Appendix: Lagrange Multipliers

Single constraint. Consider this example.

> Determine the largest volume of parallelepiped–that is, a three-dimensional figure formed by six parallelograms–whose edges parallel with the x, y, z axis inside ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The ellipsoid function above acts as constraints, that left is to determine the function that we want to optimize. This requires some clever thinking. We begin by defining point (x, y, z) be the corner of our parallelepiped. Now, this point is located in the first octant of our parallelepiped. The volume of this octant is

$$v = xyz$$

Since the parallelepiped's sides are parallel the axis, its total volume is

$$V = 8v$$

Hence, the volume of our parallelepiped is

$$V = 8xyz$$

This is the function that we want to maximize. We then construct the function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

The partial derivatives of F read as

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda}{a^2}x, \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda}{b^2}y, \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda}{c^2}z$$

To find the maximum of F, we then must solve the partial derivative equations and constraint equation

$$\begin{aligned} 8yz + \frac{2\lambda}{a^2}x &= 0 \\ 8xz + \frac{2\lambda}{b^2}y &= 0 \\ 8xy + \frac{2\lambda}{c^2}z &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \end{aligned}$$

Multiplying the first equation by x, the second by y, the third by z and adding them all together, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 24xyz + 2\lambda = 0$$

Hence

$$\lambda = -12xyz$$

Substituting this into the partial derivative equation to obtain

$$
\begin{aligned}
8yz - \frac{24yz}{a^2}x^2 = 0 &\implies x = \frac{\sqrt{3}}{3}a \\
8xz - \frac{24xz}{b^2}y^2 = 0 &\implies y = \frac{\sqrt{3}}{3}b \\
8xy - \frac{24xy}{c^2}z^2 = 0 &\implies z = \frac{\sqrt{3}}{3}c
\end{aligned}
$$

Therefore, the maximum volume of said parallelepiped is

$$V = \frac{24\sqrt{3}}{27}abc$$

Two constraints. Here's an example.

> Given two equation $z^2 = x^2 + y^2$ and $x + 2z + 3 = 0$, find the shortest and longest distance from the origin and the intersection of those two equations.

Here we want to minimize $f = x^2 + y^2 + z^2$ as usual. We construct auxiliary function

$$F = x^2 + y^2 + z^2 + \lambda_1(z^2 - x^2 - y^2) + \lambda_2(x + 2z)$$

The partial differentials of F read

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= 2x - 2\lambda_1 x + \lambda_2, \\
\frac{\partial F}{\partial y} &= 2y - 2\lambda_1 y, \\
\frac{\partial F}{\partial z} &= 2z + 2\lambda_1 z + 2\lambda_2
\end{aligned}
$$

Putting it all together, we have these equations

$$
\begin{aligned}
2x - 2\lambda_1 x + \lambda_2 &= 0 && (1) \\
2y - 2\lambda_1 y &= 0 && (2) \\
2z + 2\lambda_1 z + 2\lambda_2 &= 0 && (3) \\
z^2 - x^2 - y^2 &= 0 && (4) \\
x + 2z + 3 &= 0 && (5)
\end{aligned}
$$

By equation 2, we have two possible cases

$$2y - 2\lambda_1 y = y(1 - \lambda_1) = 0 \implies y = 0 \lor \lambda_1 = 1$$

First we consider $y = 0$. Equation 4 reads

$$z^2 = x^2 \implies z = \pm x$$

Then in the subcase $y = 0,\ z = x$; equation 5 evaluates into

$$3x + 3 = 0 \implies x = 3$$

In other hand, for subcase $y = 0,\ z = -x$; the same equation evaluates into

$$x = 3$$

Now we consider the case when $\lambda_1 = 1$. Equation 1 reduces into

$$\lambda_2 = 0$$

which means equation 5 turns into

$$4z = 0 \implies z = 0$$

and equation 5

$$x = -3$$

Using this result, equation 4 reads

$$y^2 = -9$$

which is impossible unless we are willing to take a complex value. Suppose we are willing, we have the $y = 3i$. Hence, we have three possibilities that the optimized points might take

$$\{\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}\} = \{(-1, 0, -1), (3, 0, -3), (-3, 3i, 0)\}$$

The distance from origin then evaluated by

$$\begin{aligned} d_1 &= \sqrt{\mathbf{P_1} \cdot \mathbf{P_2}} = \sqrt{2} \\ d_2 &= \sqrt{\mathbf{P_2} \cdot \mathbf{P_2}} = \sqrt{18} \\ d_3 &= \sqrt{\mathbf{P_3} \cdot \overline{\mathbf{P_3}}} = \sqrt{18} \end{aligned}$$

Hence the shortest distance is $d = \sqrt{2}$ and the longest is $d = \sqrt{18}$.