First Order

First Order ODE. Written in the form

$$y' + P(x)y = Q(x)$$

where P and Q are functions of x has the solution

$$ye^I = \int Qe^I \, dx + c$$

$$y = e^{-I} \int Qe^I \, dx + ce^{-I}$$

where

$$I = \int P \, dx$$

Bernoulli Equation. The differential equation

$$y' + P(x)y = Q(x)y^n$$

where P and Q are functions of x. It also can be written as

$$z' + (1-n)Pz = (1-n)Q$$

where

$$z = y^{1-n}$$

This is now a first-order linear equation which we can solve as we did the linear equations above.

Exact Equations. $P(x,y)dx + Q(x,y)dy$ is an exact differential [the differential of $F(x,y)$, or $Pdx + Qdy = dF$] if

$$\frac{\partial}{\partial x}P = \frac{\partial}{\partial x}Q$$

and the solution is

$$F(x,y) = \text{constant}$$

An equation which is not exact may often be made exact by multiplying it by an appropriate factor.

Homogeneous Equations. A homogeneous function of x and y of degree n means a function which can be written as $x^n f(y/x)$. An equation in the from

$$P(x,y)dx + Q(x,y)dy = 0$$

where P and Q are homogeneous functions of the same degree is called homogeneous. Thus,

$$y' = \frac{d}{dx}y = -\frac{P(x,y)}{Q(x,y)} - f(\frac{y}{x})$$

This suggests that we solve homogeneous equations by making the change of variables

$$y = xv \quad \text{with} \quad v = \frac{y}{x}$$

Second Order

Second Order with Zero Right-Hand Side. Equation of the form

$$(D-a)(D-b)y = 0, \quad a \neq b$$

has the Solution

$$y = c_1 e^{ax} + c_2 e^{bx}$$

Equation of the form

$$(D-a)(D-a)y = 0, \quad a \neq b$$

has the Solution

$$y = (Ax + B)e^{ax}$$

Now suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. The solution is now

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)}x + Be^{(\alpha-i\beta)}x \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \\ &= e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) \\ &= ce^{\alpha x} \sin(\beta x + \gamma) \end{aligned}$$

where $\alpha,\ \beta,\ \gamma,\ c,\ c_1,\ c_2$ are different constant.

Second Order with Nonzero Right-hand Side. The equation

$$\begin{aligned} a_2 \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= f(x) \\ \frac{d^2}{dx^2} y + a_1 \frac{d}{dx} y + a_0 y &= F(x) \end{aligned}$$

has the solution of the form

$$y = y_c + y_p$$

where the complementary function y_c is the general solution of the homogeneous equation (when right-hand side is equal to zero) and y_p is a particular solution, that is when the right-hand side is equal to $f(x)$ or $F(x)$. The simplest method solving them is by Inspection and Successive Integration of Two First-Order Equations.

Exponential Right-Hand Side. Suppose we have $F(x) = ke^{cx}$, or

$$(D-a)(D-b)y = ke^{cx}$$

then, we find a particular solution by assuming a solution of the form:

$$y_p = \begin{cases} Ce^{cx} & \text{if c is not equal to either a or b;} \\ Cxe^{cx} & \text{if c equals a or b, a} \neq \text{b;} \\ Cx^2e^{cx} & \text{if c = a = b.} \end{cases}$$

Complex Exponential. To find a particular solution of

$$(D-a)(D-b)y = \begin{cases} k\sin\alpha x, \\ k\cos\alpha x, \end{cases}$$

first solve

$$(D-a)(D-b)y = ke^{i\alpha x}$$

then take the real or imaginary part.

Method of Undetermined Coefficients. To find a particular solution of

$$(D-a)(D-b)y = e^{cx}P_n(x)$$

where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if c is not equal to either a or b;} \\ xe^{cx}Q_n(x) & \text{if c equals a or b, a} \neq \text{b;} \\ x^2e^{cx}Q_n(x) & \text{if c = a = b.} \end{cases}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the given differential equation.

Principle of Superposition. The easiest way of handling a complicated right-hand side: Solve a separate equation for each different exponential and add the solutions. The fact that this is correct for a linear equation is often called the principle of superposition.

Note that the principle holds only for linear equations.

Fourier Series. Suppose that the driving force $f(x)$ is periodic, we then can expand the function using Fourier Series. The equation

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = f(x) = \sum_{-\infty}^{\infty} c_n e_{inx}$$

can be solved by solving

$$a_2\frac{d^2}{dx^2}y + a_1\frac{d}{dx}y + a_0y = c_n e_{inx}$$

then add the solutions for all n (applying principle of superposition), and we have the solution of first the equation.

