Physical Mathematics

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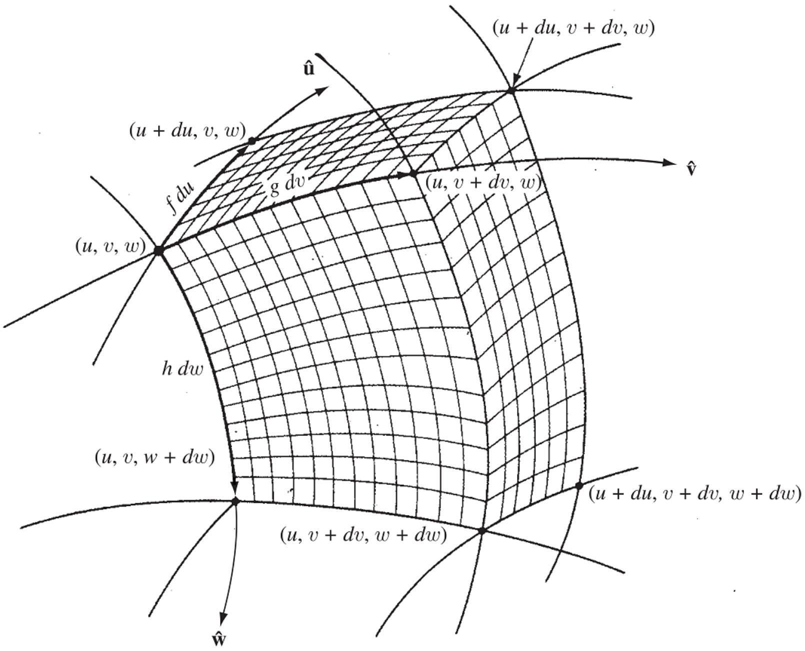
with LaTeX



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GitHub Repository

https://github.com/JohanesFaustus/PhysicalMathematics

or the hyperlink

FUNDAMENTAL

Mainly consist of precalculus, and basic Calculus.

Algebra

Laws of Exponents.

$$
\begin{aligned}
x^{\frac{m}{n}} &= \sqrt[n]{m} \\
(x^m)^n &= x^{mn} \\
x^m x^n &= x^{m+n} \\
x^a y^a &= (xy)^a
\end{aligned}
$$

Special Factorization.

$$
\begin{aligned}
x^2 - y^2 &= (x+y)(x-y) \\
x^3 - y^3 &= (x-y)(x^2+xy+y^2) \\
x^3 + y^3 &= (x+y)(x^2-xy+y^2)
\end{aligned}
$$

Quadratic formula. The following formula can be used to find the roots of quadratic equation $ax^2 + bx + c$

$$
x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \begin{cases} D > 0 & \mathrm{re}(2) \\ D = 0 & \mathrm{re}(1) \\ D < 0 & \mathrm{im}(2) \end{cases}
$$

The said quadratic equation can also be writen as

$$
ax^2 + bc + c = a(x - x_1)(x - x_2)
$$

Binomial theorem.

$$
(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k
$$

with

$$
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}
$$

Trigonometry

Trigonometry Definitions.

$$
\sin\theta = \frac{1}{\csc\theta}
$$

$$\cos\theta = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$

Law of Sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Trigonometry Double Angle Identities.

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= \cos^2\theta - \sin^2\theta \\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

Trigonometry Addition and Difference Identities.

$$\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y \\
\sin(x-y) &= \sin x\cos y - \cos x\sin y \\
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\cos(x-y) &= \cos x\cos y - \sin x\sin y \\
\tan(x+y) &= \frac{\tan x + \tan y}{1-\tan x\tan y} \\
\tan(x-y) &= \frac{\tan x - \tan y}{1+\tan x\tan y}
\end{aligned}$$

Trigonometry Product Rule.

$$\begin{aligned}
\cos x\cos y &= \frac{1}{2}[\cos(x-y)+\cos(x+y)] \\
\sin x\sin y &= \frac{1}{2}[\cos(x-y)-\cos(x+y)] \\
\sin x\cos y &= \frac{1}{2}[\sin(x+y)+\sin(x-y)]
\end{aligned}$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

Neat Mnemonics.

$$\begin{vmatrix} \mathrm{S}^+ \\ \mathrm{S}^- \\ \mathrm{C}^+ \\ \mathrm{C}^- \end{vmatrix} = \begin{vmatrix} \mathrm{SC} + \mathrm{CS} \\ \mathrm{SC} - \mathrm{CS} \\ \mathrm{CC} - \mathrm{SS} \\ \mathrm{CC} + \mathrm{SS} \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} \mathrm{CC} \\ \mathrm{SS} \\ \mathrm{SC} \\ \mathrm{CS} \end{vmatrix} = \begin{vmatrix} \mathrm{C}^- + \mathrm{C}^+ \\ \mathrm{C}^- - \mathrm{C}^+ \\ \mathrm{S}^+ + \mathrm{S}^- \\ \mathrm{S}^+ - \mathrm{S}^- \end{vmatrix}$$

Logarithm

Definition (informal). $\log_a b$ means a to the power of what equal b.

Few important log rule.

$$\begin{aligned} \log_c(ab) &= \log_c(a) + \log_c(b) \\ \log_c(\frac{a}{b}) &= \log_c(a) - \log_c(b) \\ \log_a b &= \frac{\log_c(b)}{\log_c(a)} \\ a^{\log_a b} &= b \end{aligned}$$

Limit

Few Important Limits.

$$\begin{aligned} \lim_{x \to a} c &= c \\ \lim_{x \to 0^+} \frac{1}{x} &= \infty \to \frac{1}{0^+} = \infty \\ \lim_{x \to 0^-} \frac{1}{x} &= -\infty \to \frac{1}{0^-} = -\infty \end{aligned}$$

Limit as Definition of Derivative.

$$\frac{d}{dx} y = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative

Order of calculation. How to determine the order of derivation: last computation is the first thing to do.

General Formula.

$$\begin{aligned} D\ x^n &= nx^{n-1} \\ D\ (uv) &= D\ u \cdot v + u \cdot D\ v \\ D \left(\frac{u}{v} \right) &= \frac{D\ u \cdot v - u \cdot D\ v}{v^2} \end{aligned}$$

Trigonometry Formula.

$$\begin{aligned}
D\ \sin x &= \cos x \\
D\ \cos x &= -\sin x \\
D\ \tan x &= \sec^2 x \\
D\ \cot x &= -\csc^2 x \\
D\ \sec x &= \sec x \tan x \\
D\ \csc x &= -\cot x \csc x
\end{aligned}$$

Neat Mnemonics.

$$\begin{array}{cccl}
\sec & \sec & \tan & \downarrow \text{cofunction} \\
\csc & -\csc & \cot & \\
& \leftrightarrow \text{multiply} & &
\end{array}$$

Exponential and Logarithmic Functions.

$$\begin{aligned}
D\ \ln x &= \frac{1}{x} \\
D\ a^n &= a^n \ln x \\
D\ \log_a b &= \frac{1}{b \ln a}
\end{aligned}$$

Minima and Maxima test. First derivative test:

- Determine critical points ($Dy = 0$), then divide into region;
- Pick value from each region and plug into *derivative*; and
- Do the sign-graph thing.

Second derivative test:

- Determine critical points;
- Plug critical into second derivative; and
- Positive D^2y means concave up ($\smile$) or minima, negative means concave down ($\frown$) or maxima, and 0 means inconclusive. Simply put, positive means minima, while negative means maxima.

Optimization with constrain. Elimination method:

1. Write the function f. The function itself must be in terms of one independent variable, say x, which can often be achieved by substituting our constraint, say $y(x)$, into the function $f(x)$.
2. Find the critical points.
3. Use whatever test you need.

Implicit differentiation method:

1. Write the function $f(x, y)$. This method assumes that it is not possible to solve substituting the constant y into our equation.

2. Write the differentiation with respect to independent x variable. Note that the derivative of the dependent variable often still remains; we need to solve for them too.
3. Use the following result to find the critical points.
4. Use the second derivative test. Note that you'll also need the second derivative of the constraint y evaluated at critical points to determine the second derivative of the function $f(x, y)$

Differentiation under integral sign. Differentiation under integral sign stated by Leibniz' rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\, dt = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}$$

Proof. Suppose we want dI/dx where

$$I = \int_u^v f(t)\, dt$$

By the fundamental theorem of calculus

$$I = F(v) - F(u) = \mathcal{F}(v,u)$$

or I is a function of v and u. Finding dI/dx is then a partial differentiation problem. We can write

$$\frac{dI}{dx} = \frac{\partial I}{\partial v}\frac{dv}{dx} + \frac{\partial I}{\partial u}\frac{du}{dx}$$

By the fundamental theorem of calculus, we have

$$\frac{d}{dv}\int_a^v f(x)\, dt = \frac{d}{dv}\big[F(v) - F(a)\big] = f(v)$$
$$\frac{d}{dv}\int_u^b f(x)\, dt = \frac{d}{dv}\big[F(b) - F(u)\big] = -f(u)$$

where u and v are a function of x, while a and b are a constant. This is the case when we consider $\partial I/\partial v$ or $\partial I/\partial v$; the other variable is constant. Then

$$\frac{d}{dx}\int_u^v f(t)\, dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Under not too restrictive conditions,

$$\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\, dt$$

where, as before, a and b are constant. In other words, we can differentiate under the integral sign. It is convenient to collect these formulas into one formula known as Leibniz' rule:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\, dt = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} \quad \blacksquare$$

Leibniz' rule for differentiating a product.

$$\left(\frac{d}{dx}\right)^n (fg) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx}\right)^{n-k} (f) \left(\frac{d}{dx}\right)^k (g)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Integral

Basic Formula (integration constant omitted).

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}$$

$$\int \frac{1}{x} \, dx = \ln |x|$$

$$\int u \, dv = uv - \int v \, du$$

$$\int a^x \, dx = \frac{a^x}{\ln a}$$

Trigonometry.

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \csc^2 x \, dx = -\cot x$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \csc x \tan x \, dx = -\csc x$$

Root.

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln x + \sqrt{x^2 \pm a^2}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \, dx = \frac{1}{a} \arctan \frac{x}{a}$$

Integration by part.

1. Splits the integrand. Choose u using LIATEN and let the rest be dv. (LIATEN: Log, Inverse trigonometry, Algebra, Trigonometry, ExponeN)

Table: The box thing

u	v	$\downarrow$ Differentiate
du	dv	$\uparrow$ Integrate

2. Do the box thing
3. $\int u\, dv = uv - \int v\, du$

Tabular Method. Refer to the table. Steps:

Table: The table

	Differentiate	Integrate
+	$a \searrow$	b
-	$a' \searrow$	b
+	$a'' \searrow$	b
$\vdots$	$\vdots$	$\vdots$

1. 0 in D column or use LIATEN,
2. Integrate a row, and
3. A row repeats.

Trigonometry Integral. Pythagorean Identity.

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

note that argument inside quadratic trigonometry is half of trigonometry, which means $\cos^2 2x = (1+\cos 4x)/2$. There are few cases of tricky trigonometry integral. First, if power of sin is odd and positive. The steps to evaluate it are as follows.

1. Remove one power off
2. convert remaining (even power) using Pythagorean Identity in terms of cosine
3. integrate using subs method

If the power of sine is odd and positive.

1. Same as before

If the power of sine and cosine is even and nonegative, then:

1. convert using Pythagorean Identity and solve

Trigonometry substitution. Trigonometry function and its radical pair

$$\begin{aligned}\tan\theta &= \sqrt{u^2+a^2}\\ \sin\theta &= \sqrt{a^2-u^2}\\ \sec\theta &= \sqrt{u^2-a^2}\end{aligned}$$

where u is the variable we are differentiating with respect to. Mnemonics: $+$ looks like tangent; $-$ for sin and sec; and it is a $s$in. Trigonometry substitution step is then as follows.

1. Draw a right triangle where trigonometry pair equal u/a
2. using the trigonometry pair equation*, solve for x and dx
3. find trigonometry where $\sqrt{}/a$
4. subs again if equation* still contain θ and solve

Partial Fraction.

1. Factor out denominator
2. Breakup the function and put unknown (Capital Letter) into numerator. Put numerator normally if factor is linear, put $Px+Q$ Irreducible quadratic factor IQF. In general,

$$\frac{Ax^{n-1}+Bx^{n-2}+\cdots}{x^n+x^{n-1}+\cdots}$$

3. Multiply both side by left side's denominator
4. Take the roots of the linear factors and plug them into x, and solve for the unknowns
5. Put unknowns into step 2
6. Splits Integral, then solve
7. For equating coefficients like terms, after step 3, expand equation*. Then, collect like terms and equate coefficient of like terms from both side

Appendix: Example of Optimization Problem

One variable problem. Consider this simple problem.

> What is the maximum volume of a box without top that can be made from a square plate with a side of 30 unit, given that you can cut of its corner?

We know the volume of the box is calculated by $V = lwh$, then the volume as function of height is

$$V(h) = (30 - 2h)(30 - 2h)h = 4h^3 - 120h^2 + 900$$

This is the function that we want to maximize. Setting the derivative to zero, we have

$$\begin{aligned} \frac{dV}{dh} &= 12h^2 - 240h + 900 \\ 0 &= h^2 - 20h + 75 \end{aligned} \implies h = 15 \ \vee \ h = 5$$

Putting those critical number, into $V(h)$, we obtain $V(5) = 2000$ and $V(15) = 0$. Hence, box with height 5 will maximize the volume.

Two variable problem. More complicated, but still very easy.

> Consider a floorless pup tent made from the least possible material with width $2w$, length l, creating angle θ. Find θ.

First we consider the volume and area of the shape in question.

$$\begin{aligned} V &= \frac{2w}{2} w \tan\theta \cdot l = w^2 l \tan\theta \\ A &= \frac{2w}{2} w \tan\theta \cdot 2 + \frac{wl}{\cos\theta} \cdot 2 = 2w^2 \tan\theta + 2wl \sec\theta \end{aligned}$$

Solving for l from the equation of V, we get $l = V/w^2 \tan\theta$. Then we substitute the result into A to obtain

$$A = 2w^2 \tan\theta + \frac{2wV \sec\theta}{w^2 \tan\theta} = 2w^2 \tan\theta + \frac{2V}{w} \csc\theta$$

This is the function that we want to minimize. Since A is a function of two variable $A(w, \theta)$, we need to differentiate it with respect to those two variable

$$\frac{\partial A}{\partial w} = 4w \tan\theta - \frac{2V}{w^2} \csc\theta = 0$$

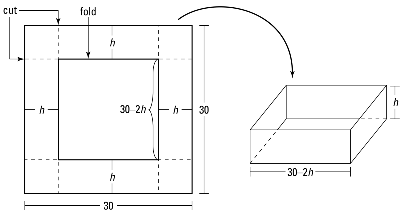


Figure: Plate and its configuration

$$\frac{\partial A}{\partial \theta} = 2w^2 \sec^2\theta - \frac{2V}{w}\csc\theta\cot\theta = 0$$

Solving those equations for w^3, we get

$$w^3 = \frac{V\csc\theta}{2\tan\theta} \ \wedge \ w^3 = \frac{V\csc\theta\cot\theta}{\sec^2\theta}$$

Equating them to obtain

$$\frac{1}{\sec^2\theta} = \frac{1}{2} \implies \theta = \frac{\pi}{4} = 45°$$

Appendix: Example of Optimization Problem with Constrain.

We try to solve this example using few methods: elimination, implicit derivative, and Lagrange multiplier. Now, consider this problem.

> What is the shortest distance from origin to curve $y = 1 - x^2$?

Elimination. What we want to minimize is the distance $d = (x^2 + y^2)^{1/2}$, however it is more convenient to minimize $f = x^2 + y^2$ instead. The function $y = 1 - x^2$ acts as constraint. Substituting the constraint into our function, we have

$$f(x) = x^2 + (1 - x^2)^2 = x^4 - x^2 + 1$$

The critical points of this function determined by

$$\frac{df}{dx} = 4x^3 - 2x = x(4x^2 - 2) = 0 \implies x = 0 \vee x = \pm\sqrt{1/2}$$

Then to determine the maxima or minima of the function, we use second test derivative

$$\frac{d^2 f}{dx^2} = 12x^2 - 2 = \begin{cases} -2, & x = 0, \quad \text{Maxima} \\ 4, & x = \pm\sqrt{1/2}, \quad \text{Minima} \end{cases}$$

Therefore, the minimum distance is

$$d = \left[\left(\sqrt{\frac{1}{2}}\right)^2 + \left(1 - \frac{1}{2}\right)^2\right]^{1/2} = \frac{\sqrt{3}}{2}$$

Implicit differentiation. We use this method if it is not possible to substitute the constraint $y(x)$ into our function f. Differentiating $f(x, y)x^2 + y^2$ with respect to x

$$\frac{df}{dx} = 2x + 2y\frac{dy}{dx}$$

From our constraint equation, we have the relation of dy in terms of dx as $dy = -2x\ dx$. Substituting this equation into df/dx while also setting it equal to zero

$$2x - 4xy = x(1 - 2y) = 0 \implies x = 0 \vee y = 1/2$$

And we obtain one of the critical points. To obtain the rest, we substitute the result $y = 1$ into our constraint equation $y = 1 - x^2$. We have then

$$x = \left(-\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}\right)$$

The second derivative test for this method is rather different from the usual. What differs is simply how to evaluate the second derivative. First we determine the second derivative of $f(x, y)$ with respect to x

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(2x + 2y\frac{dy}{dx}\right) = 2 + \left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2 y}{dx^2}$$

At $x = 0$, we have $y = 1$, $dy/dx = 0$, and $d^2y/dy^2 = -2$; while at $x = \pm\sqrt{1/2}$, we have $y = 1/2$, $dy/dx = \mp\sqrt{2}$, and $d^2y/dy^2 = -2$. Hence,

$$\left.\frac{d^2 f}{dx^2}\right|_{x=0} = -2 \text{ (Maxima)} \quad \wedge \quad \left.\frac{d^2 f}{dx^2}\right|_{x=\pm\sqrt{1/2}} = -4 \text{ (Minima)}$$

as before. Substituting the result into the equation for distance $d = (x^2 + y^2)^{1/2}$ and we have the same result

Lagrange multiplier. We have the function that we want to maximize $f(x, y) = x^2 + y^2$ and constrain $\phi(x, y) = x^2 + y = 1$. Then we construct the function

$$F(x, y) = x^2 + y^2 + \lambda(x^2 + y)$$

The partial derivative of $F(x, y)$ with respect to each variable is

$$\frac{\partial F}{\partial x} = 2x + 2\lambda x \quad \wedge \quad \frac{\partial F}{\partial y} = 2y + \lambda$$

In addition with the constraint, we then need to solve those three equations

$$\begin{aligned} 2x + 2\lambda x &= 0 \\ 2y + \lambda &= 0 \\ x^2 + y &= 1 \end{aligned}$$

Solving the first equation

$$x(1 + \lambda) = 0 \implies x = 0 \vee \lambda = -1$$

Using λ on the second equation

$$2y + \frac{1}{2} = 0 \implies y = \frac{1}{2}$$

Hence the constraint equation reads

$$x^2 + \frac{1}{2} = 1 \implies x = \pm\sqrt{1/2}$$

As before, we obtain critical points $x = \left(-\sqrt{1/2}, 0, \sqrt{1/2}\right)$. We then can move to the next step: minima test and calculating the distance, both will need not to be repeated.

Appendix: Integration Technique Example.

Trigonometry substitution. Find $\int \frac{dx}{\sqrt{9x^2+4}}$. Refer to the Mnemonics, the trigonometry pair is tangent.

$$I = \int \frac{dx}{\sqrt{(3x)^2+2^2}}$$
$$\tan\theta = \frac{3x}{2}$$

solving for x and dx

$$x = \frac{2}{3}\tan\theta$$
$$dx = \frac{2}{3}\sec^2\theta \; d\theta$$

trigonometry where $\frac{\sqrt{}}{a}$ holds is secant, solving for radical

$$\sec\theta = \frac{\sqrt{9x^2+4}}{2}$$
$$\sqrt{9x^2+4} = 2\sec\theta$$

the integral is then

$$I = \frac{1}{3}\int \sec\theta \; d\theta$$
$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

substituting the θ function

$$I = \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2} + \frac{3x}{2}\right| + C$$
$$= \frac{1}{3}\ln\left|\sqrt{9x^2+4} + 3\right| + C$$

Appendix: Moar Integral

Basic. Most common integrals.

$$\int \frac{1}{x}\; dx = \ln|x|$$
$$\int u \; dv = uv - \int v \; du$$
$$\int u \; dv = uv - \int v \; du$$
$$\int \frac{1}{ax+b}\; dx = \frac{1}{a}\ln|ax+b|$$

Rational Functions. Integrals of rational function

$$\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a}$$

$$\int (x+a)^n\,dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1$$

$$\int x(x+a)^n\,dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln|a^2+x^2|$$

$$\int \frac{x^2}{a^2+x^2}\,dx = x - a\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^3}{a^2+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{b-a}\ln\frac{a+x}{b+x}, \quad a \neq b$$

$$\int \frac{x}{(x+a)^2}\,dx = \frac{a}{a+x} + \ln|a+x|$$

$$\int \frac{x}{ax^2+bx+c}\,dx = \frac{1}{2a}\ln|ax^2+bx+c| -$$
$$\frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}$$

Roots. Integrals of roots.

$$\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2}$$

$$\int \frac{1}{\sqrt{x\pm a}}\,dx = 2\sqrt{x\pm a}$$

$$\int \frac{1}{\sqrt{a-x}}\,dx = -2\sqrt{a-x}$$

$$\int x\sqrt{x-a}\,dx = \begin{cases} \frac{2a}{3}(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}, \text{ or} \\ \frac{2}{3}x(x-a)^{3/2} - \frac{4}{15}(x-a)^{5/2}, \text{ or} \\ \frac{2}{15}(2a+3x)(x-a)^{3/2} \end{cases}$$

$$\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$$

$$\int (ax+b)^{3/2}\,dx = \frac{2}{5a}(ax+b)^{5/2}$$

$$\int \frac{x}{\sqrt{x\pm a}}\,dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a}$$

$$\int \sqrt{\frac{x}{a-x}}\,dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a}$$

$$\int \sqrt{\frac{x}{a+x}}\, dx = \sqrt{x(a+x)} - a\ln[\sqrt{x} + \sqrt{x+a}]$$

$$\int x\sqrt{ax+b}\, dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b}$$

$$\int \sqrt{x^3(ax+b)}\, dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)}$$

$$+ \frac{b^3}{8a^{5/2}} \ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}\left(x^2 \pm a^2\right)^{3/2}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}\, dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}}\, dx = -\sqrt{a^2 - x^2}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\, dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx = \frac{1}{\sqrt{a}} \ln\left|2ax + b + 2\sqrt{a(ax^2 + bx + c)}\right|$$

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Integrals with Logarithms.

$$\int \ln ax\, dx = x\ln(ax) - x$$

$$\int \frac{\ln ax}{x}\, dx = \frac{1}{2}(\ln ax)^2$$

$$\int \ln(ax+b)dx = (x + \frac{b}{a})\ln(ax+b) - x, \quad a \neq 0$$

$$\int \ln(x^2 + a^2)\, dx = x\ln(x^2 + a^2) + 2a\tan^{-1}\frac{x}{a} - 2x$$

$$\int \ln(x^2 - a^2)\, dx = x\ln(x^2 - a^2) + a\ln\frac{x+a}{x-a} - 2x$$

$$\int \ln(x^2 - a^2)\, dx = x\ln(x^2 - a^2) + a\ln\frac{x+a}{x-a} - 2x$$

$$\int \ln(ax^2 + bx + c)dx = \frac{1}{a}\sqrt{4ac - b^2}\tan^{-1}\frac{2ax + b}{\sqrt{4ac - b^2}} - 2x$$

$$+ (\frac{b}{2a} + x)\ln\,(ax^2 + bx + c)$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}(x^2 - \frac{b^2}{a^2})\ln(ax+b)$$

$$\frac{1}{2}(x^2 - \frac{a^2}{b^2})\ln(a^2 - b^2x^2)$$

Integrals with Exponential.

$$\int e^{ax}dx = \frac{1}{a}e^{ax}$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\operatorname{erf}\left(i\sqrt{ax}\right)$$

$$\int xe^{x}dx = (x-1)e^{x}$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

$$\int x^2e^{x}dx = \left(x^2 - 2x + 2\right)e^{x}$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$$

$$\int x^3e^{x}dx = \left(x^3 - 3x^2 + 6x - 6\right)e^{x}$$

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$\int x^n e^{ax}\,dx = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n, -ax]$$

$$\int e^{ax^2}\,dx = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\operatorname{erf}\left(ix\sqrt{a}\right)$$

$$\int e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\operatorname{erf}\left(x\sqrt{a}\right)$$

$$\int xe^{-ax^2}\,dx = -\frac{1}{2a}e^{-ax^2}$$

$$\int x^2e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\operatorname{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2}$$

Integrals with Trigonometry Functions.

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \sin^n ax\,dx = -\frac{1}{a}\cos ax \times {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right]$$

$$\int \sin^3 ax\,dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a}$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \cos^p ax\,dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right]$$

$$\int \cos^3 ax\,dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a}$$

$$\int \cos ax \sin bx \; dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, \quad a \neq b$$

$$\int \sin^2 ax \cos bx \; dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)}$$

$$\int \sin^2 x \cos x \; dx = \frac{1}{3} \sin^3 x$$

$$\int \cos^2 ax \sin bx \; dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)}$$

$$\int \cos^2 ax \sin ax \; dx = -\frac{1}{3a} \cos^3 ax$$

$$\int \sin^2 ax \cos^2 bx \; dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b}$$
$$- \frac{\sin[2(a+b)x]}{16(a+b)}$$

$$\int \sin^2 ax \cos^2 ax \; dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$\int \tan ax \; dx = -\frac{1}{a} \ln \cos ax$$

$$\int \tan^2 ax \; dx = -x + \frac{1}{a} \tan ax$$

$$\int \tan^n ax \; dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right)$$

$$\int \tan^3 ax \; dx = \frac{1}{a} \ln \cos ax + \frac{1}{2a} \sec^2 ax$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| = 2 \tanh^{-1}\left(\tan \frac{x}{2}\right)$$

$$\int \sec^2 ax \; dx = \frac{1}{a} \tan ax$$

$$\int \sec^3 x \; dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x|$$

$$\int \sec x \tan x \; dx = \sec x$$

$$\int \sec^2 x \tan x \; dx = \frac{1}{2} \sec^2 x$$

$$\int \sec^n x \tan x \; dx = \frac{1}{n} \sec^n x, \quad n \neq 0$$

$$\int \csc x \; dx = \ln\left|\tan \frac{x}{2}\right| = \ln|\csc x - \cot x| + C$$

$$\int \csc^2 ax \; dx = -\frac{1}{a} \cot ax$$

$$\int \csc^3 x \; dx = -\frac{1}{2} \cot x \csc x + \frac{1}{2} \ln|\csc x - \cot x|$$

$$\int \csc^n x \cot x \; dx = -\frac{1}{n} \csc^n x, n \neq 0$$

$$\int \sec x \csc x \; dx = \ln|\tan x|$$

Products of Trigonometry Functions and Monomials.

$$\int x\cos x\;dx = \cos x + x\sin x$$

$$\int x\cos ax\;dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

$$\int x^2\cos x\;dx = 2x\cos x + \left(x^2-2\right)\sin x$$

$$\int x^2\cos ax\;dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$$

$$\int x^n\cos x\;dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1,-ix)+(-1)^n\Gamma(n+1,ix)\right]$$

$$\int x^n cosax\;dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1,-iax)-\Gamma(n+1,ixa)\right]$$

$$\int x\sin x\;dx = -x\cos x + \sin x$$

$$\int x\sin ax\;dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$\int x^2\sin x\;dx = \left(2-x^2\right)\cos x + 2x\sin x$$

$$\int x^2\sin ax\;dx = \frac{2-a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2}$$

$$\int x^n\sin x\;dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix)-(-1)^n\Gamma(n+1,-ix)\right]$$

Products of Trigonometry Functions and Exponential.

$$\int e^x\sin x\;dx = \frac{1}{2}e^x(\sin x - \cos x)$$

$$\int e^{bx}\sin ax\;dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax)$$

$$\int e^x\cos x\;dx = \frac{1}{2}e^x(\sin x + \cos x)$$

$$\int e^{bx}\cos ax\;dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax)$$

$$\int xe^x\sin x\;dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x)$$

$$\int xe^x\cos x\;dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x)$$

Integrals of Hyperbolic Functions.

$$\int \cosh ax dx = \frac{1}{a} \sinh ax$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} \left[a \cosh bx - b \sinh bx \right], \quad a \neq b \\ \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2}, \quad a = b \end{cases}$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} \left[-b \cosh bx + a \sinh bx \right], \quad a \neq b \\ \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} a = b \end{cases}$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)} {}_2F_1 \left[1 + \dfrac{a}{2b}, 1, 2 + \dfrac{a}{2b}, -e^{2bx} \right] \\ \\ -\dfrac{1}{a} e^{ax} {}_2F_1 \left[\dfrac{a}{2b}, 1, 1E, -e^{2bx} \right], \quad a \neq b \\ \\ \dfrac{e^{ax} - 2 \tan^{-1}[e^{ax}]}{a}, \quad a = b \end{cases}$$

$$\int \tanh ax \; dx = \frac{1}{a} \ln \cosh ax$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[a \sin ax \cosh bx + b \cos ax \sinh bx \right]$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cos ax \cosh bx + a \sin ax \sinh bx \right]$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2 + b^2} \left[-a \cos ax \cosh bx + b \sin ax \sinh bx \right]$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2 + b^2} \left[b \cosh bx \sin ax - a \cos ax \sinh bx \right]$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a} \left[-2ax + \sinh 2ax \right]$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2 - a^2} \left[b \cosh bx \sinh ax - a \cosh ax \sinh bx \right]$$

Series

Converge Test

Geometric Series (for r less than one):

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$$

Preliminary Test:

$$\text{if } \lim_{n\to\infty} a_n \neq 0 \text{ , then series Diverges}$$

Comparation Test:

$$A \le C \text{ , A Converges}$$
$$A \ge D \text{ , A Diverges}$$

Integral Test:

$$\int^{\infty} A\, dn \begin{cases} \text{A finite} \to \text{Converges} \\ \text{A infinite} \to \text{Diverges} \end{cases}$$

Ratio Test:

$$\rho_n = \left| \frac{a_{(n+1)}}{a_n} \right|$$
$$\rho = \lim_{n\to\infty} \rho_n \begin{cases} \rho > 1 \text{ Diverge} \\ \rho = 0 \text{ Inconclusive} \\ \rho < 1 \text{ Converges} \end{cases}$$

Special Comparation:

$$\lim_{n\to\infty} \frac{A}{C} \text{ Limit finite} \to \text{A Converges}$$
$$\lim_{n\to\infty} \frac{A}{D} \text{ Limit} > 0 \to \text{A Diverges}$$

Raabe Test:

$$\rho \equiv \lim_{n\to\infty} \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) \right] \begin{cases} \rho > 1, \text{ Converge} \\ \rho = 1, \text{ Inconclusive} \\ \rho < 1, \text{ Diverge} \end{cases}$$

Root Test:

$$\rho \equiv \lim_{n\to\infty} \sqrt[n]{|a_n|}$$
$$\equiv \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \begin{cases} \rho > 1, \text{ Converge} \\ \rho = 1, \text{ Inconclusive} \\ \rho < 1, \text{ Diverge} \end{cases}$$

Alternating series Test

$$\text{if} \quad |a_{n+1}| \le |a_n| \quad \text{and} \quad \lim_{n\to\infty} a_n = 0$$

then series converges.

