallel the trigonometric functions.

We can write

$$\begin{aligned} \sin iy &= i \sinh y \\ \cos iy &= \cosh y \end{aligned}$$

Then we see that the hyperbolic functions of y are (except for one i factor) the trigonometric functions of iy.

The functions $\sin t, \cos t$, and the rest of the gang are called "circular functions" and the functions $\sinh t, \cosh t$, etc. are called "hyperbolic functions" because $x = \cos t, y = \sin t$, satisfy the equation of a circle $x^2 + y^2 = 1$, while $x = \cosh t, y = \sinh t$, satisfy the equation of a hyperbola $x^2 - y^2 = 1$.

Logarithm

If

$$z = e^w$$

then by definition

$$w = \ln z$$

We can write the law of exponents, as

$$z_1 z_2 = e^{w_1 + w_2}$$

Taking logarithms of this equation, we get

$$\ln z_1 z_2 = w_1 + w_2 = \ln z_1 + \ln z_2$$

This is the familiar law for the logarithm of a product, justified now for complex numbers. We can then find the real and imaginary parts of the logarithm of a complex number from the equation

$$\ln z = ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$$

thus

$$\ln z = \ln r + i\theta$$

Since θ has an infinite number of values (all differing by multiples of 2π), a complex number has infinitely many logarithms, differing from each other by multiples of $2\pi i$.

Complex Power. By definition, for complex a and b ($a = e$),

$$a^b = e^{b \ln a}$$

Analytic Function

Introduction to Complex Function. In general, we write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where it is understood that u and v are real functions of the real variables x and y.

Recall that functions are customarily single-valued. Does this mean that we cannot define a function by a formula such as $\ln z$? For each z, $\ln z$ has an infinite set of values. But if θ is allowed a range of only 2π , then $\ln z$ has one value for each z and this single-valued function is called a branch of $\ln z$. Thus in using formulas, we always discuss a single branch at a time so that we have a single-valued function. As a matter of terminology, however, you should know that the whole collection of branches is sometimes called a "multiple-valued function."

Definition. A function $f(z)$ is analytic–or regular or holomorphic or monogenic–in a region–region must be two dimensional, isolated points and curves are not regions–of the complex plane if it has a unique derivative at every point of the region.

The derivative of $f(z)$ is defined by the equation

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta x + i\Delta y}$$

The statement "$f(z)$ is analytic at a point $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$. When we say that $f(x)$ has a derivative at $x = x_0$, we mean that these two values are equal. When we say that $f(z)$ has a derivative at $z = z_0$, we mean that $f'(z)$ has the same value no matter how we approach z_0.

Theorems

Theorem I: Cauchy-Riemann conditions. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof. Remembering that $f(z) = f(x + iy)$, we use the rules of partial differentiation

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

and

$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}$$

Since $f = u(x, y) + iv(x, y)$, we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Combining them we have

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{df}{dz} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Since we assumed that df/dz exists and is unique (this is what analytic means), these two expressions for df/dz must be equal. Taking real and imaginary parts, we get the Cauchy-Riemann equations. ■

Theorem II. If $u(x, y)$ and $v(x, y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region–not necessarily on the boundary.

Proof (?). Although we shall not prove this, we can make it plausible by showing that it is true when we approach z_0 along any straight line. Assuming that we approach z_0 along a straight line of slope m, we will show that df/dz does not depend on m if u and v satisfy Cauchy-Riemann conditions. The equation of the straight line of slope m through the point $z_0 = x_0 + iy_0$ is

$$y - y_0 = m(x - x_0)$$

and along this line we have $dy/dx = m$. Then we find

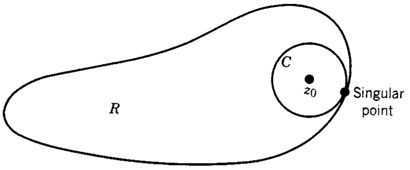
$$\begin{aligned}\frac{df}{dz} &= \frac{du + i\,dv}{dx + i\,dy} \\ &= \frac{\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy + i\left(\dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy\right)}{dx + i\,dy} \\ &= \frac{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial y}m\right)}{1 + i\,m}\end{aligned}$$

Using the Cauchy-Riemann equations, we get

$$\begin{aligned}\frac{df}{dz} &= \frac{\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial x}m + i\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}m\right)}{1 + i\,m} \\ &= \frac{\dfrac{\partial u}{\partial x}(1 + im) + i\dfrac{\partial v}{\partial y}(1 + im)}{1 + i\,m} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}\end{aligned}$$

Thus df/dz has the same value for approach along any straight line. The theorem states that it also has the same value for approach along any curve.

Theorem III. If $f(z)$ is analytic in a region R, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point C.



No proof. Some definitions are in order. A regular point of $f(z)$ is a point at which $f(z)$ is analytic. A singular point or singularity of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Theorem IV. Part 1: if $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region–that is, u and v are harmonic functions.

Part 2: any function u–or v–satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

No proof. Thus we can find solutions of Laplace's equation simply by taking the real or imaginary parts of an analytic function of z. It is also often possible, starting with a simple function which satisfies Laplace's equation, to find the explicit function $f(z)$ of which it is, say, the real part.

Theorem V: Cauchy's theorem. Let C be a simple–curve which does not cross itself–closed curve with a continuously turning tangent except possibly at a finite number of points–that is, we allow a finite number of corners, but otherwise the curve must be smooth. If $f(z)$ is analytic on and inside C, then

$$\oint_C f(z)\, dz = 0$$

Proof. We shall prove Cauchy's theorem assuming that $f'(z)$ is continuous.

$$\oint_C f(z)\, dz = \oint_C (u + iv)(dx + i\, dy)$$

or

$$\oint_C f(z)\, dz = \oint_C (u\, dx - v\, dy) + i \oint_C (v\, dx + u\, dy)$$

Green's theorem in the plane says that if $P(x, y)$, $Q(x, y)$, and their partial derivatives are continuous in a simply-connected region R, then

$$\oint_C P\, dx + Q\, dy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where C is a simple closed curve lying entirely in R and A is area inside C. Applying Green's Theorem to the first integral, we get

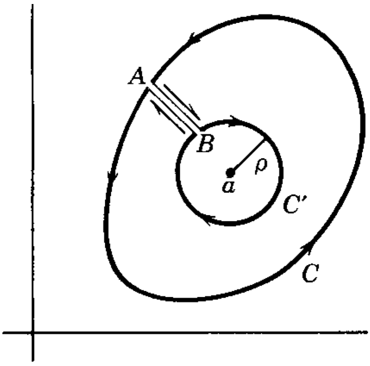
$$\oint_C (u\, dx - v\, dy) = \int_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = 0$$

where we have used the Cauchy-Riemann. In the same way the second integral in is zero

$$i \oint_C (v\, dx + u\, dy) = i \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy = i \int_A \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dx\, dy = 0$$

Theorem VI: Cauchy's integral formula. If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - a}\, dz$$



Proof. Let a be a fixed point inside the simple closed curve C and consider the function

$$\phi(z) = \frac{f(z)}{z-a}$$

where $f(z)$ is analytic on and inside C. Let C' be a small circle (inside C) with center at a and radius ρ. Thus we have

$$\oint_{C\circlearrowright} \phi(z)\,dz + \oint_{C'\circlearrowleft} \phi(z)\,dz = 0$$

and

$$\oint_{C\circlearrowleft} \phi(z)\,dz = \oint_{C'\circlearrowleft} \phi(z)\,dz$$

where both are counterclockwise. Along the circle C',

$$z = a + \rho e^{i\theta}$$
$$dz = \rho i e^{i\theta}\,d\theta$$

and the integral becomes

$$\oint_{C'\circlearrowleft} \phi(z)\,dz = \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta = \int_0^{2\pi} f(z) i\,d\theta$$

Since our calculation is valid for any (sufficiently small) value of ρ, we shall let $\rho \to 0$ (that is, $z \to a$) to simplify the formula

$$\oint_{C'\circlearrowleft} \phi(z)\,dz = \int_0^{2\pi} f(a) i\,d\theta = 2\pi i f(a)$$

thus

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz \quad \blacksquare$$

Theorem VII: Laurent's theorem. Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{z-z_0} + \ldots$$

convergent in R. Such a series is called a Laurent series. The b series is called the principal part of the Laurent series.

Definition. If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a regular point.

If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a pole of order n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a simple pole.

If there are an infinite number of b's different from zero, $f(z)$ has an essential singularity at $z = z_0$.

The coefficient b_1 of $1/(z-z_0)$ is called the residue of $f(z)$ at $z = z_0$.

Residue Theorem

Suppose we are going to find the value of $\oint f(z)\, dz$ around a simple closed curve C surrounding an isolated singular point z_0 but inclosing no other singularities. Let $f(z)$ be expanded in the Laurent series about $z = z_0$. The integral of the a series

$$\oint_C a_n(z - z0)^n\, dz = 0$$

since this pat is analytic. The integral of the b series, we replace the integrals around C by integrals around a circle C' with center at z_0 and radius ρ

$$\oint_{C'} \frac{b_n}{(z - z_0)^n}\, dz = i \int_0^{2\pi} b_n e^{i\theta(1-n)} d\theta = \frac{b_n}{1-n}\left(e^{2\pi i(1-n)} - 1\right) = 0$$

for all $n > 1$. For $n = 1$

$$\oint_{C'} \frac{b_1}{(z - z_0)}\, dz = ib_1 \int_0^{2\pi} d\theta = 2\pi i b_1$$

Then

$$\oint_{C'} f(z)\, dz = 2\pi i b_1$$

since b_1 is called the residue of $f(z)$ at $z = z_0$, we can say

$$\oint_c f(z)\, dz = 2\pi i \times \text{residue of } f(z) \text{ at the singular point inside } C$$

The only term of the Laurent series which has survived the integration process is the b_1 term; you can see the reason for the term "residue." Thus we have the residue theorem:

$$\oint_C f(z)\, dz = 2\pi i\, R(z)$$

where $R(z)$ sum of the residues of $f(z)$ inside C.

Method of Finding Residue

Laurent Series. If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0, then the residue is just the coefficient b_1 of the term $1/(z - z0)$. **Caution**: Be sure you have the expansion about $z = z_0$.

