Introduction

Sinusoidsal wave Equation.

$$y = A \sin \frac{2\pi}{\lambda}(x - vt)$$

where λ represent wavelength, but mathematically it is the same as the period of this function of x. Wave equation in single variable.

$$\begin{aligned} y(x) &= A \sin kx &&= A \sin 2\pi f x = A \sin \frac{2\pi}{\lambda} x \\ y(t) &= A \sin \omega t &&= A \sin 2\pi v t = A \sin \frac{2\pi}{T} t \end{aligned}$$

Average Value. Average of $f(x)$ on (a, b) is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Here are some usefull integrals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \; dx = 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \sin nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos mx \cos nx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \neq 0 \\ 1, & m = n = 0 \end{cases}$$

Fourier Series

2π period. Fourier Series for function of period 2π:

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} \left(a_n \cos nx + b_n \sin nx \right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with coefficients:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx \\ c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \; dx \end{aligned}$$

Proof. Multiply both sides of Fourier series by $\cos nx$ and find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum\big(a_n\cos nx+b_n\sin nx\big)\right]\cos mx\, dx$$

All terms on the right are zero except the a_n term then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{a_n}{2} \qquad \blacksquare$$

Notice that $\cos mx$ now turns into $\cos nx$–this is because the integral picks the value of n such that $m = n$. For b_n, we multiply we multiply both sides of by $\sin nx$ and take average values just as we did before

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum\big(a_n\cos nx+b_n\sin nx\big)\right]\sin mx\, dx$$

and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{b_n}{2} \qquad \blacksquare$$

To find c_n, we multiply Fourier series by $\exp(-imx)$ and again find the average value of each term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(imx) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} c_n \exp inx\right]\exp(-imx)\, dx$$

All these terms are zero except the one where $n = m$. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n \exp ix(n-m)\, dx$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(inx) \qquad \blacksquare$$

Other period. Fourier Series for function of period $2l$:

$$f(x) = \frac{a_0}{2}+\sum_{1}^{\infty}\left(a_n\cos\frac{\pi nx}{l}+b_n\sin\frac{\pi nx}{l}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n \exp\frac{in\pi x}{l}$$

with coefficients:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi nx}{l}\, dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi nx}{l}\, dx$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\exp\frac{-in\pi x}{l}\, dx$$

Fourier Transform

The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set. Fourier transforms are defined as follows

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x}\, d\alpha$$
$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx$$

$g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int$ corresponds to $\sum$. This agrees with our discussion of the physical meaning and use of Fourier integrals.

Fourier Sine Transforms. We define $f_s(x)$ and $g_s(\alpha)$ as pair of Fourier sine transforms representing odd functions.

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin \alpha x\, d\alpha$$
$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin \alpha x\, dx$$

Fourier Cosine Transforms. We define $f_c(x)$ and $g_c(\alpha)$ as pair of Fourier cosine transforms representing even functions.

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\alpha) \cos \alpha x\, d\alpha$$
$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \cos \alpha x\, dx$$

Proof (?). We rewrite Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp i\alpha_n x$$
$$c_n = \frac{1}{2l} \int_{-l}^{l} f(u) \exp(-i\alpha_n u)\, du$$

where

$$\frac{n\pi}{l} = \alpha_n$$
$$\alpha_{n+1} - \alpha_n = \Delta\alpha = \frac{\pi}{l}$$

Then

$$c_n = \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp(-\alpha_n u)\, du$$

Subsituting c_n into $f(x)$

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(x) \exp(-\alpha_n x)\, du \right] \exp \alpha_n x \\
&= \sum_{n=-\infty}^{\infty} \frac{\Delta\alpha}{2\pi} \int_{-l}^{l} f(u) \exp i\alpha_n (x-u)\, du \\
f(x) &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha
\end{aligned}
$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u) \exp i\alpha_n (x-u)\, du$$

If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity],

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) \exp i\alpha(x-u)\, du$$

then $\Delta\alpha \to 0$

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \exp i\alpha(x-u)\, du\, d\alpha
\end{aligned}
$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-i\alpha x)\, dx$$

then

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) \exp i\alpha x\, d\alpha$$

Now we expand $\exp(-i\alpha x)$ inside $g(\alpha)$ expression

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)(\cos \alpha x - i \sin \alpha x)\, dx$$

If we assume that $f(x)$ is odd, we get

$$g(x) = -\frac{i}{\pi} \int_{0}^{\infty} f(x) i \sin \alpha x\, dx$$

since the product of odd function $f(x)$ and even function $cos\alpha x$ is odd, thus the integral is zero. Then expanding the exponential in $f(x)$

$$f(x) = 2i \int_{0}^{\infty} g(\alpha) \sin \alpha x\, d\alpha$$

If we substitute $g(\alpha)$ into $f(x)$, we obtain

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(x) \sin^2 \alpha x\, dx\, d\alpha$$

and we see that the numerical factor is $2/\pi$, thus the imaginary factors are not needed. We may as well write $\sqrt{2/\pi}$ instead. Now suppose that $g(\alpha)$ is even. As before, we have

$$g(x) = \frac{1}{\pi} \int_0^\infty f(x) i \cos \alpha x \, dx$$

and

$$f(x) = 2 \int_0^\infty g(\alpha) \cos \alpha x \, d\alpha$$

Subsituting $g(x)$ into $f(x)$

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x) \cos^2 \alpha x \, dx \, d\alpha$$

We also see that it has the same numerical factor and all.

Even and Odd Function

Definition.

$$f(x) = \begin{cases} f(x) = f(-x) & \text{even} \\ f(-x) = -f(x) & \text{odd} \end{cases}$$

Integral of Even and Odd Function.

$$\int_{-l}^{l} f(x) \, dx \begin{cases} 0 & \text{odd} \\ 2 \displaystyle\int_0^l f(x) \, dx & \text{even} \end{cases}$$

Fourier expansion for odd function.

$$\text{odd } f(x), \begin{cases} a_n & = 0 \\ b_n & = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \frac{\pi n x}{l} \, dx \end{cases}$$

Fourier expansion for even function.

$$\text{even } f(x), \begin{cases} a_n & = \dfrac{2}{l} \displaystyle\int_0^l f(x) \cos \frac{\pi n x}{l} \, dx \\ b_n & = 0 \end{cases}$$

Theorem

Dirichlet Condition. If $f(x)$:

1. periodic,
2. x single valued,
3. finite number of discontinuities,
4. finite min max, and
5. $\int_{-\pi}^{\pi} |f(x)| \, dx = \text{finite}$

then the Fourier series converges to the midpoint of the jump.

Parseval's theorem. For Fourier expansions

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

we have

$$\text{The average of } [f(x)]^2 \text{ is } \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx$$

with the value of each coefficients

$$\begin{array}{rcr} \text{The average of } (\frac{1}{2}a_0)^2 & \text{is} & (\frac{1}{2}a_0)^2 \\ \text{The average of } (a_n \cos nx)^2 & \text{is} & \frac{1}{2}a_n^2 \\ \text{The average of } (b_n \sin nx)^2 & \text{is} & \frac{1}{2}b_n^2 \end{array}$$

then we have

$$\text{The average of } [f(x)]^2 = \left(\frac{1}{2} 1_0 \right)^2 + \frac{1}{2} \sum_{1}^{\infty} a_n^2 + \frac{1}{2} \sum_{1}^{\infty} b_n^2$$

or in complex expansion

$$\text{The average of } [f(x)]^2 = \sum_{-\infty}^{\infty} |c_n|^2$$