Function Expansion

Taylor Series about $x = a$:

$$\frac{1}{n!}(x-a)^n f^n(a) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots$$

Maclaurin Series:

$$\frac{1}{n!}(x)^n f^n(0) = f(0) + (x)f'(0) + \frac{1}{2!}(x)^2 f''(0) + \frac{1}{3!}(x)^3 f'''(0) + \cdots$$

Maclaurin Expansion for basic function:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } -1 < x \leq 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n)} \quad \text{for all } x$$

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \quad \text{for } |x| < 1$$

or more explicitly

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$(1+x)^p = +px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

Complex Analysis

Introduction

Complex Number. Complex number may be written in the rectangular form or polar form

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

The quantity r is called the modulus or absolute value of z, and θ is called the angle of z (or the phase, or the argument, or the amplitude of z). In symbols

$$\begin{array}{ll} \mathrm{Re}\ z = x & |z| = \mathrm{mod}\ z = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} \\ \mathrm{Im}\ z = y \text{ (not } iy) & \text{angle of } z = \theta \end{array}$$

The values of θ should be found from a diagram rather than a formula, although we do sometimes write $\theta = \arctan(y/x)$. Word of caution however, the domain of $\arctan x$ is restricted to $(-\pi/2, \pi/2)$. here's another useful (?) operator formula

$$\mathrm{Re}\ z = \frac{z+\bar{z}}{z}$$
$$\mathrm{Im}\ z = \frac{z-\bar{z}}{zi}$$

Conjugate. Complex numbers come in conjugate pairs; for such pairs are mirror images of each other with the x axis as the mirror. If we write $z = r(\cos\theta + i\sin\theta)$, then

$$\bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

Euler's Formula.

Using series expansion, we write $e^{i\theta}$, where θ is real

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(+\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right) \end{aligned}$$

We then have the very useful result known as Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Remembering that any complex number can be written in the form $re^{i\theta}$, we get

$$z_1 \times z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \quad \text{and} \quad z_1 \div z_2 = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Using the rules for multiplication and division of complex numbers, we have

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

for any integral n. The case $r = 1$, the equation becomes DeMoivre's theorem:

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

Function of Complex Number

. We define e^z by the power series

$$e^z = \sum_0^\infty \frac{z}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

then we can write

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

We then write Euler's formula, in θ and $-\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

These two equations can be solved for $\sin\theta$ and $\cos\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

These formulas are useful in evaluating integrals since products of exponentials are easier to integrate than products of sines and cosines.

We could also define $\sin z$ and $\cos z$ for complex z by their power series as we did for e^z, however it is simpler to use the complex equations we just obtained to define $\sin z$ and $\cos z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

The rest of the trigonometric functions of z are defined in the usual way in terms of these; for example, $\tan z = \sin z / \cos z$

Hyperbolic Function

Hyperbolic sine (abbreviated sinh) and the hyperbolic cosine (abbreviated cosh) is defined from pure imaginary, that is, $z = iy$. Their definitions for all z are

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^z}{2}$$

As before, the other hyperbolic functions are named and defined in a similar way to parallel the trigonometric functions.

We can write

$$\sin iy = i \sinh y$$
$$\cos iy = \cosh y$$

Then we see that the hyperbolic functions of y are (except for one i factor) the trigonometric functions of iy.

The functions $\sin t, \cos t$, and the rest of the gang are called "circular functions" and the functions $\sinh t, \cosh t$, etc. are called "hyperbolic functions" because $x = \cos t, y = \sin t$, satisfy the equation of a circle $x^2 + y^2 = 1$, while $x = \cosh t, y = \sinh t$, satisfy the equation of a hyperbola $x^2 - y^2 = 1$.

Logarithm

If

$$z = e^w$$

then by definition

$$w = \ln z$$

We can write the law of exponents, as

$$z_1 z_2 = e^{w_1 + w_2}$$

Taking logarithms of this equation, we get

$$\ln z_1 z_2 = w_1 + w_2 = \ln z_1 + \ln z_2$$

This is the familiar law for the logarithm of a product, justified now for complex numbers. We can then find the real and imaginary parts of the logarithm of a complex number from the equation

$$\ln z = ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$$

thus

$$\ln z = \ln r + i\theta = \ln r + i(\theta + 2n\pi)$$

Since θ has an infinite number of values (all differing by multiples of 2π), a complex number has infinitely many logarithms, differing from each other by multiples of $2\pi i$.

Complex Power. By definition, for complex a and b ($a = e$),

$$a^b = e^{b \ln a}$$

Analytic Function

Introduction to Complex Function. In general, we write

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

where it is understood that u and v are real functions of the real variables x and y.

Recall that functions are customarily single-valued. Does this mean that we cannot define a function by a formula such as $\ln z$? For each z, $\ln z$ has an infinite set of values. But if θ is allowed a range of only 2π , then $\ln z$ has one value for each z and this single-valued function is called a branch of $\ln z$. Thus in using formulas, we always discuss a single branch at a time so that we have a single-valued function. As a matter of terminology, however, you should know that the whole collection of branches is sometimes called a "multiple-valued function."

Definition. A function $f(z)$ is analytic–or regular or holomorphic or monogenic–in a region–region must be two dimensional, isolated points and curves are not regions–of the complex plane if it has a unique derivative at every point of the region.

The derivative of $f(z)$ is defined by the equation

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta x + i\Delta y}$$

The statement "$f(z)$ is analytic at a point $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$. When we say that $f(x)$ has a derivative at $x = x_0$, we mean that these two values are equal. When we say that $f(z)$ has a derivative at $z = z_0$, we mean that $f'(z)$ has the same value no matter how we approach z_0.

Theorems

Theorem I: Cauchy-Riemann conditions. If $f(z) = u(x,y) + iv(x,y)$ is analytic in a region, then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof. Remembering that $f(z) = f(x+iy)$, we use the rules of partial differentiation

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

and

$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}$$

Since $f = u(x,y) + iv(x,y)$, we also have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Combining them we have

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{df}{dz} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Since we assumed that df/dz exists and is unique (this is what analytic means), these two expressions for df/dz must be equal. Taking real and imaginary parts, we get the Cauchy-Riemann equations. ■

Theorem II. If $u(x, y)$ and $v(x, y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region–not necessarily on the boundary.

Proof (?). Although we shall not prove this, we can make it plausible by showing that it is true when we approach z_0 along any straight line. Assuming that we approach z_0 along a straight line of slope m, we will show that df/dz does not depend on m if u and v satisfy Cauchy-Riemann conditions. The equation of the straight line of slope m through the point $z_0 = x_0 + iy_0$ is

$$y - y_0 = m(x - x_0)$$

and along this line we have $dy/dx = m$. Then we find

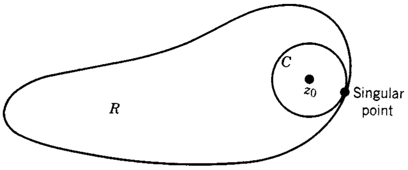
$$\begin{aligned}\frac{df}{dz} &= \frac{du + i\,dv}{dx + i\,dy} \\ &= \frac{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)}{dx + i\,dy} \\ &= \frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}m + i\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}m\right)}{1 + i\,m}\end{aligned}$$

Using the Cauchy-Riemann equations, we get

$$\begin{aligned}\frac{df}{dz} &= \frac{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}m + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}m\right)}{1 + i\,m} \\ &= \frac{\frac{\partial u}{\partial x}(1 + im) + i\frac{\partial v}{\partial y}(1 + im)}{1 + i\,m} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}\end{aligned}$$

Thus df/dz has the same value for approach along any straight line. The theorem states that it also has the same value for approach along any curve.

Theorem III. If $f(z)$ is analytic in a region R, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges inside the circle about z_0 that extends to the nearest singular point C.



No proof. Some definitions are in order. A regular point of $f(z)$ is a point at which $f(z)$ is analytic. A singular point or singularity of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an isolated singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Theorem IV. Part 1: if $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region–that is, u and v are harmonic functions.

Part 2: any function u–or v–satisfying Laplace's equation in a simply-connected region, is the real or imaginary part of an analytic function $f(z)$.

No proof. Thus we can find solutions of Laplace's equation simply by taking the real or imaginary parts of an analytic function of z. It is also often possible, starting with a simple function which satisfies Laplace's equation, to find the explicit function $f(z)$ of which it is, say, the real part.

Theorem V: Cauchy's theorem. Let C be a simple–curve which does not cross itself–closed curve with a continuously turning tangent except possibly at a finite number of points–that is, we allow a finite number of corners, but otherwise the curve must be smooth. If $f(z)$ is analytic on and inside C, then

$$\oint_C f(z)\, dz = 0$$

Proof. We shall prove Cauchy's theorem assuming that $f'(z)$ is continuous.

$$\oint_C f(z)\, dz = \oint_C (u + iv)(dx + i\, dy)$$

or

$$\oint_C f(z)\, dz = \oint_C (u\, dx - v\, dy) + i \oint_C (v\, dx + u\, dy)$$

Green's theorem in the plane says that if $P(x, y)$, $Q(x, y)$, and their partial derivatives are continuous in a simply-connected region R, then

$$\oint_C P\, dx + Q\, dy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where C is a simple closed curve lying entirely in R and A is area inside C. Applying Green's Theorem to the first integral, we get

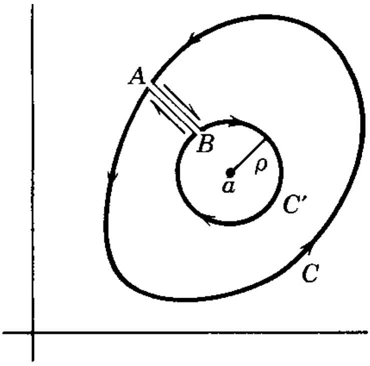
$$\oint_C (u\ dx - v\ dy) = \int_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy = \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy = 0$$

where we have used the Cauchy-Riemann. In the same way the second integral in is zero

$$i \oint_C (v\ dx + u\ dy) = i \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\ dy = i \int_A \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\ dy = 0$$

Theorem VI: Cauchy's integral formula. If $f(z)$ is analytic on and inside a simple closed curve C, the value of $f(z)$ at a point $z = a$ inside C is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - a}\, dz$$



Proof. Let a be a fixed point inside the simple closed curve C and consider the function

$$\phi(z) = \frac{f(z)}{z - a}$$

where $f(z)$ is analytic on and inside C. Let C' be a small circle (inside C) with center at a and radius ρ. Thus we have

$$\oint_{C\circlearrowleft} \phi(z)\ dz + \oint_{C'\circlearrowright} \phi(z)\ dz = 0$$

and

$$\oint_{C\circlearrowleft} \phi(z)\ dz = \oint_{C'\circlearrowleft} \phi(z)\ dz$$

where both are counterclockwise. Along the circle C',

$$z = a + \rho e^{i\theta}$$
$$dz = \rho i e^{i\theta}\ d\theta$$

and the integral becomes

$$\oint_{C'\circlearrowleft} \phi(z)\ dz = \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}}\, \rho i e^{i\theta}\ d\theta = \int_0^{2\pi} f(z) i\ d\theta$$

Since our calculation is valid for any (sufficiently small) value of ρ, we shall let $\rho \to 0$ (that is, $z \to a$) to simplify the formula

$$\oint_{C'\circlearrowleft} \phi(z)\, dz = \int_0^{2\pi} f(a) i \, d\theta = 2\pi i f(a)$$

thus

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz \quad \blacksquare$$

Theorem VII: Laurent's theorem. Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots$$

convergent in R. Such a series is called a Laurent series. The b series is called the principal part of the Laurent series.

Definition. If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a regular point.

If $b_n \neq 0$, but all the b's after b_n are zero, $f(z)$ is said to have a pole of order n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a simple pole.

If there are an infinite number of b's different from zero, $f(z)$ has an essential singularity at $z = z_0$.

The coefficient b_1 of $1/(z-z_0)$ is called the residue of $f(z)$ at $z = z_0$.

Residue Theorem

Suppose we are going to find the value of $\oint f(z)\, dz$ around a simple closed curve C surrounding an isolated singular point z_0 but inclosing no other singularities. Let $f(z)$ be expanded in the Laurent series about $z = z_0$. The integral of the a series

$$\oint_C a_n(z - z0)^n \, dz = 0$$

since this pat is analytic. The integral of the b series, we replace the integrals around C by integrals around a circle C' with center at z_0 and radius ρ

$$\oint_{C'} \frac{b_n}{(z-z_0)^n}\, dz = i \int_0^{2\pi} b_n e^{i\theta(1-n)} d\theta = \frac{b_n}{1-n}\left(e^{2\pi i(1-n)} - 1\right) = 0$$

for all $n > 1$. For $n = 1$

$$\oint_{C'} \frac{b_1}{(z-z_0)}\, dz = i b_1 \int_0^{2\pi} d\theta = 2\pi i b_1$$

Then

$$\oint_{C'} f(z)\, dz = 2\pi i b_1$$

since b_1 is called the residue of $f(z)$ at $z = z_0$, we can say

$$\oint_c f(z)\, dz = 2\pi i \times \text{residue of } f(z) \text{ at the singular point inside } C$$

The only term of the Laurent series which has survived the integration process is the b_1 term; you can see the reason for the term "residue." Thus we have the residue theorem:

$$\oint_C f(z)\, dz = 2\pi i\; R(z)$$

where $R(z)$ sum of the residues of $f(z)$ inside C.

Method of Finding Residue

Laurent Series. If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid near z_0, then the residue is just the coefficient b_1 of the term $1/(z - z0)$. **Caution**: Be sure you have the expansion about $z = z_0$.

Simple Pole. If $f(z)$ has a simple pole at $z = z_0$, we find the residue by multiplying $f(z)$ by $(z - z_0)$ and evaluating the result at $z = z_0$. In general we write

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \bigg|_{z=z_0}$$

when z_0 is a simple pole. If, $f(z)$ can be written as $g(z)/h(z)$, where $g(z)$ is analytic and not zero at z_0 and $h(z_0) = 0$, then

$$R(z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = g(z_0) \lim_{z \to z_0} \frac{(z - z_0)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$$

In another words

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \begin{cases} (z) = g(z)/h(z), \\ g(z_0) = C \neq 0, \\ h(z_0) = 0, \\ h'(z_0) \neq 0 \end{cases}$$

Perhaps the simplest way–without finding the Laurent series–to find if a function has a simple pole is that if the limit obtained is some constant not 0 or ∞, then $f(z)$ does have a simple pole and the constant is the residue. If the limit is equal to 0, the function is analytic and the residue is 0. If the limit is infinite, the pole is of higher order.

Suppose $f(z)$ is written in the form $g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic. Then you can think of $g(z)$ and $h(z)$ as power series in $(z - z_0)$. If the denominator has the factor $(z - z_0)$ to one higher power than the numerator, then $f(z)$ has a simple pole at z_0.

Multiple Poles. When $f(z)$ has a pole of order n, we can use the following method of finding residues.

$$R(z_0) = \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} \left((z - z_0)^m f(z)\right) \bigg|_{z=z_0}$$

where m is an integer greater than or equal to the order n of the pole. To compute the derivative quickly, use Leibniz' rule for differentiating a product

$$\left(\frac{d}{dx}\right)^n fg = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx}\right)^{n-k} f \left(\frac{d}{dx}\right)^k g$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Appendix I: DeMoivre's theorem

Example 1.

$$\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)^{25}=(e^{i\pi/10})^{25}=e^{2\pi i}e^{i\pi/2}=1\times i=i$$

Example 2. Find the cube roots of 8. We write the cube roots of 8 in polar form

$$z^{1/3}=8^{1/3}e^{i(0+2n\pi)/3}$$

with $z=8e^{i\cdot(0+2n\pi)}$. These values are

$$\sqrt[3]{8}=2,\quad 2e^{i2\pi/3},\quad 2e^{i4\pi/3},\quad 2e^{i2\pi},\ldots$$

or in rectangular coordinate

$$2,\quad -1+i\sqrt{3},\quad -1-\sqrt{3}$$

Example 3. Find and plot all values of $\sqrt[4]{-64}$. As, before

$$z^{1/4}=64^{1/4}e^{i(\pi+2n\pi)/4}$$

These values are

$$\sqrt[4]{-64}=2\sqrt{2},\quad 2\sqrt{2}e^{i3\pi/4},\quad 2\sqrt{2}e^{i3\pi/4},\quad 2\sqrt{2}e^{i5\pi/4},\quad 2\sqrt{2}e^{i7\pi/4},\ldots$$

or in rectangular coordinate

$$\sqrt[4]{-64}=\pm2\pm2i$$

Example 4. Find and plot all values of $\sqrt[6]{-8i}$. We have uour complex number

$$z=8e^{i(1.5\pi+2n\pi)}$$

raised to the power of 1/6

$$z^{1/6}=\sqrt[6]{8}e^{i(1.5\pi+2n\pi)/6}=\sqrt{2}e^{i(\pi/4+n\pi/3)}$$

We can do this one root at a time or more simply by using a computer to solve the equation $z^6=-8i$.

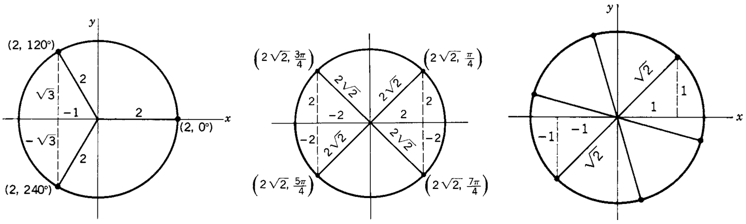


Figure: sketch of example 2, 3 and 4

Appendix II: Inverse Trigonometric

Ex. 1: Find z that satisfy $z = \arccos 2$ Since $\cos z = 2$, we have

$$\frac{e^{iz} + e^{-iz}}{2} = 2$$

To simplify the algebra, let $u = e^{iz}$ and $u^{-1} = e^{-iz}$

$$u + u^{-1} = 4$$

Multiply by u, then we get quadratic equation

$$u^2 - 4u + 1 = 0$$

Solving this equation

$$e^{iz} = 2 \pm \sqrt{3}$$

Take logarithms of both sides of this equation, and solve for z:

$$iz = \ln(2 \pm \sqrt{3}) + 2n\pi i$$
$$z = 2n\pi - i\ln(2 \pm \sqrt{3})$$

Ex. 2: Evaluating Integral. In integral tables or from your computer you may find for the indefinite integral

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^1 \frac{x}{a}$$

or

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

How are these related? Put $z = \sinh^1 x/a$, thus

$$\sinh z = \frac{x}{a} = \frac{e^z - e^{-z}}{2}$$

Let $e^z = u, e^{-z} = 1/u$. Then

$$au^2 - 2xu - a = 0$$

We solve for u, or rather z, as in the previous example

$$e^z = \frac{x \pm \sqrt{x^2 - a^2}}{a}$$

For real integrals, that is, for real z, $e^z > 0$, so we must use the positive sign. Then, taking the logarithm

$$z = \ln\left(x + \sqrt{x^2 + a^2}\right) - \ln a$$

We see that the two answers differ only by the constant $\ln a$, which is a constant of integration.

Appendix III: Laplace's equation

Consider the function $u(x, y) = x^2 - y^2$. We find that

$$\nabla^2 u = 2 - 2 = 0$$

that is, u satisfies Laplace's equation (or u is a harmonic function). Let us find the function $v(x, y)$ such that $u + iv$ is an analytic function of z. By the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$$

Integrating partially with respect to y, we get

$$v(x, y) = 2xy + g(x)$$

where $g(x)$ is a function of x to be found. Unlike our usual integration, the constant we get from partial integration not a simple constant, but rather a function of x. Differentiating partially with respect to x and again using the Cauchy-Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
$$2y + g'(x) = 2y$$

Thus we find

$$g'(x) = 0 \quad \text{and} \quad g(x) = C$$

Then

$$f(z) = x^2 - y^2 + 2ixy + C = z^2 + C$$

The pair of functions u, v are called conjugate harmonic functions.

Appendix IV: Residue Theorem

Ex. 1 Find

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

If we make the change of variable of

$$z = e^{i\theta}$$

then

$$dz = ie^{i\theta} d\theta$$
$$\cos\theta = \frac{z + 1/z}{2}$$

As θ goes from 0 to 2π, z traverses the unit circle $|z| = 1$. Making these substitutions in I, we get

$$I = \oint_C = \frac{dz/iz}{5 + 2(z + 1/z)} = \frac{1}{i}\oint_C \frac{dz}{(2z + 1)(z + 2)}$$

where C is the unit circle. The integrand has poles at $z = -1/2$ and $z = -2$. Since only $z = -1/2$ is inside the contour C, the residue of the integrand is

$$R(-1/2) = \lim_{z \to -\frac{1}{2}} (z - \frac{1}{2}) \frac{1}{(2z+1)(z+2)} \bigg|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Then by the residue theorem

$$I = \frac{1}{i} 2\pi i R(-1/2) = \frac{2\pi}{3}$$

Ex. 2 Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Altough we could evaluate I by elementary method, we will use the residue theorem by considering

$$\oint_C \frac{dz}{1+z^2}$$

where C is the closed boundary of the semicircle with radius $\rho > 1$. C incloses the singular point $z = i$ and no others, thus

$$R(i) = \lim_{z \to i} (z - i) \frac{1}{1+z^2} \bigg|_{z=i} = \frac{1}{2i}$$

Then the value of the contour integral is

$$I = 2\pi i R(i) = \pi$$

Let us write the integral in two parts: (1) an integral along the x axis, for this part $z = x$; (2) an integral along the semicircle where $z = \rho e^{i\theta}$

$$\oint_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}}{1+\rho^2 e^{2i\theta}} d\theta = \pi$$

Let $\rho \to \infty$; then the second integral tends to zero since the numerator contains ρ and the denominator ρ^2. We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

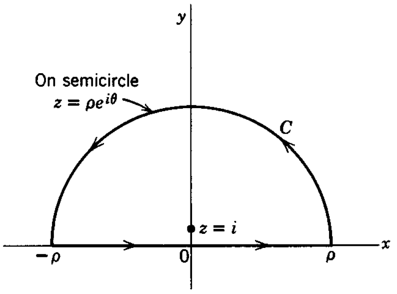


Figure: Semicircle radius ρ

Ex. 3 Evaluate

$$I = \int_0^\infty \frac{\cos x}{1+x^2}dx$$

We consider the contour integral

$$\oint_C \frac{e^{iz}}{1+z^2}dz$$

where C is the same semicircular contour as before. The singular point inclosed is again $z = i$, and the residue there is

$$R(i) = \lim_{z\to i}(z-i)\frac{e^{iz}}{1+z^2}\bigg|_{z=i} = \frac{1}{2ei}$$

The value of the contour integral is then πe. As in before, we write the contour integral as a sum of two integrals

$$\oint_C \frac{e^{iz}}{1+z^2}dz = \int_{-\rho}^{\rho} \frac{e^{ix}}{1+x^2}dx + \int_0^\pi \frac{e^{iz}}{1+z^2}dz = \frac{\pi}{e}$$

The integral along the semicircle tends to zero as the radius $\rho \to \infty$. We have then

$$\int_{-\infty}^\infty \frac{e^{ix}}{1+x^2}dx = \frac{\pi}{e}$$

Taking the real part of both sides of this equation

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$$

Since the integrand is an even function, we have

$$\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$$

Ex. 4 Evaluate

$$I = \int_{-\infty}^\infty \frac{\sin x}{x}dx$$

Here we consider

$$I = \oint \frac{e^{iz}}{z}dz$$

To avoid the singular point at $z = 0$, we integrate around the contour where it is on the straight line boundary. We then let the radius r shrink to zero so that in effect we are integrating straight through the simple pole at the origin. Since the pole is on the straight line boundary, its contribution is just halfway between zero and $2\pi i\cdot$ residue. Observing that, the integral along the large semicircle tends to zero as R tends to infinity,

$$I = \int_{-\infty}^\infty \frac{e^{ix}}{x}dx = 2\pi i \cdot \frac{1}{2}R(0) = i\pi$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^\infty \frac{\sin x}{x}dx = \pi$$

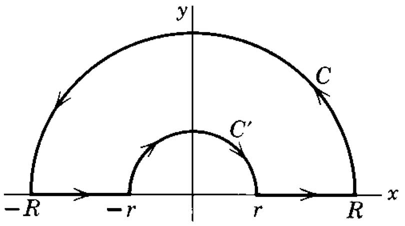


Figure: contour of example 4

Linear Algebra

Introduction

Elementary row operation. Rule as follows.

1. Interchange two rows
2. Multiply (or divide) a row by a (nonzero) constant
3. Add a multiple of one row to another

Rank. Definition as follows.

> The number of nonzero rows remaining when a matrix has been row reduced is called the rank of the matrix.

Or:

> The order of the largest nonzero determinant is the rank of the matrix.

Consider m equation with n constants. We define matrix M and A, where M has m rows and n column–which corresponds to m equation and n unknown–while A has m rows and $n+1$–which corresponds to unknown plus the constant. There are few possible cases.

1. If (rank M) $<$ (rank A), the equations are inconsistent and there is no solution.
2. If (rank M) $=$ (rank A) $= n$ (number of unknowns), there is one solution.
3. If (rank M) $=$ (rank A) $= R < n$, then R unknowns can be found in terms of the remaining $n - R$ unknowns.

Cramer's Rule. The equations

$$\begin{cases} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{cases}$$

has the solution:

$$x = \frac{1}{D}\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \text{and} \qquad y = \frac{1}{D}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Dot Product.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}||\mathbf{B}|\cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned}$$

The following applies if vector perpendicular:

$$\mathbf{A} \cdot \mathbf{B} = 0$$

The following applies if vector parallel:

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

Orthogonality. Matrix, in context of linear transformation, that preserve the length of vector is said to be orthogonal. Matrix M is orthogonal if

$$M^{-1} = M^T$$

with determinant

$$\det M = \pm 1$$

det M $= 1$ corresponds geometrically to a rotation, and det M $= -1$ means that a reflection is involved.

Cross Product.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= |\mathbf{A}||\mathbf{B}| \sin\theta \\ &= \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}$$

The following applies if vector parallel or antiparallel:

$$\mathbf{A} \cdot \mathbf{B} = 0$$

Homogeneous equations. The definition is as follows.

> Sets of linear equations when the constants on the right-hand sides are all zero are called homogeneous equations.

Homogeneous equations are never inconsistent; they always have the solution of zero–often called the trivial solution. If the number of independent equations–that is, the rank of the matrix–is the same as the number of unknowns, this is the only solution. If the rank of the matrix is less than the number of unknowns, there are infinitely many solutions.

Consider set of n homogeneous equations in n unknowns. These equations have only the trivial solution unless the rank of the matrix is less than n. This means that at least one row of the row reduced n by n matrix of the coefficients is a zero row. Which mean that the determinant D of the coefficients is zero. This fact will be used in eigenvalue problem.

> A system of n homogeneous equations in n unknowns has solutions other than the trivial solution if and only if the determinant of the coefficients is zero.

Vector in Braket Notation

Vector space. Linear vector space $\mathbb{V}$ is a collection of vectors $|1\rangle, \ldots, |n\rangle$ for which there exists definitive rule for addition and multiplication. Said rules are as follows.

1. **Closure:** $|V\rangle + |W\rangle \in \mathbb{V}$
2. **Distributive in the vector**: $a\left(|V\rangle + |W\rangle\right) = a\,|V\rangle + a\,|W\rangle$

3. **Distributive in the scalar**: $(a+b)\left|V\right\rangle = a\left|V\right\rangle + b\left|V\right\rangle$

4. **Associative in the scalar**: $a(b\left|V\right\rangle = ab\left|V\right\rangle)$

5. **Commutative in the addition**: $\left|V\right\rangle + \left|W\right\rangle + \left|W\right\rangle + \left|V\right\rangle$

6. **Associative in the addition**: As follows.

$$\left|V\right\rangle + (\left|W\right\rangle + \left|P\right\rangle) = (\left|V\right\rangle + \left|W\right\rangle) + \left|P\right\rangle$$

7. **Null vector**: $\left|V\right\rangle + \left|0\right\rangle + \left|V\right\rangle$

8. **Inverse under addition**: $\left|-V\right\rangle + \left|V\right\rangle = \left|0\right\rangle$

Vector space has n dimension if it can accommodate n linear independent vectors. We denote $\mathbb{V}^n(R)$ if the field–that is the scalar used to scale the vector–and $\mathbb{V}^n(C)$ if it is complex.

Dual space. Column vectors are concrete manifestations of an abstract vector $\left|V\right\rangle$ ket in a basis, while row vector are bra's $\left\langle V\right|$. They are adjoint of each other. Thus, there are two vector space: space of KET $\left|V\right\rangle$ and dual space of bra $\left\langle V\right|$.

Vector expansion in an orthonormal base. Suppose we are to expand vector $\left|V\right\rangle$ in an orthonormal base. First, we take the dot product of said vector with an orthonormal base $\left\langle j\right|$

$$\langle j|V\rangle = \sum_i v_i \langle j|i\rangle = v_j$$

and obtain the j-th component of the vector. Using this, we then write

$$\left|V\right\rangle = \sum_i \left|i\right\rangle \langle i|V\rangle$$

Lines and Plane

Suppose we have vector $\mathbf{A} = a\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}} + c\,\hat{\mathbf{z}}$ and vector $\mathbf{r} - \mathbf{r_0} = (x - x_0)\,\hat{\mathbf{x}} + (y - y_0)\,\hat{\mathbf{y}} + (z - z_0)\,\hat{\mathbf{z}}$, which parallel to $\mathbf{A}$. We can write:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which is the symmetric equations of a straight line. Note that $\mathbf{r}$ and $\mathbf{r_0}$ is not necessarily parallel with $\mathbf{A}$, but $\mathbf{r} - \mathbf{r_0}$ do. The parameter equation is:

$$\begin{aligned} \mathbf{r} - \mathbf{r_0} &= \mathbf{A}t \\ \mathbf{r} &= \mathbf{r_0} + \mathbf{A}t \end{aligned}$$

The previous equation is obtained by the dot identity of parallel vector

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

also called the equation of plane.

