

Gamma Function

Factorial. The factorial is defined by integral

$$\int_0^{\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ we get

$$\int_0^{\infty} x^n e^{-x} dx = n!$$

Thus we have a definite integral whose value is $n!$ for positive integral n . We can also give a meaning to $0!$; by putting $n = 0$, we get $0! = 1$. By the way, the integral can be evaluated using differentiation under integral sign.

Gamma function definition. Gamma function is used to define the factorial function for noninteger n . We define, for any $p > 0$

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx$$

From this we have

$$\begin{aligned}\Gamma(p) &= \int_0^{\infty} x^{p-1} e^{-x} dx = (p-1)! \\ \Gamma(p+1) &= \int_0^{\infty} x^p e^{-x} dx = p!\end{aligned}$$

Recursion relation. The recursion for gamma function is

$$\Gamma(p+1) = p\Gamma(p)$$

Proof. Let us integrate $\Gamma(p+1)$ by parts. Calling $u = x^p$, and $dv = e^{-x} dx$; then we get $du = px^{p-1}$, and $v = -e^{-x}$. Thus

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} px^{p-1} dx \\ &= p \int_0^{\infty} x^{p-1} e^{-x} dx \\ \Gamma(p+1) &= p\Gamma(p) \quad \blacksquare\end{aligned}$$

Negative numbers. We shall now define gamma function for $p \leq 0$ by the recursion relation

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

From this and the successive use of it, it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers.



Figure 1: Gaussian integral solved by polar method.

Important formula. We state here important formula

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

We can calculate the value of $\Gamma(1/2)$ using this equation, however we will instead try to derive it using another method. First we consider the definition

$$\Gamma(1/2) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

then we substitute $t = x^2$ and $dt = 2x dx$

$$\Gamma(1/2) = 2 \int_0^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$$

This is the famous Gaussian integral. Refer to figure 1 on how to solve it by polar coordinate.

Since everybody and their grandma already know how to solve Gaussian integral by polar coordinate, I will instead try to solve it by Feynman's trick. First consider the function

$$I(\alpha) = \left(\int_0^{\alpha} e^{-t^2} dt \right)^2$$

where I is a function of parameter fish α . Then, to evaluate the actual Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \lim_{\alpha \rightarrow \infty} \sqrt{I(\alpha)}$$

Before that, I need to evaluate the function $I(\alpha)$ first. To do that, first I differentiate I with respect parameter fish α

$$\begin{aligned}\frac{dI}{d\alpha} &= 2 \int_0^\alpha e^{-t^2} dt \left(\int_0^\alpha \frac{\partial e^{-t^2}}{d\alpha} dt + e^{-\alpha^2} \frac{d\alpha}{d\alpha} - e^{-0^2} \frac{d(0)}{d\alpha} \right) \\ \frac{dI}{d\alpha} &= \int_0^\alpha 2e^{-(t^2+\alpha^2)} dt\end{aligned}$$

where I have used Leibniz' rule for differentiating under integral sign. Then, I introduce the variable $u = t/\alpha$ and $du = dt/\alpha$

$$\frac{dI}{d\alpha} = \int_0^1 2e^{-(u^2\alpha^2+\alpha^2)} \alpha du = \int_0^1 2\alpha e^{-\alpha^2(u^2+1)} du$$

Using the fact that

$$\frac{\partial}{\partial \alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} = -2\alpha e^{-\alpha^2(u^2+1)}$$

I can rewrite the integrand as

$$\frac{dI}{d\alpha} = - \int_0^1 \frac{\partial}{\partial \alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} du$$

Since the integrand is continous, I can move the partial differentiation outside the integral and turning it into total differentiation

$$\frac{dI}{d\alpha} = - \frac{d}{d\alpha} \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} du$$

Hence

$$I(\alpha) = - \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} du + C$$

All that remains is to find the value of C . Considering the initial definition of $I(\alpha)$ and evaluating at $\alpha = 0$, I get

$$I(0) = \left(\int_0^0 e^{-t^2} dt \right)^2 = 0$$

Therefore

$$\begin{aligned}I(0) &= - \int_0^1 \frac{1}{u^2+1} du + C = 0 \\ C &= \arctan u \Big|_0^1 = \frac{\pi}{4}\end{aligned}$$

And I obtain the complete expression for the fish function

$$I(\alpha) = - \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} du + \frac{\pi}{4}$$

