Gamma Function

Factorial. The factorial is defined by integral

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ we get

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can also give a meaning to $0!$; by putting $n = 0$, we get $0! = 1$. By the way, the integral can be evaluated using differentiation under integral sign.

Gamma function definition. Gamma function is used to define the factorial function for noninterger n. We define, for any $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx$$

From this we have

$$\begin{aligned}\Gamma(p) &= \int_0^\infty x^{p-1} e^{-x}\,dx = (p-1)! \\ \Gamma(p+1) &= \int_0^\infty x^{p} e^{-x}\,dx = p!\end{aligned}$$

Recursion relation. The recursion for gamma function is

$$\Gamma(p+1) = p\Gamma(p)$$

Proof. Let us integrate $\Gamma(p+1)$ by parts. Calling $u = x^p$, and $dv = e^{-x}\,dx$; then we get $du = px^{p-1}$, and $v = -e^{-x}$. Thus

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} p x^{p-1}\,dx \\ &= p\int_0^\infty x^{p-1} e^{-x}\,dx \\ \Gamma(p-1) &= p\Gamma(p) \quad \blacksquare\end{aligned}$$

Negative numbers. We shall now define gamma function for $p \leq 0$ by the recursion relation

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

From this and the successive use of it, it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers.

Figure 1: Gaussian integral solved by polar method.

Important formula. We state here important formula

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

We can calculate the value of $\Gamma(1/2)$ using this equation, however we will instead try to derive it using another method. First we consider the definition

$$\Gamma(1/2) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\, dt$$

then we substitute $t = x^2$ and $dt = 2x\, dx$

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

This is the famous Gaussian integral. Refer to figure 1 on how to solve it by polar coordinate.

Since everybody and their grandma already know how to solve Gaussian integral by polar coordinate, I will instead try to solve it by Feynman's trick. First consider the function

$$I(\alpha) = \left(\int_0^{\alpha} e^{-t^2}\, dt\right)^2$$

where I is a function of parameter fish α. Then, to evaluate the actual Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\lim_{\alpha\to\infty} \sqrt{I(\alpha)}$$

Before that, I need to evaluate the function $I(\alpha)$ first. To do that, first I differentiate I with respect parameter fish α

$$\frac{dI}{d\alpha} = 2\int_0^\alpha e^{-t^2}\, dt \left(\int_0^\alpha \frac{\partial e^{-t^2}}{d\alpha}\, dt + e^{-\alpha^2} \frac{d\alpha}{d\alpha} - e^{-0^2} \frac{d(0)}{d\alpha} \right)$$
$$\frac{dI}{d\alpha} = \int_0^\alpha 2e^{-(t^2+\alpha^2)}\, dt$$

where I have used Leibniz' rule for differentiating under integral sign. Then, I introduce the variable $u = t/\alpha$ and $du = dt/\alpha$

$$\frac{dI}{d\alpha} = \int_0^1 2e^{-(u^2\alpha^2+\alpha^2)}\alpha\, du = \int_0^1 2\alpha e^{-\alpha^2(u^2+1)}\, du$$

Using the fact that

$$\frac{\partial}{\partial\alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} = -2\alpha e^{-\alpha^2(u^2+1)}$$

I can rewrite the integrand as

$$\frac{dI}{d\alpha} = -\int_0^1 \frac{\partial}{\partial\alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du$$

Since the integrand is continous, I can move the partial differentiation outside the integral and turning it into total differentiation

$$\frac{dI}{d\alpha} = -\frac{d}{d\alpha} \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du$$

Hence

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du + C$$

All that remains is to find the value of C. Considering the initial definition of $I(\alpha)$ and evaluating at $\alpha = 0$, I get

$$I(0) = \left(\int_0^0 e^{-t^2}\, dt \right)^2 = 0$$

Therefore

$$I(0) = -\int_0^1 \frac{1}{u^2+1}\, du + C = 0$$
$$C = \arctan u \Big|_0^1 = \frac{\pi}{4}$$