Matrix Operation

Multiplication. Matrix AB can be multiplied if they are conformable, that is if column A = row B. Matrix multiplication in index notation is:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

where i denote row and j denote column. For 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Commutator. In general, matrix do not commute. We define the commutator of the matrices A and B by

$$[A, B] = AB - BA$$

Two identity involving commutators are

$$[\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta$$
$$[\Lambda\Omega, \Theta] = \Lambda[\Omega, \theta] + [\Lambda, \theta]\Omega$$

Inverse. If a matrix has an inverse we say that it is invertible; if it doesn't have an inverse, it is called singular.

$$M^{-1} = \frac{1}{\det M} C^T$$

where C_{ij} is cofactor of m_{ij} or the checker thing you use on determining determinant. The inverse of a product follows

$$(\Lambda\Omega)^{-1} = \Lambda^{-1}\Omega^{-1}$$

By thin we can obtain the desired result

$$(\Omega\Lambda)(\Omega\Lambda)^{-1} = \Omega\Lambda\Lambda^{-1}\Omega^{-1} = \Omega\Omega^{-1} = I$$

Proof. Consider 3×3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

The row of M is thought the element of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ vector instead of its column

$$\mathbf{A} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

and so on. We define next the reciprocal triplet vector

$$\mathbf{A}_R = \mathbf{B} \times \mathbf{C}$$
$$\mathbf{B}_R = \mathbf{C} \times \mathbf{A}$$

$$\mathbf{C}_R = \mathbf{A} \times \mathbf{B}$$

which have the relation with the original vector

$$\mathbf{A} \cdot \mathbf{A}_R \neq 0, \qquad \mathbf{A} \cdot \mathbf{B}_R = \mathbf{A} \cdot \mathbf{C}_R = 0$$

and so on. From the triplet vector, we construct the cofactor transpose matrix of M

$$\bar{M} = \begin{bmatrix} a_{R1} & b_{R1} & c_{R1} \\ a_{R2} & b_{R2} & c_{R2} \\ a_{R3} & b_{R3} & c_{R3} \end{bmatrix}$$

Then

$$\begin{aligned} M \cdot \bar{M} &= \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_{R1} & b_{R1} & c_{R1} \\ a_{R2} & b_{R2} & c_{R2} \\ a_{R3} & b_{R3} & c_{R3} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} \cdot \mathbf{A}_R & \mathbf{A} \cdot \mathbf{B}_R & \mathbf{A} \cdot \mathbf{C}_R \\ \mathbf{B} \cdot \mathbf{A}_R & \mathbf{B} \cdot \mathbf{B}_R & \mathbf{B} \cdot \mathbf{C}_R \\ \mathbf{C} \cdot \mathbf{A}_R & \mathbf{C} \cdot \mathbf{B}_R & \mathbf{C} \cdot \mathbf{C}_R \end{bmatrix} \\ M \cdot \bar{M} &= \begin{bmatrix} \mathbf{A} \cdot \mathbf{A}_R & & \\ & \mathbf{B} \cdot \mathbf{B}_R & \\ & & \mathbf{C} \cdot \mathbf{C}_R \end{bmatrix} \end{aligned}$$

All three diagonal elements are equal

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A}_R \quad = \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \quad &= \quad \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \quad = \quad \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) \\ &= \quad \mathbf{B} \cdot \mathbf{B}_R \qquad\qquad = \quad \mathbf{C} \cdot \mathbf{C}_R \end{aligned}$$

We can write the product of each element as

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det M$$

Therefore, we write

$$M \cdot \bar{M} = \det M I$$

$$M^{-1} = \frac{\bar{M}}{\det M}$$

Derivative (with respect to parameter). Consider the function $\theta(\lambda) = e^{\lambda\Omega}$. Its derivative with respect to parameter λ is evaluated in the usual sense

$$\frac{d}{d\lambda}\theta(\lambda) = \Omega e^{\lambda\Omega} = e^{\lambda\Omega}\Omega = \theta(\lambda)\Omega$$

The second and third terms holds true because $[\Omega, e^{\Omega}] = 0$. If we are presented with the differential equation with this form, the solution is given by

$$\theta(\lambda) = ce^{\Omega\lambda}$$

with c as operator constant of integration.

Proof. We can proof the evaluated method by writing the exponential function as an operator then applying the derivative operator

$$\frac{d}{d\lambda}e^{\lambda\Omega} = \frac{d}{d\lambda}\begin{bmatrix} e^{\lambda\omega_1} & & \\ & \ddots & \\ & & e^{\lambda\omega_n} \end{bmatrix} = \begin{bmatrix} \omega_1 e^{\lambda\omega_1} & & \\ & \ddots & \\ & & \omega_n e^{\lambda\omega_n} \end{bmatrix}$$

$$\frac{d}{d\lambda}e^{\lambda\Omega} = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix}\begin{bmatrix} \lambda e^{\lambda\omega_1} & & \\ & \ddots & \\ & & \lambda e^{\lambda\omega_n} \end{bmatrix} = \Omega e^{\lambda\Omega}$$

If Ω is not a hermitian however, we can produce the same result by working using the power series representation

$$\frac{d}{d\lambda}e^{\lambda\Omega} = \frac{d}{d\lambda}\sum_{m=0}^{\infty}\frac{\lambda^n\Omega^n}{n!} = \sum_{m=1}^{\infty}\frac{n\lambda^{n-1}\Omega^n}{n!} = \Omega\sum_{m=1}^{\infty}\frac{\lambda^{n-1}\Omega^{n-1}}{(n-1)!}$$

Shifting the index

$$\frac{d}{d\lambda}e^{\lambda\Omega} = \Omega\sum_{m=0}^{\infty}\frac{\lambda^n\Omega^n}{n!} = \Omega e^{\lambda\Omega}$$

Element representation. The i-th row and j-th column of a matrix can be represented as

$$\langle i|\, A\, |j\rangle = A_{ij}$$

As an example, consider 2×2 matrix A and simple kets

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

All of its element can be represented as

$$\langle 1|\, A\, |1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = A_{11}$$

$$\langle 1|\, A\, |2\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = A_{12}$$

$$\langle 2|\, A\, |1\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = A_{21}$$

$$\langle 2|\, A\, |2\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = A_{22}$$

Determinant

For 2×2 matrix:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Here are some determinant rule.

$$\det(kA) = k^2 \det A \qquad (2 \times 2)$$

$$\det(kA) = k^3 \det A \qquad (3 \times 3)$$

$$\det(AB) = \det(BA) = \det(A) \times \det(B)$$

Minor. Minor of element a_{ij} is the determinant of submatrix order $(n-1)$ you get after crossing i-th row and h-th column from order n matrix. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each minor is then

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$
$$M_{21} = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad M_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix},$$
$$M_{31} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \quad M_{33} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Cofactor Cofactor is expressed is the minor that includes sign factors

$$\mathrm{C}_{ij} = (-1)^{i+j} M_{ij}$$

As a mnemonic, use the sign pattern to check the sign of the cofactor

$$(-1)^{i+j} \sim \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Special Matrices and Operator

Theorem.

1. $(ABC)^T = C^T B^T A^T$
2. $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$
3. $Tr(ABC) = Tr(BCA) = Tr(CAB)$. Trace is the sum of main diagonal. It is a theorem that the trace of a product of matrices is not changed by permuting them in cyclic order.
4. If H is a Hermitian matrix, then $U = e^{iH}$ is a unitary matrix.

Table of special matrices. Consider this.

Identity matrix. A matrix, who also acts as an operator, which leave the operated vector unchanged

$$I \left| V \right\rangle = \left| V \right\rangle \qquad \left\langle V \right| I = \left\langle V \right|$$

Its element may be written as

$$I_{ij} = \langle i | I | k \rangle = \langle i | j \rangle = \delta_{ij}$$

or using the Kronecker delta. Then $n \times n$, using the Kronecker delta, identity matrix may be written

$$I = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1n} \\ \vdots & \ddots & \vdots \\ \delta_{n1} & \cdots & \delta_{nn} \end{bmatrix}$$

Definition	Condition
Real	$A = \bar{A}$
Symmetric	$A = A^T$
Antisymmetric	$A = -A^T$
Orthogonal	$A^{-1} = A^T$
Pure Imaginary	$A = -\bar{A}$
Hermitian	$A = A^\dagger$
Antihermitian	$A = -A^\dagger$
Unitary	$A^{-1} = A^\dagger$
Normal	$AA^\dagger = A^\dagger A$

Projection Operator. The projection operator is defined as

$$\mathbb{P}_i = |i\rangle \langle i|$$

This operator can be used to write the expansion of vector

$$|V\rangle = \sum_i |i\rangle \langle i|V\rangle = \sum_i \mathbb{P}_i |V\rangle$$

Or the identity matrix, which is called the completeness relation

$$I = \sum_i |i\rangle \langle i| = \int_i |x\rangle \langle i| \; dx = \sum_i \mathbb{P}_i$$

Its action on bra is all the same

$$\langle V| \mathbb{P}_i = \langle V|i\rangle \langle i| = v_i^* \langle i|$$

Projection operator obey

$$\mathbb{P}_i \mathbb{P}_j = \delta_{ij} \mathbb{P}_j$$

Hermitian and anti Hermitian. An operator is called Hermitian fit satisfies $\Omega = \Omega^\dagger$, in other hand an operator is called anti-Hermitian if $\Omega = -\Omega^\dagger$. In the world of linear algebra, Hermitian and anti-Hermitian play the role of pure real and pure imaginary number. Just like how we can decompose every number into a sum of pure real and pure imaginary

$$\begin{aligned}
\alpha &= \frac{\alpha + \alpha^*}{2} + \frac{\alpha - \alpha^*}{2} \\
&= \frac{(a+ib) + (a-ib)}{2} + \frac{(a+ib) - (a-ib)}{2} \\
\alpha &= a + ib
\end{aligned}$$

we can decompose every operator into its Hermitian and anti Hermitian

$$\Omega = \frac{\Omega + \Omega^\dagger}{2} + \frac{\Omega - \Omega^\dagger}{2}$$

Transformation matrix. Matrix that rotate vector $\vec{r} = (x, y)$ into $\vec{R} = (X, Y)$ (in 2D)is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the one that rotate its axis instead

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

in 3D

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$
$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unitary operator. Unitary operator is defined as

$$UU^\dagger = I$$

This definition also implies that U and its Hermitian conjugate are inverse of each. As a comparison with complex number, Unitary operator is a unit modulus $e^{i\theta}$, just as $e^{i\theta}e^{-i\theta} = 1$.

Inner product. This theorem state that unitary operator preserves the inner product between the vectors they act on. This can be proved by considering two vectors

$$|V_1\rangle = U\,|W_1\rangle \qquad |V_2\rangle = U\,|W_2\rangle$$

Then

$$\langle V_2|V_1\rangle = \langle W_2|\,U^\dagger U\,|W_1\rangle = \langle W_2|W_2\rangle$$

Unitary operator is the generalization of rotation operator from $\mathbb{V}^3(R)$ to $\mathbb{V}^n(C)$, for its preserves the inner product of the vector like the rotation matrix.

For $n \times n$ unitary operator, the column, or the row really, can be seen as the component of n vector, just like operator in general. These vector, then, is orthonormal to each other. The reason for this is that the operator preserve the inner product, the transformed set of vector is also orthonormal.

Dirac delta. Under the integral sign, Dirac delta is defined as

$$\int \delta(x - x')f(x')\,dx = f(x)$$

where it samples the value of function $f(x')$ at one point x. This expression also used to define continuous orthogonal basis

$$\langle x|x'\rangle = \delta(x - x')$$

Dirac delta is an even function proved by

$$\delta(x - x') = \langle x|x'\rangle = \langle x'|x\rangle^* = \delta(x' - x)^* = \delta(x' - x)$$

Dirac delta is normalized such that

$$\int_{-\infty}^{\infty} \delta(x - x')\, dx = 1$$

Dirac delta is also defined by the limit of a Gaussian function

$$\delta(x - x') = \lim_{\epsilon \to 0} \frac{1}{\epsilon\sqrt{\pi}} \exp\left[-\frac{(x - x')^2}{\epsilon^2}\right]$$

The derivative is defined as following

$$\delta'(x - x') = \frac{d}{dx}\delta(x - x') = -\frac{d}{dx}\delta(x' - x)$$

The action of this operator is

$$\int \delta'(x - x')f(x')\, dx = \frac{d}{dx}f(x)$$

which can be proved by

$$\int \delta'(x - x')f(x')\, dx = \frac{d}{dx}\int \delta(x - x')f(x')\, dx = \frac{d}{dx}f(x)$$

In general, the action of n-th order of derivative is

$$\delta^{(n)}(x - x') = \delta(x - x')\frac{d^n}{dx^n}$$

The integral representation of Dirac delta is

$$\delta(x' - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x' - x)}\, dk$$

This is obtained from Fourier transformation of given function

$$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

and its inverse

$$f(x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'} f(k)\, dk$$

Substituting the transformation into the inverse

$$f(x') = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x' - x)\, dk}\right) f(x)\, dx$$

The term inside parenthesis is the Dirac delta function.

Differential operator. The action of this operator is described by the following equation

$$\langle x|D|f\rangle = \left\langle x\middle|\frac{df}{dx}\right\rangle = \frac{df(x)}{dx}$$

The operator also can act as the differential Dirac delta function under different basis

$$D_{xx'} = \langle x|D|x'\rangle = \delta'(x - x') = \delta(x - x')\frac{d}{dx'}$$

This can be seen by inserting the completion identity

$$\int \langle x|D|x'\rangle \langle x'|f\rangle \; dx' = \int D_{xx'} f(x') dx' = \frac{df(x)}{dx}$$

and comparing it with the action of the derivative of the Dirac delta

$$\int \delta'(x-x') f(x') \; dx = \frac{d}{dx} f(x)$$

The differential operator is not a Hermitian. For D to be a Hermitian, it must satisfy

$$D_{xx'} = D^{\dagger}_{x'x} \qquad \text{or} \qquad \langle x|D|x'\rangle = (\langle x'|D|x\rangle)^{\dagger}$$

The proof is as following

$$\langle x|D|x'\rangle = \delta'(x-x')$$

and

$$(\langle x'|D|x\rangle)^{\dagger} = \delta'(x'-x)^{*} = \delta'(x'-x) - \delta'(x-x')$$

Wave number operator. On the x basis, K acts as differential operator. This is misleading, actually. The correct term is, if K acts on a state $|\psi\rangle$, and then we project onto the position basis $|x\rangle$, the resulting expression is given by a differential operator acting on the projected state

$$\langle x|K|\psi\rangle = -i\frac{d}{dx}\psi(x)$$

On the other hand, K acts multiplicatively at the k basis

$$K\,|k\rangle = k\,|k\rangle$$

since $|k\rangle$ is an eigenstate of K.

Just like the differential operator, the wave number operator maybe expressed in terms of Dirac delta function

$$K_{xx'} = \langle x|K|x'\rangle = -i\delta'(x-x')$$

which is obtained the following expression with the definition of the Dirac delta function

$$\int \langle x|K|x'\rangle \langle x'|f\rangle \; dx' = \int K_{xx'} f(x') dx' = -i\frac{df(x)}{dx}$$

Let $|f\rangle$ and $|g\rangle$ be two kets in function space within $[a,b]$ Suppose we transform $|f\rangle$ by K and project it onto $\langle g|$. In this basis, K is a Hermitian if the surface term vanishes

$$-ig^{*}(x)f(x)\Big|_{a}^{b} = 0$$

The expression is obtained by considering the Hermitian requirement

$$\langle g|K|f\rangle = (\langle f|K|g\rangle)^{\dagger}$$

For K to be a Hermitian, it must also obey

$$\int_a^b \int_a^b \langle g|x\rangle \langle x|K|x'\rangle \langle x'|f\rangle \; dx\, dx' =$$
$$\left(\int_a^b \int_a^b \langle f|x\rangle \langle x|K|x'\rangle \langle x'|g\rangle \; dx\, dx' \right)^{\dagger}$$

Rewriting the left-hand side and using integration by part

$$\begin{aligned}
\int_a^b \int_a^b g^*(x) K_{xx'} f(x')\, dx\, dx' &= \int_a^b g^*(x) \left[-i \frac{d}{dx} f(x) \right] dx \\
&= i \int_a^b f(x) \frac{d}{dx} g^*(x)\, dx - i g^*(x) f(x) \Big|_a^b
\end{aligned}$$

We are then equating it with the right-hand side, which we write as

$$\begin{aligned}
\left[\int_a^b \int_a^b f^*(x) K_{xx'} f(x')\, dx\, dx' \right]^{\dagger} &= \left[\int_a^b \int_a^b f^*(x) \left(-i \frac{d}{dx} g(x) \right) \; dx \right]^{\dagger} \\
&= i \int_a^b f(x) \frac{d}{dx} g^*(x)\, dx
\end{aligned}$$

Clearly, both side are equal if the surface term is zero.

The basis of k has the form of

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

where $\psi_k(x) \equiv \langle k|\psi\rangle$ position representation of the abstract momentum eigenbasis $|k\rangle$. This expression can be derived by considering the act of K on the k basis and project it on x basis

$$\begin{aligned}
\langle x|K|k\rangle &= k \langle x|k\rangle \\
\int \langle x|K|x'\rangle \langle x'|k\rangle \; dx' k &= \int K_{xx'} k(x')\, dx' = \psi_k(x) \\
i \frac{d}{dx} \psi_k(x) &= k \psi_k(x)
\end{aligned}$$

This differential equation is solved by simple integration

$$\begin{aligned}
\int \frac{1}{\psi_k(x)} \; d\psi_k(x) &= \int ik\, dx \\
\ln \psi_k(x) &= ik + A \\
\psi_k(x) &= A e^{ik}
\end{aligned}$$

For continuous function, normalize any function into Dirac delta function. So

$$\int_{-\infty}^{\infty} \psi_k^*(x) \psi_k'(x)\, dx = \delta(k - k')$$

The integral gives

$$\int_{-\infty}^{\infty} A^2 e^{i(k-k')x}\, dx = A^2 \pi \delta(k - k')$$

So the value of A that normalize $\psi_k(x)$ is

$$A = \frac{1}{2\pi} \qquad \text{so that} \qquad \langle x|k\rangle = \psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

On using this basis, the Fourier transforms is just the passage from one basis $|x\rangle$ to another $|k\rangle$

$$f(k) = \langle k|f\rangle = \int_{-\infty}^{\infty} \langle k|x\rangle \langle x|f\rangle \; dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx$$
$$f(x) = \langle x|f\rangle = \int_{-\infty}^{\infty} \langle x|k\rangle \langle k|f\rangle \; dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(k) \, dk$$

Position operator. The action in x basis is described as follows

$$X\,|x\rangle = x\,|x\rangle$$

while is the k basis

$$\langle k|X|\psi\rangle = i\frac{d}{dk}\psi(k)$$

Inner Product

Finite dimension. The inner product is defined in such way to obey the following requirement.

1. **Skew-symmetry**: $\langle V|W\rangle = \langle W|V\rangle^*$
2. **Positive semidefinitness**: $\langle V|V\rangle \geq 0$. If $|V\rangle = 0\,|0\rangle$, then $\langle V|V\rangle = |0\rangle$
3. **Linearity in ket** $\langle V|(a\,|W\rangle + b\,|P\rangle)\rangle = a\,\langle V|W\rangle + b\,\langle V|P\rangle$

In following this axiom, we arrive at the following definition for inner product

$$\langle V|W\rangle = \sum_{i,j} v_i^* w_i \,\langle i|j\rangle = \begin{bmatrix} v_1^* & \dots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

If we think function as element of vector space, two functions can be said to be orthogonal on (a, b) if

$$\int_a^b A(x)B(x) \, dx = 0$$

Infinite dimension. The inner product of two continuous function along some interval is defined as

$$\langle f|g\rangle = \int_a^b f^*(x)g(x) \, dx$$

This definition comes after recalling the completeness relation for infinite space

$$\langle f|g\rangle = \langle f|\,I\,|g\rangle = \int_a^b \langle f|x\rangle \langle x|g\rangle \; dx = \int_a^b f^*(x)g(x) \, dx$$

Norm

Norm of a vector is defined as

$$|V| \equiv \sqrt{\langle V|V\rangle}$$

Two vectors with zero inner product are said to be orthogonal. If said vector has unit norm instead, it is referred as normalized. Set of basis with orthogonal and normalized condition are called orthonormal basis.

Continuous function. The norm of a continuous function is defined as

$$\int_a^b A^*(x)A(x)\; dx = N^2$$

We also said the function $N^{-1}A(x)$ to be normalized and has the norm of one.

Adjoint Operation

The following is a summary on how to perform adjoint in braket notation.

> Reverse the order of all factors and make the substitutions $\Omega \leftrightarrow \Omega^\dagger, |\rangle \leftrightarrow \langle|\,, a \leftrightarrow a^*$.

From this we also obtain the conjugate expressions

$$\begin{aligned} |aV\rangle &= a\,|V\rangle \quad &\leftrightarrow \quad \langle aV| &= \langle V|\, a^* \\ |\Omega V\rangle &= \Omega\,|V\rangle \quad &\leftrightarrow \quad \langle \Omega V| &= \langle V|\, \Omega^\dagger \end{aligned}$$

Bra and ket. Here is how the method in action. The following vector is an adjoint of each other

$$|V\rangle = \sum_i v_i\,|i\rangle \qquad \langle V| = \sum_i \langle i|\, v_i^*$$

We can also write it in terms of projection operation by recalling $v_i = \langle i|V\rangle$ and $v_i^* = \langle V|i\rangle$

$$|V\rangle = \sum_i |i\rangle\,\langle i|V\rangle \qquad \langle V| = \sum_i \langle V|i\rangle\,\langle i|$$

Operator. The matrix $\Omega^\dagger$–also called Hermitian adjoint–represent transpose conjugate of Ω

$$\Omega^\dagger = (\Omega^*)^T = (\Omega^T)^*$$

Our general rule of adjoint state

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$$

This can however be proved in another method. Consider $\langle \Omega\Lambda V|$. First treat $\Omega\Lambda$ as one operator

$$\langle (\Omega\Lambda)V| = \langle V|\,(\Omega\Lambda)^\dagger$$

Then, treat both as separate operator and pull them out of the bra one by one

$$\langle \Omega\Lambda V| = \langle \Lambda V| \Omega^\dagger = \langle V| \Lambda^\dagger\Omega^\dagger$$

Thus, we have

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$$

Equation. Suppose we have the equation involving

$$a_1 |V_1\rangle = a_2 |V_2\rangle + a_3 |V_3\rangle \langle V_4|V_5\rangle + a_4\Omega\Lambda |V_6\rangle$$

and we want to take its adjoint

$$\langle V_1| a_1^* = \langle V_2| a_2^* + \langle V_4|V_5\rangle \langle V_3| a_3^* + \langle V_6| \Lambda^\dagger\Omega^\dagger a_4^*$$

Gram-Schmidt Theorem

The Gram-Schmidt procedure is used to convert linearly independent basis $|I\rangle, \dots, |N\rangle$ into an orthonormal one $|1\rangle, \dots, |n\rangle$. We begin with the first basis vector and normalize it

$$|1\rangle = \frac{|I\rangle}{|I|} = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}}$$

For our i-th basis, we create the projection along all $(i-1)$ vector

$$|i'\rangle = |I_i\rangle - \sum_{j=1}^{i-1} |j\rangle \langle j|I_i\rangle$$

and normalize it to obtain the orthonormal basis

$$|i\rangle = \frac{|i'\rangle}{|i'|}$$

Three basis. Let $|I\rangle, |II\rangle, |III\rangle$ be linearly independent basis. The first orthonormal vector is

$$|1\rangle = \frac{|I\rangle}{|I|}$$

For the second basis

$$|2'\rangle = |II\rangle - |1\rangle \langle 1|II\rangle$$

The second term is the projection of $|II\rangle$ along the first orthonormal basis. By subtracting $|II\rangle$ by this, the only thing that remains is the perpendicular partial. Then we normalize the vector

$$|2\rangle = \frac{|2'\rangle}{|2'|}$$

Same goes for the third basis. We construct

$$|3'\rangle = |II\rangle - |1\rangle \langle 1|III\rangle - |2\rangle \langle 2|III\rangle$$

then normalize it

$$|3\rangle = \frac{|3'\rangle}{|3'|}$$

As a sanity check, we see that the projection of $|1\rangle$ along itself is norm

$$\langle 1|1\rangle = \frac{I|I}{|I|^2} = 1$$

and that the $|2'\rangle$ or $|2\rangle$ along the first is orthogonal

$$\begin{aligned}
\langle 1|2'\rangle &= \langle 1|II\rangle - \langle 1|1\rangle \langle 1|II\rangle = 0 \\
\langle 1|2\rangle &= \frac{1|2'}{|I||2'|} = 0
\end{aligned}$$

Schwarz Inequality

Theorem that ensure the magnitude of inner product never exceed the product of vector magnitude

$$|\langle V|W\rangle| \leq |V||W|$$

Another related theorem is the triangle inequality, which state that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides

$$|V+W| \leq |V| + |W|$$

Schwarz inequality proof. First we define

$$|Z\rangle = |V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle$$

Its inner product is

$$\begin{aligned}
\langle P|P\rangle &= \left\langle V - \frac{\langle W|V\rangle}{|W|^2}W \middle| V - \frac{\langle W|V\rangle}{|W|^2}W \right\rangle \\
&= \langle V|V\rangle - \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2} - \frac{\langle V|W\rangle \langle W|V\rangle}{|W|^2} \\
&- \frac{\langle V|W\rangle \langle W|V\rangle \langle W|W\rangle}{|W^4|} \\
&= \langle V|V\rangle - \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2}
\end{aligned}$$

According to semidefinitness axiom

$$\langle V|V\rangle \geq \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2}$$

Multiply by $|W|^2$ and taking the square root to obtain

$$\begin{aligned}
|V|^2|W|^2 &\geq |\langle V|W\rangle|^2 \\
|V||W| &\geq |\langle V|W\rangle|
\end{aligned}$$

Subspace

Given a vector space $\mathbb{V}$, a subset of its elements that form a vector space among themselves is called a subspace. We will denote a particular subspace i of dimensionality n by $\mathbb{V}_i^n$.

Given two subspaces $\mathbb{V}_i^{n_i}$ and $\mathbb{V}_j^{m_j}$, we define their sum $\mathbb{V}_i^{n_i} \oplus \mathbb{V}_j^{m_j}$ as the set containing

1. All vector in the $\mathbb{V}_i^{n_i}$.
2. All vector in the $\mathbb{V}_j^{m_j}$.
3. All linear combination of them.

Vector along three spatial dimension is denoted as $\mathbb{V}_3(R)$, while its component is $\mathbb{V}_x^1$ on the x direction and the same convention on other two. All vector along the xy plane is denoted $\mathbb{V}_{xy}^2$. Adding the x and y subspace will result in the same subspace $\mathbb{V}_x^1 \oplus \mathbb{V}_y^1 = \mathbb{V}_{xy}^2$, which makes sense since the plane is a linear combination of two basis.

Linear Independence

Linear combination of $\mathbf{A}$ and $\mathbf{B}$ means $a\mathbf{A} + b\mathbf{B}$ where a and b are scalars.

Vector. The vector $\mathbf{r}r = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}$ with tail at the origin is a linear combination of the unit basis vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ where a is a scalar. Set of vectors are said to independent if the only solution to the following equation is trivial

$$\sum_i a_i \left|i\right\rangle = \left|0\right\rangle$$

If the basis of the vector are independent, we can expand into

$$\left|V\right\rangle = \sum_i v_i \left|i\right\rangle$$

A function of a vector, say $f(r)$, is called linear if

$$f(r_1 + r_2) = f(r_1) + f(r_2) \qquad \text{and} \qquad f(ar) = a\ f(r)$$

Operator. Linear operator obey the following rule

$$\begin{aligned}
\Omega\alpha \left|V\right\rangle &= \alpha\Omega \left|V\right\rangle \\
\Omega \left[\alpha \left|V_1\right\rangle + \beta \left|V_2\right\rangle\right] &= \Omega\alpha \left|V_1\right\rangle + \beta\Omega \left|V_2\right\rangle \\
\left\langle V\right| \alpha\Omega &= \left\langle V\right| \Omega\alpha \\
\left[\left\langle V_1\right| \alpha + \left\langle V_2\right| \beta\right] \Omega &= \alpha \left\langle V_1\right| \Omega + \beta \left\langle V_2\right| \Omega
\end{aligned}$$

Ω is a linear operator if

$$\Omega(r_1 + r_2) = \Omega(r_1) + \Omega(r_2) \qquad \text{and} \qquad \Omega(ar) = a\ \Omega(r)$$

Complete function. A set of function $f_n(x)$ is said to be complete if any other function $f(x)$ can be expressed as linear combination of them

$$f(y) = \sum_{n=1}^{\infty} C_n f(y)$$

Linear dependence. $f_1(x), f_2(x), \cdots, f_n(x)$ have derivatives of order $n-1$, and if the Determinant

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix} \neq 0$$

then the functions are linearly independent.

Eigenvalue Problem

Consider linear operator Ω transforming non-trivial $|V\rangle$

$$\Omega |V\rangle = |W\rangle$$

Each operator has certain ket on which its transformation is simply recalling

$$\Omega |V\rangle = \omega |V\rangle$$

This is the eigenvalue equation which state that $|V\rangle$ is an eigenket of Ω with eigenvalue ω. We begin to solve the problem by writing the eigenvalue equation as

$$(\Omega - \omega I) |V\rangle = |0\rangle$$

Operating $(\Omega - \omega I)^{-1}$ on both side

$$|V\rangle = (\Omega - \omega I)^{-1} |0\rangle$$

Any operator acting on null vector can only give null vector, not arbitrary vector $|V\rangle$. Therefore, our assumption that operator $(\Omega - \omega I)^{-1}$ exist is false. Recalling the inverse of matrix M

$$M^{-1} = \frac{1}{\det M} \mathrm{C}^T$$

we see that the condition for non-existent inverse is zero determinant. Thus, the condition for non-zero eigenvector is

$$\det(\Omega - \omega I)^{-1} = 0$$

In practice, this equation is enough to determine the eigenvalue and eigenvector. For theoretical purpose however, we can determine what form does the solution take. To do so, we project the equation with basis bra $\langle i|$

$$\langle i| \Omega - \omega I |V\rangle = 0$$

or in summation form

$$\sum_j (\Omega_{ij} - \omega\delta_{ij})v_j = 0$$

Setting the determinant to zero gives us the characteristic equation

$$\sum_{m=0}^{n} c_m \omega^m = 0$$

Once the eigenvalue ω are found, we can move to the next step of determining the eigenvector. We do this by substituting the value of ω into the equation

$$(\Omega - \omega I) |V\rangle = |0\rangle$$

Then we will obtain set of equations that the equation must obey. It is conventional to normalize the eigenvector also. In the case of degeneracy, we choose the eigenvector such that they are orthogonal to each other.

Propagator

Propagator is an operator that propagate a state forward in space or time. It can be used as a solution to a differential equation. For an equation with the form

$$|\ddot{x}\rangle = \Omega |x(t)\rangle$$

have the solution written as

$$|x(t)\rangle = U(t) |x(0)\rangle$$

The propagator is constructed as

$$U(t) = \sum_i |i\rangle \langle i| \cos \omega_i t$$

with $|i\rangle$ as the i-th eigenvector and ω_i as the i-th eigenvalue.

The steps of solving initial value problem are as follows.

1. Consider the equation $|\ddot{\psi}(t)\rangle = \Omega |\psi(0)\rangle$
2. Solve the eigenvalue of operator Ω
3. Find its eigenvector
4. Find the propagator $U(t)$ in terms of the eigenvectors and eigenvalues of Ω
$$U(t) = \sum_i |i\rangle \langle i| \cos \omega_i t$$
5. Write the solution
$$|\psi(t)\rangle = U(t) |\psi(0)\rangle$$

Function of Operator

We shall consider the function, such as e^x, where x is an operator and determine its meaning. We restrict ourselves to those function that can be written as power series an operator that are hermitian.

