

# Appendix: Differential Equation Study Guide

**First Order Equations.** General Form of ODE

$$\frac{dy}{dx} = f(x, y)$$

Initial Value Problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

**Linear Equations.** General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\begin{aligned}\mu(x) &= e^{\int p(x)dx} \\ \implies \frac{d}{dx}(\mu(x)y) &= \mu(x)f(x)\end{aligned}$$

General Solution

$$y = \frac{1}{\mu(x)} \left( \int \mu(x)f(x)dx + C \right)$$

**Homogeneous Equations.** General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in  $z$ :

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

**Bernoulli Equations.** General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in  $z$ :

$$z' + (1-n)p(x)z = (1-n)q(x)$$

**Exact Equations.** General Form

$$M(x, y)dx + N(x, y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

### Method for Solving Exact Equations.

1. Let  $\phi = \int M(x, y)dx + h(y)$
2. Set  $\frac{\partial \phi}{\partial y} = N(x, y)$
3. Simplify and solve for  $h(y)$
4. Substitute the result for  $h(y)$  in the expression for  $\phi$  from step 1 and then set  $\phi = 0$ . This is the solution.

Alternatively:

1. Let  $\phi = \int N(x, y)dy + g(x)$
2. Set  $\frac{\partial \phi}{\partial x} = M(x, y)$
3. Simplify and solve for  $g(x)$ .
4. Substitute the result for  $g(x)$  in the expression for  $\phi$  from step 1 and then set  $\phi = 0$ . This is the solution.

**Integrating Factors.** Case 1. If  $P(x, y)$  depends only on  $x$ , where

$$P(x, y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0$$

is exact.

Case 2. If  $Q(x, y)$  depends only on  $y$ , where

$$Q(x, y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$$

is exact.

### Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{1}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{2}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{3}$$

**General Solution.** The general solution of (1) or (3) is

$$y = C_1y_1(t) + C_2y_2(t) + y_p(t) \tag{4}$$

where  $y_1(t)$  and  $y_2(t)$  are linearly independent solutions of (2).

**Linear Independence and The Wronskian.** Two functions  $f(x)$  and  $g(x)$  are linearly dependent if there exist numbers  $a$  and  $b$ , not both zero, such that  $af(x) + bg(x) = 0$  for all  $x$ . If  $y_1$  and  $y_2$  are two solutions of (2), then Wronskian

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1.  $\{y_1, y_2\}$  are linearly independent.
2.  $\{y_1, y_2\}$  are a fundamental set of solutions.
3.  $W(y_1, y_2)(t_0) \neq 0$  at some point  $t_0$ .
4.  $W(y_1, y_2)(t) \neq 0$  for all  $t$ .

**Initial Value Problem.** The initial value problem includes two initial conditions at the same point in time, one condition on  $y(t)$  and one condition on  $y'(t)$ .

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

The initial conditions are applied to the entire solution  $y = y_h + y_p$ .

**Linear Equation With Constant Coefficients.** The general form of the homogeneous equation is

$$ay'' + by' + cy = 0 \tag{5}$$

Non-homogeneous

$$ay'' + by' + cy = g(t) \tag{6}$$

Characteristic Equation

$$ar^2 + br + c = 0$$

Quadratic Roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

The solution of (5) of Real Roots ( $r_1 \neq r_2$ )

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{8}$$

Repeated ( $r_1 = r_2$ )

$$y_h = (C_1 + C_2 t) e^{r_1 t} \tag{9}$$

Complex ( $r = \alpha \pm i\beta$ )

$$y_H = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t) \tag{10}$$

The solution of (6) is  $y = y_p + y_h$  where  $y_h$  is given by (8) through (10) and  $y_p$  is found by undetermined coefficients or reduction of order.

**Heuristics for Undetermined Coefficients.** Also called Trial and Error

If $f(t) =$	then guess that a particular solution $y_p =$ .
$P_n(t)$	$t^s(A_0 + A_1t + \cdots + A_nt^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1t + \cdots + A_nt^n)e^{at}$
$P_n(t)e^{at} \sin bt$ or $P_n(t)e^{at} \cos bt$	$t^s e^{at}[(A_0 + A_1t + \cdots + A_nt^n) \cos bt$ $+ (A_0 + A_1t + \cdots + A_nt^n) \sin bt]$

**Method of Reduction of Order.** When solving (2), given  $y_1$ , then  $y_2$  can be found by solving

$$y_1 y_2' - y_1' y_2 = C e^{-\int p(t) dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x) dx} dx}{y_1(x)^2} \quad (11)$$

**Method of Variation of Parameters.** If  $y_1(t)$  and  $y_2(t)$  are a fundamental set of solutions to (2) then a particular solution to (3) is

$$y_P(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)f(t)}{W(t)} dt \quad (12)$$

**Cauchy-Euler Equation.** For ODE

$$ax^2 y'' + bxy' + cy = 0 \quad (13)$$

with Auxilliary Equation

$$ar(r-1) + br + c = 0 \quad (14)$$

The solutions of (13) depend on the roots  $r_{1,2}$  of (14). For Real Roots

$$y = C_1 x^{r_1} + C_2 x^{r_2}$$

Repeated Root

$$y = C_1 x^r + C_2 x^r \ln x$$

Complex

$$y = x^\alpha [C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \quad (15)$$

In (15)  $r_{1,2} = \alpha \pm i\beta$ , where  $\alpha, \beta \in \mathbb{R}$

**Series Solutions.**

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0 \quad (16)$$

If  $x_0$  is a regular point of (16) then

$$y_1(t) = (x - x_0)^n \sum_{k=0}^{\infty} a_k (x - x_k)^k$$

At a Regular Singular Point  $x_0$ , the Indicial Equation

$$r^2 + (p(0) - 1)r + q(0) = 0 \quad (17)$$

First Solution

$$y_1 = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_k)^k$$

Where  $r_1$  is the larger real root if both roots of (17) are real or either root if the solutions are complex.