Simple Pole. If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$. In general we write

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \Big|_{z = z_0}$$

when z_0 is a simple pole. If, $f(z)$ can be written as $g(z)/h(z)$, where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then

$$R(z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{(z - z_0)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$$

In another words

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \begin{cases} (z) = g(z)/h(z), \\ g(z_0) = C \neq 0, \\ h(z_0) = 0, \\ h'(z_0) \neq 0 \end{cases}$$

Perhaps the simplest way–without finding the Laurent series–to find if a function has a simple pole is that if the limit obtained is some constant not 0 or ∞, then $f(z)$ does have a simple pole and the constant is the residue. If the limit is equal to 0, the function is analytic and the residue is 0. If the limit is infinite, the pole is of higher order.

Suppose $f(z)$ is written in the form $g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic. Then you can think of $g(z)$ and $h(z)$ as power series in $(z - z_0)$. If the denominator has the factor $(z - z_0)$ to one higher power than the numerator, then $f(z)$ has a simple pole at z_0.

Multiple Poles. When $f(z)$ has a pole of order n, we can use the following method of finding residues.

$$R(z_0) = \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right) \Big|_{z = z_0}$$

where m is an integer greater than or equal to the order n of the pole. To compute the derivative quickly, use Leibniz' rule for differentiating a product

$$\left(\frac{d}{dx} \right)^n fg = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx} \right)^{n-k} f \left(\frac{d}{dx} \right)^k g$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Appendix I: DeMoivre's theorem

Example 1.

$$\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)^{25} = (e^{i\pi/10})^{25} = e^{2\pi i}e^{i\pi/2} = 1 \times i = i$$

Example 2. Find the cube roots of 8. We write the cube roots of 8 in polar form

$$z^{1/3} = 8^{1/3}e^{i(0+2n\pi)/3}$$

with $z = 8e^{i\cdot(0+2n\pi)}$. These values are

$$\sqrt[3]{8} = 2, \quad 2e^{i2\pi/3}, \quad 2e^{i4\pi/3}, \quad 2e^{i2\pi}, \ldots$$

or in rectangular coordinate

$$2, \quad -1 + i\sqrt{3}, \quad -1 - \sqrt{3}$$

Example 3. Find and plot all values of $\sqrt[4]{-64}$. As, before

$$z^{1/4} = 64^{1/4}e^{i(\pi+2n\pi)/4}$$

These values are

$$\sqrt[4]{-64} = 2\sqrt{2}, \quad 2\sqrt{2}e^{i3\pi/4}, \quad 2\sqrt{2}e^{i3\pi/4}, \quad 2\sqrt{2}e^{i5\pi/4}, \quad 2\sqrt{2}e^{i7\pi/4}, \ldots$$

or in rectangular coordinate

$$\sqrt[4]{-64} = \pm 2 \pm 2i$$

Example 4. Find and plot all values of $\sqrt[6]{-8i}$. We have uour complex number

$$z = 8e^{i(1.5\pi+2n\pi)}$$

raised to the power of 1/6

$$z^{1/6} = \sqrt[6]{8}e^{i(1.5\pi+2n\pi)/6} = \sqrt{2}e^{i(\pi/4+n\pi/3)}$$

We can do this one root at a time or more simply by using a computer to solve the equation $z^6 = -8i$.

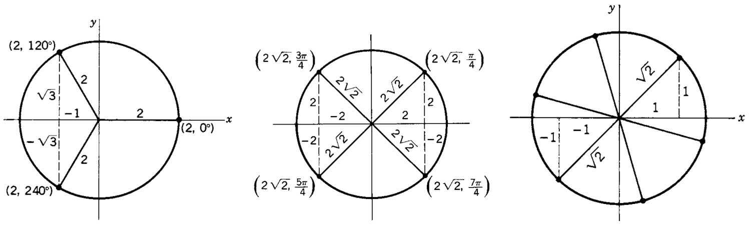


Figure: sketch of example 2, 3 and 4

Appendix II: Inverse Trigonometric

Ex. 1: Find z that satisfy $z = \arccos 2$ Since $\cos z = 2$, we have

$$\frac{e^{iz} + e^{-iz}}{2} = 2$$

To simplify the algebra, let $u = e^{iz}$ and $u^{-1} = e^{-iz}$

$$u + u^{-1} = 4$$

Multiply by u, then we get quadratic equation

$$u^2 - 4u + 1 = 0$$

Solving this equation

$$e^{iz} = 2 \pm \sqrt{3}$$

Take logarithms of both sides of this equation, and solve for z:

$$iz = \ln(2 \pm \sqrt{3}) + 2n\pi i$$
$$z = 2n\pi - i \ln(2 \pm \sqrt{3})$$

Ex. 2: Evaluating Integral. In integral tables or from your computer you may find for the indefinite integral

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^1 \frac{x}{a}$$

or

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

How are these related? Put $z = \sinh^1 x/a$, thus

$$\sinh z = \frac{x}{a} = \frac{e^z - e^{-z}}{2}$$

Let $e * z = u, e^{-z} = 1/u$. Then

$$au^2 - 2xu - a = 0$$

We solve for u, or rather z, as in the previous example

$$e^z = \frac{x \pm \sqrt{x^2 - a^2}}{z}$$

For real integrals, that is, for real z, $e^z > 0$, so we must use the positive sign. Then, taking the logarithm

$$z = \ln\left(x + \sqrt{x^2 + a^2}\right) - \ln a$$

We see that the two answers differ only by the constant $\ln a$, which is a constant of integration.

Appendix III: Laplace's equation

Consider the function $u(x,y) = x^2 - y^2$. We find that

$$\nabla^2 u = 2 - 2 = 0$$

that is, u satisfies Laplace's equation (or u is a harmonic function). Let us find the function $v(x,y)$ such that $u + iv$ is an analytic function of z. By the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$$

Integrating partially with respect to y, we get

$$v(x,y) = 2xy + g(x)$$

where $g(x)$ is a function of x to be found. Unlike our usual integration, the constant we get from partial integration not a simple constant, but rather a function of x. Differentiating partially with respect to x and again using the Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$2y + g'(x) = 2y$$

Thus we find

$$g'(x) = 0 \quad \text{and} \quad g(x) = C$$

Then

$$f(z) = x^2 - y^2 + 2ixy + C = z^2 + C$$

The pair of functions u, v are called conjugate harmonic functions.

Appendix IV: Residue Theorem

Ex. 1 Find

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

If we make the change of variable of

$$z = e^{i\theta}$$

then

$$dz = ie^{i\theta} d\theta$$

$$\cos\theta = \frac{z + 1/z}{2}$$

As θ goes from 0 to 2π, z traverses the unit circle $|z| = 1$. Making these substitutions in I, we get

$$I = \oint_C = \frac{dz/iz}{5 + 2(z + 1/z)} = \frac{1}{i}\oint_C \frac{dz}{(2z+1)(z+2)}$$

where C is the unit circle. The integrand has poles at $z = -1/2$ and $z = -2$. Since only $z = -1/2$ is inside the contour C, the residue of the integrand is

$$R(-1/2) = \lim_{z \to -\frac{1}{2}} (z - \frac{1}{2}) \frac{1}{(2z+1)(z+2)}\Bigg|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Then by the residue theorem

$$I = \frac{1}{i} 2\pi i R(-1/2) = \frac{2\pi}{3}$$

Ex. 2 Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Altough we could evaluate I by elementary method, we will use the residue theorem by considering

$$\oint_C \frac{dz}{1+z^2}$$

where C is the closed boundary of the semicircle with radius $\rho > 1$. C incloses the singular point $z = i$ and no others, thus

$$R(i) = \lim_{z \to i} (z-i) \frac{1}{1+z^2} \bigg|_{z=i} = \frac{1}{2i}$$

Then the value of the contour integral is

$$I = 2\pi i R(i) = \pi$$

Let us write the integral in two parts: (1) an integral along the x axis, for this part $z = x$; (2) an integral along the semicircle where $z = \rho e^{i\theta}$

$$\oint_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}}{1+\rho^2 e^{2i\theta}} d\theta = \pi$$

Let $\rho \to \infty$; then the second integral tends to zero since the numerator contains ρ and the denominator ρ^2. We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

Ex. 3 Evaluate

$$I = \int_0^{\infty} \frac{\cos x}{1+x^2} dx$$

We consider the contour integral

$$\oint_C \frac{e^{iz}}{1+z^2} dz$$

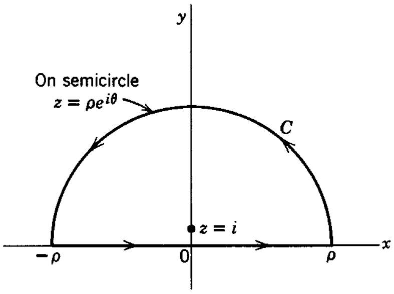


Figure: Semicircle radius ρ

where C is the same semicircular contour as before. The singular point inclosed is again $z = i$, and the residue there is

$$R(i) = \lim_{z \to i}(z-i)\frac{e^{iz}}{1+z^2}\bigg|_{z=i} = \frac{1}{2ei}$$

The value of the contour integral is then πe. As in before, we write the contour integral as a sum of two integrals

$$\oint_C \frac{e^{iz}}{1+z^2}dz = \int_{-\rho}^{\rho} \frac{e^{ix}}{1+x^2}dx + \int_0^{\pi} \frac{e^{iz}}{1+z^2}dz = \frac{\pi}{e}$$

The integral along the semicircle tends to zero as the radius $\rho \to \infty$. We have then

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx = \frac{\pi}{e}$$

Taking the real part of both sides of this equation

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$$

Since the integrand is an even function, we have

$$\int_0^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$$

Ex. 4 Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}dx$$

Here we consider

$$I = \oint \frac{e^{iz}}{z}dz$$

To avoid the singular point at $z = 0$, we integrate around the contour where it is on the straight line boundary. We then let the radius r shrink to zero so that in effect we are integrating straight through the simple pole at the origin. Since the pole is on the straight line boundary, its contribution is just halfway between zero and $2\pi i\cdot$ residue. Observing that, the integral along the large semicircle tends to zero as R tends to infinity,

$$I = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx = 2\pi i \cdot \frac{1}{2}R(0) = i\pi$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi$$

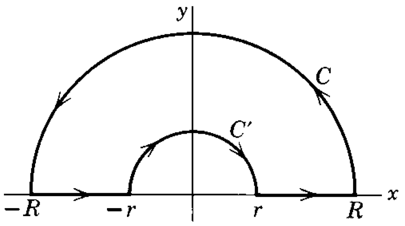


Figure: contour of example 4

Linear Algebra

Introduction

Elementary row operation:

1. Interchange two rows
2. Multiply (or divide) a row by a (nonzero) constant
3. Add a multiple of one row to another

Rank definition:

Definition 0.1 (Rank)**.** The number of nonzero rows remaining when a matrix has been row reduced is called the rank of the matrix

Definition 0.2 (Rank)**.** The order of the largest nonzero determinant is the rank of the matrix.