Laplace Transform

We define $\mathcal{L}(f)$, the Laplace transform of $f(t)$ [also written $F(p)$ since it is a function of p], by the equation

$$\mathcal{L}(f) = F(P) = \int_0^\infty f(t)e^{-pt}; dt$$

Laplace transform 101. How 2 Laplace transform in 5 steps!

1. Transform!
2. Do algebra!
3. Inverse!
4. ...
5. Profit!

Convolution

Definition. The integral

$$g * h = \int_0^t g(t-\tau)h(\tau)d(\tau) = \int_0^t g(\tau)h(t-\tau)d(\tau)$$

is called the convolution of g and h (or the resultant or the Faltung). Now suppose that we have

$$Ay' + By' + Cy = f(t), \quad y0 = y'0 = 0$$

take the Laplace transform of each term, substitute the initial conditions, and solve for Y

$$Y = \frac{F(p)}{A(p+a)(p+b)} = T(p)F(p)$$

Then y the inverse transform of Y in is the inverse transform of a product of two functions whose inverse transforms we know. Let $G(p)$ and $H(p)$ be the transforms of $g(t)$ and $h(t)$

$$G(p)H(p) = \mathcal{L}(g(t) \cdot h(p)) = \mathcal{L}(g * h)$$

Thus

$$y = \int_0^t g(t-\tau)h(\tau)d(\tau)$$

Observe from $\mathcal{L}34$ that we may use either $g(t-\tau)h(\tau)$ or $g(\tau)h(t-\tau)$ in the integral. It is well to choose whichever form is easier to integrate; it is best to put$(t-\tau)$ in the simpler function.

Fourier Transform of a Convolution. Let $g_1(\alpha)$ and $g_2(\alpha)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(v)e^{-i\alpha v}dv \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(u)e^{-i\alpha u}du \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(v)f_2(u)e^{-i\alpha(v+u)}dvdu \end{aligned}$$

Next we make the change of variables $x = v + u$, $dx = dv$, in the v integral

$$g_1(\alpha) \cdot g_2(\alpha) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x-u)f_2(u)e^{-i\alpha x}dvdu$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{-i\alpha x} \left[\int_{-\infty}^{\infty} f_1(x-u) f_2(u)\, du\right] dx$$

if we define the term in the square parenthesis as convolution, we get

$$\begin{aligned} g_1(\alpha) \cdot g_2(\alpha) &= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} f_1 * f_2 e^{-i\alpha x} dx \right) \\ &= \frac{1}{2\pi} \cdot \text{Fourier transform of } f_1 * f_2 \end{aligned}$$

In other words

$$g_1 \cdot g_2 \text{ and } f_1 * f_2 \text{ are a pair of Fourier transforms}$$

and by symmetry

$$g_1 * g_2 \text{ and } f_1 \cdot f_2 \text{ are a pair of Fourier transforms}$$

Frobenius Method

By using this method, we assume that the solution has the form of power series

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

We also assume that the first coefficient, that is a_0, is not zero. Computing the derivative of y, we obtain

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+s} \\ y' &= \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \\ y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} \end{aligned}$$

Frobenius 101. How 2 solve differential equation using generalized power series in 5 steps!

1. Tabulate!
2. Find the column in terms of x^{n+s} $x^s \rightarrow$!
3. Factor the coefficients that contain $a_0 \rightarrow$ and solve the indicial equation!
4. Solve it in terms of $a_n = -a_{n-2}$! (not factorial!)
5. As a check, put $n = 2$ at a_n not $n = 0$! (also not factorial!)

Bessel Function

The first kind of Bessel function is written as

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{\Gamma(n+1)\Gamma(n+p+1)}$$

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{\Gamma(n+1)\Gamma(n-p+1)}$$

While the second kind is

$$N_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}$$

The Bessel function is used to solve the Bessel's equation of order p

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

with the solution written as

$$y = AJ_p(x) + BN_p(x)$$

Another form of Bessel's equation is

$$x(xy')' + (K^2x^2 - p^2)y = 0$$

and the solution is

$$y = AJ_p(Kx) + BN_p(Kx)$$

Another equation that can be solved by Bessel function

$$y'' + \frac{1-2a}{x} y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2} \right]$$

The solution is

$$y = x^a Z_p(bx^c)$$

where a, b, c, p are constant and Z denote J or N or any linear combination of them.

