First Order

First Order ODE

$$y' + py = Qy \tag{1}$$

where P and Q are functions of x has the solution

$$ye^I = \int Qe^I \; dx + c$$

$$y = e^{-I} \int Qe^I \; dx + ce^{-I}$$

where

$$I = \int P \; dx$$

Bernoulli Equation. The differential equation

$$y' + Py = Qy^n$$

where P and Q are functions of x can be written as

$$z' + (1-n)Pz = (1-n)Q \tag{2}$$

where

$$z = y^{1-n}$$

This is now a first-order linear equation which we can solve as we did the linear equations above.

Exact Equations. $P(x,y)dx + Q(x,y)dy$ is an exact differential [the differential of $F(x,y)$, or $Pdx + Qdy = dF$] if

$$\frac{\partial}{\partial x} P = \frac{\partial}{\partial x} Q$$

and the solution is

$$F(x,y) = \text{constant}$$

An equation which is not exact may often be made exact by multiplying it by an appropriate factor.

Homogeneous Equations. A homogeneous function of x and y of degree n means a function which can be written as $x^n f(y/x)$. An equation of the from

$$P(x,y)dx + Q(x,y)dy = 0$$

where P and Q are homogeneous functions of the same degree is called homogeneous. Thus

$$y' = \frac{d}{dx} y = -\frac{P(x,y)}{Q(x,y)} - f(\frac{y}{x})$$

This suggests that we solve homogeneous equations by making the change of variables

$$y = xv \quad \text{with} \quad v = \frac{y}{x}$$

Second Order

Second Order with Zero Right Hand Side. Equation of the from

$$(D-a)(D-b)y = 0, \quad a \neq b$$

has the Solution

$$y = c_1 e^{ax} + c_2 e^{bx}$$

Equation of the from

$$(D-a)(D-a)y = 0, \quad a \neq b$$

has the Solution

$$y = (Ax + B)e^{ax}$$

Now suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. The solution is now

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)}x + Be^{(\alpha-i\beta)}x \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \\ &= e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) \\ &= ce^{\alpha x} \sin(\beta x + \gamma) \end{aligned}$$

where $\alpha,\ \beta,\ \gamma,\ c,\ c_1,\ c_2$ are different constant.

Second Order with Nonzero Right Hand Side. The equation

$$\begin{aligned} a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= f(x) \\ \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= F(x) \end{aligned}$$

has the solution of the from

$$y = y_c + y_p$$

where the complementary function y_c is the general solution of the homogeneous equation (when right hand side is equal to zero) and y_p is a particular solution (when the right hand side is equal to f(x) or F(x)). The simplest method solving them is by Inspection and Successive Integration of Two First-Order Equations.

Exponential Right-Hand Side. Suppose we have $F(x) = ke^{cx}$, or

$$(D-a)(D-b)y = ke^{cx}$$

then, we find a particular solution by assuming a solution of the form:

$$y_p = \begin{cases} Ce^{cx} & \text{if c is not equal to either a or b;} \\ Cxe^{cx} & \text{if c equals a or b, a} \neq \text{b;} \\ Cx^2e^{cx} & \text{if c = a = b.} \end{cases}$$

Complex Exponentials. To find a particular solution of

$$(D-a)(D-b)y = \begin{cases} k \sin \alpha x, \\ k \cos \alpha x, \end{cases}$$

first solve

$$(D-a)(D-b)y = ke^{i\alpha x}$$

then then take the real or imaginary part.

Method of Undetermined Coefficients. To find a particular solution of

$$(D-a)(D-b)y = e^{cx}P_n(x)$$

where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if c is not equal to either a or b;} \\ xe^{cx}Q_n(x) & \text{if c equals a or b, a} \neq \text{b;} \\ x^2e^{cx}Q_n(x) & \text{if c = a = b.} \end{cases}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the given differential equation.

Several Terms on the Right-Hand Side: Principle of Superposition. The easiest way of handling a complicated right-hand side: Solve a sep- arate equation for each different exponential and add the solutions. The fact that this is correct for a linear equation is often called the principle of superposition. Note that the principle holds only for linear equations.

Use of Fourier Series in Finding Particular Solutions. Suppose that the driving force f(x) is periodic, we then can expand the function using Fourier Series. The equation

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = f(x) = \sum_{-\infty}^{\infty} c_n e_{inx}$$

can be solved by solving

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = c_n e_{inx}$$

then add the solutions for all n (applying principle of superposition), and we have the solution of first the equation.

Laplace Transform

We define $\mathcal{L}(f)$, the Laplace transform of $f(t)$ [also written $F(p)$ since it is a function of p], by the equation

$$\mathcal{L}(f) = F(P) = \int_0^\infty f(t)e^{-pt}; dt$$

Convolution

Definition. The integral

$$g * h = \int_0^t g(t-\tau)h(\tau)d(\tau) = \int_0^t g(\tau)h(t-\tau)d(\tau)$$

is called the convolution of g and h (or the resultant or the Faltung). Now suppose that we have

$$Ay' + By' + Cy = f(t), \quad y0 = y'0 = 0$$

take the Laplace transform of each term, substitute the initial conditions, and solve for Y

$$Y = \frac{F(p)}{A(p+a)(p+b)} = T(p)F(p)$$

Then y the inverse transform of Y in is the inverse transform of a product of two functions whose inverse transforms we know. Let $G(p)$ and $H(p)$ be the transforms of $g(t)$ and $h(t)$

$$G(p)H(p) = \mathcal{L}(g(t) \cdot h(p)) = \mathcal{L}(g * h)$$

Thus

$$y = \int_0^t g(t-\tau)h(\tau)d(\tau)$$

Observe from $\mathcal{L}34$ that we may use either $g(t-\tau)h(\tau)$ or $g(\tau)h(t-\tau)$ in the integral. It is well to choose whichever form is easier to integrate; it is best to put$(t-\tau)$ in the simpler function.

Fourier Transform of a Convolution. Let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(v) e^{-i\alpha v} dv \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(u) e^{-i\alpha u} du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(v) f_2(u) e^{-i\alpha(v+u)} dv du \end{aligned}$$

Next we make the change of variables $x = v + u$, $dx = dv$, in the v integral

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x-u) f_2(u) e^{-i\alpha x} dv du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{-i\alpha x} \left[\int_{-\infty}^{\infty} f_1(x-u) f_2(u)\, du \right] dx \end{aligned}$$

if we define the term in the square parenthesis as convolution, we get

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} f_1 * f_2 e^{-i\alpha x} dx \right) \\ &= \frac{1}{2\pi} \cdot \text{Fourier transform of } f_1 * f_2 \end{aligned}$$

In other words

$$g_1 \cdot g_2 \text{ and } f_1 * f_2 \text{ are a pair of Fourier transforms}$$

and by symmetry

$$g_1 * g_2 \text{ and } f_1 \cdot f_2 \text{ are a pair of Fourier transforms}$$