In a M matrix with m equation (rows) and n unknown (collumn), with and A has one more column (the constants):

1. If (rank M) $<$ (rank A), the equations are inconsistent and there is no solution.
2. f (rank M) $=$ (rank A) $=$ n (number of unknowns), there is one solution.
3. If (rank M) $=$ (rank A) $= R < n$, then R unknowns can be found in terms of the remaining $n - R$ unknowns

Determinant of 2×2 matrix:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Some determinant rule:

$$\begin{aligned} \det(kA) &= k^2 \det A && (2 \times 2) \\ \det(kA) &= k^3 \det A && (3 \times 3) \\ \det(AB) &= \det(BA) = \det(A) \times \det(B) \end{aligned}$$

Cramer's Rule:
the equations

$$\begin{cases} a_1 x + b_1 y & = c_1 \\ a_2 x + b_2 y & = c_2 \end{cases}$$

has the solution:

$$x = \frac{1}{D} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \text{and} \qquad y = \frac{1}{D} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Scalar Product:

$$\begin{aligned} \vec{A} \cdot \vec{B} &= |\vec{A}||\vec{B}| \cos\theta \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned}$$

The following applies if vector perpendicular:

$$\vec{A} \cdot \vec{B} = 0$$

The following applies if vector parallel:

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

Vector Product:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= |\vec{A}||\vec{B}| \sin\theta \\ &= \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}$$

The following applies if vector parallel or antiparalel:

$$\vec{A} \cdot \vec{B} == 0$$

Lines and Plane

Suppose we have vector $\vec{A} = a\hat{i} + b\hat{j}$ and vector $\vec{r} - \vec{r_0} = (x - x_0)\hat{i} + (y - y_0)\hat{j}$, which parallel to $\vec{A}$. We can write:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which is the symmetric equations of a straight line. Note that $\vec{r}$ and $\vec{r_0}$ is not necessarily parallel with $\vec{A}$, but $\vec{r} - \vec{r_0}$ do. The parameter equation is:

$$\begin{aligned}\vec{r} - \vec{r_0} &= \vec{A}t \\ \vec{r} &= \vec{r_0} + \vec{A}t\end{aligned}$$

Now suppose that $N = a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular with $\vec{r} - \vec{r_0} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$. We then have equation of plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Matrix Operation

Matrix AB can be multiplied if thay are conformable, that is if row A = row B. Matrix multiplication in index notation is:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Matrix in general do not commute. We define the commutator of the matrices A and B by

$$[A, B] = AB - BA$$

If a matrix has an inverse we say that it is invertible; if it doesn't have an inverse, it is called singular.

$$M^{-1} = \frac{1}{\det M} C^T \qquad \text{where } C_{ij} \text{ is cofactor of } m_{ij}$$

Linear Algebra

Holy shit! He said it! He said the section title! Truly one of the paper ever. Anyways, a linear combination of $\vec{A}$ and $\vec{B}$ means $a\vec{A}+b\vec{B}$ where a and b are scalars. The vector $r = \hat{\imath}x + \hat{\jmath}y + \hat{k}$z with tail at the origin is a linear combination of the unit basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ A function of a vector, say $f(r)$, is called linear if

$$f(r_1 + r_2) = f(r_1) + f(r_2) \qquad \text{and} \qquad f(ar) = a\ f(r)$$

where a is a scalar. O is a linear operator if

$$O(r_1 + r_2) = O(r_1) + O(r_2) \qquad \text{and} \qquad O(ar) = a\ O(r)$$

Derivative, for example, is linear operator, while square root is not.

Matrix, in context of linear tarnsformation, that preserve the length of vector is said to be orthogonal. Matrix M is orthogonal if

$$M^{-1} = M^T$$

with determinant

$$\det M = \pm 1$$

det M = 1 corresponds geometrically to a rotation, and det M = -1 means that a reflection is involved.

Matrix that rotate vector $\vec{r} = (x, y)$ into $\vec{R} = (X, Y)$ (in 2D)is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the one that rotate its axis instead

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

in 3D

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $f_1(x), f_2(x), \cdots, f_n(x)$ have derivatives of order $n-1$, and if the Determinant

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix} \neq 0$$

then the functions are linearly independent.

Homogeneous Equations

Definition 0.3 (Homogeneous Equations)**.** Sets of linear equations when the constants on the right hand sides are all zero are called homogeneous equations.

> Homogeneous equations are never inconsistent; they always have the solution "all unknowns = 0" (often called the "trivial solution"). If the number of independent equations (that is, the rank of the matrix) is the same as the number of unknowns, this is the only solution. If the rank of the matrix is less than the number of unknowns, there are infinitely many solutions.

Consider set of n homogeneous equations in n unknowns. These equations have only the trivial solution unless the rank of the matrix is less than n. This means that at least one row of the row reduced n by n matrix of the coefficients is a zero row. Which mean that the determinant D of the coefficients is zero. This fact will be used in eigenvalue problem.

> A system of n homogeneous equations in n unknowns has solutions other than the trivial solution if and only if the determinant of the coefficients is zero.

Special Matrices

Table of special matrices:

Definition	Condition
Real	$A = \bar{A}$
Symmetric	$A = A^T$
Antisymmetric	$A = -A^T$
Orthogonal	$A^{-1} = A^T$
Pure Imaginary	$A = -\bar{A}$
Hermitian	$A = A^\dagger$
Antihermitian	$A = -A^\dagger$
Unitary	$A^{-1} = A^\dagger$
Normal	$AA^\dagger = A^\dagger A$

Few theorem:

1. $(ABC)^T = C^T B^T A^T$
2. $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$
3. $Tr(ABC) = Tr(BCA) = Tr(CAB)$. Trace is the sum of main diagonal.It is a theorem that the trace of a product of matrices is not changed by permuting them in cyclic order.
4. If H is a Hermitian matrix, then $U = e^{iH}$ is a unitary matrix.

Partial Differentitation

Identity Involving Partial Derivative

The Jacobian of $[u(x,y),v(x,y)]$ with respect to (x,y) is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Here are some identity relating the Jacobian with partial derivative.

Unity. Unity as in one

$$\frac{\partial(u,v)}{\partial(x,y)} = 1$$

Proof. Trivial

$$\frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1 \quad \blacksquare$$

Change of order. It can be proved that change of order cost the minus sign

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(v,u)}{\partial(x,y)} = -\frac{\partial(u,v)}{\partial(y,x)}$$

Proof. Those three terms literally have the same value when evaluated

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$-\frac{\partial(v,u)}{\partial(x,y)} = -\begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{vmatrix} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$\frac{\partial(u,v)}{\partial(y,x)} = -\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

See? $\blacksquare$

Jacobian. In terms of Jacobian, partial derivative of u with respect to x can be written as

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{\partial(u,y)}{\partial(x,y)}$$

Proof. Just evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} \quad \blacksquare$$

Chain rule for partial derivative. The expression is

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)}$$

Proof. The total differential of u and v as function w and z read

$$du = \frac{\partial u}{\partial w}\,dw + \frac{\partial u}{\partial v}\,dz \quad \wedge \quad dv = \frac{\partial v}{\partial w}\,dw + \frac{\partial v}{\partial z}\,dz$$

We can therefore evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \\ \frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix}$$

$$= \left| \begin{pmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \right| = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix} \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)} \quad \blacksquare$$

The real chain rule. We have

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y = 1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(x,y)} = \left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y \quad \blacksquare$$

Yet another chain rule... Even more chain rule...

$$\left.\frac{\partial x}{\partial y}\right|_w = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Proof. Trivial

$$\left.\frac{\partial x}{\partial y}\right|_w = \frac{\partial(x,w)}{\partial(y,w)} = \frac{\partial(x,w)}{\partial(z,w)}\frac{\partial(z,w)}{\partial(y,w)} = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Cyclic rule. This is chain rule all over again...

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_z = -1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(x,y)} = -\frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(y,z)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(y,x)}$$
$$= -\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial y}{\partial x}\right|_z \left.\frac{\partial z}{\partial y}\right|_x \quad \blacksquare$$

Application in Thermodynamics

Here we will derive some useful intensive parameter used in thermodynamics. We assumed entropy function S has the form of

$$S = S(U, V, N_{i|r})$$

where N is number of chemical potential and $N_{i|r} \equiv N_1, \ldots N_r$. Therefore, its total differential is

$$dS = \left.\frac{\partial S}{\partial U}\right|_{V,N_{i|r}} dU + \left.\frac{\partial S}{\partial V}\right|_{U,N_{i|r}} dV + \sum_{j=1}^{r} \left.\frac{\partial S}{\partial N_j}\right|_{U,V,N_{i\neq r}} dN_j$$

We also assume the following quantities

$$T = \left.\frac{\partial U}{\partial S}\right|_{V,N_i} \quad ; P = -\left.\frac{\partial U}{\partial V}\right|_{S,N_i} \quad ; \mu_j = \left.\frac{\partial U}{\partial N}\right|_{S,V,N_{i\neq j}}$$

First identity. As follows

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T}$$

Proof. We use chain rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = 1 \implies \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \left(\left.\frac{\partial U}{\partial S}\right|_{V,N_i}\right)^{-1}$$

Then, from the definition of temperature

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T} \quad \blacksquare$$

Second identity. The identity written as

$$\left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T}$$

Proof. We invoke cyclic rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial V}\right|_{U,N_i} \left.\frac{\partial V}{\partial U}\right|_{U,N_i}$$

Then, from the first identity and the definition of pressure

$$1 = T\left.\frac{\partial S}{\partial V}\right|_{U,N_i} \frac{1}{P} \implies \left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T} \quad \blacksquare$$

Third Identity. Expressed as

$$\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{P}{T}$$

Proof. We again invoke cyclic with $x \to U, y \to Nj, z \to S$; while keeping V and all N except N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \left.\frac{\partial N_j}{\partial U}\right|_{U,N_{i\neq j}}$$

Then, from the definition of temperature and chemical potential

$$1 = -T\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \frac{1}{\mu_j} \implies \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{\mu_j}{T} \quad \blacksquare$$

Vector Analysis

Vector Operation

There are four vector operation: Addition, Multiplication by a scalar, Dot product, and Cross Product. (i) Addition of two vectors. Addition is commutative and associative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

(ii) Multiplication by a scalar. Scalar multiplication is distributive.

