Appendix: Differential Equation Study Guide

First Order Equations. General Form of ODE

$$\frac{dy}{dx} = f(x, y)$$

Initial Value Problem

$$y' = f(x, y), \; y(x_0) = y_0$$

Linear Equations. General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\mu(x) = e^{\int p(x)dx}$$
$$\implies \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)f(x)$$

General Solution

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

Homogeneous Equations. General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in z:

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations. General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in z:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations. General Form

$$M(x, y)dx + N(x, y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations.

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors. Case 1. If $P(x,y)$ depends only on x, where

$$P(x,y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact.

Case 2. If $Q(x,y)$ depends only on y, where

$$Q(x,y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact.

Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{1}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{2}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{3}$$

General Solution. The general solution of (1) or (3) is

$$y = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \tag{4}$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (2).

Linear Independence and The Wronskian. Two functions $f(x)$ and $g(x)$ are linearly dependent if there exist numbers a and b, not both zero, such that $af(x)+bg(x)=0$ for all x. If y_1 and y_2 are two solutions of (2), then Wronskian

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t) = Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.
2. $\{y_1, y_2\}$ are a fundamental set of solutions.
3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem. The initial value problem includes two initial conditions at the same point in time, one condition on $y(t)$ and one condition on $y'(t)$.

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

The initial conditions are applied to the entire solution $y = y_h + y_p$.

Linear Equation With Constant Coefficients. The general form of the homogeneous equation is

$$ay'' + by' + cy = 0 \tag{5}$$

Non-homogeneous

$$ay'' + by' + cy = g(t) \tag{6}$$

Characteristic Equation

$$ar^2 + br + c = 0$$

Quadratic Roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

The solution of (5) of Real Roots $(r_1 \neq r_2)$

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{8}$$

Repeated $(r_1 = r_2)$

$$y_h = (C_1 + C_2 t)e^{r_1 t} \tag{9}$$

Complex $(r = \alpha \pm i\beta)$

$$y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) \tag{10}$$

The solution of (6) is $y = y_p + y_h$ where y_h is given by (8) through (10) and y_p is found by undetermined coefficients or reduction of order.

Heuristics for Undetermined Coefficients. Also called Trial and Error

If $f(t) =$	then guess that a particular solution $y_p =$.
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at}\sin bt$ or $P_n(t)e^{at}\cos bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n)\cos bt$ $+(A_0 + A_1 t + \cdots + A_n t^n)\sin bt]$

Method of Reduction of Order. When solving (2), given y_1, then y_2 can be found by solving

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}dx}{y_1(x)^2} \tag{11}$$

Method of Variation of Parameters. If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (2) then a particular solution to (3) is

$$y_P(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}dt \tag{12}$$

Cauchy-Euler Equation. For ODE

$$ax^2y'' + bxy' + cy = 0 \tag{13}$$

with auxiliary Equation

$$ar(r-1) + br + c = 0 \tag{14}$$

The solutions of (13) depend on the roots $r_{1,2}$ of (14). For Real Roots

$$y = C_1x^{r_1} + C_2x^{r_2}$$

Repeated Root

$$y = C_1x^r + C_2x^r \ln x$$

Complex

$$y = x^\alpha[C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)] \tag{15}$$

In (15) $r_{1,2} = \alpha \pm i\beta$, where $\alpha,\beta \in \mathbb{R}$

Series Solutions.

$$(x-x_0)^2y'' + (x-x_0)p(x)y' + q(x)y = 0 \tag{16}$$

If x_0 is a regular point of (16) then

$$y_1(t) = (x-x_0)^n\sum_{k=0}^{\infty} a_k(x-x_k)^k$$

At a Regular Singular Point x_0, the indicial Equation

$$r^2 + (p(0)-1)r + q(0) = 0 \tag{17}$$

First Solution

$$y_1 = (x-x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x-x_k)^k$$

Where r_1 is the larger real root if both roots of (17) are real or either root if the solutions are complex.