Appendix: Differential Equation Study Guide

First Order Equations. General Form of ODE

$$\frac{dy}{dx} = f(x, y)$$

Initial Value Problem

$$y' = f(x, y), \; y(x_0) = y_0$$

Linear Equations. General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\mu(x) = e^{\int p(x)dx}$$
$$\Longrightarrow \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)f(x)$$

General Solution

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

Homogeneous Equations. General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in z:

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations. General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in z:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations. General Form

$$M(x, y)dx + N(x, y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations.

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors. Case 1. If $P(x,y)$ depends only on x, where

$$P(x,y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact.

Case 2. If $Q(x,y)$ depends only on y, where

$$Q(x,y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact.

Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{1}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{2}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{3}$$

General Solution. The general solution of (1) or (3) is

$$y = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \tag{4}$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (2).

Linear Independence and The Wronskian. Two functions $f(x)$ and $g(x)$ are linearly dependent if there exist numbers a and b, not both zero, such that $af(x) + bg(x) = 0$ for all x. If y_1 and y_2 are two solutions of (2), then Wronskian

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.

2. $\{y_1, y_2\}$ are a fundamental set of solutions.

3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.

4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem. The initial value problem includes two initial conditions at the same point in time, one condition on $y(t)$ and one condition on $y'(t)$.

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

The initial conditions are applied to the entire solution $y = y_h + y_p$.

Linear Equation With Constant Coefficients. The general form of the homogeneous equation is

$$ay'' + by' + cy = 0 \tag{5}$$

Non-homogeneous

$$ay'' + by' + cy = g(t) \tag{6}$$

Characteristic Equation

$$ar^2 + br + c = 0$$

Quadratic Roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

The solution of (5) of Real Roots ($r_1 \neq r_2$)

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{8}$$

Repeated ($r_1 = r_2$)

$$y_h = (C_1 + C_2 t)e^{r_1 t} \tag{9}$$

Complex ($r = \alpha \pm i\beta$)

$$y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) \tag{10}$$

The solution of (6) is $y = y_p + y_h$ where y_h is given by (8) through (10) and y_p is found by undetermined coefficients or reduction of order.

Heuristics for Undetermined Coefficients. Also called Trial and Error

If $f(t) =$	then guess that a particular solution $y_p =$.
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at} \sin bt$ or $P_n(t)e^{at} \cos bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n) \cos bt$ $+(A_0 + A_1 t + \cdots + A_n t^n) \sin bt]$

Method of Reduction of Order. When solving (2), given y_1, then y_2 can be found by solving

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx} dx}{y_1(x)^2} \tag{11}$$

Method of Variation of Parameters. If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (2) then a particular solution to (3) is

$$y_P(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)f(t)}{W(t)} dt \tag{12}$$

Cauchy-Euler Equation. For ODE

$$ax^2y'' + bxy' + cy = 0 \tag{13}$$

with Auxilliary Equation

$$ar(r-1) + br + c = 0 \tag{14}$$

The solutions of (13) depend on the roots $r_{1,2}$ of (14). For Real Roots

$$y = C_1x^{r_1} + C_2x^{r_2}$$

Repeated Root

$$y = C_1x^r + C_2x^r \ln x$$

Complex

$$y = x^\alpha[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \tag{15}$$

In (15) $r_{1,2} = \alpha \pm i\beta$, where $\alpha,\beta \in \mathbb{R}$

Series Solutions.

$$(x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0 \tag{16}$$

If x_0 is a regular point of (16) then

$$y_1(t) = (x - x_0)^n \sum_{k=0}^{\infty} a_k(x - x_k)^k$$

At a Regular Singular Point x_0, the Indicial Equation

$$r^2 + (p(0) - 1)r + q(0) = 0 \tag{17}$$

First Solution

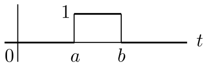
$$y_1 = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k(x - x_k)^k$$

Where r_1 is the larger real root if both roots of (17) are real or either root if the solutions are complex.

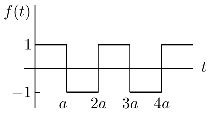
Appendix: Laplace Table

$f(t) = \mathcal{L}^{-1}F(s) = F(s)$		$\mathcal{L}f(t) = F(s)$	
$\mathcal{L}1$	1	$\dfrac{1}{p+a}$	$\text{Re } p > 0$
$\mathcal{L}2$	e^{-at}	$\dfrac{1}{p}$	$\text{Re } p > 0$
$\mathcal{L}3$	$\sin at$	$\dfrac{a}{p^2+a^2}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}4$	$\cos at$	$\dfrac{p}{p^2+a^2}$	$\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}5$	$t^k, k > -1$	$\dfrac{k!}{p^{k+1}}$ or $\dfrac{\Gamma(k+1)}{p^{k+1}}$	$\text{Re } p > 0$
$\mathcal{L}6$	$t^k e^{-at}, k > -1$	$\dfrac{k!}{(p+a)^{k+1}}$ or $\dfrac{\Gamma(k+1)}{(p+a)^{k+1}}$	$\text{Re } (p+a) > 0$
$\mathcal{L}7$	$\dfrac{e^{-at} - e^{-bt}}{b-a}$	$\dfrac{1}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$

