Appendix: Using Gamma Function to Evaluate Integral

Ex. 1. Consider

$$\int_0^3 x^3 e^{-4x^2} = \frac{\gamma(4,32)}{2\cdot 4^2}$$

Ex. 2. Consider

$$\int_1^2 x^3 e^{-x}\, dx = \gamma(4,2) - \gamma(4,1)$$

Ex. 3. The distribution function for classical partial with respect to its energy is given by

$$n(\epsilon) = \frac{2\pi N}{(\pi kT)^{3/2}} \epsilon^{1/2} \exp\left(-\frac{\epsilon}{kT}\right)$$

Suppose we want to determine the fraction of particle with energy more than $\epsilon \geq 3kT$. Said fraction is determined first find the number of particle within the range of energy

$$n(\epsilon \geq 3kT) = \int_0^{3kT} \frac{2\pi N}{(\pi kT)^{3/2}} \epsilon^{1/2} \exp\left(-\frac{\epsilon}{kT}\right)\, d\epsilon$$

By substituting $\epsilon/kT = x$, we have $\epsilon = kTx$ and $d\epsilon = kT\, dx$

$$n(\epsilon \geq 3kT) = \frac{2N}{\sqrt{\pi}} \int_0^3 x^{1/2} \exp(-x)\, dx = \frac{2N}{\sqrt{\pi}} = \frac{2N}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, 3\right)$$

Then the fraction

$$F = \frac{n(\epsilon \geq 3kT)}{N} = \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, 3\right)$$

Ex. 4. Now suppose we want to find the fraction of particle whose speed lies within $(\sqrt{2kT/m}, \sqrt{8kT/m})$ where its distribution function is given by

$$n(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

The number of particle within said range is

$$n\left(\sqrt{\frac{2kT}{m}} \leq v \leq \sqrt{\frac{8kT}{m}}\right) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{\sqrt{2kT/m}}^{\sqrt{8kT/m}} v^2 \exp\left(-\frac{mv^2}{2kT}\right)\, dv$$

By substituting $mv^2/2kT = x^2$, we have

$$v^2 = \frac{2kT}{m} x^2 \quad \text{and} \quad dv = \sqrt{\frac{2kT}{m}}\, dx$$

and limit of

$$v = \sqrt{\frac{2kT}{m}} \implies x = 1$$

$$v = \sqrt{\frac{8kT}{m}} \implies x = 2$$

Then

$$n = \frac{4N}{\sqrt{\pi}} \frac{m}{2kT} \int_1^2 \frac{2kT}{m} x^2 e^{-x^2} \sqrt{\frac{2kT}{m}}\, dx = \frac{4N}{\sqrt{\pi}} \left[\gamma(1.5, 1) - \gamma(1.5, 2)\right]$$

Appendix: Using Beta Function to Evaluate Integral

Ex. 1. Consider

$$I = \int_0^{\pi/2} \cos^n \theta \, d\theta$$

In terms of beta function

$$I = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

For the case of odd n, we write

$$\Gamma\left(\frac{n+1}{2}\right) = \frac{n-1}{2}\frac{n-3}{2}\dots 1 = \frac{(n-1)!!}{2^{(n-1)/2}}$$

where its argument is an even number. We also write

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\left(\frac{n-2}{2}\right)\dots\sqrt{\pi} = \frac{n!!\sqrt{\pi}}{2^{(n+1)/2}}$$

where its argument is an odd number. Hence,

$$I = \frac{(n-1)!!}{n!!}$$

For the case of even n, we then write

$$\Gamma\left(\frac{n+1}{2}\right) = \frac{n-1}{2}\frac{n-3}{2}\dots\sqrt{\pi} = \frac{\sqrt{\pi}(n-1)!!}{2^{(n+1)/2}}$$

where its argument is an odd number. We also write

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\left(\frac{n-2}{2}\right)\dots 1 = \frac{n!!}{2^{(n-1)/2}}$$

where its argument is an even number. Hence,

$$I = \frac{\pi}{2}\frac{(n-1)!!}{n!!}$$

Putting it all together

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \begin{cases} \dfrac{(n-1)!!}{n!!}, & \text{for odd } n \\[2ex] \dfrac{\pi}{2}\dfrac{(n-1)!!}{n!!}, & \text{for even } n \end{cases}$$

Ex. 2 Consider

$$I = \int_{-1}^{1} x^{2n}(1-x^2)^{1/2}\, dx$$

Since the integrand is an even function, we write

$$I = 2\int_{0}^{1} x^{2n}(1-x^2)^{1/2}\, dx$$

Substituting $t = x^2$ and $dt = 2\sqrt{t}\, dx$ and writing the resulting integral in terms of beta function

$$I = \int_0^1 t^{(2n-1)/2}(1-t)^{1/2}\, dt = B\left(n+\frac{1}{2}, \frac{3}{2}\right)$$

In terms of gamma function

$$I = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+2)}$$

For any even and odd n, we write

$$\Gamma\left(n+\frac{1}{2}\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\dots\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

and

$$\Gamma(n+2) = \frac{(2n+2)!!}{2^{n+1}}$$

Therefore

$$I = \frac{(2n-1)!!}{(2n+1)!!}\pi$$

Since the result differ for different n, we then evaluate for both case and write

$$\int_{-1}^{1} x^{2n}(1-x^2)^{1/2}\, dx = \begin{cases} \dfrac{\pi}{2}, & \text{for } n = 0 \\ \dfrac{(2n-1)!!}{(2n+1)!!}\pi, & \text{for interger } n > 0 \end{cases}$$