And I obtain the complete expression for the fish function

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du + \frac{\pi}{4}$$

Now I can evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2 \lim_{\alpha\to\infty} \left(-\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du + \frac{\pi}{4} \right)^{1/2} = 2\frac{\sqrt{\pi}}{2}$$

and I find

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Much to my chagrin, it is actually more trouble some than the polar method. Let's us try it for comparison

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dx\ dy\right)^{1/2}$$

Doing the change of coordinate thing

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\ dr\ d\theta\right)^{1/2}$$
$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(2\pi\int_{0}^{\infty} e^{-r^2} r\ dr\right)^{1/2}$$

That integral can by easily evaluated using u substitution; making the substitution $u = -r^2$

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \left(2\pi\int_{-\infty}^{0} \frac{e^u}{2}\ du\right)^{1/2} = \left(2\pi\frac{e^u}{2}\bigg|_{-\infty}^{0}\right)^{1/2}$$

And I get the same result

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Damn, it is really more shrimple.

Another form of Gaussian integral. Here we state without proof.

$$\int_{-\infty}^{\infty} \exp\left(-\alpha x^2 + \beta x\right)\ dx = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right)$$
$$\int_{0}^{\infty} x^m \exp\left(-\alpha x^2\right)\ dx = \frac{1}{2\alpha^{(m+1)/2}}\Gamma\left(\frac{m+1}{2}\right)$$

Here's another one, not really a Gaussian integral, but since it involves natural number it counts

$$\sum_{n=0}^{\infty} n^k e^{-nk} = (-1)^k\frac{d^k}{dk^k}\sum_{n=0}^{\infty} e^{-nx} = (-1)\frac{d^k}{dk^k}\frac{1}{1-e^{-x}}$$

Beta Function

Definition. The beta function is also defined by a definite integral

$$B(p,q)\int_0^1 x^{p-1}(1-x)^{q-1}\ dx$$

for $p > 0$,and $q > 0$.

Change of order. It is easy to show that

$$B(p,q) = B(q,p)$$

Proof. Putting $x = 1 - y$ and $dx = -dy$

$$B(p,q) = -\int_1^0 (1-y)^{p-1} y^{q-1}\, dy = \int_0^1 y^{q-1}(1-y)^{p-1}\, dy$$
$$B(p,q) = B(q,p) \quad \blacksquare$$

Integration Range. The range of integration can be changed with

$$B(p,q) = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1}(a-y)^{q-1}\, dy$$

Another form is

$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy$$

Proof. Putting $x = y/a$ and $dx = dy/a$

$$B = \int_0^a (\frac{y}{a})^{p-1}(1-\frac{y}{a})^{q-1}\frac{1}{a}\, dy = \frac{1}{a^{p+1-1}}\int_0^a y^{p-1}(a-y)^{q-1}\, dy \quad \blacksquare$$

For the second form, we put $x = y/(1+y)$ and $dx = dy/(1+y)^2$

$$B(p,q) = \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1}\left(\frac{(1+y)-y}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}\, dy$$
$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy \quad \blacksquare$$

Trigonometric form. In terms of sine and cosine, the beta function reads

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta$$

Proof. Putting $x = \sin^2\theta$ and $dx = 2\cos\theta\sin\theta\, d\theta$

$$B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos\theta)^{q-1}\cos\theta\sin\theta\, d\theta$$
$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta \quad \blacksquare$$

Gamma Function. Beta functions are easily expressed in terms of gamma functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof. First we consider the gamma function of p

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\, dt$$

Then we make the substitution $t = y^2$ and $dt = 2y\, dy$

$$\Gamma(p) = \int_0^\infty y^{2p-2}e^{-y^2}2y\, dy = 2\int_0^\infty y^{2p-1}e^{-y^2}\, dy$$

Next we calculate the product of two gamma function p and q

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty x^{2p-1}e^{-x^2} y^{2q-1}e^{-y^2}\ dx\ dy$$