Exponential function. Has the following form

$$e^{\Omega} = \begin{bmatrix} e_1^{\omega} & & \\ & \ddots & \\ & & e_n^{\omega} \end{bmatrix}$$

Proof. By going to the eigenbasis of hermitian operator and raising it to the power of m

$$\Omega = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix}, \qquad \Omega^m = \begin{bmatrix} \omega_1^m & & \\ & \ddots & \\ & & \omega_n^m \end{bmatrix}$$

Inserting this into the series representation of exponential function

$$e^{\Omega} = \sum_{m=0}^{\infty} \frac{\Omega^m}{m!}$$

we have

$$e^{\Omega} = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{bmatrix} \omega_1^m & & \\ & \ddots & \\ & & \omega_n^m \end{bmatrix} = \begin{bmatrix} \sum_{m=0}^{\infty} \frac{\omega_1^m}{m!} & & \\ & \ddots & \\ & & \sum_{m=0}^{\infty} \frac{\omega_n^m}{m!} \end{bmatrix}$$

$$e^{\Omega} = \begin{bmatrix} e^{\omega_1} & & \\ & \ddots & \\ & & e^{\omega_n} \end{bmatrix}$$

Geometric series. For hermitian Ω

$$\sum_{m=0}^{\infty} \Omega^n = (1-\Omega)^{-1}$$

Proof. We write the power series as

$$f(\Omega) = \sum_{m=0}^{\infty} \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix} = \begin{bmatrix} \sum_{m=0}^{\infty} \omega_1^n & & \\ & \ddots & \\ & & \sum_{m=0}^{\infty} \omega_n^m \end{bmatrix}$$

$$f(\Omega) = \begin{bmatrix} (1-\omega_1)^{-1} & & \\ & \ddots & \\ & & (1-\omega_n)^{-1} \end{bmatrix} = (1-\Omega)^{-1}$$

Application: Gram-Schmidt Theorem

Convert these linearly independent basis into orthonormal basis

$$|I\rangle = \begin{bmatrix}3\\0\\0\end{bmatrix} \quad |II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} \qquad |III\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix}$$

First normalize

$$|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{3}\begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Then we construct

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\2\\5\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

and normalize

$$|2\rangle = \frac{|2'\rangle}{|2'|} = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$$

Doing the something for the third base

$$\begin{aligned}|3'\rangle &= |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle \\ &= \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\begin{bmatrix}0 & 1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}\begin{bmatrix}0\\2\\5\end{bmatrix} \\ &= \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}0\\12/5\\24/5\end{bmatrix} = \begin{bmatrix}0\\-2/5\\1/5\end{bmatrix}\end{aligned}$$

And normalize it

$$|3\rangle = \frac{|3'\rangle}{|3'|} = \sqrt{5}\begin{bmatrix}0\\-2/5\\1/5\end{bmatrix} = \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

Application: Determining determinant

Consider the matrix

$$A = \begin{bmatrix}2 & -5 & 2\\7 & 3 & 4\\2 & 1 & 5\end{bmatrix}$$

We use the elements of third column first

$$\begin{aligned}\det A = \begin{vmatrix}2 & -5 & 2\\7 & 3 & 4\\2 & 1 & 5\end{vmatrix} &= 2\begin{vmatrix}7 & 3\\2 & 1\end{vmatrix} - 4\begin{vmatrix}2 & -5\\2 & 1\end{vmatrix} + 5\begin{vmatrix}2 & -5\\7 & 3\end{vmatrix} \\ &= 2\cdot 1 - 4\cdot 11 + 5\cdot 38 = 148\end{aligned}$$

Then, as a check we use the first row's

$$\det A = 1 \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} + 5 \begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix} + 2 \begin{vmatrix} 7 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 11 + 135 + 2 =$$

Application: Inverse matrix

We use inverse matrix to solve the following equation

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 0 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -10 \end{bmatrix}$$

Notice the equation has the following form

$$\Omega \left| V \right\rangle = \omega$$
$$\left| V \right\rangle = \Omega^{-1}\omega$$

To find vector $|V\rangle$ that satisfy the equation, we need to determine the inverse of Ω. The minor of each element are

$$\begin{aligned} &M_{11} = 6, \quad M_{12} = -4, \quad M_{13} = 3, \\ &M_{21} - 3 =, \quad M_{22} = 3, \quad M_{23} = -3, \\ &M_{31} = 3, \quad M_{32} = -2, \quad M_{33} = 3 \end{aligned}$$

Thus, the cofactor is

$$\mathrm{C} = \begin{bmatrix} 6 & 4 & 3 \\ 3 & 3 & 3 \\ 3 & 2 & 3 \end{bmatrix}$$

Next, using the first row to find the determinant

$$\det \Omega = 1 \cdot 6 + 1 \cdot (6 - 3) = 3$$

Finally the inverse is

$$\Omega^{-1} = \frac{1}{\det \Omega} \mathrm{C}^T = \frac{1}{3} \begin{bmatrix} 6 & 3 & 3 \\ 4 & 3 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Now we can use the inverse to find the value of the vector

$$\left| V \right\rangle = \frac{1}{3} \begin{bmatrix} 6 & 3 & 3 \\ 4 & 3 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ -10 \end{bmatrix} = \begin{bmatrix} 10 + 1 - 10 \\ \frac{20+3-20}{3} \\ 5 + 1 - 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Application: Eigenvalue Problem

We shall find the eigenvalue of the hermitian matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

First write the eigenvalue equation

$$(\Omega - \omega I)\,|\omega\rangle = \begin{bmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{bmatrix} |\omega\rangle = 0$$

The characteristic equation is

$$-\omega^3 + \omega = \omega(\omega^2 + 1) = 0$$

This implies the eigenvalues are

$$\omega = 0, \pm 1$$

Next, we substitute the eigenvalue into the eigenvalue equation. First consider the eigenvalue $\omega = 0$

$$(\Omega - \omega I)\,|\omega_0\rangle = 0 \implies \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 = 0 \\ 0 = 0 \\ v_1 = 0 \end{bmatrix}$$

And we get arbitrary v_2, to normalize the eigenvector we choose

$$\langle \omega_0 | = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Next is the case of $\omega = 1$

$$(\Omega - \omega I)\,|\omega_1\rangle = 0 \implies \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_1 + v_3 = 0 \\ -v_2 = 0 \\ v_1 - v_3 = 0 \end{bmatrix}$$

the eigenvector corresponds to the eigenvalue is

$$|\omega_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Finally the last eigenvalue $\omega = -1$

$$(\Omega - \omega I)\,|\omega_{-1}\rangle = 0 \implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_3 = 0 \\ v_2 = 0 \\ v_1 + v_3 = 0 \end{bmatrix}$$

with the eigenvector of

$$\langle \omega_{-1} | = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

We can also use these eigenvectors to diagonal the operator Ω. From the eigenvector, we construct the unitary matrix

$$U = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

As a check, we can also confirm the unitary identity of unitary matrix

$$U^\dagger U = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As expected. We can now jump to the diagonalization

$$
\begin{aligned}
U^\dagger \Omega U &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
U^\dagger \Omega U &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

Which is just the eigenvalue as the diagonal element.

Application: Eigenvalue Problem With Degeneracy

Now consider operator with matrix element

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The eigenvalue equation is

$$(\Omega - \omega I) \left|\omega\right\rangle = 0 \implies \begin{bmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{bmatrix} \left|\omega\right\rangle = 0$$

while the characteristic equation

$$(2-\omega)[(1-\omega)^2 - 1] = (2-\omega)[\omega^2 - 2\omega] = (2-\omega)\omega(\omega-2) = 0$$

which result in eigenvalues of

$$\omega = 0, 2, 2$$

with $\omega = 2$ as degenerate. First we consider the eigenvalue $\omega = 0$

$$(\Omega - \omega I) \left|\omega_0\right\rangle = 0 \implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 = 0 \\ 2v_2 = 0 \\ v_1 + v_3 = 0 \end{bmatrix}$$

with eigenvector of

$$\left\langle\omega_0\right| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

Next is the degenerate eigenvalue of $\omega = 2$

$$(\Omega - \omega I) \left|\omega_2\right\rangle = 0 \implies \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_1 + v_2 = 0 \\ 0 = 0 \\ -v_1 + v_3 = 0 \end{bmatrix}$$

The eigenvector, which are also degenerate, has the arbitrary component v_2. This mean that the two degenerate eigenvector lies on the same plane. For the first degenerate eigenvector, let us just choose the simplest normalized vector that also satisfies the equation above

$$|\omega_2, \alpha\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For the second degenerate eigenvector, we choose in a way such that it is orthonormal with the first. The orthogonal condition state that

$$\langle \omega_2, \alpha |^{\circ}_{2}, \beta\rangle = 0$$

Since the degenerate vectors lies in the same plane on arbitrary v_2, we shall determine the value of v_2 such that $|\omega_2, \alpha\rangle$ is orthogonal with $|\omega_2, \beta\rangle$

$$\langle \omega_2, \alpha | \omega_2', \beta\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ 1 \end{bmatrix} = \beta + 2$$

or $\beta = -2$. All that left is normalizing it

$$|\omega_2, \beta\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Application: Solving Second Order Coupled ODE Using Propagator

---

Consider the case of two masses m that are coupled to each other and to the wall by spring force constant k. We denote x_1 and x_2 as the masses' displacement from the equilibrium. Doing Newtonian analysis to the first mass, we have the equation of motion

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - k\xi_2 \\ \ddot{x}_1 &= -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \end{aligned}$$

where $\xi_2 = x_1 - x_2$ is the displacement of the second spring. For the second mass,

$$\begin{aligned} m\ddot{x}_2 &= -kx_2 - k\xi_1 \\ \ddot{x}_2 &= -\frac{2k}{m}x_2 + \frac{k}{m}x_1 \end{aligned}$$

where $\xi_1 = x_2 - x_1$. Writing these equation in matrix form

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k/m & k/m \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Or

$$|\ddot{x}(t)\rangle = \Omega\, |x(t)\rangle$$

As per our usual steps, we first solve the eigenvalue problem of matrix Ω. We write the eigenvalue equation as

$$(\Omega + \omega^2 I)\, |x(t)\rangle = 0$$

The eigenvalue of Ω is written as $-\omega^2$ in anticipation of the fact that ω being real. Setting the determinant to zero

$$\det(\Omega + \omega^2 I) = \det \begin{vmatrix} -2k/m + \omega^2 & k/m \\ k/m & -2k/m + \omega^2 \end{vmatrix}$$

$$0 = \left(\omega^2 - \frac{2k}{m}\right)^2 = \left(\frac{k}{m}\right)^2$$

$$\omega^2 = \frac{2k}{m} \pm \frac{k}{m}$$

$$\omega = \pm\sqrt{\frac{2k}{m} \pm \frac{k}{m}}$$

Since we are taking the square of ω, both positive and negative value of ω produce the same eigenvalue. We're then taking the positive quantity only

$$\omega = \sqrt{\frac{3k}{m}}, \sqrt{\frac{k}{m}}$$

The equation governing eigenvector corresponding to the eigenvalue of $\sqrt{3k/m}$ is

$$(\Omega - \omega^2 I) \left|\omega_{\sqrt{3k/m}}\right\rangle = \begin{bmatrix} k/m & k/m \\ k/m & k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m}(x_1 + x_2) \\ \frac{k}{m}(x_1 + x_2) \end{bmatrix}$$

The normalized eigenvector is

$$\left|\omega_{\sqrt{3k/m}}\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The equation for the second eigenvalue

$$(\Omega - \omega^2 I) \left|\omega_{\sqrt{k/m}}\right\rangle = \begin{bmatrix} -k/m & k/m \\ k/m & -k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m}(-x_1 + x_2) \\ \frac{k}{m}(x_1 - x_2) \end{bmatrix}$$

With the eigenvector of

$$\left|\omega_{\sqrt{3k/m}}\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We now construct the propagator

$$\begin{aligned}
U(t) &= \sum_i \left|\omega_i\right\rangle \left\langle\omega_i\right| \cos\omega_i t \\
&= \left|\omega_{\sqrt{3k/m}}\right\rangle \left\langle\omega_{\sqrt{3k/m}}\right| \cos\sqrt{\frac{3k}{m}} + \left|\omega_{\sqrt{k/m}}\right\rangle \left\langle\omega_{\sqrt{k/m}}\right| \cos\sqrt{\frac{k}{m}} \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \cos\sqrt{\frac{3k}{m}}t \\
&= \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \cos\sqrt{\frac{k}{m}}t \\
&= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cos\sqrt{\frac{3k}{m}}t + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cos\sqrt{\frac{k}{m}}t \\
U(t) &= \begin{bmatrix} \cos\sqrt{\frac{3k}{m}} + \cos\sqrt{\frac{3k}{m}} & \cos\sqrt{\frac{3k}{m}} - \cos\sqrt{\frac{3k}{m}} \\ \cos\sqrt{\frac{3k}{m}} - \cos\sqrt{\frac{3k}{m}} & \cos\sqrt{\frac{3k}{m}} + \cos\sqrt{\frac{3k}{m}} \end{bmatrix}
\end{aligned}$$

For given $|x(0)\rangle$ the solution of said ODE is

$$|x(t)\rangle = \begin{bmatrix} \cos\sqrt{\frac{3k}{m}}t + \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{3k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{3k}{m}}t - \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{3k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Application: Yet Another Propagator Problem

Consider a string of length L clamped at its two ends $x = 0$ and L. The displacement $\psi(x,t)$ obeys the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}$$

We recognize the differential operator as $-K^2$ which is Hermitian since $\psi(0) = \psi(L) = 0$.

Our general strategy is still the same as previously, that is we find the solution as a function of time $\psi(t)$. There is one more step, however, to find the solution as a function of both space and time: we project the solution $\psi(t)$ to x basis.

We move to the first step of solving differential equation using propagator by solving the eigenvalue problem of

$$|\psi\rangle = K^2 |\psi\rangle$$

If we project this in the x basis, the action reads

$$-\frac{\partial^2 \psi_k(x)}{\partial x^2} = k^2 \psi_k(x)$$

with $\psi_k(x) = \langle x|\psi_k\rangle$. The general solution to this differential equation is

$$\psi_k(x) = A\cos kx + B\sin kx$$

We have also the boundaries condition of

$$\psi_k(x) = \begin{cases} \psi_k(0) = 0 \\ \psi_k(L) = 0 \end{cases}$$

Applying the first condition return

$$A = 0$$

For the second condition, if we do not consider trivial solution of $B \neq 0$, we obtain the eigenvalue of

$$k = \frac{m\pi}{L}$$

with $k = [1, -\infty]$. We do not consider the zero and the negative value since they do not

Thus we obtain discrete set of eigenvector, or rather eigenfunction, with label m

$$\psi_m(x) = B\sin\left(\frac{m\pi}{L}x\right)$$

For discrete set of continuous, the normalization equation reads

$$\int_0^\lambda \psi_m^*(x)\psi_{m'}(x)\,dx = \delta_{mm'}$$

Substituting the expression for the solution

$$\begin{aligned}\int_0^L B^2 \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{m'\pi}{L}x\right)\,dx &= \frac{B^2L}{m\pi}\int_0^{m\pi}\sin^2 u\,du\\ &= \frac{B^2L}{m\pi}\int_0^{m\pi}\frac{1-\cos u}{2}\,du\\ &= \frac{B^2L}{m\pi}\left[\frac{u}{2}\bigg|_0^{m\pi} - \frac{\cos 2u}{4}\bigg|_0^{2m\pi}\right]\\ \int_0^L B^2 \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{m'\pi}{L}x\right)\,dx &= \frac{B^2L}{2}\end{aligned}$$

which give the normalization constant of

$$B = \sqrt{\frac{2}{L}}$$

The normalized eigenfunction now reads

$$\psi_k(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)$$

Let us associate each solution by integer m an abstract $|m\rangle$, which on x basis reads as the said solution

$$\langle x|m\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)$$

Instead of writing the propagator, we'll just skip to writing the solution for the differential equation since we do not know the expression for the $|m\rangle$. As stated previously, we project the solution $\psi(t)$ into x basis to obtain

$$\begin{aligned}\psi(x,t) = \langle x|\psi(t)\rangle &= \langle x|\,U(t)\,|\psi(0)\rangle\\ &= \sum_{m=1}^{\infty}\langle x|m\rangle\,\langle m|\psi(0)\rangle\cos\omega_m t\end{aligned}$$

Using the relation $\omega = kv$, we may the express the angular frequency as the spatial frequency. Although feeding $\omega_m = m\pi/L$ directly to the argument of the cosine term does not give a dimensionless quantity, it gives one if one consider that ω_m is equal to k_m times v which happens to be one in this case. Next, we insert the completeness relation

$$\psi(x,t) = \int_0^\lambda \sum_{m=1}^{\infty}\langle x|m\rangle\,\langle m|x'\rangle\,\langle x'|\psi(0)\rangle\cos\omega_m t$$

Finally, we obtain the solution

$$\psi(x,t) = \sum_{m=1}^{\infty}\frac{2}{L}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m\pi}{L}t\right)\int_0^L \sin\left(\frac{m\pi}{L}x'\right)\psi(x',0)\,dx$$

Partial Differentitation

Total Diferential

For a function $f = f(x, y, z, \dots)$, its total derivative is defined as

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

Identity Involving Partial Derivative

The Jacobian of $[u(x, y), v(x, y)]$ with respect to (x, y) is defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Here are some identity relating the Jacobian with partial derivative.

Unity. Unity as in one

$$\frac{\partial(u, v)}{\partial(x, y)} = 1$$

Proof. Trivial

$$\frac{\partial(x, y)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1 \quad \blacksquare$$

Change of order. It can be proved that change of order cost the minus sign

$$\frac{\partial(u, v)}{\partial(x, y)} = -\frac{\partial(v, u)}{\partial(x, y)} = -\frac{\partial(u, v)}{\partial(y, x)}$$

Proof. Those three terms literally have the same value when evaluated

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$-\frac{\partial(v, u)}{\partial(x, y)} = -\begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{vmatrix} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$\frac{\partial(u,v)}{\partial(y,x)} = -\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \\ \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

See? ■

Jacobian. In terms of Jacobian, partial derivative of u with respect to x can be written as

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{\partial(u,y)}{\partial(x,y)}$$

Proof. Just evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} \qquad ■$$

Chain rule for partial derivative. The expression is

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)}$$

Proof. The total differential of u and v as function w and z read

$$du = \frac{\partial u}{\partial w}\,dw + \frac{\partial u}{\partial v}\,dz \quad \wedge \quad dv = \frac{\partial v}{\partial w}\,dw + \frac{\partial v}{\partial z}\,dz$$

We can therefore evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial x} & \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial y} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial y} \\ \\ \dfrac{\partial v}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial x} & \dfrac{\partial v}{\partial w}\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial y} \end{vmatrix}$$

$$= \left| \begin{pmatrix} \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial z} \\ \\ \dfrac{\partial v}{\partial w} & \dfrac{\partial v}{\partial z} \end{pmatrix} \begin{pmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix} \right| = \begin{vmatrix} \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial z} \\ \\ \dfrac{\partial v}{\partial w} & \dfrac{\partial v}{\partial z} \end{vmatrix} \begin{vmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)} \qquad ■$$

The real chain rule. We have

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y = 1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(x,y)} = \left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y \qquad ■$$

Yet another chain rule... Even more chain rule...

$$\left.\frac{\partial x}{\partial y}\right|_w = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Proof. Trivial

$$\left.\frac{\partial x}{\partial y}\right|_w = \frac{\partial(x,w)}{\partial(y,w)} = \frac{\partial(x,w)}{\partial(z,w)}\frac{\partial(z,w)}{\partial(y,w)} = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Cyclic rule. This is chain rule all over again...

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_z = -1$$

Proof. Trivial

$$\begin{aligned} 1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(x,y)} &= -\frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(y,z)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(y,x)} \\ &= -\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial y}{\partial x}\right|_z \left.\frac{\partial z}{\partial y}\right|_x \quad \blacksquare \end{aligned}$$

Application in Thermodynamics

Here we will derive some useful intensive parameter used in thermodynamics. We assumed entropy function S has the form of

$$S = S(U, V, N_{i|r})$$

where N is number of chemical potential and $N_{i|r} \equiv N_1, \ldots N_r$. Therefore, its total differential is

$$dS = \left.\frac{\partial S}{\partial U}\right|_{V,N_{i|r}} dU + \left.\frac{\partial S}{\partial V}\right|_{U,N_{i|r}} dV + \sum_{j=1}^{r} \left.\frac{\partial S}{\partial N_j}\right|_{U,V,N_{i\neq r}} dN_j$$

We also assume the following quantities

$$T = \left.\frac{\partial U}{\partial S}\right|_{V,N_i} \quad ; P = -\left.\frac{\partial U}{\partial V}\right|_{S,N_i} \quad ; \mu_j = \left.\frac{\partial U}{\partial N}\right|_{S,V,N_{i\neq j}}$$

First identity. As follows.

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T}$$

Proof. We use chain rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = 1 \implies \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \left(\left.\frac{\partial U}{\partial S}\right|_{V,N_i}\right)^{-1}$$

Then, from the definition of temperature

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T} \quad \blacksquare$$

Second identity. The identity written as

$$\left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T}$$

Proof. We invoke cyclic rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial V}\right|_{U,N_i} \left.\frac{\partial V}{\partial U}\right|_{U,N_i}$$

Then, from the first identity and the definition of pressure

$$1 = T\left.\frac{\partial S}{\partial V}\right|_{U,N_i} \frac{1}{P} \implies \left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T} \quad \blacksquare$$

Third Identity. Expressed as

$$\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{P}{T}$$

Proof. We again invoke cyclic with $x \to U, y \to Nj, z \to S$; while keeping V and all N except N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \left.\frac{\partial N_j}{\partial U}\right|_{U,N_{i\neq j}}$$

Then, from the definition of temperature and chemical potential

$$1 = -T\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \frac{1}{\mu_j} \implies \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{\mu_j}{T} \quad \blacksquare$$

Lagrange Multipliers

Let $f(x, y, z)$ be our function that we want to optimize and $\phi(x, y, z) =$ const be our constraint. We then set the total differential of $f(x, y, z)$ and $\phi(x, y, z)$ equal to zero

$$\begin{aligned} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial y}dz = 0 \\ \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial y}dz = 0 \end{aligned}$$

Next, we construct the function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

and set its total derivative to zero

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

It follows that, for any value of dx, dy, dz, we choose λ such that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

Putting it all together, to optimize $f(x, y, z)$ with constraint $\phi(x, y, z)$, we need to optimize $F(x, y, z)$, which obtained by solving three partial derivative equations and constraint equation $\phi(x, y, z) = \text{const}$. The equations in question are

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0, & \frac{\partial F}{\partial y} &= 0, \\ \frac{\partial F}{\partial z} &= 0, & \phi &= \text{const.} \end{aligned}$$

Multiple constraint. If there are multiple constraints, say ϕ_1 and ϕ_2, we function F we construct instead is

$$F(x, y, z) = f(x, y, z) + \lambda_1 \phi_1(x, y, z) + \lambda_2 \phi_2(x, y, z)$$

As aside, the function that we want to optimize need not to a function of three variable x, y, z. The previous derivation can be justified for any number of variable. Of course, with more variable there are more variables.

Leibniz' rule for Integral

Differentiation under integral sign stated by Leibniz' rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x, t)\, dt = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x, v) \frac{dv}{dx} - f(x, u) \frac{du}{dx}$$

Proof. Suppose we want dI/dx where

$$I = \int_u^v f(t)\, dt$$

By the fundamental theorem of calculus

$$I = F(v) - F(u) = \mathcal{F}(v, u)$$

or I is a function of v and u. Finding dI/dx is then a partial differentiation problem. We can write

$$\frac{dI}{dx} = \frac{\partial I}{\partial v} \frac{dv}{dx} + \frac{\partial I}{\partial u} \frac{du}{dx}$$

By the fundamental theorem of calculus, we have

$$\begin{aligned} \frac{d}{dv} \int_a^v f(x)\, dt &= \frac{d}{dv} \big[F(v) - F(a)\big] = f(v) \\ \frac{d}{dv} \int_u^b f(x)\, dt &= \frac{d}{dv} \big[F(b) - F(u)\big] = -f(u) \end{aligned}$$

where u and v are a function of x, while a and b are a constant. This is the case when we consider $\partial I/\partial v$ or $\partial I/\partial v$; the other variable is constant. Then

$$\frac{d}{dx} \int_u^v f(t)\, dt = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$$

Under not too restrictive conditions,

$$\frac{d}{dx} \int_a^b f(x,t)\, dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\, dt$$

where, as before, a and b are constant. In other words, we can differentiate under the integral sign. It is convenient to collect these formulas into one formula known as Leibniz' rule:

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x,t)\, dt = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} \qquad \blacksquare$$

Appendix: Lagrange Multipliers

Single constraint. Consider this example.

> Determine the largest volume of parallelepiped–that is, a three-dimensional figure formed by six parallelograms–whose edges parallel with the x, y, z axis inside ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The ellipsoid function above acts as constraints, that left is to determine the function that we want to optimize. This requires some clever thinking. We begin by defining point (x, y, z) be the corner of our parallelepiped. Now, this point is located in the first octant of our parallelepiped. The volume of this octant is

$$v = xyz$$

Since the parallelepiped's sides are parallel the axis, its total volume is

$$V = 8v$$

Hence, the volume of our parallelepiped is

$$V = 8xyz$$

This is the function that we want to maximize. We then construct the function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

The partial derivatives of F read as

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda}{a^2}x, \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda}{b^2}y, \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda}{c^2}z$$

To find the maximum of F, we then must solve the partial derivative equations and constraint equation

$$\begin{aligned} 8yz + \frac{2\lambda}{a^2}x &= 0 \\ 8xz + \frac{2\lambda}{b^2}y &= 0 \\ 8xy + \frac{2\lambda}{c^2}z &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \end{aligned}$$

Multiplying the first equation by x, the second by y, the third by z and adding them all together, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 24xyz + 2\lambda = 0$$

Hence

$$\lambda = -12xyz$$

Substituting this into the partial derivative equation to obtain

$$8yz - \frac{24yz}{a^2}x^2 = 0 \implies x = \frac{\sqrt{3}}{3}a$$
$$8xz - \frac{24xz}{b^2}y^2 = 0 \implies y = \frac{\sqrt{3}}{3}b$$
$$8xy - \frac{24xy}{c^2}z^2 = 0 \implies z = \frac{\sqrt{3}}{3}c$$

Therefore, the maximum volume of said parallelepiped is

$$V = \frac{24\sqrt{3}}{27}abc$$

Two constraints. Here's an example.

> Given two equation $z^2 = x^2 + y^2$ and $x + 2z + 3 = 0$, find the shortest and longest distance from the origin and the intersection of those two equations.

Here we want to minimize $f = x^2 + y^2 + z^2$ as usual. We construct auxiliary function

$$F = x^2 + y^2 + z^2 + \lambda_1(z^2 - x^2 - y^2) + \lambda_2(x + 2z)$$

The partial differentials of F read

$$\frac{\partial F}{\partial x} = 2x - 2\lambda_1 x + \lambda_2,$$
$$\frac{\partial F}{\partial y} = 2y - 2\lambda_1 y,$$
$$\frac{\partial F}{\partial z} = 2z + 2\lambda_1 z + 2\lambda_2$$

Putting it all together, we have these equations

$$2x - 2\lambda_1 x + \lambda_2 = 0 \tag{1}$$
$$2y - 2\lambda_1 y = 0 \tag{2}$$
$$2z + 2\lambda_1 z + 2\lambda_2 = 0 \tag{3}$$
$$z^2 - x^2 - y^2 = 0 \tag{4}$$
$$x + 2z + 3 = 0 \tag{5}$$

By equation 2, we have two possible cases

$$2y - 2\lambda_1 y = y(1 - \lambda_1) = 0 \implies y = 0 \ \vee \ \lambda_1 = 1$$

First we consider $y = 0$. Equation 4 reads

$$z^2 = x^2 \implies z = \pm x$$

Then in the subcase $y = 0,\ z = x$; equation 5 evaluates into

$$3x + 3 = 0 \implies x = 3$$

In other hand, for subcase $y = 0,\ z = -x$; the same equation evaluates into

$$x = 3$$

Now we consider the case when $\lambda_1 = 1$. Equation 1 reduces into

$$\lambda_2 = 0$$

which means equation 5 turns into

$$4z = 0 \implies z = 0$$

and equation 5

$$x = -3$$

Using this result, equation 4 reads

$$y^2 = -9$$

which is impossible unless we are willing to take a complex value. Suppose we are willing, we have the $y = 3i$. Hence, we have three possibilities that the optimized points might take

$$\{\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}\} = \{(-1, 0, -1), (3, 0, -3), (-3, 3i, 0)\}$$

The distance from origin then evaluated by

$$d_1 = \sqrt{\mathbf{P_1} \cdot \mathbf{P_2}} = \sqrt{2}$$
$$d_2 = \sqrt{\mathbf{P_2} \cdot \mathbf{P_2}} = \sqrt{18}$$
$$d_3 = \sqrt{\mathbf{P_3} \cdot \overline{\mathbf{P_3}}} = \sqrt{18}$$

Hence the shortest distance is $d = \sqrt{2}$ and the longest is $d = \sqrt{18}$.

Vector Analysis

Vector Operation

There are four vector operation: Addition, Multiplication by a scalar, Dot product, and Cross Product. (i) Addition of two vectors. Addition is commutative and associative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

(ii) Multiplication by a scalar. Scalar multiplication is distributive.

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$$

(iii) Dot product of two vectors. The dot product of two vectors is defined

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta$$

where θ is the angle they form. Note that dot product is commutative and distributive.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$
$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB\sin\theta\ \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The cross product is distributive, but not commutative.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$
$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$$

Few rule for manipulating vector. (i): To add vectors, add like components

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

(ii): To multiply by a scalar, multiply each component.

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$$

Rule (iii): To calculate the dot product, multiply like components, and add.

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Rule (iv): To calculate the cross product, form the determinant whose first row is unit vector, whose second row is A (in component form), and whose third row is B.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Triple Product

(i) Scalar triple product.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = B \cdot (\mathbf{C} \times A) = C \cdot (\mathbf{A} \times B)$$

They are cyclic and in component form

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector triple product.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

The product is linear combination of vector in parentheses.

Separation Vector

Separation vector defined as vector from the source point $\bar{r'}$ to the field point $\bar{r}$

$$\mathscr{r} \equiv \bar{\mathbf{r}} - \bar{\mathbf{r}'}.$$

Del Operator

Vector operator defined as follows.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Operation Involving Del Operator

There are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the divergence);

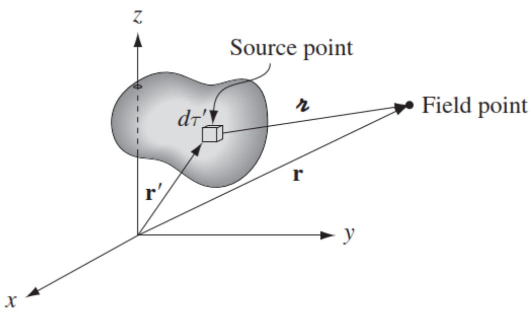


Figure: Separation vector

3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the curl).

Gradient of scalar function $T(x, y, z)$

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

can be used to define partial derivative of T

$$\begin{aligned} dT &= \left(\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= \nabla T \cdot \hat{\mathbf{u}} \end{aligned}$$

Note that ∇T is a vector quantity, with three components. The gradient ∇T points in the direction of maximum increase of the function T. Moreover, The magnitude ∇T gives the slope (rate of increase) along this maximal direction.

Divergence of vector function $\mathbf{V}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is a scalar. Divergence is a measure of how much the vector V spreads out (diverges) from the point in question.

Curl of vector function $\mathbf{V}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

The name curl is also well-chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.