Derivation. First we write the Bessel's equation as

$$x(xy')' + (x^2 - p^2)y = 0$$

By the Frobenius' method

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$$

$$(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

and

$$
\begin{aligned}
x(xy')' &= \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} \\
x^2 y &= \sum_{n=0}^{\infty} a_n x^{n+s+2} \\
-p^2 y &= -\sum_{n=0}^{\infty} a_n p^2 x^{n+s}
\end{aligned}
$$

Tabulate them

	x^{n+s}	x^s	x^{s+1}
$x(xy')'$	$a_n(n+s)^2$	$a_0 s^2$	$a_1(s+1)^2$
x^2y	a_{n-2}	$-$	$-$
$-p^2y$	$-a_n p^2$	$-a_0 p^2$	$-a_1 p^2$

From this we have the indicial equation

$$s^2 - p^2 = 0 \implies s = \pm p$$

And the general formula of the coefficient

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2}$$

For $s = \pm p$ and odd n, the coefficient is zero; proved by

$$a_1 \left[(s+1)^2 - p^2 \right] = a_1 \left[2p+1 \right] = 0 \implies a_1 = 0$$

We begin first for the case $s = p$. The coefficient is given by

$$a_n = -\frac{a_{n-2}}{(n+p)^2 - p^2} = -\frac{a_{n-2}}{n^2 - 2np} = -\frac{a_{n-2}}{n(n+2p)}$$

For even n, we write

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n+2p)} = -\frac{a_{2n-2}}{2^2 n(n+p)}$$

The coefficients for few odd n are as follows.

$$
\begin{aligned}
a_2 &= -\frac{a_0}{2^2(p+1)} = -\frac{a_0\Gamma(p+1)}{2^2\Gamma(p+2)} \\
a_4 &= -\frac{a_2}{2^2 2(p+2)} = -\frac{a_2\Gamma(p+2)}{2^2 2\Gamma(p+3)} = \frac{a_0\Gamma(p+1)}{2^4 2!\Gamma(p+3)} \\
a_6 &= -\frac{a_4}{2^2 3(p+3)} = -\frac{a_4\Gamma(p+3)}{2^2 3\Gamma(p+4)} = -\frac{a_0\Gamma(p+1)}{2^6 3!\Gamma(p+4)}
\end{aligned}
$$

The solution is written

$$
\begin{aligned}
y = &\sum_{n=0}^{\infty} a_n x^{n+p} = a_0 x^p + a_2 x^{p+2} + a_4 x^{p+4} + a_6 x^{p+6} \\
&= a_0 x^p \Gamma(p+1) \left[\frac{1}{\Gamma(p+1)} - \frac{(x/2)^2}{\Gamma(p+2)} + \frac{(x/2)^4}{2!\Gamma(p+3)} \right. \\
&\qquad \left. - \frac{(x/2)^6}{3!\Gamma(p+4)} + \dots \right]
\end{aligned}
$$

$$= a_0 2^p \Gamma(p+1)\left(\frac{x}{2}\right)^p \left[\frac{1}{\Gamma(1)\Gamma(p+1)} - \frac{(x/2)^2}{\Gamma(2)\Gamma(p+2)} + \frac{(x/2)^4}{\Gamma(3)\Gamma(p+3)} - \frac{(x/2)^6}{\Gamma(4)\Gamma(p+4)} + \dots\right]$$

If we define

$$a_0 = \frac{1}{2p\Gamma(p+1)}$$

then the solution, which is defined as $J_p(x)$, is written

$$J_p(x) = \frac{(x/2)}{\Gamma(1)\Gamma(p+2)} - \frac{(x/2)^{p+2}}{\Gamma(3)\Gamma(p+3)} + \frac{(x/2)^{p+4}}{\Gamma(3)\Gamma(p+3)} - \frac{(x/2)^{p+6}}{\Gamma(4)\Gamma(p+4)} + \dots$$

or

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{\Gamma(n+1)\Gamma(n+p+1)}$$

Next we consider the solution for $s = -p$. Since the steps are the same, we only need to change the sign of p. The solution is written

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{\Gamma(n+1)\Gamma(n-p+1)}$$

As an aside, for the Bessel equation written in the form

$$x^2 y'' + xy' + (K^2x^2 - p^2)y = 0$$

All the terms are unchanged except the term

$$K^2x^2y = \sum_{n=0}^{\infty} a_n K^2 x^{n+s+2}$$

This will result the change of argument in the Bessel equation from $Z(x)$ into $Z(Kx)$.

Recursion relation. Here are few relations of Bessel function with its derivative.

$$\begin{aligned}
&\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \\
&\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p}_{p+1}(x) \\
&J_{p-1}(x) + J(p+1)(x) = \frac{2p}{x} J_p(x) \\
&J_{p-1}(x) - J_{p+1}(x) = 2J'(x) \\
&J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)
\end{aligned}$$

And bonus relation that only apply for integral p

$$J_{-p}(x) = (-1)^p J_p x, \quad J_p(-x) = (-1)^n J_n(x)$$