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$$

(iii) Dot product of two vectors. The dot product of two vectors is defined

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos\theta$$

where θ is the angle they form. Note that dot product is commutative and distributive.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$
$$A \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv |A||B| \sin\theta \, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The cross product is distributive, but not commutative.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$
$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$$

Few rule for manipulating vector. (i): To add vectors, add like components

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

(ii): To multiply by a scalar, multiply each component.

$$a\mathbf{A} = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}.$$

Rule (iii): To calculate the dot product, multiply like components, and add.

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Rule (iv): To calculate the cross product, form the determinant whose first row is unit vector, whose second row is A (in component form), and whose third row is B.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \, A_x & A_y & A_z \\ B_x & B_y & B_z & & \end{vmatrix}$$

Triple Product

(i) Scalar triple product.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = B \cdot (\mathbf{C} \times A) = C \cdot (\mathbf{A} \times B)$$

They are cyclic and in component form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector triple product.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

The product is linear combination of vector in parentheses.

Separation Vector

Separation vector defined as vector from the source point $\bar{r'}$ to the field point $\bar{r}$

$$\mathscr{r} \equiv \bar{\mathbf{r}} - \bar{\mathbf{r}'}.$$

Del Operator

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Operation Involving Del Operator

There are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the divergence);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the curl).

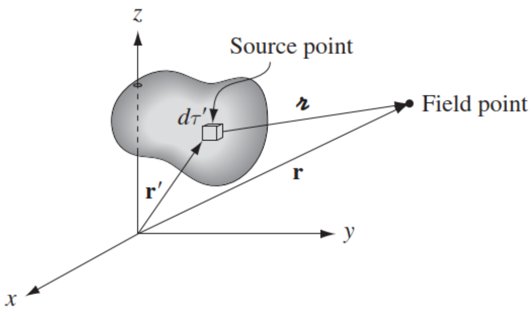


Figure: Separation vector

Gradient of scalar function $T(x, y, z)$

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

can be used to define partial derivative of T

$$\begin{aligned} dT &= \left(\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= \nabla T \cdot \hat{\mathbf{u}} \end{aligned}$$

Note that ∇T is a vector quantity, with three components. The gradient ∇T points in the direction of maximum increase of the function T. Moreover, The magnitude ∇T gives the slope (rate of increase) along this maximal direction.

Divergence of vector function $\mathbf{V}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is a scalar. Divergence is a measure of how much the vector V spreads out (diverges) from the point in question.

Curl of vector function $\mathbf{V}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial zx} \\ V_x & V_y & V_z \end{vmatrix}$$

The name curl is also well-chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.

Product Rule

There are two ways to construct a scalar as the product of two functions

$$\begin{aligned} fg & \quad \text{(product of two scalar functions)} \\ \mathbf{A} \cdot \mathbf{B} & \quad \text{(dot product of two vector functions)} \end{aligned}$$

and two ways to make a vector

$$\begin{aligned} f\mathbf{A} & \quad \text{(scalar times vector)} \\ \mathbf{A} \times \mathbf{B} & \quad \text{(cross product of two vectors)} \end{aligned}$$

Accordingly, there are six product rule, two for gradients

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

two for divergences

$$\nabla(f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

and two for curls

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivative

(1) Divergence of gradient: $\nabla\cdot(\nabla T)$. Called Laplacian of T. Notice that the Laplacian of a scalar T is a scalar.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y} + \frac{\partial^2 T}{\partial z}$$

Occasionally, we shall speak of the Laplacian of a vector, $\nabla^2\mathbf{v}$. By this we mean a vector quantity whose x-component is the Laplacian of V_x, and so on:

$$\nabla^2\mathbf{v} \equiv (\nabla^2 V_x)\hat{\mathbf{x}} + (\nabla^2 V_y)\hat{y} + (\nabla^2 V_z)\hat{\mathbf{z}}$$

(2) The curl of a gradient: $\nabla\times(\nabla T)$. Always zero.

$$\nabla\cdot(\nabla T) = 0$$

(3) Gradient of divergence: $\nabla(\nabla\cdot\mathbf{v})$. $\nabla(\nabla\cdot\mathbf{v})$ is not the same as the Laplacian of a vector.

$$\nabla(\nabla\cdot\mathbf{v}) \neq \nabla^2\mathbf{v} = (\nabla\cdot\nabla)\mathbf{v}$$

(4) The divergence of a curl: $\nabla\cdot(\nabla\times\mathbf{v})$. Always zero.

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$

(5) Curl of curl:$\nabla\times(\nabla\times\mathbf{v})$. From the definition of ∇,

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$$

Fundamental Theorem of Calculus

The fundamental theorem of calculus says the integral of a derivative over some region is given by the value of the function at the end points (boundaries).

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

Gradient. The fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (line integral) of a derivative (gradient) is given by the value of the function at the boundaries (a and b).

$$\int_a^b (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Corollary 1: $\int_a^b(\nabla T)\cdot d\mathbf{l}$ is independent of the path.

Corollary 2: $\oint(\nabla T)\cdot d\mathbf{l} = 0$ since the beginning and end points are identical.

Divergences . Like the other "fundamental theorems," it says that the integral of a derivative (divergence) over a region (volume V) is equal to thevalue of the function at the boundary (surface S).

$$\int_V (\nabla\cdot\mathbf{v})\,d\tau = \oint_S \mathbf{v}\cdot d\mathbf{a}$$

If v represents the flow of an incompressible fluid, then the flux of $\mathbf{v}$ is the total amount of fluid passing out through the surface, per unit time. There are two ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. Alternatively,

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Curl. As always, the integral of a derivative (curl) over a region (patch of surface, S) is equal to the value of the function at the boundary (perimeter of the patch, P). Now, the integral of the curl over some surface (flux of the curl) represents the "total amount of swirl," and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Corollary 1. $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line. It doesn't matter which way you go as long as you are consistent. For a closed surface (divergence theorem), $d\mathbf{a}$ points in the direction of the outward normal; but for an open surface is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$.

Corollary 2. $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point.

Integration by Parts

It applies to the situation in which you are called upon to integrate the product of one function (f) and the derivative of another (g); it says you can transfer the derivative from g to f, at the cost of a minus sign and a boundary term.

$$\int_a^b f\left(\frac{dg}{dx}\right) dx = -\int_a^b g\left(\frac{df}{dx}\right) dx + fg\Big|_a^b$$

Curvilinear Coordinates

I shall use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. Infinitesimal displacement vector can be written

$$d\mathbf{l} = f\ du\ \hat{\mathbf{u}} + g\ dv\ \hat{\mathbf{v}} + h\ dw\ \hat{\mathbf{w}}$$

where f, g, and h are functions of position characteristic of the particular coordinate system. While infinitesimal volume is

$$d\tau = fgh\ du\ dv\ dw$$

Use table 1 for references.

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	r $\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Table 1

Gradient. The gradient of t is

$$\nabla \mathbf{t} \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$

Divergence. The divergence of **A** in curvilinear coordinates:

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$

Curl.

$$\begin{aligned}\nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} \\ + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}\end{aligned}$$

Laplacian.

$$\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]$$

Spherical.

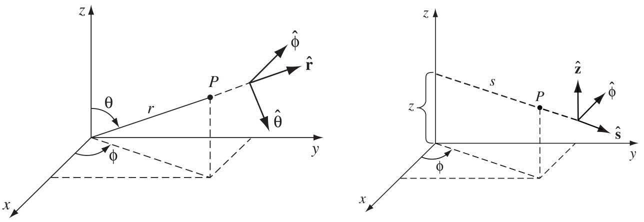
$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\sqrt{x^2+y^2}/z \\ \phi &= \arctan y/z \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\boldsymbol{y}} \end{cases}$$

Cylindrical.

$$\begin{cases} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan y/z \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$



Spherical Coordinates and Cylindrical Coordinates

Dirac Delta

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1. That is to say

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

It follows that

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Since the product is zero anyway except at $x = 0$, we may as well replace $f(x)$ by the value it assumes at the origin. In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$$

It's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\, dx = \int_{-\infty}^{\infty} f(x)D_2(x)\, dx$$

It is easy to generalize the delta function to three dimensions

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

with $\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, and it's integral

$$\int_{\text{all space}} \delta^3(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\, dxdydz = 1$$

Generalizing Delta function, we get

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r\text{-}a})\, d\tau = f(\mathbf{a})$$

Few Dirac delta function

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Fourier Transform of a δ function. Using the definition of a Fourier transform, we write

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\alpha x}\, dx = \frac{1}{2\pi}e^{-i\alpha x}$$

and its inverse transform

$$\delta(x-a) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-a)}\, d\alpha$$

The integral however does not converge. If we replace the limits by $-n, n$, we obtain a set of functions which are increasingly peaked around $x = a$ as n increases, but all have area 1.