Product Rule

There are two ways to construct a scalar as the product of two functions

$$\begin{aligned} fg &\quad \text{(product of two scalar functions)} \\ \mathbf{A} \cdot \mathbf{B} &\quad \text{(dot product of two vector functions)} \end{aligned}$$

and two ways to make a vector

$$\begin{aligned} f\mathbf{A} &\quad \text{(scalar times vector)} \\ \mathbf{A} \times \mathbf{B} &\quad \text{(cross product of two vectors)} \end{aligned}$$

Accordingly, there are six product rule, two for gradients

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

two for divergences

$$\nabla(f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

and two for curls

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivative

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$. Called Laplacian of T. Notice that the Laplacian of a scalar T is a scalar.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y} + \frac{\partial^2 T}{\partial z}$$

Occasionally, we shall speak of the Laplacian of a vector, $\nabla^2 \mathbf{v}$. By this we mean a vector quantity whose x-component is the Laplacian of V_x, and so on:

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 V_x)\hat{\mathbf{x}} + (\nabla^2 V_y)\hat{y} + (\nabla^2 V_z)\hat{\mathbf{z}}$$

(2) The curl of a gradient: $\nabla \times (\nabla T)$. Always zero.

$$\nabla \cdot (\nabla T) = 0$$

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$. $\nabla(\nabla \cdot \mathbf{v})$ is not the same as the Laplacian of a vector.

$$\nabla(\nabla \cdot \mathbf{v}) \neq \nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v}$$

(4) The divergence of a curl: $\nabla \cdot (\nabla \times \mathbf{v})$. Always zero.

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

(5) Curl of curl:$\nabla \times (\nabla \times \mathbf{v})$. From the definition of ∇,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Fundamental Theorem of Calculus

The fundamental theorem of calculus says the integral of a derivative over some region is given by the value of the function at the end points (boundaries).

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Gradient. The fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (line integral) of a derivative (gradient) is given by the value of the function at the boundaries (a and b).

$$\int_a^b (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Corollary 1: $\int_a^b (\nabla T) \cdot d\mathbf{l}$ is independent of the path.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$ since the beginning and end points are identical.

Divergences . Like the other "fundamental theorems," it says that the integral of a derivative (divergence) over a region (volume V) is equal to the value of the function at the boundary (surface S).

$$\int_V (\nabla \cdot \mathbf{v})\, d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

If v represents the flow of an incompressible fluid, then the flux of $\mathbf{v}$ is the total amount of fluid passing out through the surface, per unit time. There are two ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. Alternatively,

$$\int \text{(faucets within the volume)} = \oint \text{(flow out through the surface)}$$

Curl. As always, the integral of a derivative (curl) over a region (patch of surface, S) is equal to the value of the function at the boundary (perimeter of the patch, P). Now, the integral of the curl over some surface (flux of the curl) represents the "total amount of swirl," and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Corollary 1. $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line. It doesn't matter which way you go as long as you are consistent. For a closed surface (divergence theorem), $d\mathbf{a}$ points in the direction of the outward normal; but for an open surface is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$.

Corollary 2. $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point.

Integration by Parts

It applies to the situation in which you are called upon to integrate the product of one function (f) and the derivative of another (g); it says you can transfer the derivative from g to f, at the cost of a minus sign and a boundary term.

$$\int_a^b f\left(\frac{dg}{dx}\right) dx = -\int_a^b g\left(\frac{df}{dx}\right) dx + fg\Big|_a^b$$

Curvilinear Coordinates

I shall use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. Infinitesimal displacement vector can be written

$$d\mathbf{l} = f\ du\ \hat{\mathbf{u}} + g\ dv\ \hat{\mathbf{v}} + h\ dw\ \hat{\mathbf{w}}$$

Table 1: Table

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

where f, g, and h are functions of position characteristic of the particular coordinate system. While infinitesimal volume is

$$d\tau = fgh\ du\ dv\ dw$$

Use table 1 for references.

Gradient. The gradient of t is

$$\nabla \mathbf{t} \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$

Divergence. The divergence of $\mathbf{A}$ in curvilinear coordinates:

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$

Curl.

$$\nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}$$

Laplacian.

$$\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]$$

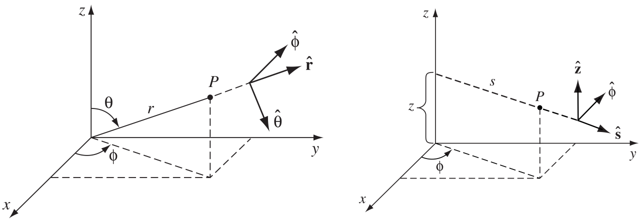
Spherical.

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\ \hat{\mathbf{r}} + \cos\theta\cos\phi\ \hat{\boldsymbol{\theta}} - \sin\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\ \hat{\mathbf{r}} + \cos\theta\sin\phi\ \hat{\boldsymbol{\theta}} + \cos\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\sqrt{x^2+y^2}/z \\ \phi &= \arctan y/z \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\boldsymbol{y}} + \cos\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\ \hat{\mathbf{x}} + \cos\theta\sin\phi\ \hat{\boldsymbol{y}} - \sin\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\boldsymbol{y}} \end{cases}$$

Cylindrical.

$$\begin{cases} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\ \hat{\mathbf{s}} - \sin\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\ \hat{\mathbf{s}} + \cos\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$



Spherical Coordinates and Cylindrical Coordinates

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan y/z \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

Dirac Delta

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1. That is to say

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x-a)\; dx = 1$$

It follows that

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Since the product is zero anyway except at $x = a$, we may as well replace $f(x)$ by the value it assumes at the origin. In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\; dx = f(a)$$

It's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\; dx = \int_{-\infty}^{\infty} f(x)D_2(x)\; dx$$

It is easy to generalize the delta function to three dimensions

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

with $\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, and it's integral

$$\int_{\text{all space}} \delta^3(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\; dxdydz = 1$$

Generalizing Delta function, we get

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r\text{-}a})\; d\tau = f(\mathbf{a})$$

Few Dirac delta function

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Fourier Transform of a δ function. Using the definition of a Fourier transform, we write

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\alpha x}\,dx = \frac{1}{2\pi}e^{-i\alpha x}$$

and its inverse transform

$$\delta(x-a) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-a)}\,d\alpha$$

The integral however does not converge. If we replace the limits by $-n, n$, we obtain a set of functions which are increasingly peaked around $x = a$ as n increases, but all have area 1.

Derivative of a δ function. Using repeated integrations by parts gives

$$\int_{-\infty}^{\infty} \phi(x)\delta^{(n)}(x-a)\,dx = (-1)^n\phi^{(n)}(a)$$

Few formulas involving δ function. For step function

$$u(x-a) = \begin{cases} 1, x > a \\ 0, x < a \end{cases}$$

$$u'(x-a) = \delta(x-a)$$

It is easy to see how the derivative of step function is equal to delta function.

Helmholtz Theorem

Suppose we are told that the divergence of a vector function F(r) is a specified scalar function D(r):

$$\nabla \cdot \mathbf{F} = D$$

and the curl of F(r) is a specified vector function C(r):

$$\nabla \times \mathbf{F} = \mathbf{C}$$

For consistency, C must be divergenceless $\nabla \cdot \mathbf{C} = 0$. Helmholtz theorem state if the divergence D(r) and the curl C(r) of a vector function F(r) are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$ and if F(r) goes to zero as $r \to \infty$, then F is given uniquely by

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$$

Potential Theorem

Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, F satisfies one if and only if it satisfies all the others):

- $\nabla \times \mathbf{F} = 0$ everywhere.
- $\int_a^b \mathbf{F} \cdot \, d\mathbf{l}$ is independent of path, for any given end points.
- $\oint \mathbf{F} \cdot \, d\mathbf{l} = 0$ for any closed loop.
- $\mathbf{F}$ is the gradient of some scalar function: $F = -\nabla V$.

Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

- $\nabla \cdot \mathbf{F} = 0$ everywhere.
- $\int \mathbf{F} \cdot \, d\mathbf{a}$ is independent of surface, for any given boundary line.
- $\oint \mathbf{F} \cdot \, d\mathbf{a} = 0$ for any closed surface.
- $\mathbf{F}$ is the curl of some scalar function: $\mathbf{F} = -\nabla \mathbf{A}$.

State Function and Conservative Field

State function, such as internal energy U and entropy S, can be thought as conservative field. The condition that must be satisfied by conservative field $\mathbf{V}$ is

$$\nabla \times \mathbf{V} = 0$$

Suppose we actually evaluate the curl of vector function $\mathbf{V}(x, y, z)$, we get

$$\nabla \times \mathbf{V} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix}$$

$$\nabla \times \mathbf{V} = \hat{\mathbf{i}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Since $\mathbf{V}$, as a conservative field, has curl of zero, those term inside parenthesis can be evaluated into

$$\frac{\partial V_z}{\partial y} = \frac{\partial V_y}{\partial z}, \quad \frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}, \quad \frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y}$$

For state function $U(S, V, N)$, the equation reads

$$\frac{\partial U_N}{\partial V} = \frac{\partial U_V}{\partial N}, \quad \frac{\partial U_S}{\partial N} = \frac{\partial U_N}{\partial S}, \quad \frac{\partial U_V}{\partial S} = \frac{\partial U_S}{\partial V}$$

Of course you can't evaluate the curl of state function, but hear me out. What we consider is not the function U itself, but rather, the differential dU. Its total differential may be written as

$$dU(S,V,N) = \left.\frac{\partial U}{\partial S}\right|_{V,N} dS + \left.\frac{\partial U}{\partial V}\right|_{S,N} dV + \left.\frac{\partial U}{\partial N}\right|_{S,V} dN$$

Here, the differentials (dS, dT, dN) act like unit vector, thus we can pretend that dU is a vector field with components of

$$U_S = \left.\frac{\partial U}{\partial S}\right|_{V,N}, \quad U_V = \left.\frac{\partial U}{\partial V}\right|_{S,N}, \quad U_N = \left.\frac{\partial U}{\partial N}\right|_{S,V}$$

Therefore

$$\frac{\partial}{\partial V}\frac{\partial U}{\partial N} = \frac{\partial}{\partial N}\frac{\partial U}{\partial V}, \quad \frac{\partial}{\partial N}\frac{\partial U}{\partial S} = \frac{\partial}{\partial S}\frac{\partial U}{\partial N}, \quad \frac{\partial}{\partial S}\frac{\partial U}{\partial V} = \frac{\partial}{\partial V}\frac{\partial U}{\partial S}$$

This is what it means to be an exact differential.

Fourier Series and Transform

Introduction

Sinusoidsal wave Equation.

$$y = A \sin \frac{2\pi}{\lambda}(x - vt)$$

where λ represent wavelength, but mathematically it is the same as the period of this function of x. Wave equation in single variable.

$$\begin{aligned} y(x) &= A \sin kx & &= A \sin 2\pi f x = A \sin \frac{2\pi}{\lambda} x \\ y(t) &= A \sin \omega t & &= A \sin 2\pi v t = A \sin \frac{2\pi}{T} t \end{aligned}$$

Average Value. Average of $f(x)$ on (a, b) is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Here are some usefull integrals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \sin nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos mx \cos nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 1, & m = n = 0 \end{cases}$$

Fourier Series

2π period. Fourier Series for function of period 2π:

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with coefficients:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \end{aligned}$$

Proof. Multiply both sides of Fourier series by $\cos nx$ and find the average value of each term

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} & f(x)\cos mx\; dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum\big(a_n\cos nx+b_n\sin nx\big)\right]\cos mx\; dx\end{aligned}$$

All terms on the right are zero except the a_n term then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\; dx = \frac{a_n}{2} \qquad \blacksquare$$

Notice that $\cos mx$ now turns into $\cos nx$–this is because the integral picks the value of n such that $m = n$. For b_n, we multiply we multiply both sides of by $\sin nx$ and take average values just as we did before

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} & f(x)\sin mx\; dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum\big(a_n\cos nx+b_n\sin nx\big)\right]\sin mx\; dx\end{aligned}$$

and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\; dx = \frac{b_n}{2} \qquad \blacksquare$$

To find c_n, we multiply Fourier series by $\exp(-imx)$ and again find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(imx) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} c_n \exp inx\right]\exp(-imx)\; dx$$

All these terms are zero except the one where $n = m$. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n \exp ix(n-m)\; dx$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) \qquad \blacksquare$$

Other period. Fourier Series for function of period $2l$:

$$\begin{aligned} f(x) &= \frac{a_0}{2}+\sum_{1}^{\infty}\left(a_n\cos\frac{\pi n x}{l}+b_n\sin\frac{\pi n x}{l}\right) \\ &= \sum_{n=-\infty}^{\infty} c_n \exp\frac{in\pi x}{l}\end{aligned}$$

with coefficients:

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi n x}{l}\; dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi n x}{l}\; dx \\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)\exp\frac{-in\pi x}{l}\; dx\end{aligned}$$

Fourier Transform

The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set. Fourier transforms are defined as follows

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x}\, d\alpha$$
$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx$$

$g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int$ corresponds to $\sum$. This agrees with our discussion of the physical meaning and use of Fourier integrals.

Fourier Sine Transforms. We define $f_s(x)$ and $g_s(\alpha)$ as pair of Fourier sine transforms representing odd functions.

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin \alpha x\, d\alpha$$
$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin \alpha x\, dx$$

Fourier Cosine Transforms. We define $f_c(x)$ and $g_c(\alpha)$ as pair of Fourier cosine transforms representing even functions.

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\alpha) \cos \alpha x\, d\alpha$$
$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \cos \alpha x\, dx$$

Proof (?). We rewrite Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp i\alpha_n x$$
$$c_n = \frac{1}{2l} \int_{-l}^{l} f(u) \exp(-i\alpha_n u)\, du$$

where

$$\frac{n\pi}{l} = \alpha_n$$
$$\alpha_{n+1} - \alpha_n = \Delta\alpha = \frac{\pi}{l}$$

Then

$$c_n = \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp(-\alpha_n u)\, du$$

Subsituting c_n into $f(x)$

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(x)\exp(-\alpha_n x)\; du \right] \exp \alpha_n x \\
&= \sum_{n=-\infty}^{\infty} \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp i\alpha_n(x-u)\; du \\
f(x) &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha
\end{aligned}
$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u)\exp i\alpha_n(x-u)\; du$$

If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity],

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)\exp i\alpha(x-u)\; du$$

then $\Delta\alpha \to 0$

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\; d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\exp i\alpha(x-u)\; du\; d\alpha
\end{aligned}
$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\exp(-i\alpha x)\; dx$$

then

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)\exp i\alpha x\; d\alpha$$

Now we expand $\exp(-i\alpha x)$ inside $g(\alpha)$ expression

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha x)\; dx$$

If we assume that $f(x)$ is odd, we get

$$g(x) = -\frac{i}{\pi} \int_{0}^{\infty} f(x) i \sin\alpha x\; dx$$

since the product of odd function $f(x)$ and even function $cos\alpha x$ is odd, thus the integral is zero. Then expanding the exponential in $f(x)$

$$f(x) = 2i \int_{0}^{\infty} g(\alpha)\sin\alpha x\; d\alpha$$

If we substitute $g(\alpha)$ into $f(x)$, we obtain

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(x)\sin^2 \alpha x\; dx\; d\alpha$$

and we see that the numerical factor is $2/\pi$, thus the imaginary factors are not needed. We may as well write $\sqrt{2/\pi}$ instead. Now suppose that $g(\alpha)$ is even. As before, we have

$$g(x) = \frac{1}{\pi}\int_0^\infty f(x) i \cos \alpha x \, dx$$

and

$$f(x) = 2\int_0^\infty g(\alpha)\cos \alpha x \, d\alpha$$

Subsituting $g(x)$ into $f(x)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(x)\cos^2 \alpha x \, dx \, d\alpha$$

We also see that it has the same numerical factor and all.

Even and Odd Function

Definition.

$$f(x) = \begin{cases} f(x) = f(-x) & \text{even} \\ f(-x) = -f(x) & \text{odd} \end{cases}$$

Integral of Even and Odd Function.

$$\int_{-l}^{l} f(x)\, dx \begin{cases} 0 & \text{odd} \\ 2\displaystyle\int_0^l f(x)\, dx & \text{even} \end{cases}$$

Fourier expansion for odd function.

$$\text{odd } f(x), \begin{cases} a_n & = 0 \\ b_n & = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{\pi n x}{l}\, dx \end{cases}$$

Fourier expansion for even function.

$$\text{even } f(x), \begin{cases} a_n & = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos\frac{\pi n x}{l}\, dx \\ b_n & = 0 \end{cases}$$

Theorem

Dirichlet Condition. If $f(x)$:

1. periodic,
2. x single valued,
3. finite number of discontinuities,
4. finite min max, and
5. $\int_{-\pi}^{\pi} |f(x)|\, dx = \text{finite}$

then the Fourier series converges to the midpoint of the jump.

Parseval's theorem. For Fourier expansions

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

we have

$$\text{The average of } [f(x)]^2 \text{ is } \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx$$

with the value of each coefficients

$$\begin{aligned} \text{The average of } (\tfrac{1}{2}a_0)^2 &\quad \text{is} \quad & (\tfrac{1}{2}a_0)^2 \\ \text{The average of } (a_n \cos nx)^2 &\quad \text{is} \quad & \tfrac{1}{2}a_n^2 \\ \text{The average of } (b_n \sin nx)^2 &\quad \text{is} \quad & \tfrac{1}{2}b_n^2 \end{aligned}$$

then we have

$$\text{The average of } [f(x)]^2 = \left(\frac{1}{2}1_0\right)^2 + \frac{1}{2}\sum_{1}^{\infty} a_n^2 + \frac{1}{2}\sum_{1}^{\infty} b_n^2$$

or in complex expansion

$$\text{The average of } [f(x)]^2 = \sum_{-\infty}^{\infty} |c_n|^2$$

Ordinary Diferential Equation

First Order

First Order ODE. Written in the form

$$y' + P(x)y = Q(x)$$

where P and Q are functions of x has the solution

$$ye^I = \int Qe^I \, dx + c$$
$$y = e^{-I} \int Qe^I \, dx + ce^{-I}$$

where

$$I = \int P \, dx$$

Bernoulli Equation. The differential equation

$$y' + P(x)y = Q(x)y^n$$

where P and Q are functions of x. It also can be written as

$$z' + (1-n)Pz = (1-n)Q$$

where

$$z = y^{1-n}$$

This is now a first-order linear equation which we can solve as we did the linear equations above.

Exact Equations. $P(x,y)dx + Q(x,y)dy$ is an exact differential [the differential of $F(x,y)$, or $Pdx + Qdy = dF$] if

$$\frac{\partial}{\partial x}P = \frac{\partial}{\partial x}Q$$

and the solution is

$$F(x,y) = \text{constant}$$

An equation which is not exact may often be made exact by multiplying it by an appropriate factor.

Homogeneous Equations. A homogeneous function of x and y of degree n means a function which can be written as $x^n f(y/x)$. An equation in the form

$$P(x,y)dx + Q(x,y)dy = 0$$

where P and Q are homogeneous functions of the same degree is called homogeneous. Thus,

$$y' = \frac{d}{dx}y = -\frac{P(x,y)}{Q(x,y)} - f(\frac{y}{x})$$

This suggests that we solve homogeneous equations by making the change of variables

$$y = xv \quad \text{with} \quad v = \frac{y}{x}$$

Second Order

Second Order with Zero Right-Hand Side. Equation of the form

$$(D-a)(D-b)y = 0, \quad a \neq b$$

has the Solution

$$y = c_1 e^{ax} + c_2 e^{bx}$$

Equation of the form

$$(D-a)(D-a)y = 0, \quad a \neq b$$

has the Solution

$$y = (Ax + B)e^{ax}$$

Now suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. The solution is now

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)}x + Be^{(\alpha-i\beta)}x \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \\ &= e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) \\ &= ce^{\alpha x} \sin(\beta x + \gamma) \end{aligned}$$

where $\alpha,\ \beta,\ \gamma,\ c,\ c_1,\ c_2$ are different constant.

Second Order with Nonzero Right-hand Side. The equation

$$\begin{aligned} a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= f(x) \\ \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= F(x) \end{aligned}$$

has the solution of the form

$$y = y_c + y_p$$

where the complementary function y_c is the general solution of the homogeneous equation (when right-hand side is equal to zero) and y_p is a particular solution, that is when the right-hand side is equal to $f(x)$ or $F(x)$. The simplest method solving them is by Inspection and Successive Integration of Two First-Order Equations.

Exponential Right-Hand Side. Suppose we have $F(x) = ke^{cx}$, or

$$(D-a)(D-b)y = ke^{cx}$$

then, we find a particular solution by assuming a solution of the form:

$$y_p = \begin{cases} Ce^{cx} & \text{if c is not equal to either a or b;} \\ Cxe^{cx} & \text{if c equals a or b, a} \neq \text{b;} \\ Cx^2e^{cx} & \text{if c = a = b.} \end{cases}$$

Complex Exponential. To find a particular solution of

$$(D-a)(D-b)y = \begin{cases} k\sin\alpha x, \\ k\cos\alpha x, \end{cases}$$

first solve

$$(D-a)(D-b)y = ke^{i\alpha x}$$

then take the real or imaginary part.

Method of Undetermined Coefficients. To find a particular solution of

$$(D-a)(D-b)y = e^{cx}P_n(x)$$

where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if c is not equal to either a or b;} \\ xe^{cx}Q_n(x) & \text{if c equals a or b, a} \neq \text{b;} \\ x^2e^{cx}Q_n(x) & \text{if c = a = b.} \end{cases}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the given differential equation.

Euler's equation. Has the form of

$$a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = f(x)$$

By substituting $x = e^z$, we obtain the following equation

$$a_2\frac{d^2y}{dx^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = f(e^z)$$

Proof. First, we compute the first derivative of y

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = e^z\frac{dy}{dx} = x\frac{dy}{dx}$$

Then the second

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(e^z\frac{dy}{dx}\right) = e^z\frac{dy}{dx} + e^z\frac{d}{dz}\left(\frac{dy}{dx}\right)$$
$$\frac{d^2y}{dz^2} = \frac{dy}{dz} + x\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dz} = \frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

And substituting it

$$a_2\left(\frac{d^2y}{dz^2} - \frac{dy}{dx}\right) + a_1\frac{dy}{dz} + a_0y = f(e^z)$$
$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = f(e^z)$$

Principle of Superposition. The easiest way of handling a complicated right-hand side: Solve a separate equation for each different exponential and add the solutions. The fact that this is correct for a linear equation is often called the principle of superposition.

Note that the principle holds only for linear equations.

Fourier Series. Suppose that the driving force $f(x)$ is periodic, we then can expand the function using Fourier Series. The equation

$$a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y = f(x) = \sum_{-\infty}^{\infty} c_n e_{inx}$$

can be solved by solving

$$a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y = c_n e_{inx}$$

then add the solutions for all n (applying principle of superposition), and we have the solution of first the equation.

Laplace Transform

We define $\mathcal{L}(f)$, the Laplace transform of $f(t)$ [also written $F(p)$ since it is a function of p], by the equation

$$\mathcal{L}(f) = F(P) = \int_0^\infty f(t)e^{-pt}; dt$$

Laplace transform 101. How 2 Laplace transform in 5 steps!

1. Transform!
2. Do algebra!
3. Inverse!
4. ...
5. Profit!

Convolution

Definition. The integral

$$g * h = \int_0^t g(t-\tau)h(\tau)d(\tau) = \int_0^t g(\tau)h(t-\tau)d(\tau)$$

is called the convolution of g and h (or the resultant or the Faltung). Now suppose that we have

$$Ay' + By' + Cy = f(t), \quad y0 = y'0 = 0$$

take the Laplace transform of each term, substitute the initial conditions, and solve for Y

$$Y = \frac{F(p)}{A(p+a)(p+b)} = T(p)F(p)$$

Then y the inverse transform of Y in is the inverse transform of a product of two functions whose inverse transforms we know. Let $G(p)$ and $H(p)$ be the transforms of $g(t)$ and $h(t)$

$$G(p)H(p) = \mathcal{L}(g(t) \cdot h(t)) = \mathcal{L}(g * h)$$

Thus

$$y = \int_0^t g(t-\tau)h(\tau)d(\tau)$$

Observe from $\mathcal{L}34$ that we may use either $g(t-\tau)h(\tau)$ or $g(\tau)h(t-\tau)$ in the integral. It is well to choose whichever form is easier to integrate; it is best to put$(t-\tau)$ in the simpler function.

Fourier Transform of a Convolution. Let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(v) e^{-i\alpha v} dv \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(u) e^{-i\alpha u} du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(v) f_2(u) e^{-i\alpha(v+u)} dv du \end{aligned}$$

Next we make the change of variables $x = v + u$, $dx = dv$, in the v integral

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x-u) f_2(u) e^{-i\alpha x} dv du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{-i\alpha x} \left[\int_{-\infty}^{\infty} f_1(x-u) f_2(u)\, du \right] dx \end{aligned}$$

if we define the term in the square parenthesis as convolution, we get

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} f_1 * f_2 e^{-i\alpha x} dx \right) \\ &= \frac{1}{2\pi} \cdot \text{Fourier transform of } f_1 * f_2 \end{aligned}$$

In other words

$$g_1 \cdot g_2 \text{ and } f_1 * f_2 \text{ are a pair of Fourier transforms}$$

and by symmetry

$$g_1 * g_2 \text{ and } f_1 \cdot f_2 \text{ are a pair of Fourier transforms}$$

Rodrigues' Formula

The second order Sturm-Liouville differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda_n w(x) y$$

Has the solution $y = y_n$, with

$$y_n = \frac{1}{w(x)} \frac{d^n}{dx^n}[w(x)p(x)^n]$$

Alternatively, the Sturm-Liouville equation can be written as

$$a(x)y'' + b(x)y' + c(x)y = -\lambda_n \tilde{w}(x) y$$

then we define the integrating factor such that $w(x) = \mu(x)\tilde{w}(x)$, where

$$\mu(x) = \frac{1}{a(x)} \exp\left(\int_0^x \frac{b(x)}{a(x)} dx\right)$$

Using this definition, the solution reads

$$y_n = \frac{1}{\mu(x)\tilde{w}(x)} \frac{d^n}{dx^n}[\mu(x)\tilde{w}(x)a(x)^n]$$

In most cases $c = 0, \tilde{w}(x) = 1$ and the equation takes the form of

$$a(x)y'' + b(x)y' + \lambda_n y = 0$$

Then the solution reads

$$y_n = \frac{1}{\mu(x)} \frac{d^n}{dx^n}[\mu(x)a(x)^n]$$

Rewriting. First we begin with the form

$$ay'' + by' + cy' = -\lambda\tilde{w}y$$

Multiply by the integrating factor

$$\mu(ay'' + by' + cy') = -\lambda\mu\tilde{w}y$$

To equating this form with the Sturm-Liouville form

$$py'' + p'y' + qy = -\lambda wy$$

we define

$$p \equiv \mu a, \; p' \equiv \mu b, \; q \equiv \mu c, \; w = \mu\tilde{w}$$

From this definition, we see

$$\frac{d}{dx}\mu a = \mu' a + \mu a' = \mu b$$

Dividing by the integrating factor and solving for the integrating factor

$$\begin{aligned}
\frac{\mu'}{\mu}a + a' &= b \\
\frac{\mu'}{\mu} &= \frac{b - a'}{a} \\
\int \frac{1}{\mu} d\mu &= \int \frac{b - a'}{a} dx \\
\mu &= \exp\left(\int \frac{b - a'}{a} dx\right)
\end{aligned}$$

Since

$$\int \frac{a'}{a} dx = \int \frac{1}{a} da = \ln a$$

we can write

$$\mu = \frac{1}{a} \exp\left(\int \frac{b}{a} dx\right)$$

Therefore, we have the solution

$$y_n = \frac{1}{\mu\tilde{w}} \frac{d^n}{dx^n} [\mu\tilde{w}(\mu a)^n]$$

This should be the correct solution, however it is not. I don't know why, but the solution is actually

$$y_n = \frac{1}{\mu\tilde{w}} \frac{d^n}{dx^n} [\mu\tilde{w}a^n]$$

I wonder why the term μ^n disappears...

Frobenius Method

By using this method, we assume that the solution has the form of power series

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

We also assume that the first coefficient, that is a_0, is not zero. Computing the derivative of y, we obtain

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+s} \\ y' &= \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \\ y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} \end{aligned}$$

Frobenius 101. How 2 solve differential equation using generalized power series in 5 steps!

1. Tabulate!
2. Find the column in terms of x^{n+s} $x^s \rightarrow$!
3. Factor the coefficients that contain $a_0 \rightarrow$ and solve the indicial equation!
4. Solve it in terms of $a_n = -a_{n-2}$! (not factorial!)
5. As a check, put $n = 2$ at a_n not $n = 0$! (also not factorial!)

Bessel Function

The first kind of Bessel function is written as

$$\begin{aligned} J_p(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{\Gamma(n+1)\Gamma(n+p+1)} \\ J_{-p}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{\Gamma(n+1)\Gamma(n-p+1)} \end{aligned}$$

While the second kind is

$$N_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}$$

The Bessel function is used to solve the Bessel's equation of order p

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

with the solution written as

$$y = AJ_p(x) + BN_p(x)$$

Another form of Bessel's equation is

$$x(xy')' + (K^2x^2 - p^2)y = 0$$

and the solution is

$$y = AJ_p(Kx) + BN_p(Kx)$$

Another equation that can be solved by Bessel function

$$y'' + \frac{1-2a}{x} y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] y = 0$$

The solution is

$$y = x^a Z_p(bx^c)$$

where a, b, c, p are constant and Z denote J or N or any linear combination of them.