Now I can evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \lim_{\alpha \rightarrow \infty} \left(- \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} du + \frac{\pi}{4} \right)^{1/2} = 2 \frac{\sqrt{\pi}}{2}$$

and I find

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Much to my chagrin, it is actually more trouble some than the polar method. Let's us try it for comparison

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \right)^{1/2}$$

Doing the change of coordinate thing

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2} dx &= \left(\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \right)^{1/2} \\ \int_{-\infty}^{\infty} e^{-x^2} dx &= \left(2\pi \int_0^{\infty} e^{-r^2} r dr \right)^{1/2} \end{aligned}$$

That integral can be easily evaluated using u substitution; making the substitution $u = -r^2$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \left(2\pi \int_{-\infty}^0 \frac{e^u}{2} du \right)^{1/2} = \left(2\pi \frac{e^u}{2} \Big|_{-\infty}^0 \right)^{1/2}$$

And I get the same result

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Damn, it is really more shrimple.

Another form of Gaussian integral. Here we state without proof.

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-\alpha x^2 + \beta x) dx &= \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \\ \int_0^{\infty} x^m \exp(-\alpha x^2) dx &= \frac{1}{2\alpha^{(m+1)/2}} \Gamma\left(\frac{m+1}{2}\right) \end{aligned}$$

Here's another one, not really a Gaussian integral, but since it involves natural number it counts

$$\sum_{n=0}^{\infty} n^k e^{-nx} = (-1)^k \frac{d^k}{dx^k} \sum_{n=0}^{\infty} e^{-nx} = (-1)^k \frac{d^k}{dx^k} \frac{1}{1 - e^{-x}}$$

Beta Function

Definition. The beta function is also defined by a definite integral

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

for $p > 0$, and $q > 0$.

Change of order. It is easy to show that

$$B(p, q) = B(q, p)$$

Proof. Putting $x = 1 - y$ and $dx = -dy$

$$B(p, q) = - \int_1^0 (1 - y)^{p-1} y^{q-1} dy = \int_0^1 y^{q-1} (1 - y)^{p-1} dy$$

$$B(p, q) = B(q, p) \quad \blacksquare$$

Integration Range. The range of integration can be changed with

$$B(p, q) = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1} (a - y)^{q-1} dy$$

Another form is

$$B(p, q) = \int_0^\infty \frac{y^{p-1}}{(1 + y)^{p+q}} dy$$

Proof. Putting $x = y/a$ and $dx = dy/a$

$$B = \int_0^a \left(\frac{y}{a}\right)^{p-1} \left(1 - \frac{y}{a}\right)^{q-1} \frac{1}{a} dy = \frac{1}{a^{p+1-1}} \int_0^a y^{p-1} (a - y)^{q-1} dy \quad \blacksquare$$

For the second form, we put $x = y/(1 + y)$ and $dx = dy/(1 + y)^2$

$$B(p, q) = \int_0^\infty \left(\frac{y}{1 + y}\right)^{p-1} \left(\frac{(1 + y) - y}{1 + y}\right)^{q-1} \frac{1}{(1 + y)^2} dy$$

$$B(p, q) = \int_0^\infty \frac{y^{p-1}}{(1 + y)^{p+q}} dy \quad \blacksquare$$

Trigonometric form. In terms of sine and cosine, the beta function reads

$$B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} d\theta$$

Proof. Putting $x = \sin^2 \theta$ and $dx = 2 \cos \theta \sin \theta d\theta$

$$B(p, q) = \int_0^{\pi/2} (\sin^2 \theta)^{p-1} (\cos \theta)^{q-1} \cos \theta \sin \theta d\theta$$

$$B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-1} (\cos \theta)^{2q-1} d\theta \quad \blacksquare$$

Gamma Function. Beta functions are easily expressed in terms of gamma functions

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}$$

Proof. First we consider the gamma function of p

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

Then we make the substitution $t = y^2$ and $dt = 2y dy$

$$\Gamma(p) = \int_0^\infty y^{2p-2} e^{-y^2} 2y dy = 2 \int_0^\infty y^{2p-1} e^{-y^2} dy$$

Next we calculate the product of two gamma function p and q

$$\Gamma(p)\Gamma(q) = 4 \int_0^\infty \int_0^\infty x^{2p-1} e^{-x^2} y^{2q-1} e^{-y^2} dx dy$$