$\mathcal{L}8$	$\dfrac{ae^{-at} - be^{-bt}}{b-a}$	$\dfrac{p}{(p+a)(p+b)}$	$\mathrm{Re}\ (p+a) > 0$ $\mathrm{Re}\ (p+b) > 0$
$\mathcal{L}9$	$\sinh at$	$\dfrac{a}{p^2 - a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Re}\ a\rvert$
$\mathcal{L}10$	$\cosh at$	$\dfrac{p}{p^2 - a^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Re}\ a\rvert$
$\mathcal{L}11$	$t \sin at$	$\dfrac{2ap}{(p^2 + a^2)^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$\mathcal{L}12$	$t \cos at$	$\dfrac{p^2 - a^2}{(p^2 + a^2)^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$\mathcal{L}13$	$e^{-at} \sin bt$	$\dfrac{b}{(p+a)^2 + b^2}$	$\mathrm{Re}\ (p+a)$ $> \lvert\mathrm{Im}\ b\rvert$
$\mathcal{L}14$	$e^{-at} \cos bt$	$\dfrac{p+a}{(p+a)^2 + b^2}$	$\mathrm{Re}\ (p+a)$ $> \lvert\mathrm{Im}\ b\rvert$
$\mathcal{L}15$	$1 - \cos at$	$\dfrac{a^2}{p(p^2 + a^2)}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$\mathcal{L}16$	$at - \sin at$	$\dfrac{a^3}{p^2(p^2 + a^2)}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$
$\mathcal{L}17$	$\sin at - at \cos at$	$\dfrac{2a^3}{(p^2 + a^2)^2}$	$\mathrm{Re}\ p > \lvert\mathrm{Im}\ a\rvert$

$\mathcal{L}18$	$e^{-at}(1-at)$	$\dfrac{p}{(p+a)^2}$	Re $(p{+}a) > 0$
$\mathcal{L}19$	$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{p}$	Re $p > \lvert\text{Im } a\rvert$
$\mathcal{L}20$	$\dfrac{1}{t}\sin at \cos bt$	$\dfrac{1}{2}\left(\arctan \dfrac{a+b}{p} + \arctan \dfrac{a-b}{p}\right)$	Re $p > 0$
$\mathcal{L}21$	$\dfrac{e^{at}-e^{-bt}}{t}$	$\ln \dfrac{p+b}{p+a}$	Re $(p{+}a) > 0$ Re $(p{+}b) > 0$
$\mathcal{L}22$	$1-\operatorname{erf}\left(\dfrac{a}{2\sqrt{t}}\right), a > 0$	$\dfrac{1}{p}e^{-a\sqrt{p}}$	Re $p > 0$
$\mathcal{L}23$	$J_0(at)$	$(p^2+a^2)^{-1/2}$	Re $p > \lvert\text{Im } a\rvert$ or Re $p \geq 0$ for real $a \neq 0$
$\mathcal{L}24$	unit step, or Heaviside function $u(t{-}a) = \begin{cases} 1, \\ t > a > 0 \\ 0, t < a \end{cases}$	$\dfrac{1}{p}e^{-pa}$	Re $p > 0$
$\mathcal{L}25$	$f(t) = u(t{-}a){-}u(t{-}b)$ 	$\dfrac{e^{-ap}-e^{-}bp}{p}$	All p

$\mathcal{L}26$



$$\frac{1}{p}\tanh\frac{ap}{2}$$

All p

$\mathcal{L}27$

$\delta(t-a), a \geq 0$ $\qquad$ e^{-pa}

$\mathcal{L}28$

$$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$$
$$= g(t-a)u(t-a)$$

$$e^{-pa}G(p)$$

$G(p)$ means $\mathcal{L}(g)$.
Therefore $e^{-pa}\mathcal{L}[g(t-a)]$

$\mathcal{L}29$

$e^{-at}g(t)$ $\qquad$ $G(p+a)$

$\mathcal{L}30$

$g(at), a > 0$ $\qquad$ $\dfrac{1}{a}G\left(\dfrac{p}{a}\right)$

$\mathcal{L}31$

$$\frac{g(t)}{t}$$

if integrable

$$\int_p^\infty G(u)\,du$$

$\mathcal{L}32$

$t^n g(t)$ $\qquad$ $(-1)^n\left(\dfrac{d}{dp}\right)^n(G(p))$

$\mathcal{L}33$

$$\int_0^t g(\tau)\, d\tau \qquad\qquad \frac{1}{p}G(p)$$

$\mathcal{L}34$ Convolution of g and h, often written as

$$g * h$$

$$\int_0^t g(t-\tau)h(\tau)\, d\tau = \int_0^t g(\tau)h(t-\tau)\, d\tau \qquad\qquad G(p)H(p)$$

$\mathcal{L}35$ Transforms of derivatives of y

$$\begin{aligned}
\mathcal{L}(y) &= Y \\
\mathcal{L}(y') &= pY - y \\
\mathcal{L}(y'') &= p^2Y - py_0 - y'_0 \\
\mathcal{L}(y''') &= p^3Y - p^2y_0 - py'_0 - y_0 \\
\mathcal{L}(y^n) &= p^nY - p^{n-1}y_0 - p^{n-2}y'_0 - \cdots - y_0^{n-1}
\end{aligned}$$