The generating function, expression that encodes an infinite sequence, for Bessel function is

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{p=-\infty}^{\infty} J_n(x) t^p$$

Series representation derivation. First we write the Bessel's equation as

$$x(xy')' + (x^2 - p^2)y = 0$$

By the Frobenius' method

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$$

$$(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

and

$$x(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}$$

Table 2: Table

	x^{n+s}	x^s	x^{s+1}
$x(xy')'$	$a_n(n+s)^2$	$a_0 s^2$	$a_1(s+1)^2$
x^2y	a_{n-2}	$-$	$-$
$-p^2y$	$-a_n p^2$	$-a_0 p^2$	$-a_1 p^2$

$$x^2y = \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$-p^2y = -\sum_{n=0}^{\infty} a_n p^2 x^{n+s}$$

Tabulate them

From this we have the indicial equation

$$s^2 - p^2 = 0 \implies s = \pm p$$

And the general formula of the coefficient

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2}$$

For $s = \pm p$ and odd n, the coefficient is zero; proved by

$$a_1\left[(s+1)^2 - p^2\right] = a_1\left[2p+1\right] = 0 \implies a_1 = 0$$

We begin first for the case $s = p$. The coefficient is given by

$$a_n = -\frac{a_{n-2}}{(n+p)^2 - p^2} = -\frac{a_{n-2}}{n^2 - 2np} = -\frac{a_{n-2}}{n(n+2p)}$$

For even n, we write

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n+2p)} = -\frac{a_{2n-2}}{2^2 n(n+p)}$$

The coefficients for few odd n are as follows.

$$a_2 = -\frac{a_0}{2^2(p+1)} = -\frac{a_0\Gamma(p+1)}{2^2\Gamma(p+2)}$$

$$a_4 = -\frac{a_2}{2^2 2(p+2)} = -\frac{a_2\Gamma(p+2)}{2^2 2\Gamma(p+3)} = \frac{a_0\Gamma(p+1)}{2^4 2!\Gamma(p+3)}$$

$$a_6 = -\frac{a_4}{2^2 3(p+3)} = -\frac{a_4\Gamma(p+3)}{2^2 3\Gamma(p+4)} = -\frac{a_0\Gamma(p+1)}{2^6 3!\Gamma(p+4)}$$

The solution is written

$$y = \sum_{n=0}^{\infty} a_n x^{n+p} = a_0 x^p + a_2 x^{p+2} + a_4 x^{p+4} + a_6 x^{p+6}$$

$$= a_0 x^p \Gamma(p+1)\left[\frac{1}{\Gamma(p+1)} - \frac{(x/2)^2}{\Gamma(p+2)} + \frac{(x/2)^4}{2!\Gamma(p+3)} - \frac{(x/2)^6}{3!\Gamma(p+4)} + \dots\right]$$

$$= a_0 2^p \Gamma(p+1) \left(\frac{x}{2}\right)^p \left[\frac{1}{\Gamma(1)\Gamma(p+1)} - \frac{(x/2)^2}{\Gamma(2)\Gamma(p+2)} + \frac{(x/2)^4}{\Gamma(3)\Gamma(p+3)} - \frac{(x/2)^6}{\Gamma(4)\Gamma(p+4)} + \ldots\right]$$

If we define

$$a_0 = \frac{1}{2p\Gamma(p+1)}$$

then the solution, which is defined as $J_p(x)$, is written

$$J_p(x) = \frac{(x/2)}{\Gamma(1)\Gamma(p+2)} - \frac{(x/2)^{p+2}}{\Gamma(3)\Gamma(p+3)} + \frac{(x/2)^{p+4}}{\Gamma(3)\Gamma(p+3)} - \frac{(x/2)^{p+6}}{\Gamma(4)\Gamma(p+4)} + \ldots$$

or

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{\Gamma(n+1)\Gamma(n+p+1)}$$

Next we consider the solution for $s = -p$. Since the steps are the same, we only need to change the sign of p. The solution is written

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{\Gamma(n+1)\Gamma(n-p+1)}$$

As an aside, for the Bessel equation written in the form

$$x^2 y'' + xy' + (K^2 x^2 - p^2)y = 0$$

All the terms are unchanged except the term

$$K^2 x^2 y = \sum_{n=0}^{\infty} a_n K^2 x^{n+s+2}$$

This will result the change of argument in the Bessel equation from $Z(x)$ into $Z(Kx)$.

Generating function proof. Proved by writing it out.

$$\exp\left[\frac{xt}{2}\right] \exp\left[-\frac{x}{2t}\right] = \sum_{n=0}^{\infty} \frac{(xt/2)^n}{n!} \sum_{m=0}^{\infty} \frac{(-x/2t)^m}{m!}$$

$$\exp\left[\frac{xt}{2}\right] \exp\left[-\frac{x}{2t}\right] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{t^{n-m}}{n!m!} \left(\frac{x}{2}\right)^{n+m}$$

We then define $p \equiv n - m$ to shift the indices

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{p=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{p+2m}}{(p+m)!m!} t^p = \sum_{p=-\infty}^{\infty} J_p(x) t^p$$

Recursive relation. Here are few relations of Bessel function with its derivative.

$$
\begin{aligned}
J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x) \\
J_{p-1}(x) - J_{p+1}(x) &= 2J_p'(x) \\
\frac{d}{dx}[x^p J_p(x)] &= x^p J_{p-1}(x) \\
\frac{d}{dx}[x^{-p} J_p(x)] &= -x^{-p} J_{p+1}(x) \\
J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) &= \frac{p}{x} J_p(x) - J_{p+1}(x)
\end{aligned}
$$

And bonus relation that only apply for integral p

$$
\begin{aligned}
J_{-p}(x) &= (-1)^p J_p(x) \\
J_p(-x) &= (-1)^p J_p(x)
\end{aligned}
$$

First relation proof. Differentiate the expression for generating function with respect to t

$$
\begin{aligned}
\frac{x}{2}\left(1 + \frac{1}{t^2}\right) \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] &= \sum_{p=-\infty}^{\infty} pJ_p(x)t^{p-1} \\
\sum_{p=-\infty}^{\infty} \left[J_p(x) + J_{p-2}(x)\right] t^p &= \sum_{p=-\infty}^{\infty} \frac{2p}{x} J_{p+1} t^p
\end{aligned}
$$

Taking the constant for the term t^p, we have

$$
J_p(x) + J_{p-2}(x) = \frac{2p}{x} J_{p+1}
$$

Second relation proof. Differentiate with respect to x instead

$$
\begin{aligned}
\frac{1}{2}\left(t - \frac{1}{t}\right) \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] &= \sum_{p=-\infty}^{\infty} J_p'(x)t^p \\
\sum_{p=-\infty}^{\infty} \left[J_{p+1}(x) + J_{p-1}(x)\right] t^p &= \sum_{p=-\infty}^{\infty} 2J_p' t^p
\end{aligned}
$$

and as before, we have

$$
J_{p+1}(x) + J_{p-1}(x) = 2J_p'
$$

Third relation proof. Simply evaluate the derivative and use both first and second relation

$$
\begin{aligned}
\frac{d}{dx}\left[x^p J_p(x)\right] &= \frac{x^p}{2}\left[J_{p-1}(x) + J_{p+1}(x)\right] + \frac{x^p}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right] \\
\frac{d}{dx}\left[x^p J_p(x)\right] &= x^p J_{p-1}(x)
\end{aligned}
$$

Fourth relation proof. The same as the third

$$
\begin{aligned}
\frac{d}{dx}\left[x^{-p} J_p(x)\right] &= -\frac{x^{-p}}{2}\left[J_{p-1}(x) + J_{p+1}(x)\right] + \frac{x^p}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right] \\
\frac{d}{dx}\left[x^{-p} J_p(x)\right] &= x^{-p} J_{p+1}(x)
\end{aligned}
$$

Fifth relation proof. Add both first and second relation to obtain the middle side

$$\begin{aligned} 2J_{p-1}(x) &= \frac{2p}{x}J_p(x) + 2J_p'(x) \\ J_p'(x) &= J_{p-1}(x) - \frac{p}{x}J_p(x) \end{aligned}$$

and subtract to obtain the right side

$$\begin{aligned} 2J_{p+1}(x) &= \frac{2p}{x}J_p(x) - 2J_p'(x) \\ J_p'(x) &= \frac{p}{x}J_p(x) - J_{p+1}(x) \end{aligned}$$

Orthogonality. Suppose α and β are the zeros of the Bessel function order p. We can say that the function $\sqrt{x}J_p(\alpha x)$ is orthogonal with itself on $(0,1)$. We can also say that the functions $J_p^2(\alpha x)$ are orthogonal with respect with weight function x. Thus, we write

$$\int_0^1 xJ_p(\alpha x)J_p(\beta x)\,dx = \begin{cases} 0 & \alpha \neq \beta \\ \frac{1}{2}J_p'^2(\alpha) = \frac{1}{2}J_{p+1}^2(\alpha) = \frac{1}{2}J_{p-1}^2(\alpha) & \alpha = \beta \end{cases}$$

We can change the integration limit by substituting $x = r/a$

$$\int_0^a rJ_p\left(\alpha\frac{r}{a}\right)J_p\left(\beta\frac{r}{a}\right)\,dr = \begin{cases} 0 \\ \quad \alpha \neq \beta \\ \frac{a^2}{2}J_p'^2(\alpha) = \frac{a^2}{2}J_{p+1}^2(\alpha) = \frac{a^2}{2}J_{p-1}^2(\alpha) \\ \quad \alpha = \beta \end{cases}$$

Proof. To prove the relation orthogonality of the Bessel function on $(0,1)$ with respect to the weight function x, consider the equations

$$\begin{aligned} x(xy')' + (\alpha^2x^2 - p^2)y &= 0 \\ x(xy')' + (\beta^2x^2 - p^2)y &= 0 \end{aligned}$$

which are solved by the functions $J_p(\alpha x)$ and $J_p(\beta x)$ respectively. For brevity's sake, we define $J_p(\alpha x) \equiv u$, $J_p(\beta x) \equiv v$ and we write

$$\begin{aligned} x(xu')' + (\alpha^2x^2 - p^2)u &= 0 \\ x(xv')' + (\beta^2x^2 - p^2)v &= 0 \end{aligned}$$

Multiplying the first equation with u and the second with v

$$\begin{aligned} xv(xu')' + (\alpha^2x^2 - p^2)uv &= 0 \\ xu(xv')' + (\beta^2x^2 - p^2)vu &= 0 \end{aligned}$$

Subtracting them

$$\begin{aligned} xv(xu')' - xu(xv')' + (\alpha^2 - p^2)x^2uv &= 0 \\ v(xu')' - u(xv')' + (\alpha^2 - p^2)xuv &= 0 \end{aligned}$$

Note that we can write the first two terms as

$$\frac{d}{dx}\left(vxu' - uxv'\right) = x'xu' + v(xu)' - u'xv' - u(xv')'$$

$$= v(xu')' - u(xv')'$$

On integrating it within $(0,1)$

$$(vxu' - uxv')\Big|_0^1 + \int_0^1 (\alpha^2 - p^2)xuv\ dx = 0$$

By the definition

$$J_p(\beta)J_p'(\alpha) - J_p(\alpha)j_p'(\beta) + \int_0^1 (\alpha^2 - p^2)xJ_p(\alpha)j_p(\beta)\ dx = 0$$

where the value at lower limit of those two terms are zero. Since both α and β are the zeros of the Bessel functions

$$\int_0^1 (\alpha^2 - p^2)xJ_p(\alpha)J_p(\beta)\ dx = 0$$

If $\alpha \neq \beta$, the terms inside parenthesis are not equal to zero. Hence,

$$\int_0^1 xJ_p(\alpha)J_p(\beta)\ dx = 0$$

If $\alpha \neq \beta$, the terms inside parenthesis are not equal to zero. Hence, we can simply divide both side by it and the integral is not zero. To find its value, suppose that β is not a zero, unlike α. We can write

$$\int_0^1 xJ_p(\alpha)J_p(\beta)\ dx = \frac{\alpha J_p(\beta)J_p(\alpha)}{\beta^2 - \alpha^2}$$

Now we let $\beta \to \alpha$ and evaluate the right term using L'Hôpital's rule to find

$$\lim_{\beta\to\alpha} \frac{\alpha J_p(\beta)J_p'(\alpha)}{\beta^2 - \alpha^2} = \lim_{\beta\to\alpha} \frac{\alpha J_p'(\beta)J_p'(\alpha)}{2\beta} = \frac{1}{2}J_p'(\alpha)$$

Hankel's function. If the first and the second kind of Bessel function are analog to sin and cos, the Hankel's function is an analog to $\exp(\pm ix) = \cos x \pm i \sin x$. The function is defined as

$$H_p^{(1)}(x) = J_p(x) + iN_p(x)$$
$$H_p^{(2)}(x) = J_p(x) - iN_p(x)$$

Hyperbolic Bessel functions. This function is the solution of

$$x^2y'' + xy' - (x^2 + p^2)y = 0$$

and defined as

$$I_p = i^{-p}J_p(ix)$$
$$K_p(x) = \frac{\pi}{2}i^{p+1}H_p^{(1)}(ix)$$

They are analog to $\sinh x = -i\sin(ix)$ and $\cosh x = \cos(ix)$ respectively.

Spherical Bessel functions. If p is a half integer then the Bessel function $Z_p(x)$ is defined to be spherical Bessel function and defined as follows

$$\begin{aligned}
j_n(x) &= \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) = \left(-\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \\
y_n(x) &= \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) = \left(\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right) \\
h_n^{(1)}(x) &= j_n(x) + iy_n(x) \\
h_n^{(2)}(x) &= j_n(x) - iy_n(x)
\end{aligned}$$

From this we can obtain

$$J_{1/2}(x) = \frac{2}{\pi x} \sin x, \quad J_{-1/2}(x) = \frac{2}{\pi x} \cos x$$

Legendre Function

The Legendre differential equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$$

has the solution of Legendre polynomial, which by the Rodrigues formula is

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

The Legendre polynomial is defined such that $y(1) = P_l(1) = 1$.

Another closely related equation is

$$(1 - x^2)y'' - 2xy' + \left[l(l + 1) - \frac{m^2}{1 - x^2}\right] y = 0$$

or by substituting $x = \cos\theta$

$$\frac{d^2\Phi}{d\theta^2} + \frac{\cos\theta}{\sin\theta} \frac{d\Theta}{d\theta} + \left[l(l + 1) - \frac{m^2}{\sin^2\theta}\Theta\right] \Theta = 0$$

The solution of said function is the associated Legendre function, which defined by the Rodrigues formula as

$$P_l^m(x) = (1 - x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_l(x)$$

or

$$P_l^m(x) = \frac{1}{2^l l!} (1 - x^2)^{m/2} \left(\frac{d}{dx}\right)^{l+m} (x^2 - 1)^l$$

For negative value of m, we have

$$P_l^{-m}(x) = (-1)^m \frac{(l - m)!}{(l + m)!} P_l^m(x)$$

Negative value of m has the same polynomial order with the positive one, they only differ in constant. Whereas for negative x

$$P_l^{-m}(-x) = (-1)^{l+m} P_l^m(x)$$

Laplace integral representation. The integral representation for the Legendre polynomial is

$$P_l(x) = \frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\cos\theta\right)^l\,d\theta$$

where the domain is $|x| > 1$ due to the square root term.

Series derivation. By assuming the solution has the form of power series, we can use the Frobenius method. We also assume that $s = 0$ for simplification. Thus, each term can be represented as power series

			x^n
y''	$=$	$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$	$(n+2)(n+1)a_{n+2}$
$-x^2y''$	$=$	$\sum_{n=0}^{\infty} n(n-1)a_n x^n$	$-n(n-1)a_n$
$-2xy'$	$=$	$\sum_{n=0}^{\infty} n a_n x^n$	$-na_n$
$l(l+1)y$	$=$	$\sum_{n=0}^{\infty} l(l+1)a_n x^n$	$l(l+1)a_n$

From the x^n coefficient

$$(n+2)(n+1)a_{n+2} = -a_n[-n(n-1) - n + l(l+1)]$$

We write the coefficient of the a_s as follows

$$\begin{aligned}-n(n-1) - n + l(l+1) &= -n^2 - 2n + l^2 + l = l^2 - n^2 + l - n \\ &= (l+n)(l-n) + l - n = (l-n)(l+n+1)\end{aligned}$$

The formula for $n+2$ term is then

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n$$

Here we have few terms

$$\begin{aligned}a_2 &= -\frac{l(l+1)}{2!}a_0 \\ a_3 &= -\frac{(l-1)(l+2)}{3!}a_1 \\ a_4 &= -\frac{(l-2)(l+3)}{4\cdot 3}a_2 = \frac{l(l+1)(l-2)(l+3)}{4!}a_0 \\ a_5 &= -\frac{(l-3)(l+4)}{5\cdot 4}a_3 = \frac{(l-1)(l+2)(l-3)(l+4)}{5!}a_1\end{aligned}$$

Since neither a_0 and a_1 not zero, the solution is a superposition of two series in terms of a_0 and a_1

$$y = a_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \dots\right]$$

$$+ a_1 \left[x - \frac{(l-1)(l+2)}{3!} x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!} x^5 - \dots \right]$$

Now consider $l = 0$. The solution takes the form

$$y = a_0 + a_1 \left[x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \dots \right]$$

At $x^2 = 1$ the a_1 series is divergent by the integral test

$$\int^{\infty} \frac{1}{2n+1} \, dn = \frac{1}{2} \ln(2n+1) \bigg|^{\infty} = \infty$$

Therefore we throw the a_1 series out. By the definition of Legendre polynomial we have $a_0 = 1$, thus $P_0(x) = 1$. In simple terms, we cans say that for odd value of l, we throw the even value of constant a_0; and for even value of l, we throw away the odd value of constant a_1.

We can use this method to determine the value of $P_l(x)$ for other l, but this method is simply terrible to use. There are other method that are more efficient, Rodrigues formula for example.

Rodrigues formula proof. Consider the function $v = (x^2-1)^l$. Differentiate it with respect to x

$$\frac{dv}{dx} = l(x^2-1)^{l-1} 2x$$

Multiply it with $(x^2 - 1)$

$$(x^2-1) \frac{dv}{dx} = 2lxv$$

Differentiate $l+1$ times, which according to the Leibniz' rule for differentiation

$$\begin{aligned} \frac{d^{l+1}}{dx^{l+1}} fg &= \sum_{k=0}^{l+1} \frac{(l+1)!}{k!(l+1-k)!} \left(\frac{d}{dx} \right)^{l+1-k} f \left(\frac{d}{dx} \right)^k g \\ &= \left(\frac{d}{dx} \right)^{l+1} f \left(\frac{d}{dx} \right)^0 g + (l+1) \left(\frac{d}{dx} \right)^l f \left(\frac{d}{dx} \right)^1 g \\ &\quad + \frac{l(l+1)}{2} \left(\frac{d}{dx} \right)^{l-1} f \left(\frac{d}{dx} \right)^2 g + \dots \end{aligned}$$

For the left side, we take $f = dv/dx$ and $g = (x^2 - 1)$

$$\frac{d^{l+1}}{dx^{l+1}} \left[\frac{dv}{dx} (x^2-1) \right] = (x^2-1) \frac{d^{l+2} v}{dx^{l+2}} + 2(l+1) x \frac{d^{l+1} v}{dx^{l+1}} + \frac{2l(l+1)}{2!} \frac{d^l v}{dx^l}$$

As for the right side, we take $f = v$ and $g = x$

$$\frac{d^{l+1}}{dx^{l+1}} [2lvx] = 2lx \frac{d^{l+1} v}{dx^{l+1}} + 2l(l+1) \frac{d^l v}{dx^l}$$

Equating both side

$$(1-x^2) \frac{d^{l+2} v}{dx^{l+2}} + [2xl - 2x(l+1)] \frac{d^{l+1} v}{dx^{l+1}} + [2l(l+1) - l(l+1)] \frac{d^l v}{dx^l} = 0$$

$$(1-x^2)\left(\frac{d^l v}{dx^l}\right)'' - 2x\left(\frac{d^l v}{dx^l}\right)' + l(l+1)\frac{dv}{dx} = 0$$

This is the Legendre's equation if

$$y = \frac{d^l v}{dx^l} = \frac{d^l}{dx^l}(x^2-1)^l$$

The next step is to apply the definition of Legendre polynomial $P_l(1) = 1$. This can be achieved by determining the value of constant C such that

$$y(1) = C\frac{d^l}{dx^l}(x^2-1)^l\bigg|_{x=1} = 1$$

We use the relation

$$\frac{d^l}{dx^l}(x^2-1)^l\bigg|_{x=1} = 2^l l!$$

which can be proofed by the induction method. First, consider the base case of $l = 0$. According to the hypothesis,

$$\frac{d^0}{dx^0}(x^2-1)^0\bigg|_{x=1} = 2^0 0!$$
$$1 = 1$$

Then consider the inductive case $l+1$

$$\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1}\bigg|_{x=1} = 2^{l+1}(l+1)!$$

By the Leibniz' rule, we set $f = (x^2-1)^l$ and $g = (x^2-1)$

$$\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1} = (x^2-1)\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l + 2(l+1)x\frac{d^l}{dx^l}(x^2-1)^l + \frac{2l(l+1)}{2!}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l$$

On evaluating it at $x = 1$, we have the first and the third term to be zero. The first one is obvious enough; but for the third term, note that the term (x^2-1) will survive, thus evaluating at $x = 1$ will result in said term to be zero also. All that remain is the second term

$$\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1}\bigg|_{x=1} = 2(l+1)x\frac{d^l}{dx^l}(x^2-1)^l$$

By using the hypothesis, we have completed our proof

$$\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1}\bigg|_{x=1} = 2(l+1)x2^l l! = 2^{l+1}(l+1)!$$

Since the relation have been proofed, we can use it to obtain

$$C\frac{d^l}{dx^l}(x^2-1)^l\bigg|_{x=1} = 1 \implies C = \frac{1}{2^l l!}$$

Thus

$$y(x) = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

We begin the proof of the Rodrigues formula for associated Legendre polynomial by substituting

$$y = (1-x^2)^{\frac{m}{2}} u$$

Now we evaluate the first derivative

$$\begin{aligned} y' &= u\frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x) + (1-x^2)^{\frac{m}{2}} u' \\ y' &= (1-x^2)^{\frac{m}{2}} u' - mx(1-x^2)^{\frac{m}{2}-1} u \end{aligned}$$

Then the second

$$\begin{aligned} y'' &= (1-x^2)^{\frac{m}{2}} u'' - mx(1-x^2)^{\frac{m}{2}-1} u' - mx(1-x^2)^{\frac{m}{2}-1} u' \\ &\quad - mu\left[(1-x^2)^{\frac{m}{2}-1} + x\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}(-2x)\right] \\ y'' &= (1-x^2)^{\frac{m}{2}} u'' - 2mx(1-x^2)^{\frac{m}{2}-1} u' \\ &\quad - m\left[(1-x^2)^{\frac{m}{2}-1} - 2x^2\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}\right] u \end{aligned}$$

Now consider the equation that we are going to solve

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0$$

On using the substituted value of y and its derivatives, we can write the first term as

$$\begin{aligned} (1-x^2)y'' &= (1-x^2)^{\frac{m}{2}+1} u'' - 2mx(1-x^2)^{\frac{m}{2}} u' \\ &\qquad - m\left[(1-x^2)^{\frac{m}{2}} - 2x^2\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-1}\right] u \end{aligned}$$

then the second

$$-2xy' = 2mx^2(1-x^2)^{\frac{m}{2}-1} u - 2x(1-x^2)^{\frac{m}{2}} u'$$

and the third

$$\left[l(l+1) - \frac{m^2}{1-x^2}\right] y = \left[l(l+1)(1-x^2)^{\frac{m}{2}} - m^2(1-x^2)^{\frac{m}{2}-1}\right] u$$

Hence we have

$$\begin{gathered} (1-x^2)^{\frac{m}{2}+1} u'' - 2(m+1)x(1-x^2)^{\frac{m}{2}} u' \\ + \Big[l(l+1)(1-x^2)^{\frac{m}{2}} - m^2(1-x^2)^{\frac{m}{2}-1} \\ + 2mx^2(1-x^2)^{\frac{m}{2}-1} \\ -m(1-x^2)^{\frac{m}{2}} + 2mx^2\left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-1} \Big] u = 0 \end{gathered}$$

Terms inside square bracket can be simplified into

$$l(l+1)(1-x^2)^{\frac{m}{2}} - m^2(1-x^2)^{\frac{m}{2}-1} - m(1-x^2)^{\frac{m}{2}} + m^2x^2(1-x^2)^{\frac{m}{2}-1}$$

Then

$$l(l+1)(1-x^2)^{\frac{m}{2}} - m(1-x^2)^{\frac{m}{2}} - m^2(1-x^2)(1-x^2)^{\frac{m}{2}-1}$$

Finally

$$l(l+1)(1-x^2)^{\frac{m}{2}} - m(m+1)(1-x^2)^{\frac{m}{2}}$$

Substituting back and multiplying by $(1-x^2)^2/m$

$$(1-x^2)u'' - 2(m+1)xu' + [l(l+1) - m(m+1)]\, u = 0$$

Now the associated Legendre equation turns into the Legendre equation if $m = 0$ and has solution of $u = P_l$ or $(1-x^2)^{\frac{m}{2}} P_l$, the solution is. For the case of general integer m, first differentiate it, to obtain

$$\begin{gathered} -2xu'' + (1-x^2)u''' - 2(m+1)(u' + xu'') \\ + [l(l+1) - m(m+1)]\, u' = 0 \end{gathered}$$

Or

$$\begin{gathered} (1-x^2)(u')'' - [2(m+1)+2]x(u')' \\ + [l(l+1) - m(m+1) - 2(m+1)]\, u' = 0 \end{gathered}$$

This just the previous Legendre equation with $u \to u'$ and $m \to m+1$. If $u = P_l$ is the solution at $m = 0$ and $u = P_l'$ is the solution at $m+1$, then by the induction method we can say that for integer $0 \le m \le l$, $u = (P_l)^{(m)}$ is the solution. To put it in other words, the solution of associated Legendre equation is

$$\begin{aligned} y &= (1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^m P_l \\ y &= \frac{1}{2^l l!}(1-x^2)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^{l+m} (x^2-1)^l \end{aligned}$$

Generating function. The function $\Phi(x,h)$ below is the generating function for the Legendre polynomial

$$(1 - 2xh + h^2)^{-1/2} = \sum_{l=0}^{\infty} h^l P_l(x)$$

Proof. We first show that the function indeed can be expressed as series. For brevity, we take $u = 2xh - h^2$

$$\begin{aligned} (1-u)^{-1/2} &= \sum_{k=0}^{\infty} \frac{\Gamma(1/2)(-u)^k}{\Gamma(k+1)\Gamma(1/2-k)} \\ &= 1 + \frac{1}{2}u + \frac{(-1/2)(-3/2)}{\Gamma(3)}u^2 + \dots \\ &= 1 + \frac{1}{2}(2xh - h^2) + \frac{3}{8}(4x^2 - 4xh^3 + h^2) + \dots \\ &= 1 + xh + \left(\frac{3}{2}x^2 - \frac{1}{2}\right)h^2 + \dots \\ \Phi(x,h) &= P_0(x) + hP_1(x) + h^2P_2(x) + \dots \end{aligned}$$

By evaluating the series at $x = 1$, we have the expression in terms of h

$$\Phi(1,h) = P_0(1) + hP_1(1) + h^2P_2(1) + \dots$$

thus, the function does indeed have the identity $P_l(1) = 1$. Next we will show that the generating function satisfies the Legendre equation by the following formula

$$(1-x^2)\frac{\partial^2 \Phi}{\partial x^2} - 2x\frac{\partial \Phi}{\partial x} + h\frac{\partial^2 h\Phi}{\partial h^2} = 0$$

Substituting the series representation of the generating function

$$(1-x^2)\sum_{l=0}^{\infty} h^l P_l''(x) - 2x\sum_{l=0}^{\infty} h^l P_l'(x) + \sum_{l=0}^{\infty}(l+1)lh^l P_l(x) = 0$$

Taking the coefficient of h^l, we obtain the Legendre function

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$$

Recursion relations. Some examples of recursion relations areas follows.

$$lP_l(x) = (2l-l)xP_{l-1}(x) - l(-1)P_{l-2}(x)$$
$$xP_l' - P_{l-1}' = lP_l(x)$$
$$P_l'(x)xP_{l-1}' = lP_{l-1}(x)$$
$$(1-x^2)P_l'(x) = lP_{l-1}(x)lxP_l(x)$$
$$(2l+1)P_l(x) = P_{l+1}'(x) - P_{l-1}'x$$
$$(1-x^2)P_{l-1}'(x = lxP_{l-1})(x) - lP_l(x)$$

Also some identity

$$P_l(-x) = (-1)^l P_l(x)$$

First relation proof. Differentiate the generating function with respect to h

$$\frac{\partial \Phi}{\partial h} = -\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h)$$

Or

$$(1-2xh+h^2)\frac{\partial \Phi}{\partial h} = (x-h)\Phi$$

First write it as power series

$$(1-2xh+h^2)\sum_{l=0}^{\infty} lh^{l-1}P_l(x) = (x-h)\sum_{l=0}^{\infty} h^l P_l$$

then take the h^{l-1} coefficient

$$lP_l - 2x(l-1)P_{l-1} + (l-2)P_{l-2} = xP_{l-1} - P_{l-2}$$

Rearrange it

$$lP_l = [2x(l-1)+x]\,P_{l-1} - [l-2+1]\,P_{l-2}$$
$$lP_l = (2l-l)xP_{l-1}(x) - l(-1)P_{l-2}(x)$$

Second relation proof. 404.

Third relation proof. 404.

Fourth relation proof. 404.

Fifth relation proof. 404.

Sixth relation proof. 404.

Bonus. Consider the case of negative value x and t for the $\Phi(x,t)$. Substituting it to the function representation

$$\Phi(-x,-t) = (1 - 2(-x)(-h) + (-h)^2)^{-1/2} = (1 - 2xh + h^2)^{-1/2}$$

We see that this is simply $\Phi(x,t) = \sum h^l P_l(x)$. Now, substituting it to the series representation

$$\Phi(-x,-t) = \sum_{l=0}^{\infty} (-h)^l P_l(-x) = \sum_{l=0}^{\infty} (-1)^l h^l P_l(-x)$$

Now taking the coefficient of h^l

$$(-1)^l P_l(-x) = P_l(x) \implies P_l(-x) = (-1)^l P_l(x)$$

Orthogonality. The following equation state the orthogonality of Legendre polynomial on $(-1,1)$

$$\int_{-1}^{1} P_l(x) P_m(x)\ dx = \frac{2}{2l+1}\delta_{lm}$$

On using this, we also obtain the following theorem.

$$\int_{-1}^{1} P_l(x) \cdot (\text{polynomial degree } l < 0)\ dx = 0$$

For the case of associated Legendre polynomial, we have

$$\int_{-1}^{1} [P_l^m(x)]^2\ dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$

Proof. Consider two Legendre polynomials for two different value of l

$$\frac{d}{dx}\left[(1-x^2)P_l'\right] + l(l+1)P_l = 0$$
$$\frac{d}{dx}\left[(1-x^2)P_m'\right] + l(l+1)P_m = 0$$

Multiply the first equation with P_m and the second with P_l

$$P_m \frac{d}{dx}\left[(1-x^2)P_l'\right] + l(l+1)P_l P_m = 0$$
$$P_l \frac{d}{dx}\left[(1-x^2)P_m'\right] + m(m+1)P_l P_m = 0$$

Subtract both

$$P_m \frac{d}{dx}\left[(1-x^2)P_l'\right] - P_l \frac{d}{dx}\left[(1-x^2)P_m'\right]$$

$$+\left[(l(l+1))-m(m+1)\right]P_lP_m$$

The first two terms can be written as

$$\begin{aligned}\frac{d}{dx}\left[(1-x^2)(P_mP_l'-P_lP_m')\right] &= \frac{d}{dx}\left[(1-x^2)P_mP_l'-(1-x^2P_lP_m')\right]\\ &= (1-x^2)P_l'P_m' + P_m\frac{d}{dx}\left[(1-x^2)P_l'\right]\\ &\quad -(1-x^2)P_l'P_m' - P_l\frac{d}{dx}\left[(1-x^2)P_m'\right]\end{aligned}$$

which is what those two terms are. Then integrate between $(-1,1)$

$$\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)(P_mP_l'-P_lP_m')\right]\,dx \\ +\int_{-1}^{1}\left[(l(l+1))-m(m+1)\right]P_lP_m\,dx=0$$

Evaluate it

$$(1-x^2)(P_mP_l'-P_lP_m')\Big|_{-1}^{1}+\left[(l(l+1))-m(m+1)\right]\int_{-1}^{1}P_lP_m\,dx=0$$

The first term is zero due to the term of $(1-x^2)$; while the second term is also zero if $m=l$, which due to the constant term outside integral. Hence, it is proved that for $m=l$, the following integral is true

$$\int_{-1}^{1}P_lP_m\,dx=0$$

By considering P_m as a polynomial order m, we can also state

$$\int_{-1}^{1}P_l\cdot(\text{polynomial degree } l<0)\,dx=0$$

To prove the value of the integral for the same order l, first recall the recursive relation of

$$lP_l = xP_l' - P_{l-1}'$$

Multiply by P_l and integrate on $(-1,1)$

$$l\int_{-1}^{1}[P_l]^2\,dx=\int_{-1}^{1}xP_lP_l'\,dx-\int_{-1}^{1}P_lP_{l-1}'\,dx$$

The second term is zero since P_{l-1}' is a polynomial order $l-2$. We evaluate the remaining integral using integration by part

$$\int_{-1}^{1}xP_lP_l'\,dx = x[P_l]^2\Big|_{-1}^{1}-\int_{-1}^{1}P_l\frac{d}{dx}\left[xP_l\right]\,dx$$

Recalling the $[P_l(-1)]^2=(-1)^{2l}=1$ and using the derivative product rule for the second integral

$$\int_{-1}^{1}xP_lP_l'\,dx=2-\int_{-1}^{1}[P_l]^2\,dx-\int_{-1}^{1}xP_lP_l'\,dx$$

$$\int_{-1}^{1} xP_lP_l'\,dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_l]^2\,dx$$

Then substituting back into our original equation

$$l\int_{-1}^{1}[P_l]^2\,dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_l]^2\,dx$$
$$\int_{-1}^{1}[P_l]^2\,dx = \frac{1}{l+1/2} = \frac{2}{2l+1}$$

Or by using Kronecker delta, we can also express the result as

$$\int_{-1}^{1} P_lP_m\,dx = \frac{2}{2l+1}\delta_{lm}$$

Another form of associated Legendre equation. We shall prove

$$\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0$$

is the associated Legendre equation by substituting $x = \cos\theta$. First we evaluate the differential

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{dy}{dx} = -(1-x^2)^{1/2}\frac{dy}{dx}$$

Applying the operator one more

$$\begin{aligned}\frac{d^2y}{d\theta^2} &= -(1-x^2)^{1/2}\frac{d}{dx}\left[-(1-x^2)^{1/2}\frac{dy}{dx}\right]\\ &= -(1-x^2)^{1/2}\left[\frac{x}{(1-x^2)^{1/2}}\frac{dy}{dx} - (1-x^2)^{1/2}\frac{d^2y}{dx^2}\right]\\ \frac{d^2y}{d\theta^2} &= (1-x^2)^{1/2}\frac{d^2y}{dx^2} - x\frac{dy}{dx}\end{aligned}$$

Then we simply substituted both into the another form of the associated Legendre equation

$$(1-x^2)^{1/2}\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{(1-x^2)^{1/2}}\right]y = 0$$

Hermite Function

The differential equation

$$y_n'' - x^2y_n = -(2n+1)y_n$$

has the solution called the Hermite function

$$y_n = e^{x^2/2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

Another closely related equation is called the Hermite equation

$$y'' - 2xy' + 2ny = 0$$

which is solved by the Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$$

Hermite polynomial can also be obtained by multiplying the Hermite function by $(-1)^n e^{x^2/2}$. This multiplication is performed such that the polynomial is raising from negative $-\infty$ and orthogonal on $(-\infty, \infty)$.