Like Gaussian integral, this is easier to evaluate in polar coordinate

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^{\pi/2}\int_0^\infty (r\cos\theta)^{2p-1}(r\sin\theta)^{2q-21}e^{-r^2} r\ dx\ dy \\ &= 2\int_0^\infty r^{2(p+q)-1}e^{-r^2}\ dr\cdot 2\int_0^{\pi/2}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\ d\theta \\ \Gamma(p)\Gamma(q) &= \Gamma(p+q)B(p,q) \quad \blacksquare\end{aligned}$$

Error Function

We define error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\ dt$$

There is also closely related integrals which are used and sometimes referred to as the error function called standard normal or Gaussian cumulative distribution function $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\ dt$$

Here are some of their relations.

$$\begin{aligned}\Phi(x) &= \frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \\ \Phi(x)-\frac{1}{2} &= \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \\ \operatorname{erf}(x) &= 2\Phi\left(x\sqrt{2}\right)-1\end{aligned}$$

Proof. Consider the definition of $\Phi(x)$. Making the substitution of $u = t/\sqrt{2}$

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{u=-\infty}^{u=x/\sqrt{2}} e^{-u^2}\sqrt{2}\ du \\ &= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^0 e^{-u^2}\ du + \int_0^\infty e^{-u^2}\ du\right) \\ \Phi(x) &= \frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \quad \blacksquare\end{aligned}$$

To prove the third relation, we first rewrite the equation as

$$\operatorname{erf}\left(x/\sqrt{2}\right) = 2\Phi(x)-1$$

then we make the substitution $u = x/\sqrt{2}$

$$\operatorname{erf}(u) = 2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u\sqrt{2}} e^{-t^2/2}\ dt - 1 = 2\Phi\left(x\sqrt{2}\right)-1 \quad \blacksquare$$

Complementary error function. Defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Its relations with the actual error function are as follows.

$$\operatorname{erfc}(x) = 1 - \operatorname{erfc}(x)$$
$$\operatorname{erfc}\left(x/\sqrt{2}\right) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt$$

Proof. The first relation is quite easy to prove. Consider

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-t^2}\, dt = 1$$

then

$$\frac{2}{\sqrt{\pi}} \left(\int_{-\infty}^x e^{-t^2} + \int_x^\infty e^{-t^2} \right) = 1$$
$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \quad \blacksquare$$

To proof the second relation, we substitute the limit of integration from $t = x/\sqrt{2}$ into $x = t\sqrt{2}$

$$\operatorname{erfc}\left(x/\sqrt{2}\right) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{e^{t^2/2}}{\sqrt{2}}\, dt = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \quad \blacksquare$$

Imaginary error function. We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2}\, dt$$

Here are some relation to the actual error function.

$$\operatorname{erf}(ix) = i\operatorname{erfi}(x)$$
$$\operatorname{erf}\left(\frac{1-i}{\sqrt{2}} x\right) = (1-i)\sqrt{\frac{2}{\pi}} \int_0^x (\cos^2 u + i \sin^2 u)\, du$$

Riemann zeta function

The Riemann zeta function $\zeta(p)$ is defined by

$$\zeta(p) = \sum_{n=0}^\infty \frac{1}{k^p}$$

for real $p > 1$. Here are some value of the Riemann zeta function

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6}; & \zeta(4) &= \frac{\pi^4}{90}; & \zeta(6) &= \frac{\pi^6}{945} \\ \zeta(3) &= 1.202; & \zeta(5) &= 1.036; & \zeta(7) &= 1.008 \end{aligned}$$

Integrals. Here are some integral in terms of gamma function and Riemann zeta function.

$$\int_0^\infty \frac{x^p}{e^x - 1} dx = \Gamma(p+1)\zeta(p+1)$$
$$\int_0^\infty \frac{x^p e^x}{(e^x - 1)^2} dx = \Gamma(p+1)\zeta(p)$$
$$\int_0^\infty \frac{x^{p-1}}{e^x + 1} dx = \left(1 - 2^{1-p}\right)\Gamma(p)\zeta(p)$$