Permutation and Combination

Permutation. Consider finite set A with n elements. An r-permutation is an ordered selection of r elements from A, with $1 \leq r \leq n$. In permutation, order does matter, unlike combination, and that all arrangements are distinct. r-permutation of an n elements set is defined as

$$P(n, r) = n(n-1)\dots(n-r+1)$$

or simply

$$P(n, r) = \frac{n!}{(n-r)!}$$

Combination. Combination counts the number of ways to chose r object form finite set A with n elements where order of selection does not matter. For all integer n and $1 \leq r \leq n$, the number of combination when r elements are chosen out of finite set with n elements $C(n, r)$ is

$$C(n, r) = \frac{P(n, r)}{r!} = \binom{n}{r}$$

or

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Difference. Suppose we are choosing 2 people out of 4 to be president and vice-president. Here order matter, thus we say that there are

$$P(4, 2) = \frac{4!}{(4-2)!} = 12$$

ways to choose 2 people out of 4 to be president and vice-president. Now, we change the situation into choosing 2 out of 4 people to be given a gift. Here, order does not matter, hence we say that there are

$$C(4, 2) = \frac{4!}{2!(4-2)!} = 6$$

ways to choose 2 people out of 4 to be given gift.

Restricted Partition Generating Functions

Definition. To find the number of ways distributing, called configuration, L identical object in N distinct boxes subject to condition that not more that P object are in one box, we use

$$D(N, P, L) = \frac{1}{L!}\frac{d^L}{dx^L}f(x)\bigg|_{x=0}$$

where

$$f(x) = \left(1 + x + x^2 + \dots + x^P\right)^N = \left(\sum_{i=0}^{P} x^i\right)^N$$

In other hands, the number of configuration of certain set $n_{k|P}$ is given by

$$D(N,P,n_{k|P}) = \frac{N!}{\prod_{i=0}^{P} n_i!}$$

while the total number of configuration form all possible set is

$$D_T(N,P) = (P+1)^N$$

For a special case when $L \leq N$, the expression for $D(N,P,L)$ simplify into

$$D(N,P,L) = \frac{1}{L!}\frac{(N+L-1)!}{(N-1)!}$$

This equation gives the number of ways to distribute L indistinguishable objects in P distinguishable box.

Derivation. Let the boxes be numbered $1,\ldots,N$ and p_i as number of objects in i-th box, then

$$\sum_{i=1}^{N} p_i = L$$

with $0 \leq p_i \leq P$. Set obtained form interchanging p_i and p_k, with $p_i \neq p_k$ is counted as different set, however the same exchange with $p_i = p_k$, does not count as different set. Let also n_k as the number of boxed having k number object, hence we have these two restricted for our combination

$$\sum_{k=0}^{P} n_k = N, \quad \sum_{k=0}^{P} k n_k = L$$

which we will denote as restriction R_I and R_{II} respectively.

Now consider the number of configuration $D(N,P,n_{k|P})$ obtained by counting different ways to choose the set of $n_{k|P} \equiv (n_1,\ldots,n_P)$, which is evaluated by choosing n_0 from N boxes, followed by choosing n_1 from $N-n_0$ boxes, and so on. Hence,

$$\begin{aligned} D(N,P,n_{k|P}) &= \binom{N}{n_0}\cdots\binom{N-\cdots-n_{p-1}}{n_p} \\ &= \frac{N!}{n_0!(N-n_0)!}\cdots\frac{(N-\cdots-n_{p-1}!)}{n_p!(N-\cdots-n_p)!} \\ D(N,P,n_{k|P}) &= \frac{N!}{\prod_{i=0}^{P} n_i!} \end{aligned}$$

The number of configuration satisfies the first condition, however it does not satisfy the second condition since the number of objects in the set $n_{k|P}$ is

$$\sum_{k=0}^{P} k n_k \equiv M(n_{k|P})$$

is not necessarily L. Our task is then to find the configuration D which satisfies our restriction, formally

$$D(N,P,L) = \sum_{R_I \text{ and } R_{II}} D(N,P,n_{k|P})$$

To find the number of configuration that satisfy those two restriction, we first consider the value of summing $D(N,P,n_{k|P})$ over all possible value of n_k; this makes it so that D satisfies the first condition, but not the second. Formally

$$D_T(N,P) \equiv \sum_{R_I} D(N,P,n_{k|P})$$

Using the result that we derived previously

$$D_T(N,P) = \frac{N!}{\prod_{i=0}^{P} n_i!}$$

Recall the multinomial theorem

$$\left(\sum_{i=0}^{P} x_i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} \prod_{i=0}^{P} x_i^{n_i}$$

Let $x_i = 1$ for all i, and we get

$$(P+1)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!}$$

Therefore

$$D_T(N,P) = (P+1)^N$$

This is the number of ways to distribute $M = 0,\ldots,NP$ objects in N boxes, with each box only having maximum P objects. What we want however is $M = L$. To do that, we put $x_i = x^i$ in to multinomial theorem

$$\left(\sum_{i=0}^{P} x^i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} \prod_{i=0}^{P} x^{i \cdot n_i} = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} x^{\sum_{i=0}^{P} i \cdot n_i}$$

$$\left(\sum_{i=0}^{P} x^i\right)^N = \sum_{R_I} \frac{N}{\prod_{i=0}^{P} n_i!} x^{M(n_{i|P})}$$

Clearly,

$$D_T(N,P,L) = \text{Coefficient of } x^L \text{ in } \left(\sum_{i=0}^{P} x^i\right)^N$$

which is obtained by

$$D(N,P,L) = \frac{1}{L!}\frac{d^L}{dx^L}\left(\sum_{i=0}^{P} x^i\right)^N \Bigg|_{x=0} \quad \blacksquare$$

Now we consider special case when $L \leq P$. Note that the L-th derivative of $f(x)$, especially x^{L+k} with $k \equiv 1, 2. \ldots$ at $x = 0$ is also zero. We can expand the definition of $f(x)$ as polynomial degree P into degree infinity and write

$$f(x) = \left(\sum_{i=0}^{\infty} x^i\right)^N$$

Evaluating it

$$f(x) = (1-x)^{-N}$$

Substituting it into the expression for $D(N,P,L)$, we see that

$$\begin{aligned} D(N,P,L) = \frac{1}{L!}\frac{d^L}{dx^L}(1-x)^N &= \frac{1}{L!}N\frac{d^{L-1}}{dx^{L-1}}(1-x)^{N-1} \\ &= \frac{1}{L!}N(N+1)\frac{d^{L-2}}{dx^{L-2}}(1-x)^{N-2} \end{aligned}$$

Hence, in general

$$D(N,P,L) = \frac{1}{L!}\frac{(N+L-1)!}{(N-1)!} \quad \blacksquare$$