The generating function $\phi(x, h)$ for Hermite polynomial is

$$\exp(2xh - h^2) = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!}$$

Using the generating function we obtain the following recursion relation

$$H_n'(x) = 2xH_{n-1}(x)$$
$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Derivation. Using the operator D, we have

$$\begin{aligned}(D-x)(D+x)y &= (D-x)(Dy+xy)\\ &= D^2y + y + xDy - xDy - x^2y\\ &= D^2 - x^2y_n + y\end{aligned}$$

and

$$\begin{aligned}(D+x)(D-x)y &= (D-x)(Dy-xy)\\ &= D^2y - y - xDy + xDy - x^2y\\ &= D^2 - x^2y_n - y\end{aligned}$$

Then, we can write

$$(D-x)(D+x)y_n = D^2 - x^2y_n + y_n = -2ny_n$$
$$(D+x)(D-x)y_n = D^2 - x^2y_n - y_n = -2(n+1)y_n$$

Operating $(D+x)$ on the first equation and $(D-x)$ on the second

$$(D+x)(D-x)(D+x)y_n = -2n(D+x)y_m$$
$$(D-x)(D+x)(D-x)y_n = -2(n+1)(D-x)y_n$$

If the equations are identical, that is $y_n = (D-x)y_m$ and $n+m+1$ for the first n equation and second m equations

$$-2ny_n = -2(m+1)y_m$$
$$y_{m+1} = (D-x)y_m$$

If $y_n = (D = x)y_m$ and $n+1 = m$ for the second n equations and the first m equations

$$-2(n+1)y_n = -2m(D+x)y_m$$
$$y_{m-1} = (D+x)y_m$$

Mathematically, these operators are called the raising operator $(D-x)$ and lowering operator $(D+x)$. Quantum mechanics definition of raising operator $a^\dagger$ and lowering operator a is different to accommodate adjoint $[a, a^\dagger] = 1$

$$a^\dagger = \frac{1}{\sqrt{2}}(x - D)$$
$$a = \frac{1}{\sqrt{2}}(x + D)$$

We can derive the solution for Hermite equations

$$y'' - 2xy' + 2ny = 0$$

using the Rodrigues formula. First, the integral factor is evaluated as follows

$$\mu = \exp\left(\int_0^x -2x\, dx\right) = e^{-x^2}$$

The solution is the

$$y = \frac{1}{e^{-x^2}}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

The Hermite polynomial is obtained by multiplying the solution with factor of $(-1)^n$

$$H_n = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

Orthogonality. We state the orthogonality of Hermite function or Hermite polynomial as follows.

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi}2^n n!\delta_{nm}$$

The Hermite polynomial is orthogonal on $(-\infty, \infty)$ with respect to weight function e^{-x^2}. This is why the Hermite function is defined in such way to allow the function to be orthogonal on $(-\infty, \infty)$ with respect to itself

$$\int_{-\infty}^{\infty} y_n y_m\, dx = \sqrt{\pi}2^n n!\delta_{nm}$$

Recursive relation derivation. Derivation the generating function

$$\exp(2xh - h^2) = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}$$

with respect either x or h will result in the recursive relation. The first relation is obtained by performing derivative to the $\phi(x, h)$ with respect to x

$$2h \exp(2xh - h^2) = \sum H_n' \frac{h^n}{n!}$$
$$2\sum \frac{H_{n-1}}{(n-1)!}h^n = \sum \frac{H_n'}{n!}h^n$$

and equating the h^n term

$$2\frac{H_{n-1}}{(n-1)!} = \frac{H'_n}{n!}$$
$$2nH_{n-1} = H'_n$$

In other hand, the second relation is obtained by derivating with respect to h

$$(2x-2h)\exp(2xh-h^2) = \sum \frac{nH_n}{n!}h^{n-1}$$
$$2x\sum \frac{H_n}{n!}h^n - 2\sum \frac{H_{n-1}}{(n-1)!}h^n = \sum \frac{H_(n+1)}{n!}$$

and, as before, equation the h^n terms

$$2x\frac{H_n}{n!} - 2\frac{H_{n-1}}{(n-1)!} = \frac{H_{n+1}}{n!}$$
$$2xH_n - 2nH_{n-1} = H_{n+1}$$

Laguerre function

The differential equation

$$xy'' + (1-x)y' + ny = 0$$

has the solution of Laguerre polynomials

$$L_n(x) = \frac{1}{n!}e^x\left(\frac{d}{dx}\right)^n(x^ne^{-x})$$

or if we were to carry the differentiation using Leibniz's rule

$$L_n(x) = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{x^m}{m!}$$

Another closely related equation is

$$xy'' + (k+1-x)y' + ny = 0$$

which is solved by the associated Laguerre polynomials

$$L_n^k(x) = (-1)^k\left(\frac{d}{dx}\right)^k L_{n+k}(x)$$

Derivation. The Laguerre equation

$$xy'' + (1-x)y' + ny = 0$$

can be solved using Rodrigues formula, where the integral factor is

$$\mu = \frac{1}{x}\exp\left(\int_0^x \frac{1-x}{x}\,dx\right) = \frac{\exp(\ln x - x)}{x} = e^{-x}$$

The solution then reads

$$y = \frac{1}{e^{-x}}\left(\frac{d}{dx}\right)^n(x^ne^{-x})$$

On multiplying with $1/n!$ we obtain the Laguerre polynomials

$$L_n(x) = \frac{1}{n!}e^x\left(\frac{d}{dx}\right)^n(x^ne^{-x})$$

Laplace's Equation

Consider scalar function u, which may represent gravitational potential in a region containing no mass, the electrostatic potential in a charge-free region, the steady-state temperature in a region containing no sources of heat, or the velocity potential for an incompressible fluid with no vortices and no sources or sinks. Laplace's equation state that

$$\nabla^2 u = 0$$

Cylindrical domain. Suppose we evaluate Laplace equation in cylindrical domain. The Laplacian reads

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

By the separation of variables, we assume the solution of $u = R(r)\Phi(\phi)Z(z)$. Thus, the Laplace's equation now reads

$$\frac{\Phi R}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{RZ}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + R\Phi\frac{\partial^2 Z}{\partial z^2} = 0$$

Then we divide by $u = R(r)\Phi(\phi)Z(z)$,

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

Since only last term is the function of z alone, we can safely say that it is a constant. Therefore, we define

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = K^2 \implies Z = Ae^{Kz} + Be^{-KZ}$$

Substituting this value and multiplying by r^2, we have

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} + K^2 r^2 = 0$$

Now we see that the second term is constant. We define

$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi} = -n^2 \implies \Phi = C\sin n\theta + D\cos n\theta$$

where n is an integer. The reason for said separation constant is due to periodicity. For a position in polar coordinate, we denote them as $\theta + 2n\pi$. Hence, for a most physical reason, the position θ and $\theta + 2n\pi$ must give the same result, which is possible if the solution is periodic with period of 2π. Finally, we substitute this and multiplying with R to obtain the radial solution

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + (K^2 r^2 - n^2)R = 0$$

This is Bessel function, in particular

$$R = \sum_{m=0}^{\infty} E_m J_m(Kr) + \sum_{m=0}^{\infty} F_m N_m(Kr)$$

Now, putting it all together, we have the most general solution for the Laplace's equation in cylindrical domain

$$u = \sum_{m=0}^{\infty} \left[A_m e^{Kz} + B_m e^{-KZ}\right] \left[C_m \sin n\theta + D_m \cos n\theta\right] \\ \left[E_m J_m(Kr) + F_m N_m(Kr)\right]$$

Spherical domain. The Laplacian reads

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

We assume the solution of $u = R(r)\Phi(\phi)Z(z)$

$$\frac{\Phi Z}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R\Phi}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{R\Theta}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

As before, we multiply it by $u = R(r)\Phi(\phi)Z(z)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi \sin^2\phi}\frac{d^2\Phi}{d\phi^2} = 0$$

If we multiply by $\sin^2\theta$, we see that the last terms is a function of ϕ, while the other two are not. Hence, we can say

$$\frac{1}{\Phi}\frac{d^2\Phi}{2\phi^2} = -m^2 \implies \phi = Ae^{im\phi} + Be^{-im\phi}$$

The constant is chosen to be negative due to periodicity, as before. With this, we can write

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

The first term is a Function of r while the other two is a function of θ, so

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + l(l+1)\Theta = 0$$

We first consider the first equation

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$$

By the Euler's method of substituting $r = e^z$, we see that $a_1 = 2$ and $a_2 = 1$, thus we write

$$R''(e^z) + R'(e^z) - l(l+1)R(e^z) = 0$$

Using the method of writing differential as operator

$$[D^2 + D - l(l+1)]R(e^z) = 0$$
$$[D + (l+1)][D - l]R(e^z) = 0$$

This has the solution

$$R(e^z) = Ae^{lz} + Be^{-l(l+1)z}$$

Substituting back $r = e^z$, we obtain the solution in terms of r

$$R(r) = Ar^l + Br^{-l(l+1)}$$

Next we consider the θ equation

$$\frac{d^2\Phi}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

This is the associated Legendre equation if we substitute $x = \cos\theta$. Thus, we have the solution

$$\Theta = P_l^m(\cos\theta)$$

Putting them all together, we have the general solution to the Laplacian

$$u = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_l^m(\theta,\phi)$$

where $Y_l^m(\theta,\phi)$ is the spherical harmonics.

Poisson's Equation

Consider the same scalar function u as the case of Laplace's equation, however we have a region containing mass, electric charge, or sources of heat or fluid denoted by $f(x,y,z)$. Poisson's equation is written as

$$\nabla^2 u = f(x,y,z)$$

Heat Flow or Diffusion Equation

Now suppose that the temperature is non-steady. The flow of temperature is governed by the equation

$$\nabla^2 u = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

where α is a constant called the diffusivity.

Wave Equation

Here u may represent the displacement from equilibrium; in electricity it may be the current or potential along a transmission line; or it may be a component of **E** or **B** in an electromagnetic wave. The equation is written as

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Helmholtz Equation

Helmholtz's equation is the spatial part of either diffusion or wave equation

$$\nabla^2 F + k^2 F = 0$$

Schrödinger's Equation

Also known as the wave function equation of quantum mechanics

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

Appendix: Frobenius' Method

I will demonstrate this technique. Consider the following differential equation.

$$x^2y'' + 4xy' + (x^2 + 2)y = 0$$

The solution will take the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

Substituting this into each term, we have

$$\begin{aligned}
x^2y'' &= \sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s} \\
4xy' &= \sum_{n=0}^{\infty} 4(n+s)a_n x^{n+s} \\
xy &= \sum_{n=0}^{\infty} a_n x^{n+s+2} \\
2y &= \sum_{n=0}^{\infty} 2a_n x^{n+s}
\end{aligned}$$

Then we put them into table.

Table 3: Table

	x^{n+s}	x^s	x^{s+1}
x^2y''	$(n+s)(n+s-1)a_n$	$s(s-1)a_0$	$s(s+1)a_1$
$4xy'$	$4(n+s)a_n$	$4sa_0$	$4(s+1)a_1$
x^2y	a_{n-2}	–	–
$2y$	$2a_n$	$2a_0$	$2a_1$

Using the terms on x^s column, we have the following indicial equation.

$$\begin{aligned}
s(s-1)a_0 + 4sa_0 + 2a_0 &= 0 \\
a_0\left[s(s+3) + 2\right] &= 0
\end{aligned}$$

Since a_0 cannot be zero, we write

$$s^2 + 3s + 2 = 0$$

By solving the indicial equation we obtain $s = (-1, -2)$. From the x^{n+s}, we obtain the general formula for a_n in terms of a_{n-2}

$$a_n\left[(n+s)(n+s+3) + 2\right] = -a_{n-2}$$

We also obtain the fact the value of a_1 is zero, proved by the terms in x^{s+1} column

$$\left.\begin{array}{l} a_1\left[(s+1)(s+4)+2\right] = 0 \\ s = (-1,-2) \end{array}\right\} \implies a_0 = 0$$

Since we have two value of s, we first consider the case for $s = -1$. The general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-1)(n+2)+2} = -\frac{a_{n-2}}{n^2+n} = -\frac{a_{n-2}}{n(n+1)}$$

The values of a_n for few n are as follows

$$\begin{aligned} a_2 &= -\frac{a_0}{3!} \\ a_4 &= -\frac{a_2}{4 \cdot 5} = \frac{a_0}{5!} \\ a_6 &= -\frac{a_4}{6 \cdot 7} = -\frac{a_0}{7!} \end{aligned}$$

Thus the solution for this case is

$$\begin{aligned} y_{-1} = \sum_{n=0}^{\infty} a_n x^{n-1} &= \frac{a_0}{x} - \frac{a_0}{3!}x + \frac{a_0}{5!}x^3 - \frac{a_0}{7!}x^5 + \dots \\ &= \frac{a_0}{x^2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = \frac{a_0}{x^2}\sin x \end{aligned}$$

For the case of $s = -2$, the general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-2)(n+1)+2} = -\frac{a_{n-2}}{n^2-n} = -\frac{a_{n-2}}{n(n-1)}$$

The values of a_n for few n are as follows

$$\begin{aligned} a_2 &= -\frac{a_0}{2!} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!} \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!} \end{aligned}$$

Thus the solution for this case is

$$\begin{aligned} y_{-2} = \sum_{n=0}^{\infty} a_n x^{n-2} &= \frac{a_0}{x^2} - \frac{a_0}{2!} + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^4 + \dots \\ &= \frac{a_0}{x^2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) = \frac{a_0}{x^2}\cos x \end{aligned}$$

Hence, the complete form of the solution is

$$y = \frac{a_0}{x^2}(\cos x + \sin x)$$

Bessel Equation

Ex. 1. Suppose we are going to solve

$$y'' + 9xy = 0$$

We know that the equation has no y' factor, then

$$\frac{1-2a}{x} = 0 \implies a = \frac{1}{2}$$

By assuming

$$2c - 2 = 1 \implies c = \frac{3}{2}$$

We can equate the first x coefficient

$$(bc)^2 = 9 \implies b = 2$$

And

$$\frac{a^2 - p^2c^2}{x^2} = 0 \implies p = \sqrt{\frac{a^2}{c^2}} = \frac{1}{3}$$

The solution takes the form of

$$y = x^{1/2} Z_{1/3}(2x^{3/2}) = x^{1/2} \left[AJ_{1/3}(2x^{3/2}) + BN_{1/3}(2x^{3/2}) \right]$$

Laplacian in Cylindrical Coordinates.

Steady-state temperature in a cylinder. Find the steady-state temperature distribution in a semi-infinite solid cylinder of radius a if the base is held at T and the curved sides at 0°. The boundary condition are

1. $u(z = \infty) = 0$,
2. u does not depend on angular term,
3. u converges at origin,
4. $u(r = a) = 0$, and
5. $u(z = 0) = 100$

and the general solution is

$$u = \sum_{m=0}^{\infty} \left[A_m e^{Kz} + B_m e^{-KZ} \right] \left[C_m \sin n\theta + D_m \cos n\theta \right]$$
$$\left[E_m J_m(Kr) + F_m N_m(Kr) \right]$$

We see that the first condition demands that $A_m = 0$, the second demands that $n = 0$, and the third demands that $F_m = 0$. Using the fourth condition on our solution we obtained so far

$$u(r = a) = GJ_0(Ka)e^{-Kz} = 0$$

with G as our arbitrary constant. Since the exponential term cannot be zero for arbitrary z, the zero must be the Bessel function. We then define

$$Ka = k_m$$

where k_m is the m-th zeros of the Bessel function. Since there are infinite value of them, we write

$$u = \sum_{m=1}^{\infty} G_m J_0 \left(\frac{k_m r}{a} \right) \exp \left(\frac{k_m z}{a} \right)$$

All that left is to determine the constant G_n, which is obtained from the fourth condition

$$u(z=0) = \sum_{m=1}^{\infty} G_m J_0\left(\frac{k_m r}{a}\right) = T$$

Multiplying by $rJ_0(k_\mu r/a)$ and integrating them from $(0,a)$

$$\int_0^a \sum_{m=0}^{\infty} G_m J_0\left(\frac{k_m r}{a}\right) J_0\left(\frac{k_\mu r}{a}\right) r\, dr = \int_0^a TJ_0\left(\frac{k_\mu r}{a}\right) r\, dr$$

Let us consider the first integral first. By the only orthoganality of Bessel function, only the term with $m=\mu$ will survive the integral. So we might as well write

$$\int_0^a \sum_{m=0}^{\infty} G_m J_0\left(\frac{k_m r}{a}\right) J_0\left(\frac{k_\mu r}{a}\right) r\, dr = \int_0^a G_m r J_0^2\left(\frac{k_m r}{a}\right)\, dr$$

which evaluate into

$$\int_0^a G_m r\left[J_0\left(\frac{k_m r}{a}\right)\right]^2 dr = \frac{a^2}{2} G_m J_1^2$$

To evaluate the right side, we need to consider the relation

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$$

With $p=1$ and $x = k_m r/a$, the relation turns into

$$\frac{a}{k_m}\frac{d}{dr}\left[rJ_1\left(\frac{k_m r}{a}\right)\right] = rJ_0\left(\frac{k_m r}{a}\right)$$

On using this relation, the right side integral now reads

$$\int_0^a TJ_0\left(\frac{k_\mu r}{a}\right) r\, dr = \int_0^a T\frac{a}{k_m}\frac{d}{dr}\left[rJ_1\left(\frac{k_m r}{a}\right)\right] dr$$

By the fundamental theorem of calculus, we have

$$\int_0^a T\frac{a}{k_m}\frac{d}{dr}\left[rJ_1\left(\frac{k_m r}{a}\right)\right] dr = \frac{T_0 a^2}{k_m} J_1(k_m)$$

Now we equate both side

$$\frac{a^2}{2} G_m J_1^2 = \frac{T_0 a^2}{k_m} J_1(k_m)$$

and solve fo the constant

$$G_m = \frac{2T_0}{k_m J_1(k_m)}$$

Thus we have the complete solution

$$u = \sum_{m=1}^{\infty} \frac{2T_0}{k_m J_1(k_m)} J_0\left(\frac{k_m r}{a}\right) \exp\left(\frac{k_m z}{a}\right)$$

with k_m as the zeros of J_0.

Hermitian Function

The solution of Schrödinger equation in harmonic potential can be expressed as a Hermitian function. In one dimensional harmonics potential, the Schrödinger equation reads

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\psi = E\psi$$
$$\psi'' - \left(\frac{m\omega}{\hbar}x\right)^2\psi + \frac{2mE}{\hbar^2}\psi = 0$$

We define

$$\Xi = \left(\frac{m\omega}{\hbar}x\right)^{1/2} x \quad \text{and} \quad \Psi = \left(\frac{m\omega}{\hbar}\right)^2 \psi$$

Then the derivatives reads

$$\frac{d\psi}{dx} = \frac{d\psi}{d\Psi}\frac{d\Psi}{d\Xi}\frac{d\Xi}{dx} = \left(\frac{\hbar}{m\omega}\right)^{1/2}\frac{d\Psi}{d\Xi}$$
$$\frac{d^2\psi}{dx^2} = \frac{d}{d\Xi}\left(\frac{d\psi}{dx}\right)\frac{d\Xi}{dx} = \frac{d^2\Psi}{dx^2}$$

while the other terms

$$\left(\frac{m\omega}{\hbar}x\right)^2\psi = \left(\frac{m\omega}{\hbar}\right)^2\frac{\hbar}{m\omega}\Xi^2\left(\frac{\hbar}{m\omega}\right)^2\Psi = \Xi\Psi$$
$$\frac{2mE}{\hbar^2}\psi = \frac{2mE}{\hbar^2}\left(\frac{\hbar}{m\omega}\right)^2\Psi = \frac{2E}{\hbar\omega}\Psi$$

Thus

$$\Psi'' - \Xi\Psi + \frac{2E}{m\omega}\Psi = 0$$

This is Hermitian function with

$$\frac{2E}{m\omega} = 2n+1 \quad \text{or} \quad E = \left(n+\frac{1}{2}\right)h\omega$$

whose solution in terms of dummy variable Ξ is

$$\Psi_n = e^{-\Xi^2/2}H_n(\Xi)$$

In terms of our original variable x, we have

$$\psi = \frac{\hbar}{m\omega}\exp\left(\frac{m\omega}{2\hbar}x^2\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

This solution, however, has not been normalized yet. To do so, we need to multiply the solution ψ with the inverse normalization constant A^{-1}, which can be evaluated from

$$A^0 = \int_{-\infty}^{\infty}|\psi|^0\,dx = \left(\frac{\hbar}{m\omega}\right)^2\sqrt{\pi}2^n n!\left(\frac{\hbar}{m\omega}\right)^{1/2} = \left(\frac{\hbar}{m\omega}\right)^{5/2}\sqrt{\pi}n^n n!$$

which then

$$A^{-1} = \left(\frac{m\omega}{\hbar}\right)^{5/4}\frac{1}{\pi^{1/4}}\frac{1}{2^n n!}$$

Therefore, the normalized solution is

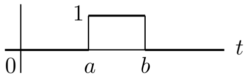
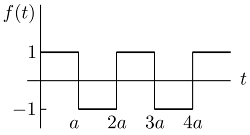
$$\psi = \left(\frac{m\omega}{\hbar}\right)^{5/4} \frac{1}{\pi^{1/4}} \frac{1}{2^n n!} \frac{\hbar}{m\omega} \exp\left(\frac{m\omega}{2\hbar} x^2\right) H_n \left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$

$$\psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{2^n n!} \exp\left(\frac{m\omega}{2\hbar} x^2\right) H_n \left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$

Appendix II: Laplace Table

	$y = f(t), \quad t > 0$ $y = f(t) = 0, t < 0$	$Y = L(y) = F(p)$
$\mathcal{L}1$	1	$\dfrac{1}{p+a}$ $\text{Re } p > 0$
$\mathcal{L}2$	e^{-at}	$\dfrac{1}{p}$ $\text{Re } p > 0$
$\mathcal{L}3$	$\sin at$	$\dfrac{a}{p^2+a^2}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}4$	$\cos at$	$\dfrac{p}{p^2+a^2}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}5$	$t^k, \ k > -1$	$\dfrac{k!}{p^{k+1}}$ or $\dfrac{\Gamma(k+1)}{p^{k+1}}$ $\text{Re } p > 0$
$\mathcal{L}6$	$t^k e^{-at}, \ k > -1$	$\dfrac{k!}{(p+a)^{k+1}}$ or $\dfrac{\Gamma(k+1)}{(p+a)^{k+1}}$ $\text{Re } (p+a) > 0$
$\mathcal{L}7$	$\dfrac{e^{-at} - e^{-bt}}{b-a}$	$\dfrac{1}{(p+a)(p+b)}$ $\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$\mathcal{L}8$	$\dfrac{ae^{-at} - be^{-bt}}{b-a}$	$\dfrac{p}{(p+a)(p+b)}$ $\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
$\mathcal{L}9$	$\sinh at$	$\dfrac{a}{p^2-a^2}$ $\text{Re } p > \lvert\text{Re } a\rvert$

$\mathcal{L}10$	$\cosh at$	$\dfrac{p}{p^2 - a^2}$ $\text{Re } p > \lvert\text{Re } a\rvert$
$\mathcal{L}11$	$t \sin at$	$\dfrac{2ap}{(p^2 + a^2)^2}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}12$	$t \cos at$	$\dfrac{p^2 - a^2}{(p^2 + a^2)^2}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}13$	$e^{-at} \sin bt$	$\dfrac{b}{(p+a)^2 + b^2}$ $\text{Re } (p+a) > \lvert\text{Im } b\rvert$
$\mathcal{L}14$	$e^{-at} \cos bt$	$\dfrac{p+a}{(p+a)^2 + b^2}$ $\text{Re } (p+a) > \lvert\text{Im } b\rvert$
$\mathcal{L}15$	$1 - \cos at$	$\dfrac{a^2}{p(p^2 + a^2)}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}16$	$at - \sin at$	$\dfrac{a^3}{p^2(p^2 + a^2)}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}17$	$\sin at - at \cos at$	$\dfrac{2a^3}{(p^2 + a^2)^2}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}18$	$e^{-at}(1 - at)$	$\dfrac{p}{(p+a)^2}$ $\text{Re } (p+a) > 0$
$\mathcal{L}19$	$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{p}$ $\text{Re } p > \lvert\text{Im } a\rvert$
$\mathcal{L}20$	$\dfrac{1}{t} \sin at \cos bt$	$\dfrac{1}{2}\left(\arctan \dfrac{a+b}{p} + \arctan \dfrac{a-b}{p}\right)$ $\text{Re } p > 0$

$\mathcal{L}21$	$\dfrac{e^{at} - e^{-bt}}{t}$	$\ln \dfrac{p+b}{p+a}$ Re $(p+a) > 0$ Re $(p+b) > 0$
$\mathcal{L}22$	$1 - \text{erf}\left(\dfrac{a}{2\sqrt{t}}\right)$ $a > 0$	$\dfrac{1}{p} e^{-a\sqrt{p}}$ Re $p > 0$
$\mathcal{L}23$	$J_0(at)$	$(p^2 + a^2)^{-1/2}$ Re $p > \lvert\text{Im } a\rvert$ Re $p \geq 0$ for real $a \neq 0$
$\mathcal{L}24$	unit step, Heaviside function $u(t-a) = \begin{cases} 1, \\ t > a > 0 \\ 0,\ t < a \end{cases}$	$\dfrac{1}{p} e^{-pa}$ Re $p > 0$
$\mathcal{L}25$	$f(t) = u(t-a) - u(t-b)$ 	$\dfrac{e^{-ap} - e^{-}bp}{p}$ All p
$\mathcal{L}26$		$\dfrac{1}{p} \tanh \frac{ap}{2}$ All p
$\mathcal{L}27$	$\delta(t-a),\ a \geq 0$	e^{-pa}
$\mathcal{L}28$	$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$ $f(t) = g(t-a)u(t-a)$	$e^{-pa}G(p)$ $G(p)$ means $\mathcal{L}(g)$ Therefore $e^{-pa}\mathcal{L}\left[g(t-a)\right]$
$\mathcal{L}29$	$e^{-at}g(t)$	$G(p+a)$
$\mathcal{L}30$	$g(at), a > 0$	$\dfrac{1}{a} G\left(\dfrac{p}{a}\right)$
$\mathcal{L}31$	$\dfrac{g(t)}{t}$ if integrable	$\displaystyle\int_p^\infty G(u)\, du$

$\mathcal{L}32$	$t^n g(t)$	$(-1)^n \left(\dfrac{d}{dp}\right)^n (G(p))$
$\mathcal{L}33$	$\int_0^t g(\tau)\, d\tau$	$\dfrac{1}{p}G(p)$
$\mathcal{L}34$	Convolution of g and h, often written as $g * h$ $= \int_0^t g(t-\tau)h(\tau)\, d\tau$ $= \int_0^t g(\tau)h(t-\tau)\, d\tau$	$G(p)H(p)$

Transforms of derivatives of y

$\mathcal{L}35$

$$\begin{aligned}
\mathcal{L}(y') &= pY - y \\
\mathcal{L}(y'') &= p^2Y - py_0 - y_0' \\
\mathcal{L}(y''') &= p^3Y - p^2y_0 - py_0' - y_0 \\
\mathcal{L}(y^n) &= p^nY - p^{n-1}y_0 - p^{n-2}y_0' - \cdots - y_0^{n-1}
\end{aligned}$$

Appendix: Differential Equation Study Guide

First Order Equations. General Form of ODE

$$\frac{dy}{dx} = f(x, y)$$

Initial Value Problem

$$y' = f(x, y),\ y(x_0) = y_0$$

Linear Equations. General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\mu(x) = e^{\int p(x)dx}$$
$$\implies \frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$$

General Solution

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

Homogeneous Equations. General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in z:

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations. General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in z:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations. General Form

$$M(x, y)dx + N(x, y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations.

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors. Case 1. If $P(x,y)$ depends only on x, where

$$P(x,y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact.

Case 2. If $Q(x,y)$ depends only on y, where

$$Q(x,y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact.

Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{6}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{7}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{8}$$

General Solution. The general solution of (6) or (8) is

$$y = C_1y_1(t) + C_2y_2(t) + y_p(t) \tag{9}$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (7).

Linear Independence and The Wronskian. Two functions $f(x)$ and $g(x)$ are linearly dependent if there exist numbers a and b, not both zero, such that $af(x) + bg(x) = 0$ for all x. If y_1 and y_2 are two solutions of (7), then Wronskian

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.
2. $\{y_1, y_2\}$ are a fundamental set of solutions.
3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem. The initial value problem includes two initial conditions at the same point in time, one condition on $y(t)$ and one condition on $y'(t)$.