Derivative of a δ function. Using repeated integrations by parts gives

$$\int_{-\infty}^{\infty} \phi(x)\delta^{(n)}(x-a)\, dx = (-1)^n\phi^{(n)}(a)$$

Few formulas involving δ function. For step function

$$u(x-a) = \begin{cases} 1, x > a \\ 0, x < a \end{cases}$$
$$u'(x-a) = \delta(x-a)$$

It is easy to see how the derivative of step function is equal to delta function.

Helmholtz Theorem

Suppose we are told that the divergence of a vector function F(r) is a specified scalar function D(r):

$$\nabla \cdot \mathbf{F} = D$$

and the curl of F(r) is a specified vector function C(r):

$$\nabla \times \mathbf{F} = \mathbf{C}$$

For consistency, C must be divergenceless $\nabla \cdot \mathbf{C} = 0$. Helmholtz theorem state if the divergence D(r) and the curl C(r) of a vector function F(r) are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$ and if F(r) goes to zero as $r \to \infty$, then F is given uniquely by

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$$

Potential Theorem

Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, F satisfies one if and only if it satisfies all the others):

- $\nabla \times \mathbf{F} = 0$ everywhere.
- $\int_a^b \mathbf{F} \cdot\, d\mathbf{l}$ is independent of path, for any given end points.
- $\oint \mathbf{F} \cdot\, d\mathbf{l} = 0$ for any closed loop.
- $\mathbf{F}$ is the gradient of some scalar function: $F = -\nabla V$.

Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

- $\nabla \cdot \mathbf{F} = 0$ everywhere.
- $\int \mathbf{F} \cdot \, d\mathbf{a}$ is independent of surface, for any given boundary line.
- $\oint \mathbf{F} \cdot \, d\mathbf{a} = 0$ for any closed surface.
- $\mathbf{F}$ is the curl of some scalar function: $\mathbf{F} = -\nabla \mathbf{A}$.

Fourier Series and Transform

Introduction

Sinusoidsal wave Equation.

$$y = A \sin \frac{2\pi}{\lambda}(x - vt)$$

where λ represent wavelength, but mathematically it is the same as the period of this function of x. Wave equation in single variable.

$$\begin{aligned} y(x) &= A \sin kx & &= A \sin 2\pi f x = A \sin \frac{2\pi}{\lambda} x \\ y(t) &= A \sin \omega t & &= A \sin 2\pi v t = A \sin \frac{2\pi}{T} t \end{aligned}$$

Average Value. Average of $f(x)$ on (a, b) is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Here are some usefull integrals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \; dx = 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \sin nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos mx \cos nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 1, & m = n = 0 \end{cases}$$

Fourier Series

2π period. Fourier Series for function of period 2π:

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with coefficients:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx \\ c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \; dx \end{aligned}$$

Proof. Multiply both sides of Fourier series by $\cos nx$ and find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos mx\;dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum \left(a_n \cos nx + b_n \sin nx\right)\right]\cos mx\;dx$$

All terms on the right are zero except the a_n term then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{a_n}{2} \qquad \blacksquare$$

Notice that $\cos mx$ now turns into $\cos nx$–this is because the integral picks the value of n such that $m = n$. For b_n, we multiply we multiply both sides of by $\sin nx$ and take average values just as we did before

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin mx\;dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum \left(a_n \cos nx + b_n \sin nx\right)\right]\sin mx\;dx$$

and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{b_n}{2} \qquad \blacksquare$$

To find c_n, we multiply Fourier series by $\exp(-imx)$ and again find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(imx) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} c_n \exp inx\right]\exp(-imx)\;dx$$

All these terms are zero except the one where $n = m$. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n \exp ix(n-m)\;dx$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) \qquad \blacksquare$$

Other period. Fourier Series for function of period $2l$:

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(a_n \cos\frac{2\pi nx}{l} + b_n \sin\frac{2\pi nx}{l}\right)$$
$$= \sum_{n=-\infty}^{\infty} c_n \exp\frac{in\pi x}{l}$$

with coefficients:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{2\pi nx}{l}\;dx$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{2\pi nx}{l}\;dx$$
$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\exp\frac{-in\pi x}{l}\;dx$$

Fourier Transform

The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set. Fourier transforms are defined as follows

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\, d\alpha$$
$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\, dx$$

$g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int$ corresponds to $\sum$. This agrees with our discussion of the physical meaning and use of Fourier integrals.

Fourier Sine Transforms. We define $f_s(x)$ and $g_s(\alpha)$ as pair of Fourier sine transforms representing odd functions.

$$f_s(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} g_s(\alpha)\sin\alpha x\, d\alpha$$
$$g_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(x)\sin\alpha x\, dx$$

Fourier Cosine Transforms. We define $f_c(x)$ and $g_c(\alpha)$ as pair of Fourier cosine transforms representing even functions.

$$f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} g_c(\alpha)\cos\alpha x\, d\alpha$$
$$g_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(x)\cos\alpha x\, dx$$

Proof (?). We rewrite Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp i\alpha_n x$$
$$c_n = \frac{1}{2l}\int_{-l}^{l} f(u)\exp(-i\alpha_n u)\, du$$

where

$$\frac{n\pi}{l} = \alpha_n$$
$$\alpha_{n+1} - \alpha_n = \Delta\alpha = \frac{\pi}{l}$$

Then

$$c_n = \frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)\exp(-\alpha_n u)\, du$$

Subsituting c_n into $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty}\left[\frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(x)\exp(-\alpha_n x)\, du\right]\exp\alpha_n x$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp i\alpha_n(x-u)\, du$$

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u) \exp i\alpha_n(x-u)\, du$$

If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity],

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) \exp i\alpha(x-u)\, du$$

then $\Delta\alpha \to 0$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \exp i\alpha(x-u)\, du\, d\alpha$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-i\alpha x)\, dx$$

then

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) \exp i\alpha x\, d\alpha$$

Now we expand $\exp(-i\alpha x)$ inside $g(\alpha)$ expression

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha x)\, dx$$

If we assume that $f(x)$ is odd, we get

$$g(x) = -\frac{i}{\pi} \int_{0}^{\infty} f(x) i \sin\alpha x\, dx$$

since the product of odd function $f(x)$ and even function $cos\alpha x$ is odd, thus the integral is zero. Then expanding the exponential in $f(x)$

$$f(x) = 2i \int_{0}^{\infty} g(\alpha) \sin\alpha x\, d\alpha$$

If we substitute $g(\alpha)$ into $f(x)$, we obtain

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(x) \sin^2\alpha x\, dx\, d\alpha$$

and we see that the numerical factor is $2/\pi$, thus the imaginary factors are not needed. We may as well write $\sqrt{2/\pi}$ instead. Now suppose that $g(\alpha)$ is even. As before, we have

$$g(x) = \frac{1}{\pi} \int_{0}^{\infty} f(x) i \cos\alpha x\, dx$$

and

$$f(x) = 2 \int_0^\infty g(\alpha) \cos \alpha x \, d\alpha$$

Subsituting $g(x)$ into $f(x)$

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x) \cos^2 \alpha x \, dx \, d\alpha$$

We also see that it has the same numerical factor and all.

Even and Odd Function

Definition.

$$f(x) = \begin{cases} f(x) = f(-x) & \text{even} \\ f(-x) = -f(x) & \text{odd} \end{cases}$$

Integral of Even and Odd Function.

$$\int_{-l}^{l} f(x) \, dx \begin{cases} 0 & \text{odd} \\ 2 \displaystyle\int_0^l f(x) \, dx & \text{even} \end{cases}$$

Fourier expansion for odd function.

$$\text{odd } f(x), \begin{cases} a_n & = 0 \\ b_n & = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \frac{2\pi n x}{l} \, dx \end{cases}$$

Fourier expansion for even function.

$$\text{odd } f(x), \begin{cases} a_n & = \dfrac{2}{l} \displaystyle\int_0^l f(x) \cos \frac{2\pi n x}{l} \, dx \\ b_n & = 0 \end{cases}$$

Theorem

Dirichlet Condition. If f(x):

1. periodic,
2. x single valued,
3. finite number of discontinuities,
4. finite min max, and
5. $\int_{-\pi}^{\pi} |f(x)| \, dx = \text{finite}$

then the Fourier series converges to the midpoint of the jump.

Parseval's theorem. For Fourier expansions

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos nx + b_n \sin nx \right)$$

we have

$$\text{The average of } [f(x)]^2 \text{ is } \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx$$

with the value of each coefficients

$$\begin{array}{rcc}
\text{The average of } (\frac{1}{2}a_0)^2 & \text{is} & (\frac{1}{2}a_0)^2 \\
\text{The average of } (a_n \cos nx)^2 & \text{is} & \frac{1}{2}a_n^2 \\
\text{The average of } (b_n \sin nx)^2 & \text{is} & \frac{1}{2}b_n^2
\end{array}$$

then we have

$$\text{The average of } [f(x)]^2 \; = \left(\frac{1}{2}1_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$$

or in complex expansion

$$\text{The average of } [f(x)]^2 \; = \sum_{-\infty}^{\infty} |c_n|^2$$

Ordinary Diferential Equation

First Order

First Order ODE

$$y' + py = Qy \tag{1}$$

where P and Q are functions of x has the solution

$$ye^I = \int Qe^I \, dx + c$$

$$y = e^{-I} \int Qe^I \, dx + ce^{-I}$$

where

$$I = \int P \, dx$$

Bernoulli Equation. The differential equation

$$y' + Py = Qy^n$$

where P and Q are functions of x can be written as

$$z' + (1-n)Pz = (1-n)Q \tag{2}$$

where

$$z = y^{1-n}$$

This is now a first-order linear equation which we can solve as we did the linear equations above.

Exact Equations. $P(x,y)dx + Q(x,y)dy$ is an exact differential [the differential of $F(x,y)$, or $Pdx + Qdy = dF$] if

$$\frac{\partial}{\partial x}P = \frac{\partial}{\partial x}Q$$

and the solution is

$$F(x,y) = \text{constant}$$

An equation which is not exact may often be made exact by multiplying it by an appropriate factor.

