$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

Sinusoidsal wave Equation.

$$y = A \sin \frac{2\pi}{\lambda}(x - vt)$$

where λ represent wavelength, but mathematically it is the same as the period of this function of x. Wave equation in single variable.

$$y(x) = A \sin kx = A \sin 2\pi f x = A$$

$$y(t) = A \sin \omega t = A \sin 2\pi v t = A$$

Average Value. Average of $f(x)$ on (a,b) is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Here are some usefull integrals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \; dx = 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx = \begin{cases} 0, & m \neq \\ \frac{1}{2}, & m = \\ 0, & m = \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx = \begin{cases} 0, & m \neq \\ \frac{1}{2}, & m = \\ 1, & m = \end{cases}$$

Fourier Series

2π period. Fourier Series for function of period 2π:

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \Big(a_n \cos nx + b_n \text{si}$$

with coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx$$

Proof. Multiply both sides of Fourier series by $\cos nx$ and find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{a_0}{2}+\sum \left(a_n \cos nx+b_n\right.\right.$$

All terms on the right are zero except the a_n term then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{a_n}{2} \qquad \blacksquare$$

Notice that $\cos mx$ now turns into $\cos nx$–this is because the integral picks the value of n such that $m = n$. For b_n, we multiply we multiply both sides of by $\sin nx$ and take average values just as we did before

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin mx\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{a_0}{2}+\sum \left(a_n \cos nx+b_n\right.\right.$$

and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{b_n}{2} \qquad \blacksquare$$

To find c_n, we multiply Fourier series by $\exp(-imx)$ and again find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(imx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Bigg[$$

All these terms are zero except the one where $n = m$. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) \qquad \blacksquare$$

Other period. Fourier Series for function of period $2l$:

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty}\left(a_n \cos\frac{2\pi nx}{l} + b_n\right.$$

$$= \sum_{n=-\infty}^{\infty} c_n \exp\frac{in\pi x}{l}$$

with coefficients:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{2\pi nx}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{2\pi nx}{l}\,dx$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\exp\frac{-in\pi x}{l}\,dx$$

The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set. Fourier transforms are defined as follows

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\, d\alpha$$

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\, dx$$

$g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int$ corresponds to $\sum$. This agrees with our discussion of the physical meaning and use of Fourier integrals.

Fourier Sine Transforms. We define $f_s(x)$ and $g_s(\alpha)$ as pair of Fourier sine transforms representing odd functions.

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin \alpha x \; d\alpha$$

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin \alpha x \; dx$$

Fourier Cosine Transforms. We define $f_c(x)$ and $g_c(\alpha)$ as pair of Fourier cosine transforms representing even functions.

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\alpha) \cos \alpha x \; d\alpha$$

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \cos \alpha x \; dx$$

Proof (?). We rewrite Fourier

series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp i\alpha_n x$$

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(u) \exp(-i\alpha_n u)\, du$$

where

$$\frac{n\pi}{l} = \alpha_n$$

$$\alpha_{n+1} - \alpha_n = \Delta\alpha = \frac{\pi}{l}$$

Then

$$c_n = \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp(-\alpha_n u)\, du$$

Subsituting c_n into $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} \Big[\frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(x) \exp(-$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp i\alpha_n$$

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u) \exp i\alpha_n(x-u)\, du$$

If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity],

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) \exp i\alpha(x-u)\, du$$

then $\Delta\alpha \to 0$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\ d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\exp i\alpha(x$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-i\alpha x)\ dx$$

then

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)\exp i\alpha x\ d\alpha$$

Now we expand $\exp(-i\alpha x)$ inside $g(\alpha)$ expression

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha$$

If we assume that $f(x)$ is odd, we get

$$g(x) = -\frac{i}{\pi}\int_{0}^{\infty} f(x) i\sin\alpha x\ dx$$

since the product of odd function $f(x)$ and even function $cos\alpha x$

is odd, thus the integral is zero. Then expanding the exponential in $f(x)$

$$f(x) = 2i \int_0^\infty g(\alpha) \sin \alpha x \; d\alpha$$

If we substitute $g(\alpha)$ into $f(x)$, we obtain

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x) \sin^2 \alpha x \; dx$$

and we see that the numerical factor is $2/\pi$, thus the imaginary factors are not needed. We may as well write $\sqrt{2/\pi}$ instead. Now suppose that $g(\alpha)$ is even. As before, we have

$$g(x) = \frac{1}{\pi} \int_0^\infty f(x) i \cos \alpha x \; dx$$

and

$$f(x) = 2 \int_0^\infty g(\alpha) \cos \alpha x \; d\alpha$$

Subsituting $g(x)$ into $f(x)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(x)\cos^2\alpha x\ dx$$

We also see that it has the same numerical factor and all.

Even and Odd Function

Definition.

$$f(x) = \begin{cases} f(x) = f(-x) & \text{even} \\ f(-x) = -f(x) & \text{odd} \end{cases}$$

Integral of Even and Odd Function.

$$\int_{-l}^{l} f(x)\ dx \begin{cases} 0 & \text{odd} \\ 2\int_0^l f(x)\ dx & \text{even} \end{cases}$$

Fourier expansion for odd function.

$$\text{odd } f(x), \begin{cases} a_n &= 0 \\ b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{2\pi
\end{cases}$$

Fourier expansion for even function.

$$\text{odd } f(x), \begin{cases} a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{2\pi \\ b_n &= 0 \end{cases}$$

Theorem

Dirichlet Condition. If f(x):

1. periodic,
2. x single valued,
3. finite number of discontinuities,
4. finite min max, and
5. $\int_{-\pi}^{\pi} |f(x)|\, dx =$ finite

then the Fourier series converges to the midpoint of the jump.

Parseval's theorem. For Fourier expansions

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos nx + b_n \sin n\right.$$

we have

The average of $[f(x)]^2$ is $\frac{1}{2\pi}\int_{-\pi}^{\pi}[f($

with the value of each coefficients

The average of $(\frac{1}{2}a_0)^2$ is $($

The average of $(a_n \cos nx)^2$ is

The average of $(b_n \sin nx)^2$ is

then we have

The average of $[f(x)]^2$ $= \left(\frac{1}{2}1_0\right)^2 +$

or in complex expansion

The average of $[f(x)]^2$ $= \sum_{-\infty}^{\infty} |c_n|^2$