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

The initial conditions are applied to the entire solution $y = y_h + y_p$.

Linear Equation With Constant Coefficients. The general form of the homogeneous equation is

$$ay'' + by' + cy = 0 \tag{10}$$

Non-homogeneous

$$ay'' + by' + cy = g(t) \tag{11}$$

Characteristic Equation

$$ar^2 + br + c = 0$$

Quadratic Roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{12}$$

The solution of (10) of Real Roots $(r_1 \neq r_2)$

$$y_h = C_1e^{r_1t} + C_2e^{r_2t} \tag{13}$$

Repeated $(r_1 = r_2)$

$$y_h = (C_1 + C_2t)e^{r_1t} \tag{14}$$

Complex $(r = \alpha \pm i\beta)$

$$y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) \tag{15}$$

The solution of (11) is $y = y_p + y_h$ where y_h is given by (13) through (15) and y_p is found by undetermined coefficients or reduction of order.

Table: Heuristics method.

If $f(t) =$	then guess that a particular solution $y_p =$.
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at} \sin bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n) \cos bt$
or $P_n(t)e^{at} \cos bt$	$+(A_0 + A_1 t + \cdots + A_n t^n) \sin bt]$

Heuristics for Undetermined Coefficients. Also called Trial and Error.

Method of Reduction of Order. When solving (7), given y_1, then y_2 can be found by solving

$$y_1 y_2' - y_1' y_2 = C e^{-\int p(t)dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx} dx}{y_1(x)^2} \tag{16}$$

Method of Variation of Parameters. If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (7) then a particular solution to (8) is

$$y_P(t) = -y_1(t) \int \frac{y_2(t) f(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t) f(t)}{W(t)} dt \tag{17}$$

Cauchy-Euler Equation. For ODE

$$ax^2 y'' + bxy' + cy = 0 \tag{18}$$

with auxiliary Equation

$$ar(r-1) + br + c = 0 \tag{19}$$

The solutions of (18) depend on the roots $r_{1,2}$ of (19). For Real Roots

$$y = C_1 x^{r_1} + C_2 x^{r_2}$$

Repeated Root

$$y = C_1 x^r + C_2 x^r \ln x$$

Complex

$$y = x^\alpha [C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \tag{20}$$

In (20) $r_{1,2} = \alpha \pm i\beta$, where $\alpha, \beta \in \Re$

Series Solutions.

$$(x - x_0)^2 y'' + (x - x_0) p(x) y' + q(x) y = 0 \tag{21}$$

If x_0 is a regular point of (21) then

$$y_1(t) = (x - x_0)^n \sum_{k=0}^{\infty} a_k (x - x_k)^k$$

At a Regular Singular Point x_0, the indicial Equation

$$r^2 + (p(0) - 1)r + q(0) = 0 \tag{22}$$

First Solution

$$y_1 = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_k)^k$$

Where r_1 is the larger real root if both roots of (22) are real or either root if the solutions are complex.

Calculus of Variation

The Euler Equation

Any problem in the calculus of variations is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and solving the resulting differential equation. When the function $F = F(r, \theta, \theta')$, the Euler's equation read

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

If $F = F(t, x, \dot{x})$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

Notice that the first derivative in the Euler equation is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative.

Proof. We will try to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

where F is a given function. Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x)$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Differentiating with respect to x, we get

$$Y(x) = y(x)' + \epsilon\eta'(x)$$

Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\, dx$$

Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x) \right] dx$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] dx$$

Assuming that y'' is continuous, we can integrate the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)\, dx = -\int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y'}\eta(x)\, dx + \left.\frac{\partial F}{\partial y'}\eta(x)\right|_{x_1}^{x_2}$$

The first term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \eta(x)\, dx$$

Since $\eta(x)$ is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \blacksquare$$

Notice carefully here that we are not saying that when an integral is zero, the integrand is also zero; this is not true. What we are saying is that the only way $\int f(x)\eta(x)\, dx$ can always be zero for every $\eta(x)$ is for $f(x)$ to be zero.

Several Variables

If there are n dependent variables in the original integral, there are n Euler-Lagrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}$$

Application: Shortest Between two points

Arbitrary path is given by

$$L = \int_1^2 \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

We factor dx from the integrand in order to make the function we are optimizing not dependent on the y variable and make the evaluation using Euler-Lagrange equation easier

$$f(y, y', x) = \sqrt{1 + y'^2}$$

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial y'}$$

$\partial f/\partial y = 0$ implies simply that $\partial f/\partial y'$ is a constant. Accordingly,

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= \frac{y'}{\sqrt{1 + y'^2}} = C \\
y'^2 &= C^2(1 + y^2) \\
y'^2(1 - C^2) = C^2 & \\
y &= \int \frac{C}{\sqrt{1 - C^2}}\, dx = \mathcal{C}x
\end{aligned}$$

which is the equation for straight line.

Application: Brachistochrone Given two points 1 and 2, with 1 higher above the ground, in what shape should we build a frictionless roller coaster track so that a car released from point 1 will reach point 2 in the shortest possible time?

The speed at which the coaster descend can be determined by the conservation energy principle

$$mgy = \frac{1}{2}mv^2 v \qquad\qquad = \sqrt{2gy}$$

Thus the time to travel between points

$$t = \int_{t_1}^{t_2} \frac{ds}{v} = \int_{t_1}^{t_2} \sqrt{\frac{dx^2 + dy^2}{2gh}}$$

Since v gives a function of y, we take it as independent variable for the same reason as previously

$$t = \frac{1}{\sqrt{2g}} \int_{t_1}^{t_2} \sqrt{\frac{1 + x'^2}{y}}\, dy$$

Ignoring the constant, the function we want to optimize is

$$f(x, x', y) = \sqrt{\frac{1 + x'^2}{y}}$$

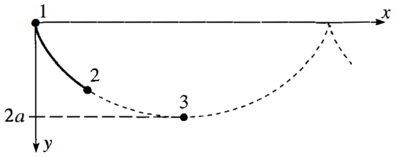


Figure: Brachistochrone problem

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$$

$\partial f/\partial x = 0$ implies simply that $\partial f/\partial x'$ is a constant. Accordingly,

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Here we take the constant as $\sqrt{1/2a}$

$$\begin{aligned}
x'^2 &= \frac{y(1+x'^2)}{2a} \\
x'^2\left(1 - \frac{y}{2a}\right) &= \frac{y}{2a} \\
x' &= \sqrt{\frac{y}{2a}\frac{1}{2-y/2a}} \\
x' &= \sqrt{\frac{y}{2a-y}}
\end{aligned}$$

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

To solve this integral, we substitute $y = a(1-\cos\alpha)$ and $dy = a\sin\alpha\, d\theta$

$$\begin{aligned}
x &= \int \left[\frac{a(1-\cos\theta)}{a(1+\cos\theta)}\right]^{1/2} a\sin\theta\ s\theta \\
&= a\int \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{1/2} [(1+\cos\theta)(1-\cos\theta)]^{1/2}\ d\theta \\
&= a\int (1-\cos\theta)\ d\theta \\
x &= a(\theta - \sin\theta) + c
\end{aligned}$$

Therefore the path of the coaster is given by the following parametric equation

$$\begin{cases} x = a(\theta - \sin\theta) + c \\ y = a(1-\cos\theta) \end{cases}$$

Special Function

Gamma Function

Factorial. The factorial is defined by integral

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ we get

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can also give a meaning to $0!$; by putting $n = 0$, we get $0! = 1$. By the way, the integral can be evaluated using differentiation under integral sign.

Gamma function definition. Gamma function is used to define the factorial function for noninterger n. We define, for any $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$$

From this we have

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx = (p-1)!$$
$$\Gamma(p+1) = \int_0^\infty x^{p} e^{-x}\, dx = p!$$

Recursion relation. The recursion for gamma function is

$$\Gamma(p+1) = p\Gamma(p)$$

Proof. Let us integrate $\Gamma(p+1)$ by parts. Calling $u = x^p$, and $dv = e^{-x}\, dx$; then we get $du = px^{p-1}$, and $v = -e^{-x}$. Thus

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} p x^{p-1}\, dx \\ &= p\int_0^\infty x^{p-1} e^{-x}\, dx \\ \Gamma(p-1) &= p\Gamma(p) \quad \blacksquare\end{aligned}$$

Incomplete gamma function. Supose the limit of the integral of gamma function is not $(0, \infty)$. We define the result function as incomplete gamma function. The first kind is the lower gamma function

$$\gamma(p, t) = \int_0^t x^{p-1} e^{-x}\, dx$$

while the second is the upper gamma function

$$\Gamma(p, t) = \int_t^\infty x^{p-1} e^{-x}\, dx$$

Figure 1: Gaussian integral solved by polar method.

Negative numbers. We shall now deifne gamma function for $p \leq 0$ by the recursion relation

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

From this and the successive use of it, it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers.

Gaussian integral. We state here important formula

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

We can calculate the value of $\Gamma(1/2)$ using this equation, however we will instead try to derive it using another method. First we consider the definition

$$\Gamma(1/2) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt$$

then we substitute $t = x^2$ and $dt = 2x\,dx$

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

This is the famous Gaussian integral. Refer to figure 1 on how to solve it by polar coordinate.

Since everybody and their grandma already know how to solve Gaussian integral by polar coordinate, I will instead try to solve it by Feynman's trick. First consider the function

$$I(\alpha) = \left(\int_0^\alpha e^{-t^2} \, dt \right)^2$$

where I is a function of parameter fish α. Then, to evaluate the actual Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \lim_{\alpha \to \infty} \sqrt{I(\alpha)}$$

Before that, I need to evaluate the function $I(\alpha)$ first. To do that, first I differentiate I with respect parameter fish α

$$\begin{aligned}
\frac{dI}{d\alpha} &= 2 \int_0^\alpha e^{-t^2} \, dt \left(\int_0^\alpha \frac{\partial e^{-t^2}}{d\alpha} \, dt + e^{-\alpha^2} \frac{d\alpha}{d\alpha} - e^{-0^2} \frac{d(0)}{d\alpha} \right) \\
\frac{dI}{d\alpha} &= \int_0^\alpha 2e^{-(t^2+\alpha^2)} \, dt
\end{aligned}$$

where I have used Leibniz' rule for differentiating under integral sign. Then, I introduce the variable $u = t/\alpha$ and $du = dt/\alpha$

$$\frac{dI}{d\alpha} = \int_0^1 2e^{-(u^2\alpha^2+\alpha^2)} \alpha \, du = \int_0^1 2\alpha e^{-\alpha^2(u^2+1)} \, du$$

Using the fact that

$$\frac{\partial}{\partial \alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} = -2\alpha e^{-\alpha^2(u^2+1)}$$

I can rewrite the integrand as

$$\frac{dI}{d\alpha} = -\int_0^1 \frac{\partial}{\partial \alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du$$

Since the integrand is continuous, I can move the partial differentiation outside the integral and turning it into total differentiation

$$\frac{dI}{d\alpha} = -\frac{d}{d\alpha} \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du$$

Hence

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du + C$$

All that remains is to find the value of C. Considering the initial definition of $I(\alpha)$ and evaluating at $\alpha = 0$, I get

$$I(0) = \left(\int_0^0 e^{-t^2} \, dt \right)^2 = 0$$

Therefore

$$I(0) = -\int_0^1 \frac{1}{u^2+1} \, du + C = 0$$

$$C = \arctan u \Big|_0^1 = \frac{\pi}{4}$$

And I obtain the complete expression for the fish function

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du + \frac{\pi}{4}$$

Now I can evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2 \lim_{\alpha\to\infty} \left(-\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1}\, du + \frac{\pi}{4} \right)^{1/2} = 2\frac{\sqrt{\pi}}{2}$$

and I find

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Much to my chagrin, it is actually more trouble some than the polar method. Let's try it for comparison

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy \right)^{1/2}$$

Doing the change of coordinate thing

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \left(\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta \right)^{1/2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \left(2\pi \int_0^{\infty} e^{-r^2} r\, dr \right)^{1/2}$$

That integral can be easily evaluated using u substitution; making the substitution $u = -r^2$

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \left(2\pi \int_{-\infty}^{0} \frac{e^u}{2}\, du \right)^{1/2} = \left(2\pi \frac{e^u}{2} \Big|_{-\infty}^{0} \right)^{1/2}$$

And I get the same result

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Damn, it is really more shrimple.

Another form of Gaussian integral. Here have few.

$$\int_0^{\infty} x^m \exp(-\alpha x)\, dx = \frac{\Gamma(m+1)}{\alpha^{m+1}}$$

$$\int_0^{\infty} x^m \exp\left(-\alpha x^2\right)\, dx = \frac{1}{2\alpha^{(m+1)/2}} \Gamma\left(\frac{m+1}{2}\right)$$

$$\int_{-\infty}^{\infty} \exp\left(-\alpha x^2 + \beta x + \gamma\right)\, dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right)$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{2} i x^2\right)\, dx = \sqrt{2\pi} \exp\left(\frac{\pi}{4} i\right)$$

And also for incomplete gamma integral

$$\int_0^\infty x^m \exp(-\alpha x)\, dx = \frac{\Gamma(m+1, s\alpha)}{\alpha^{m+1}}$$

$$\int_s^\infty x^m \exp\left(-\alpha x^2\right)\, dx = \frac{1}{2\alpha^{(m+1)/2}}\Gamma\left(\frac{m+1}{2}, s\alpha\right)$$

Here's another one, not really a Gaussian integral, but since it involves natural number it counts

$$\sum_{n=0}^\infty n^k e^{-nk} = (-1)^k \frac{d^k}{dk^k}\sum_{n=0}^\infty e^{-nx} = (-1)\frac{d^k}{dk^k}\frac{1}{1-e^{-x}}$$

Proof of the first integral. Consider

$$\int_0^\infty x^m \exp(-\alpha x)\, dx$$

Substitute $\alpha x = t$ and $dx = dt/\alpha$

$$\int_0^\infty \left(\frac{t}{\alpha}\right)^m \frac{\exp(-t)}{\alpha}\, dt = \frac{1}{\alpha^{m+1}}\int_0^\infty x^m e^{-x}\, dx = \frac{\Gamma(m+1)}{\alpha^{m+1}} \quad \blacksquare$$

Now suppose the limit is not $(0, \infty)$, say (s, ∞). Those limits transform

$$x = s \implies t = \alpha s$$
$$x = \infty \implies t = \infty$$

Thus

$$\int_{s\alpha}^\infty \left(\frac{t}{\alpha}\right)^m \frac{\exp(-t)}{\alpha}\, dt = \frac{1}{\alpha^{m+1}}\int_{\alpha s}^\infty x^m e^{-x}\, dx = \frac{\Gamma(m+1, s\alpha)}{\alpha^{m+1}} \quad \blacksquare$$

Proof of the second integral. Consider

$$\int_0^\infty x^m \exp\left(-\alpha x^2\right)\, dx$$

By substituting $t = \alpha x^2$, we have the following variable

$$x = \sqrt{\frac{t}{\alpha}} \quad \text{and} \quad dx = \frac{1}{2\sqrt{\alpha t}}\, dt$$

Thus our integral transforms into

$$\begin{aligned}\int_0^\infty \left(\frac{t}{\alpha}\right)^{m/2} \frac{\exp(-t)}{2\sqrt{\alpha t}}\, dt &= \frac{1}{2\alpha^{(m+1)/2}}\int_0^\infty t^{(m+1)/2}\exp(-t)\, dt \\ &= \frac{1}{2\alpha^{(m+1)/2}}\Gamma\left(\frac{m+1}{2}\right) \quad \blacksquare\end{aligned}$$

Now suppose the limit is not $(0, \infty)$, say (s, ∞). Those limits transform

$$x = s \implies t = \alpha s^2$$

$$x = \infty \implies t = \infty$$

Thus

$$\int_{\alpha s^2}^{\infty} \left(\frac{t}{\alpha}\right)^{m/2} \frac{\exp(-t)}{2\sqrt{\alpha t}}\, dt = \frac{1}{2\alpha^{(m+1)/2}} \int_{\alpha s^2}^{\infty} t^{(m+1)/2} \exp(-t)\, dt$$
$$= \frac{1}{2\alpha^{(m+1)/2}} \Gamma\left(\frac{m+1}{2}, \alpha s^2\right) \quad \blacksquare$$

Proof of the third integral. 404.

Proof of the fourth integral. 404.

Beta Function

Definition. The beta function is also defined by a definite integral

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$

for $p > 0$, and $q > 0$.

Change of order. It is easy to show that

$$B(p, q) = B(q, p)$$

Proof. Putting $x = 1 - y$ and $dx = -dy$

$$B(p, q) = -\int_1^0 (1-y)^{p-1} y^{q-1}\, dy = \int_0^1 y^{q-1}(1-y)^{p-1}\, dy$$
$$B(p, q) = B(q, p) \quad \blacksquare$$

Integration Range. The range of integration can be changed with

$$B(p, q) = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1}(a-y)^{q-1}\, dy$$

Another form is

$$B(p, q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy$$

Proof of the first relation. Putting $x = y/a$ and $dx = dy/a$

$$B = \int_0^a (\frac{y}{a})^{p-1} (1 - \frac{y}{a})^{q-1} \frac{1}{a}\, dy = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1}(a-y)^{q-1}\, dy \quad \blacksquare$$

Proof of the second relation. For the second form, we put $x = y/(1+y)$ and $dx = dy/(1+y)^2$

$$B(p, q) = \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1} \left(\frac{(1+y)-y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2}\, dy$$
$$B(p, q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\, dy \quad \blacksquare$$

Trigonometric form. In terms of sine and cosine, the beta function reads

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta$$

Proof. Putting $x = \sin^2\theta$ and $dx = 2\cos\theta\sin\theta\, d\theta$

$$B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos\theta)^{q-1}\cos\theta\sin\theta\, d\theta$$

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta \quad \blacksquare$$

Gamma Function. Beta functions are easily expressed in terms of gamma functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof. First we consider the gamma function of p

$$\Gamma(p) = \int_0^{\infty} t^{p-1}e^{-t}\, dt$$

Then we make the substitution $t = y^2$ and $dt = 2y\, dy$

$$\Gamma(p) = \int_0^{\infty} y^{2p-2}e^{-y^2}2y\, dy = 2\int_0^{\infty} y^{2p-1}e^{-y^2}\, dy$$

Next we calculate the product of two gamma function p and q

$$\Gamma(p)\Gamma(q) = 4\int_0^{\infty}\int_0^{\infty} x^{2p-1}e^{-x^2}y^{2q-1}e^{-y^2}\, dx\, dy$$

Like Gaussian integral, this is easier to evaluate in polar coordinate

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^{\pi/2}\int_0^{\infty} (r\cos\theta)^{2p-1}(r\sin\theta)^{2q-21}e^{-r^2} r\, dx\, dy \\ &= 2\int_0^{\infty} r^{2(p+q)-1}e^{-r^2}\, dr \cdot 2\int_0^{\pi/2} (\cos\theta)^{2p-1}(\sin\theta)^{2q-1}\, d\theta \\ \Gamma(p)\Gamma(q) &= \Gamma(p+q)B(p,q) \quad \blacksquare\end{aligned}$$

Incomplete beta function. Defined as follows.

$$B_t(p,q) = \int_0^{t} x^{p-1}(1-x)^{q-1}\, dx$$

This definition is using the first definition of beta function, so for any other definition, say the trigonometric form, we must then change it back into our first derinition using whatever substitution. There is also the ratio

$$I_t = \frac{B_t(p,q)}{B(p,q)}$$

Error Function

We define error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

There is also closely related integrals which are used and sometimes referred to as the error function called standard normal or Gaussian cumulative distribution function $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt$$

Here are some of their relations.

$$\begin{aligned} \Phi(x) &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \\ \operatorname{erf}(x) &= 2\Phi\left(x\sqrt{2}\right) - 1 \end{aligned}$$

Proof. Consider the definition of $\Phi(x)$. Making the substitution of $u = t/\sqrt{2}$

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{u=x/\sqrt{2}} e^{-u^2}\sqrt{2}\, du \\ &= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^0 e^{-u^2}\, du + \int_0^\infty e^{-u^2}\, du\right) \\ \Phi(x) &= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right) \quad \blacksquare \end{aligned}$$

Proof of the second relation. To prove the third relation, we first rewrite the equation as

$$\operatorname{erf}\left(x/\sqrt{2}\right) = 2\Phi(x) - 1$$

then we make the substitution $u = x/\sqrt{2}$

$$\operatorname{erf}(u) = 2\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u\sqrt{2}} e^{-t^2/2}\, dt - 1 = 2\Phi\left(x\sqrt{2}\right) - 1 \quad \blacksquare$$

Complementary error function. Defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Its relations with the actual error function are as follows.

$$\begin{aligned} \operatorname{erfc}(x) &= 1 - \operatorname{erfc}(x) \\ \operatorname{erfc}\left(x/\sqrt{2}\right) &= \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \end{aligned}$$

Proof of the first relation. The first relation is quite easy to prove. Consider

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 1$$

then

$$\frac{2}{\sqrt{\pi}} \left(\int_{-\infty}^{x} e^{-t^2} + \int_{x}^{\infty} e^{-t^2} \right) = 1$$

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \quad \blacksquare$$

Proof of the second relation. To proof the second relation, we substitute the limit of integration from $t = x/\sqrt{2}$ into $x = t\sqrt{2}$

$$\operatorname{erfc}\left(x/\sqrt{2}\right) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \frac{e^{t^2/2}}{\sqrt{2}}\, dt = \sqrt{\frac{2}{\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt \quad \blacksquare$$

Imaginary error function. We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{t^2}\, dt$$

Here are some relation to the actual error function.

$$\begin{aligned} \operatorname{erf}(ix) &= i \operatorname{erfi}(x) \\ \operatorname{erf}\left(\frac{1-i}{\sqrt{2}}x\right) &= (1-i)\sqrt{\frac{2}{\pi}} \int_{0}^{x} (\cos^2 u + i \sin^2 u)\, du \end{aligned}$$

Riemann Zeta Function

The Riemann zeta function $\zeta(p)$ is defined by

$$\zeta(p) = \sum_{n=0}^{\infty} \frac{1}{k^p}$$

for real $p > 1$. Here are some value of the Riemann zeta function

$$\begin{array}{cccccccc} \zeta(2) & = & \dfrac{\pi^2}{6}; & \zeta(4) & = & \dfrac{\pi^4}{90}; & \zeta(6) & = & \dfrac{\pi^6}{945} \\ \zeta(3) & = & 1.202; & \zeta(5) & = & 1.036; & \zeta(7) & = & 1.008 \end{array}$$

Integrals. Here are some integral in terms of gamma function and Riemann zeta function.

$$\begin{aligned} \int_{0}^{\infty} \frac{x^p}{e^x - 1} dx &= \Gamma(p+1)\zeta(p+1) \\ \int_{0}^{\infty} \frac{x^p e^x}{(e^x - 1)^2} dx &= \Gamma(p+1)\zeta(p) \\ \int_{0}^{\infty} \frac{x^{p-1}}{e^x + 1} dx &= \left(1 - 2^{1-p}\right)\Gamma(p)\zeta(p) \end{aligned}$$

Striling's Formula

Used to express $n!$ or its logaritm for large n.

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

And for its logaritm

$$\ln n! = n \ln n - n$$

Proof. By the definition

$$p! = \int_0^\infty x^p e^{-x} \, dx$$

We then rewrite it as

$$p! = \int_0^\infty \exp\left(p \ln x - x\right) \, dx$$

Substitute $x = p + y\sqrt{p}$ and $dx = \sqrt{p}\, dy$

$$p! = \int_{-\sqrt{p}}^\infty \exp\left[p \ln\left(p + y\sqrt{p}\right) - p - y\sqrt{p}\right] \sqrt{p}\, dy$$

The logaritm above may be expressed as

$$\ln\left(p + y\sqrt{p}\right) = \ln p + \ln\left(1 + \frac{y}{\sqrt{p}}\right)$$

Recall the infinite series representation of

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Hence

$$\ln\left(p + y\sqrt{p}\right) = \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \frac{y^3}{3p^{3/2}} - \dots$$

Then substituting back

$$p! = \int_{-\sqrt{p}}^\infty \exp\left[p \ln p + \left(\frac{py}{\sqrt{p}} - \frac{py^2}{2p} + \frac{py^3}{3p^{3/2}} - \dots\right) - p - y\sqrt{p}\right] \sqrt{p}\, dy$$

Pulling the constant out of the integral

$$p! = p^p e^{-p} \sqrt{p} \int_{-\sqrt{p}}^\infty \exp\left[-\frac{y^2}{2} + \frac{y^3}{3p^{1/2}} - \dots\right] \, dy$$

For larger p, the integral approach the gaussian integral. Therefore,

$$p! \sim p^p e^{-p} \sqrt{2\pi p} \quad \blacksquare$$

If we take its logaritm, the square root term will be negligible and ae are able to write

$$\ln p! \sim p \ln p - p \quad \blacksquare$$

Probability

Permutation and Combination

Permutation. Consider finite set A with n elements. An r-permutation is an ordered selection of r elements from A, with $1 \leq r \leq n$. In permutation, order does matter, unlike combination, and that all arrangements are distinct. r-permutation of an n elements set is defined as

$$P(n,r) = n(n-1)\dots(n-r+1)$$

or simply

$$P(n,r) = \frac{n!}{(n-r)!}$$

Combination. Combination counts the number of ways to chose r object form finite set A with n elements where order of selection does not matter. For all integer n and $1 \leq r \leq n$, the number of combination when r elements are chosen out of finite set with n elements $C(n,r)$ is

$$C(n,r) = \frac{P(n,r)}{r!} = \binom{n}{r}$$

or

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Difference. Suppose we are choosing 2 people out of 4 to be president and vice-president. Here order matter, thus we say that there are

$$P(4,2) = \frac{4!}{(4-2)!} = 12$$

ways to choose 2 people out of 4 to be president and vice-president. Now, we change the situation into choosing 2 out of 4 people to be given a gift. Here, order does not matter, hence we say that there are

$$C(4,2) = \frac{4!}{2!(4-2)!} = 6$$

ways to choose 2 people out of 4 to be given gift.

Restricted Partition Generating Functions

Definition. To find the number of ways distributing, called configuration, L identical object in N distinct boxes subject to condition that not more that P object are in one box, we use

$$D(N,P,L) = \frac{1}{L!}\frac{d^L}{dx^L}f(x)\bigg|_{x=0}$$

where

$$f(x) = \left(1 + x + x^2 + \dots + x^P\right)^N = \left(\sum_{i=0}^{P} x^i\right)^N$$

In other hands, the number of configuration of certain set $n_{k|P}$ is given by

$$D(N,P,n_{k|P}) = \frac{N!}{\displaystyle\prod_{i=0}^{P} n_i!}$$

while the total number of configuration form all possible set is

$$D_T(N,P) = (P+1)^N$$

For a special case when $L \leq N$, the expression for $D(N,P,L)$ simplify into

$$D(N,P,L) = \frac{1}{L!}\frac{(N+L-1)!}{(N-1)!}$$

This equation gives the number of ways to distribute L indistinguishable objects in P distinguishable box.

Derivation. Let the boxes be numbered $1,\ldots,N$ and p_i as number of objects in i-th box, then

$$\sum_{i=1}^{N} p_i = L$$

with $0 \leq p_i \leq P$. Set obtained form interchanging p_i and p_k, with $p_i \neq p_k$ is counted as different set, however the same exchange with $p_i = p_k$, does not count as different set. Let also n_k as the number of boxed having k number object, hence we have these two restricted for our combination

$$\sum_{k=0}^{P} n_k = N, \quad \sum_{k=0}^{P} k n_k = L$$

which we will denote as restriction R_I and R_{II} respectively.

Now consider the number of configuration $D(N,P,n_{k|P})$ obtained by counting different ways to choose the set of $n_{k|P} \equiv (n_1,\ldots,n_P)$, which is evaluated by choosing n_0 from N boxes, followed by choosing n_1 from $N-n_0$ boxes, and so on. Hence,

$$\begin{aligned}
D(N,P,n_{k|P}) &= \binom{N}{n_0}\cdots\binom{N-\cdots-n_{p-1}}{n_p} \\
&= \frac{N!}{n_0!(N-n_0)!}\cdots\frac{(N-\cdots-n_{p-1}!)}{n_p!(N-\cdots-n_p)!} \\
D(N,P,n_{k|P}) &= \frac{N!}{\displaystyle\prod_{i=0}^{P} n_i!}
\end{aligned}$$

The number of configuration satisfies the first condition, however it does not satisfy the second condition since the number of objects in the set $n_{k|P}$ is

$$\sum_{k=0}^{P} k n_k \equiv M(n_{k|P})$$

is not necessarily L. Our task is then to find the configuration D which satisfies our restriction, formally

$$D(N,P,L) = \sum_{R_I \text{ and } R_{II}} D(N,P,n_{k|P})$$

To find the number of configuration that satisfy those two restriction, we first consider the value of summing $D(N,P,n_{k|P})$ over all possible value of n_k; this makes it so that D satisfies the first condition, but not the second. Formally

$$D_T(N,P) \equiv \sum_{R_I} D(N,P,n_{k|P})$$

Using the result that we derived previously

$$D_T(N,P) = \frac{N!}{\prod_{i=0}^{P} n_i!}$$

Recall the multinomial theorem

$$\left(\sum_{i=0}^{P} x_i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} \prod_{i=0}^{P} x_i^{n_i}$$

Let $x_i = 1$ for all i, and we get

$$(P+1)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!}$$

Therefore

$$D_T(N,P) = (P+1)^N$$

This is the number of ways to distribute $M = 0,\dots,NP$ objects in N boxes, with each box only having maximum P objects. What we want however is $M = L$. To do that, we put $x_i = x^i$ in to multinomial theorem

$$\left(\sum_{i=0}^{P} x^i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} \prod_{i=0}^{P} x^{i\cdot n_i} = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} x^{\sum_{i=0}^{P} i\cdot n_i}$$

$$\left(\sum_{i=0}^{P} x^i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} x^{M(n_{i|P})}$$

Clearly,

$$D_T(N,P,L) = \text{Coefficient of } x^L \text{ in } \left(\sum_{i=0}^{P} x^i\right)^N$$

which is obtained by

$$D(N,P,L) = \frac{1}{L!}\frac{d^L}{dx^L}\left(\sum_{i=0}^{P} x^i\right)^N \Bigg|_{x=0} \quad \blacksquare$$

Now we consider special case when $L \leq P$. Note that the L-th derivative of $f(x)$, especially x^{L+k} with $k \equiv 1, 2.\ldots$ at $x = 0$ is also zero. We can expand the definition of $f(x)$ as polynomial degree P into degree infinity and write

$$f(x) = \left(\sum_{i=0}^{\infty} x^i\right)^N$$

Evaluating it

$$f(x) = (1-x)^{-N}$$

Substituting it into the expression for $D(N,P,L)$, we see that

$$\begin{aligned} D(N,P,L) = \frac{1}{L!}\frac{d^L}{dx^L}(1-x)^N &= \frac{1}{L!}N\frac{d^{L-1}}{dx^{L-1}}(1-x)^{N-1} \\ &= \frac{1}{L!}N(N+1)\frac{d^{L-2}}{dx^{L-2}}(1-x)^{N-2} \end{aligned}$$

Hence, in general

$$D(N,P,L) = \frac{1}{L!}\frac{(N+L-1)!}{(N-1)!} \quad \blacksquare$$