Like Gaussian integral, this is easier to evaluate in polar coordinate

$$\begin{aligned} \Gamma(p)\Gamma(q) &= 4 \int_0^{\pi/2} \int_0^\infty (r \cos \theta)^{2p-1} (r \sin \theta)^{2q-1} e^{-r^2} r dx dy \\ &= 2 \int_0^\infty r^{2(p+q)-1} e^{-r^2} dr \cdot 2 \int_0^{\pi/2} (\cos \theta)^{2p-1} (\sin \theta)^{2q-1} d\theta \\ \Gamma(p)\Gamma(q) &= \Gamma(p+q)B(p, q) \quad \blacksquare \end{aligned}$$

Error Function

We define error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

There is also closely related integrals which are used and sometimes referred to as the error function called standard normal or Gaussian cumulative distribution function $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

Here are some of their relations.

$$\begin{aligned} \Phi(x) &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}(x/\sqrt{2}) \\ \Phi(x) - \frac{1}{2} &= \frac{1}{2} \operatorname{erf}(x/\sqrt{2}) \\ \operatorname{erf}(x) &= 2\Phi(x\sqrt{2}) - 1 \end{aligned}$$

Proof. Consider the definition of $\Phi(x)$. Making the substitution of $u = t/\sqrt{2}$

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{u=x/\sqrt{2}} e^{-u^2} \sqrt{2} du \\ &= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^0 e^{-u^2} du + \int_0^\infty e^{-u^2} du \right) \\ \Phi(x) &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}(x/\sqrt{2}) \quad \blacksquare \end{aligned}$$

To prove the third relation, we first rewrite the equation as

$$\operatorname{erf}(x/\sqrt{2}) = 2\Phi(x) - 1$$

then we make the substitution $u = x/\sqrt{2}$

$$\operatorname{erf}(u) = 2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u\sqrt{2}} e^{-t^2/2} dt - 1 = 2\Phi(x\sqrt{2}) - 1 \quad \blacksquare$$

Complementary error function. Defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

Its relations with the actual error function are as follows.

$$\begin{aligned}\operatorname{erfc}(x) &= 1 - \operatorname{erf}(x) \\ \operatorname{erfc}(x/\sqrt{2}) &= \sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-t^2/2} dt\end{aligned}$$

Proof. The first relation is quite easy to prove. Consider

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 1$$

then

$$\begin{aligned}\frac{2}{\sqrt{\pi}} \left(\int_{-\infty}^x e^{-t^2} + \int_x^{\infty} e^{-t^2} \right) &= 1 \\ \operatorname{erf}(x) + \operatorname{erfc}(x) &= 1 \quad \blacksquare\end{aligned}$$

To proof the second relation, we substitute the limit of integration from $t = x/\sqrt{2}$ into $x = t\sqrt{2}$

$$\operatorname{erfc}(x/\sqrt{2}) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{e^{t^2/2}}{\sqrt{2}} dt = \sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-t^2/2} dt \quad \blacksquare$$

Imaginary error function. We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt$$

Here are some relation to the actual error function.

$$\begin{aligned}\operatorname{erf}(ix) &= i \operatorname{erfi}(x) \\ \operatorname{erf}\left(\frac{1-i}{\sqrt{2}}x\right) &= (1-i) \sqrt{\frac{2}{\pi}} \int_0^x (\cos^2 u + i \sin^2 u) du\end{aligned}$$

Riemann zeta function

The Riemann zeta function $\zeta(p)$ is defined by

$$\zeta(p) = \sum_{n=0}^{\infty} \frac{1}{k^p}$$

for real $p > 1$. Here are some value of the Riemann zeta function

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6}; & \zeta(4) &= \frac{\pi^4}{90}; & \zeta(6) &= \frac{\pi^6}{945} \\ \zeta(3) &= 1.202; & \zeta(5) &= 1.036; & \zeta(7) &= 1.008\end{aligned}$$

Integrals. Here are some integral in term of gamma function and Riemann zeta function.

$$\begin{aligned}\int_0^\infty \frac{x^p}{e^x - 1} dx &= \Gamma(p+1)\zeta(p+1) \\ \int_0^\infty \frac{x^p e^x}{(e^x - 1)^2} dx &= \Gamma(p+1)\zeta(p) \\ \int_0^\infty \frac{x^{p-1}}{e^x + 1} dx &= (1 - 2^{1-p}) \Gamma(p)\zeta(p)\end{aligned}$$