Homogeneous Equations. A homogeneous function of x and y of degree n means a function which can be written as $x^n f(y/x)$. An equation of the from

$$P(x,y)dx + Q(x,y)dy = 0$$

where P and Q are homogeneous functions of the same degree is called homogeneous. Thus

$$y' = \frac{d}{dx}y = -\frac{P(x,y)}{Q(x,y)} - f(\frac{y}{x})$$

This suggests that we solve homogeneous equations by making the change of variables

$$y = xv \quad \text{with} \quad v = \frac{y}{x}$$

Second Order

Second Order with Zero Right Hand Side. Equation of the from

$$(D-a)(D-b)y=0, \quad a\neq b$$

has the Solution

$$y=c_1e^{ax}+c_2e^{bx}$$

Equation of the from

$$(D-a)(D-a)y=0, \quad a\neq b$$

has the Solution

$$y=(Ax+B)e^{ax}$$

Now suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. The solution is now

$$\begin{aligned}
y &= Ae^{(\alpha+i\beta)}x+Be^{(\alpha-i\beta)}x\\
&= e^{\alpha x}(Ae^{i\beta x}+Be^{-i\beta x})\\
&= e^{\alpha x}(c_1\sin\beta x+c_2\cos\beta x)\\
&= ce^{\alpha x}\sin(\beta x+\gamma)
\end{aligned}$$

where $\alpha,\ \beta,\ \gamma,\ c,\ c_1,\ c_2$ are different constant.

Second Order with Nonzero Right Hand Side. The equation

$$\begin{aligned}
a_2\frac{d^2}{dx^2}y+a_1\frac{d}{dx}y+a_0y &= f(x)\\
\frac{d^2}{dx^2}y+a_1\frac{d}{dx}y+a_0y &= F(x)
\end{aligned}$$

has the solution of the from

$$y=y_c+y_p$$

where the complementary function y_c is the general solution of the homogeneous equation (when right hand side is equal to zero) and y_p is a particular solution (when the right hand side is equal to f(x) or F(x)). The simplest method solving them is by Inspection and Successive Integration of Two First-Order Equations.

Exponential Right-Hand Side. Suppose we have $F(x)=ke^{cx}$, or

$$(D-a)(D-b)y=ke^{cx}$$

then, we find a particular solution by assuming a solution of the form:

$$y_p=\begin{cases} Ce^{cx} & \text{if c is not equal to either a or b;}\\ Cxe^{cx} & \text{if c equals a or b, a} \neq \text{b;}\\ Cx^2e^{cx} & \text{if c = a = b.}\end{cases}$$

Complex Exponentials. To find a particular solution of

$$(D-a)(D-b)y=\begin{cases} k\sin\alpha x,\\ k\cos\alpha x,\end{cases}$$

first solve

$$(D-a)(D-b)y = ke^{i\alpha x}$$

then then take the real or imaginary part.

Method of Undetermined Coefficients. To find a particular solution of

$$(D-a)(D-b)y = e^{cx}P_n(x)$$

where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if c is not equal to either a or b;} \\ xe^{cx}Q_n(x) & \text{if c equals a or b, a} \neq \text{b;} \\ x^2e^{cx}Q_n(x) & \text{if c = a = b.} \end{cases}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the given differential equation.

Several Terms on the Right-Hand Side: Principle of Superposition. The easiest way of handling a complicated right-hand side: Solve a sep- arate equation for each different exponential and add the solutions. The fact that this is correct for a linear equation is often called the principle of superposition. Note that the principle holds only for linear equations.

Use of Fourier Series in Finding Particular Solutions. Suppose that the driving force f(x) is periodic, we then can expand the function using Fourier Series. The equation

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = f(x) = \sum_{-\infty}^{\infty} c_n e_{inx}$$

can be solved by solving

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = c_n e_{inx}$$

then add the solutions for all n (applying principle of superposition), and we have the solution of first the equation.

Laplace Transform

We define $\mathcal{L}(f)$, the Laplace transform of $f(t)$ [also written $F(p)$ since it is a function of p], by the equation

$$\mathcal{L}(f) = F(P) = \int_0^{\infty} f(t)e^{-pt}; dt$$

Convolution

Definition. The integral

$$g * h = \int_0^t g(t-\tau)h(\tau)d(\tau) = \int_0^t g(\tau)h(t-\tau)d(\tau)$$

is called the convolution of g and h (or the resultant or the Faltung). Now suppose that we have

$$Ay' + By' + Cy = f(t), \quad y0 = y'0 = 0$$

take the Laplace transform of each term, substitute the initial conditions, and solve for Y

$$Y = \frac{F(p)}{A(p+a)(p+b)} = T(p)F(p)$$

Then y the inverse transform of Y in is the inverse transform of a product of two functions whose inverse transforms we know. Let $G(p)$ and $H(p)$ be the transforms of $g(t)$ and $h(t)$

$$G(p)H(p) = \mathcal{L}(g(t) \cdot h(p)) = \mathcal{L}(g * h)$$

Thus

$$y = \int_0^t g(t-\tau)h(\tau)d(\tau)$$

Observe from $\mathcal{L}34$ that we may use either $g(t-\tau)h(\tau)$ or $g(\tau)h(t-\tau)$ in the integral. It is well to choose whichever form is easier to integrate; it is best to put$(t-\tau)$ in the simpler function.

Fourier Transform of a Convolution. Let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(v)e^{-i\alpha v}dv \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(u)e^{-i\alpha u}du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(v)f_2(u)e^{-i\alpha(v+u)}dvdu \end{aligned}$$

Next we make the change of variables $x = v + u$, $dx = dv$, in the v integral

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x-u)f_2(u)e^{-i\alpha x}dvdu \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{-i\alpha x}\left[\int_{-\infty}^{\infty} f_1(x-u)f_2(u)\, du\right]dx \end{aligned}$$

if we define the term in the square parenthesis as convolution, we get

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} f_1 * f_2 e^{-i\alpha x}dx\right) \\ &= \frac{1}{2\pi} \cdot \text{Fourier transform of } f_1 * f_2 \end{aligned}$$

In other words

$$g_1 \cdot g_2 \text{ and } f_1 * f_2 \text{ are a pair of Fourier transforms}$$

and by symmetry

$$g_1 * g_2 \text{ and } f_1 \cdot f_2 \text{ are a pair of Fourier transforms}$$

Appendix: Differential Equation Study Guide

First Order Equations. General Form of ODE

$$\frac{dy}{dx} = f(x, y)$$

Initial Value Problem

$$y' = f(x, y),\ y(x_0) = y_0$$

Linear Equations. General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\mu(x) = e^{\int p(x)dx}$$
$$\implies \frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$$

General Solution

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

Homogeneous Equations. General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in z:

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations. General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in z:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations. General Form

$$M(x, y)dx + N(x, y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations.

1. Let $\phi = \int M(x, y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x, y)$
3. Simplify and solve for $h(y)$
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x, y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x, y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors. Case 1. If $P(x, y)$ depends only on x, where

$$P(x, y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0$$

is exact.

Case 2. If $Q(x, y)$ depends only on y, where

$$Q(x, y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$$

is exact.

Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{3}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{4}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{5}$$

General Solution. The general solution of (3) or (5) is

$$y = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \tag{6}$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (4).

Linear Independence and The Wronskian. Two functions $f(x)$ and $g(x)$ are linearly dependent if there exist numbers a and b, not both zero, such that $af(x) + bg(x) = 0$ for all x. If y_1 and y_2 are two solutions of (4), then Wronskian

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.
2. $\{y_1, y_2\}$ are a fundamental set of solutions.
3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem. The initial value problem includes two initial conditions at the same point in time, one condition on $y(t)$ and one condition on $y'(t)$.

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

The initial conditions are applied to the entire solution $y = y_h + y_p$.

Linear Equation With Constant Coefficients. The general form of the homogeneous equation is

$$ay'' + by' + cy = 0 \tag{7}$$

Non-homogeneous

$$ay'' + by' + cy = g(t) \tag{8}$$

Characteristic Equation

$$ar^2 + br + c = 0$$

Quadratic Roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9}$$

The solution of (7) of Real Roots ($r_1 \neq r_2$)

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{10}$$

Repeated ($r_1 = r_2$)

$$y_h = (C_1 + C_2 t)e^{r_1 t} \tag{11}$$

Complex ($r = \alpha \pm i\beta$)

$$y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) \tag{12}$$

The solution of (8) is $y = y_p + y_h$ where y_h is given by (10) through (12) and y_p is found by undetermined coefficients or reduction of order.

Heuristics for Undetermined Coefficients. Also called Trial and Error

If $f(t) =$	then guess that a particular solution $y_p =$.
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at} \sin bt$ or $P_n(t)e^{at} \cos bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n) \cos bt$ $+(A_0 + A_1 t + \cdots + A_n t^n) \sin bt]$

Method of Reduction of Order. When solving (4), given y_1, then y_2 can be found by solving

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}dx}{y_1(x)^2} \tag{13}$$

Method of Variation of Parameters. If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (4) then a particular solution to (5) is

$$y_P(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t) \int \frac{y_1(t)f(t)}{W(t)}dt \tag{14}$$

Cauchy-Euler Equation. For ODE

$$ax^2y'' + bxy' + cy = 0 \tag{15}$$

with Auxilliary Equation

$$ar(r-1) + br + c = 0 \tag{16}$$

The solutions of (15) depend on the roots $r_{1,2}$ of (16). For Real Roots

$$y = C_1x^{r_1} + C_2x^{r_2}$$

Repeated Root

$$y = C_1x^r + C_2x^r \ln x$$

Complex

$$y = x^\alpha[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \tag{17}$$

In (17) $r_{1,2} = \alpha \pm i\beta$, where $\alpha,\beta \in \mathbb{R}$

Series Solutions.

$$(x-x_0)^2y'' + (x-x_0)p(x)y' + q(x)y = 0 \tag{18}$$

If x_0 is a regular point of (18) then

$$y_1(t) = (x-x_0)^n \sum_{k=0}^{\infty} a_k(x-x_k)^k$$

At a Regular Singular Point x_0, the Indicial Equation

$$r^2 + (p(0)-1)r + q(0) = 0 \tag{19}$$

First Solution

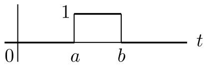
$$y_1 = (x-x_0)^{r_1} \sum_{k=0}^{\infty} a_k(x-x_k)^k$$

Where r_1 is the larger real root if both roots of (19) are real or either root if the solutions are complex.

