Appendix: Frobenius' Method

I will demonstrate this technique. Consider the following differential equation.

$$x^2y'' + 4xy' + (x^2 + 2)y = 0$$

The solution will take the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

Substituting this into each term, we have

$$\begin{aligned} x^2y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} \\ 4xy' &= \sum_{n=0}^{\infty} 4(n+s)a_n x^{n+s} \\ xy &= \sum_{n=0}^{\infty} a_n x^{n+s+2} \\ 2y &= \sum_{n=0}^{\infty} 2a_n x^{n+s} \end{aligned}$$

Then we put them into table.

	x^{n+s}	x^s	x^{s+1}
x^2y''	$(n+s)(n+s-1)a_n$	$s(s-1)a_0$	$s(s+1)a_1$
$4xy'$	$4(n+s)a_n$	$4sa_0$	$4(s+1)a_1$
x^2y	a_{n-2}	$-$	$-$
$2y$	$2a_n$	$2a_0$	$2a_1$

Using the terms on x^s column, we have the following indicial equation.

$$\begin{aligned} s(s-1)a_0 + 4sa_0 + 2a_0 &= 0 \\ a_0\left[s(s+3)+2\right] &= 0 \end{aligned}$$

Since a_0 cannot be zero, we write

$$s^2 + 3s + 2 = 0$$

By solving the indicial equation we obtain $s = (-1, -2)$. From the x^{n+s}, we obtain the general formula for a_n in terms of a_{n-2}

$$a_n\left[(n+s)(n+s+3)+2\right] = -a_{n-2}$$

We also obtain the fact the value of a_1 is zero, proved by the terms in x^{s+1} column

$$\left.\begin{aligned} &a_1\left[(s+1)(s+4)+2\right] = 0 \\ &s = (-1, -2) \end{aligned}\right\} \implies a_0 = 0$$

Since we have two value of s, we first consider the case for $s = -1$. The general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-1)(n+2)+2} = -\frac{a_{n-2}}{n^2+n} = -\frac{a_{n-2}}{n(n+1)}$$

The values of a_n for few n are as follows

$$\begin{aligned} a_2 &= -\frac{a_0}{3!} \\ a_4 &= -\frac{a_2}{4 \cdot 5} = \frac{a_0}{5!} \\ a_6 &= -\frac{a_4}{6 \cdot 7} = -\frac{a_0}{7!} \end{aligned}$$

Thus the solution for this case is

$$\begin{aligned} y_{-1} = \sum_{n=0}^{\infty} a_n x^{n-1} &= \frac{a_0}{x} - \frac{a_0}{3!}x + \frac{a_0}{5!}x^3 - \frac{a_0}{7!}x^5 + \dots \\ &= \frac{a_0}{x^2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = \frac{a_0}{x^2}\sin x \end{aligned}$$

For the case of $s = -2$, the general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-2)(n+1)+2} = -\frac{a_{n-2}}{n^2 - n} = -\frac{a_{n-2}}{n(n-1)}$$

The values of a_n for few n are as follows

$$\begin{aligned} a_2 &= -\frac{a_0}{2!} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!} \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!} \end{aligned}$$

Thus the solution for this case is

$$\begin{aligned} y_{-2} = \sum_{n=0}^{\infty} a_n x^{n-2} &= \frac{a_0}{x^2} - \frac{a_0}{2!} + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^4 + \dots \\ &= \frac{a_0}{x^2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) = \frac{a_0}{x^2}\cos x \end{aligned}$$

Hence, the complete form of the solution is

$$y = \frac{a_0}{x^2}(\cos x + \sin x)$$

Bessel Equation

Ex. 1. Suppose we are going to solve

$$y'' + 9xy = 0$$

We know that the equation has no y' factor, then

$$\frac{1-2a}{x} = 0 \implies a = \frac{1}{2}$$

By assuming

$$2c - 2 = 1 \implies c = \frac{3}{2}$$

We can equate the first x coefficient

$$(bc)^2 = 9 \implies b = 2$$

And

$$\frac{a^2 - p^2 c^2}{x^2} = 0 \implies p = \sqrt{\frac{a^2}{c^2}} = \frac{1}{3}$$

The solution takes the form of

$$y = x^{1/2} Z_{1/3}(2x^{3/2}) = x^{1/2} \left[A J_{1/3}(2x^{3/2}) + B N_{1/3}(2x^{3/2}) \right]$$