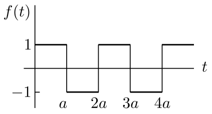
Appendix: Laplace Table

	$f(t) = \mathcal{L}^{-1}F(s) = F(s)$	$\mathcal{L}f(t) = F(s)$	
$\mathcal{L}1$	1	$\dfrac{1}{p+a}$	Re $p > 0$
$\mathcal{L}2$	e^{-at}	$\dfrac{1}{p}$	Re $p > 0$
$\mathcal{L}3$	$\sin at$	$\dfrac{a}{p^2+a^2}$	Re $p > \lvert \text{Im } a \rvert$
$\mathcal{L}4$	$\cos at$	$\dfrac{p}{p^2+a^2}$	Re $p > \lvert \text{Im } a \rvert$
$\mathcal{L}5$	$t^k, k > -1$	$\dfrac{k!}{p^{k+1}}$ or $\dfrac{\Gamma(k+1)}{p^{k+1}}$	Re $p > 0$
$\mathcal{L}6$	$t^k e^{-at}, k > -1$	$\dfrac{k!}{(p+a)^{k+1}}$ or $\dfrac{\Gamma(k+1)}{(p+a)^{k+1}}$	Re $(p+a) > 0$
$\mathcal{L}7$	$\dfrac{e^{-at} - e^{-bt}}{b-a}$	$\dfrac{1}{(p+a)(p+b)}$	Re $(p+a) > 0$ Re $(p+b) > 0$

$\mathcal{L}8$	$\dfrac{ae^{-at} - be^{-bt}}{b-a}$	$\dfrac{p}{(p+a)(p+b)}$	$\text{Re }(p+a) > 0$ $\text{Re }(p+b) > 0$
$\mathcal{L}9$	$\sinh at$	$\dfrac{a}{p^2 - a^2}$	$\text{Re } p > \lvert\text{Re } a\rvert$
$\mathcal{L}10$	$\cosh at$	$\dfrac{p}{p^2 - a^2}$	$\text{Re } p > \lvert\text{Re } a\rvert$
$\mathcal{L}11$	$t \sin at$	$\dfrac{2ap}{(p^2 + a^2)^2}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}12$	$t \cos at$	$\dfrac{p^2 - a^2}{(p^2 + a^2)^2}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}13$	$e^{-at} \sin bt$	$\dfrac{b}{(p+a)^2 + b^2}$	$\text{Re }(p+a)$ $> \lvert\text{Im } b\rvert$
$\mathcal{L}14$	$e^{-at} \cos bt$	$\dfrac{p+a}{(p+a)^2 + b^2}$	$\text{Re }(p+a)$ $> \lvert\text{Im } b\rvert$
$\mathcal{L}15$	$1 - \cos at$	$\dfrac{a^2}{p(p^2 + a^2)}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}16$	$at - \sin at$	$\dfrac{a^3}{p^2(p^2 + a^2)}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}17$	$\sin at - at \cos at$	$\dfrac{2a^3}{(p^2 + a^2)^2}$	$\text{Re } p > \lvert\text{Im } a\rvert$

$\mathcal{L}18$	$e^{-at}(1-at)$	$\dfrac{p}{(p+a)^2}$	Re $(p+a) > 0$
$\mathcal{L}19$	$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{p}$	Re $p > \lvert \mathrm{Im}\ a \rvert$
$\mathcal{L}20$	$\dfrac{1}{t} \sin at \cos bt$	$\dfrac{1}{2}\left(\arctan \dfrac{a+b}{p} + \arctan \dfrac{a-b}{p}\right)$	Re $p > 0$
$\mathcal{L}21$	$\dfrac{e^{at} - e^{-bt}}{t}$	$\ln \dfrac{p+b}{p+a}$	Re $(p+a) > 0$ Re $(p+b) > 0$
$\mathcal{L}22$	$1 - \mathrm{erf}\left(\dfrac{a}{2\sqrt{t}}\right), a > 0$	$\dfrac{1}{p} e^{-a\sqrt{p}}$	Re $p > 0$
$\mathcal{L}23$	$J_0(at)$	$(p^2 + a^2)^{-1/2}$	Re $p > \lvert \mathrm{Im}\ a \rvert$ or Re $p \geq 0$ for real $a \neq 0$
$\mathcal{L}24$	unit step, or Heaviside function $u(t-a) = \begin{cases} 1, \\ t > a > 0 \\ 0, t < a \end{cases}$	$\dfrac{1}{p} e^{-pa}$	Re $p > 0$
$\mathcal{L}25$	$f(t) = u(t-a) - u(t-b)$ 	$\dfrac{e^{-ap} - e^{-}bp}{p}$	All p

$\mathcal{L}26$



$$\frac{1}{p}\tanh\frac{ap}{2}$$

All p

$\mathcal{L}27$

$\delta(t-a), a \geq 0$

$$e^{-pa}$$

$\mathcal{L}28$

$$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$$
$$= g(t-a)u(t-a)$$

$$e^{-pa}G(p)$$

$G(p)$ means $\mathcal{L}(g)$.
Therefore $e^{-pa}\mathcal{L}[g(t-a)]$

$\mathcal{L}29$

$$e^{-at}g(t)$$

$$G(p+a)$$

$\mathcal{L}30$

$g(at), a > 0$

$$\frac{1}{a}G\left(\frac{p}{a}\right)$$

$\mathcal{L}31$

$$\frac{g(t)}{t}$$

if integrable

$$\int_p^\infty G(u)\,du$$

$\mathcal{L}32$

$$t^n g(t)$$

$$(-1)^n\left(\frac{d}{dp}\right)^n(G(p))$$

$\mathcal{L}33$

$$\int_0^t g(\tau)\,d\tau \qquad\qquad \frac{1}{p}G(p)$$

$\mathcal{L}34$ Convolution of g and h, often written as

$$g * h$$

$$\int_0^t g(t-\tau)h(\tau)\,d\tau = \int_0^t g(\tau)h(t-\tau)\,d\tau \qquad\qquad G(p)H(p)$$

$\mathcal{L}35$ Transforms of derivatives of y

$$\begin{aligned}
\mathcal{L}(y) &= Y \\
\mathcal{L}(y') &= pY - y \\
\mathcal{L}(y'') &= p^2Y - py_0 - y_0' \\
\mathcal{L}(y''') &= p^3Y - p^2y_0 - py_0' - y_0 \\
\mathcal{L}(y^n) &= p^nY - p^{n-1}y_0 - p^{n-2}y_0' - \cdots - y_0^{n-1}
\end{aligned}$$

Calculus of Variation

The Euler Equation

Any problem in the calculus of variations is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and solving the resulting differential equation. When the function $F = F(r, \theta, \theta')$, the Euler's equation read

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

If $F = F(t, x, \dot{x})$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

Notice that the first derivative in the Euler equation is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative.

Proof. We will try to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

where F is a given function. Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x)$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Differentiating with respect to x, we get

$$Y(x) = y(x)' + \epsilon\eta'(x)$$

Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\, dx$$

Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y} \eta(x) + \frac{\partial F}{\partial Y'} \eta'(x) \right] dx$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$$

Assuming that y'' is continuous, we can integrate the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = -\int_{x_1}^{x_2} \frac{d}{dx} \frac{\partial F}{\partial y'} \eta(x)\, dx + \left. \frac{\partial F}{\partial y'} \eta(x) \right|_{x_1}^{x_2}$$

The first term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \eta(x)\, dx$$

Since $\eta(x)$ is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \blacksquare$$

Notice carefully here that we are not saying that when an integral is zero, the integrand is also zero; this is not true. What we are saying is that the only way $\int f(x)\eta(x)\, dx$ can always be zero for every $\eta(x)$ is for $f(x)$ to be zero.

Several Variable

If there are n dependent variables in the original integral, there are n Euler-Lagrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du} \frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du} \frac{\partial f}{\partial y'}$$

Special Function

Gamma Function

Factorial. The factorial is defined by integral

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ we get

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can also give a meaning to $0!$; by putting $n = 0$, we get $0! = 1$. By the way, the integral can be evaluated using differentiation under integral sign.

Gamma function definition. Gamma function is used to define the factorial function for noninterger n. We define, for any $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$$

From this we have

$$\begin{aligned}\Gamma(p) &= \int_0^\infty x^{p-1} e^{-x}\, dx = (p-1)! \\ \Gamma(p+1) &= \int_0^\infty x^{p} e^{-x}\, dx = p!\end{aligned}$$

Recursion relation. The recursion for gamma function is

$$\Gamma(p+1) = p\Gamma(p)$$

Proof. Let us integrate $\Gamma(p+1)$ by parts. Calling $u = x^p$, and $dv = e^{-x}\, dx$; then we get $du = px^{p-1}$, and $v = -e^{-x}$. Thus

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} p x^{p-1}\, dx \\ &= p\int_0^\infty x^{p-1} e^{-x}\, dx \\ \Gamma(p-1) &= p\Gamma(p) \quad \blacksquare\end{aligned}$$

Negative numbers. We shall now define gamma function for $p \leq 0$ by the recursion relation

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

From this and the successive use of it, it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers.

Figure 1: Gaussian integral solved by polar method.

Important formula. We state here important formula

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

We can calculate the value of $\Gamma(1/2)$ using this equation, however we will instead try to derive it using another method. First we consider the definition

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt$$

then we substitute $t = x^2$ and $dt = 2x\, dx$

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\, dx = \int_{-\infty}^\infty e^{-x^2}\, dx$$

This is the famous Gaussian integral. Refer to figure 1 on how to solve it by polar coordinate.

Since everybody and their grandma already know how to solve Gaussian integral by polar coordinate, I will instead try to solve it by Feynman's trick. First consider the function

$$I(\alpha) = \left(\int_0^\alpha e^{-t^2}\, dt\right)^2$$

where I is a function of parameter fish α. Then, to evaluate the actual Gaussian integral

$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2\lim_{\alpha\to\infty} \sqrt{I(\alpha)}$$

Before that, I need to evaluate the function $I(\alpha)$ first. To do that, first I differentiate I with respect parameter fish α

$$
\begin{aligned}
\frac{dI}{d\alpha} &= 2\int_0^\alpha e^{-t^2}\,dt\left(\int_0^\alpha \frac{\partial e^{-t^2}}{d\alpha}\,dt + e^{-\alpha^2}\frac{d\alpha}{d\alpha} - e^{-0^2}\frac{d(0)}{d\alpha}\right)\\
\frac{dI}{d\alpha} &= \int_0^\alpha 2e^{-(t^2+\alpha^2)}\,dt
\end{aligned}
$$

where I have used Leibniz' rule for differentiating under integral sign. Then, I introduce the variable $u = t/\alpha$ and $du = dt/\alpha$

$$\frac{dI}{d\alpha} = \int_0^1 2e^{-(u^2\alpha^2+\alpha^2)}\alpha\,du = \int_0^1 2\alpha e^{-\alpha^2(u^2+1)}\,du$$

Using the fact that

$$\frac{\partial}{\partial\alpha}\frac{e^{-\alpha^2(u^2+1)}}{u^2+1} = -2\alpha e^{-\alpha^2(u^2+1)}$$

I can rewrite the integrand as

$$\frac{dI}{d\alpha} = -\int_0^1 \frac{\partial}{\partial\alpha}\frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\,du$$

Since the integrand is continous, I can move the partial differentiation outside the integral and turning it into total differentiation

$$\frac{dI}{d\alpha} = -\frac{d}{d\alpha}\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\,du$$

Hence

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\,du + C$$

All that remains is to find the value of C. Considering the initial definition of $I(\alpha)$ and evaluating at $\alpha = 0$, I get

$$I(0) = \left(\int_0^0 e^{-t^2}\,dt\right)^2 = 0$$

Therefore

$$
\begin{aligned}
I(0) &= -\int_0^1 \frac{1}{u^2+1}\,du + C = 0\\
C &= \arctan u\Big|_0^1 = \frac{\pi}{4}
\end{aligned}
$$

And I obtain the complete expression for the fish function

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\,du + \frac{\pi}{4}$$

Now I can evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\lim_{\alpha\to\infty}\left(-\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\,du + \frac{\pi}{4}\right)^{1/2} = 2\frac{\sqrt{\pi}}{2}$$

and I find

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Much to my chagrin, it is actually more trouble some than the polar method. Let's us try it for comparison

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dx\ dy\right)^{1/2}$$

Doing the change of coordinate thing

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\ dr\ d\theta\right)^{1/2}$$
$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(2\pi\int_{0}^{\infty} e^{-r^2} r\ dr\right)^{1/2}$$

That integral can by easily evaluated using u substitution; making the substitution $u = -r^2$

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(2\pi\int_{-\infty}^{0} \frac{e^u}{2}\ du\right)^{1/2} = \left(2\pi\frac{e^u}{2}\bigg|_{-\infty}^{0}\right)^{1/2}$$

And I get the same result

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Damn, it is really more shrimple.

Another form of Gaussian integral. Here we state without proof.

$$\int_{-\infty}^{\infty} \exp\left(-\alpha x^2 + \beta x\right)\ dx = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right)$$
$$\int_{0}^{\infty} x^m \exp\left(-\alpha x^2\right)\ dx = \frac{1}{2\alpha^{(m+1)/2}}\Gamma\left(\frac{m+1}{2}\right)$$

Here's another one, not really a Gaussian integral, but since it involves natural number it counts

$$\sum_{n=0}^{\infty} n^k e^{-nk} = (-1)^k \frac{d^k}{dk^k}\sum_{n=0}^{\infty} e^{-nx} = (-1)\frac{d^k}{dk^k}\frac{1}{1-e^{-x}}$$

Beta Function

Definition. The beta function is also defined by a definite integral

$$B(p,q)\int_0^1 x^{p-1}(1-x)^{q-1}\ dx$$

for $p > 0$,and $q > 0$.

Change of order. It is easy to show that

$$B(p,q) = B(q,p)$$

Proof. Putting $x = 1 - y$ and $dx = -dy$

$$B(p,q) = -\int_1^0 (1-y)^{p-1} y^{q-1}\, dy = \int_0^1 y^{q-1}(1-y)^{p-1}\, dy$$
$$B(p,q) = B(q,p) \quad \blacksquare$$

Integration Range. The range of integration can be changed with

$$B(p,q) = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1}(a-y)^{q-1}\, dy$$

Another form is

$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy$$

Proof. Putting $x = y/a$ and $dx = dy/a$

$$B = \int_0^a \big(\frac{y}{a}\big)^{p-1} \big(1 - \frac{y}{a}\big)^{q-1} \frac{1}{a}\, dy = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1}(a-y)^{q-1}\, dy \quad \blacksquare$$

For the second form, we put $x = y/(1+y)$ and $dx = dy/(1+y)^2$

$$B(p,q) = \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1} \left(\frac{(1+y)-y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2}\, dy$$
$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy \quad \blacksquare$$

Trigonometric form. In terms of sine and cosine, the beta function reads

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta$$

Proof. Putting $x = \sin^2\theta$ and $dx = 2\cos\theta\sin\theta\, d\theta$

$$B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos\theta)^{q-1} \cos\theta\sin\theta\, d\theta$$
$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta \quad \blacksquare$$

Gamma Function. Beta functions are easily expressed in terms of gamma functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof. First we consider the gamma function of p

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt$$

Then we make the substitution $t = y^2$ and $dt = 2y\, dy$

$$\Gamma(p) = \int_0^\infty y^{2p-2} e^{-y^2} 2y\, dy = 2\int_0^\infty y^{2p-1} e^{-y^2}\, dy$$

Next we calculate the product of two gamma function p and q

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty x^{2p-1}e^{-x^2}y^{2q-1}e^{-y^2}\, dx\, dy$$

Like Gaussian integral, this is easier to evaluate in polar coordinate

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^{\pi/2}\int_0^\infty (r\cos\theta)^{2p-1}(r\sin\theta)^{2q-21}e^{-r^2}r\, dx\, dy \\ &= 2\int_0^\infty r^{2(p+q)-1}e^{-r^2}\, dr \cdot 2\int_0^{\pi/2}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\, d\theta \\ \Gamma(p)\Gamma(q) &= \Gamma(p+q)B(p,q) \quad \blacksquare\end{aligned}$$

Error Function

We define error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

There is also closely related integrals which are used and sometimes referred to as the error function called standard normal or Gaussian cumulative distribution function $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\, dt$$

Here are some of their relations.

$$\begin{aligned}\Phi(x) &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \\ \Phi(x) - \frac{1}{2} &= \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \\ \operatorname{erf}(x) &= 2\Phi\left(x\sqrt{2}\right) - 1\end{aligned}$$

Proof. Consider the definition of $\Phi(x)$. Making the substitution of $u = t/\sqrt{2}$

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{u=-\infty}^{u=x/\sqrt{2}} e^{-u^2}\sqrt{2}\, du \\ &= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^0 e^{-u^2}\, du + \int_0^\infty e^{-u^2}\, du\right) \\ \Phi(x) &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \quad \blacksquare\end{aligned}$$

To prove the third relation, we first rewrite the equation as

$$\operatorname{erf}\left(x/\sqrt{2}\right) = 2\Phi(x) - 1$$

then we make the substitution $u = x/\sqrt{2}$

$$\operatorname{erf}(u) = 2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u\sqrt{2}} e^{-t^2/2}\, dt - 1 = 2\Phi\left(x\sqrt{2}\right) - 1 \quad \blacksquare$$

Complementary error function. Defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Its relations with the actual error function are as follows.

$$\operatorname{erfc}(x) = 1 - \operatorname{erfc}(x)$$

$$\operatorname{erfc}\left(x/\sqrt{2}\right) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt$$

Proof. The first relation is quite easy to prove. Consider

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-t^2}\, dt = 1$$

then

$$\frac{2}{\sqrt{\pi}} \left(\int_{-\infty}^x e^{-t^2} + \int_x^\infty e^{-t^2} \right) = 1$$

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \quad \blacksquare$$

To proof the second relation, we substitute the limit of integration from $t = x/\sqrt{2}$ into $x = t\sqrt{2}$

$$\operatorname{erfc}\big(x/\sqrt{2}\big) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{t^2/2}}{\sqrt{2}}\, dt = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \quad \blacksquare$$

Imaginary error function. We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2}\, dt$$

Here are some relation to the actual error function.

$$\operatorname{erf}(ix) = i \operatorname{erfi}(x)$$

$$\operatorname{erf}\left(\frac{1-i}{\sqrt{2}} x \right) = (1-i)\sqrt{\frac{2}{\pi}} \int_0^x (\cos^2 u + i \sin^2 u)\, du$$

Riemann zeta function

The Riemann zeta function $\zeta(p)$ is defined by

$$\zeta(p) = \sum_{n=0}^\infty \frac{1}{k^p}$$

for real $p > 1$. Here are some value of the Riemann zeta function

$$\begin{array}{llllll} \zeta(2) = \dfrac{\pi^2}{6}; & \zeta(4) = \dfrac{\pi^4}{90}; & \zeta(6) = \dfrac{\pi^6}{945} \\ \zeta(3) = 1.202; & \zeta(5) = 1.036; & \zeta(7) = 1.008 \end{array}$$

Integrals. Here are some integral in term of gamma function and Riemann zeta function.

$$\int_0^\infty \frac{x^p}{e^x - 1} dx = \Gamma(p+1)\zeta(p+1)$$
$$\int_0^\infty \frac{x^p e^x}{(e^x - 1)^2} dx = \Gamma(p+1)\zeta(p)$$
$$\int_0^\infty \frac{x^{p-1}}{e^x + 1} dx = \left(1 - 2^{1-p}\right)\Gamma(p)\zeta(p)$$

References

[1] Mary L. Boas. *Mathematical Methods in the Physical Sciences.* 3rd. Wiley, 2005.

[2] David J. Griffiths. *Introduction to Electrodynamics.* Cambridge University Press, 2017.

[3] Mark Ryan. *Calculus For Dummies.* 2nd. Wiley, 2016.

[4] B.E. Shapiro. *Table of Integrals.* 2014.