Special Function

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1. Gamma Function

1.1. Factorial.

The factorial is defined by integral

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

Putting $\alpha = 1$ we get

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can also give a meaning to $0!$; by putting $n = 0$, we get $0! = 1$. By the way, the integral can be evaluated using differentiation under integral sign.

1.2. Gamma function definition.

Gamma function is used to define the factorial function for noninterger n. We define, for any $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$$

1.3. Recursion relation.

The recursion for gamma function is

$$\Gamma(p+1) = p\Gamma(p)$$

This can be used to define gamma function for $p \leq 0$

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

Proof. Let us integrate $\Gamma(p+1)$ by parts. Calling $u = x^p$, and $dv = e^{-x}\, dx$; then we get $du = px^{p-1}$, and $v = -e^{-x}$. Thus

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} p x^{p-1}\, dx \\ &= p\int_0^\infty x^{p-1} e^{-x}\, dx \\ \Gamma(p-1) &= p\Gamma(p) \quad \blacksquare\end{aligned}$$

1.4. Important formula.

We state here important formula

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Figure 1: Gaussian integral solved by polar method.

We can calculate the value of $\Gamma(1/2)$ using this equation, however we will instead try to derive it using another method. First we consider the definition

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt$$

then we substitute $t = x^2$ and $dt = 2x\, dx$

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\, dx = \int_{-\infty}^\infty e^{-x^2}\, dx$$

This is the famous Gaussian integral. Refer to figure 1 on how to solve it solve it by polar coordinate.

Since everybody and their grandma already know how to solve Gaussian integral by polar coordinate, I will instead try to solve it by Feynmann's trick. First consider the function

$$I(\alpha) = \left(\int_0^\alpha e^{-t^2}\, dt\right)^2$$

where I is a function of parameter fish α. Then, to evaluate the actual Gaussian integral

$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2\lim_{\alpha\to\infty} \sqrt{I(\alpha)}$$

Before that, I need to evaluate the function $I(\alpha)$ first. To do that, first I differentiate I with respect parameter fish α

$$\frac{dI}{d\alpha} = 2\int_0^\alpha e^{-t^2} dt \left(\int_0^\alpha \frac{\partial e^{-t^2}}{d\alpha} \, dt + e^{-\alpha^2} \frac{d\alpha}{d\alpha} - e^0 \frac{d(0)}{d\alpha} \right)$$
$$\frac{dI}{d\alpha} = \int_0^\alpha 2e^{-(t^2+\alpha^2)} \, dt$$

where I have used Leibniz' rule for differentiating under integral sign. Then, I introduce the variable $u = t/\alpha$ and $du = dt/\alpha$

$$\frac{dI}{d\alpha} = \int_0^1 2e^{-(u^2\alpha^2+\alpha^2)} \alpha \, du = \int_0^1 2\alpha e^{-\alpha^2(u^2+1)} \, du$$

Using the fact that

$$\frac{\partial}{\partial\alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} = -2\alpha e^{-\alpha^2(u^2+1)}$$

I can rewrite the integrand as

$$\frac{dI}{d\alpha} = -\int_0^1 \frac{\partial}{\partial\alpha} \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du$$

Since the integrand is continous, I can move the partial differentiation outside the integral and turnig it into total differentiation

$$\frac{dI}{d\alpha} = -\frac{d}{d\alpha} \int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du$$

Hence

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du + C$$

All that remains is to find the value of C. Considering the initial definition of $I(\alpha)$ and evaluating at $\alpha = 0$, I get

$$I(0) = \left(\int_0^0 e^{-t^2} \, dt \right)^2 = 0$$

Therefore

$$I(0) = -\int_0^1 \frac{1}{u^2+1} \, du + C = 0$$
$$C = \arctan u \Big|_0^1 = \frac{\pi}{4}$$

And I obtain the complete expression for the fish function

$$I(\alpha) = -\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du + \frac{\pi}{4}$$

Now I can evaluate the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2 \lim_{\alpha\to\infty} \left(-\int_0^1 \frac{e^{-\alpha^2(u^2+1)}}{u^2+1} \, du + \frac{\pi}{4} \right)^{1/2}$$
$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = 2\frac{\sqrt{\pi}}{2}$$

and I find

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

Much to my chagrin, it is actually more trouble some than the polar method. Let's us try it for comparison

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy \right)^{1/2}$$

Doing the change of coordinate thing

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \left(\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta \right)^{1/2}$$
$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \left(2\pi \int_0^{\infty} e^{-r^2} r \, dr \right)^{1/2}$$

That integral can by easily evaluated using u substitution; making the substitution $u = -r^2$

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \left(2\pi \int_{-\infty}^{0} \frac{e^u}{2} \, du \right)^{1/2}$$
$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \left(2\pi \frac{e^u}{2} \Big|_{-\infty}^{0} \right)^{1/2}$$

And I get the same result

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

Damn, it is really more shrimple.

2. Error Function

We define error function as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

There is also closely closely related integrals which are used and sometimes referred to as the error function called standard normal or Gaussian cumulative distribution function $\Phi(x)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt$$

Here are some of their relations.

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\big(x/\sqrt{2}\big)$$
$$\Phi(x) - \frac{1}{2} = \frac{1}{2} \operatorname{erf}\big(x/\sqrt{2}\big)$$
$$\operatorname{erf}(x) = 2\Phi\big(x\sqrt{2}\big) - 1$$

Proof. Consider the definition of $\Phi(x)$. Making the substitution of $u = t/\sqrt{2}$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{u=x/\sqrt{2}} e^{-u^2} \sqrt{2} \, du$$
$$= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{0} e^{-u^2} \, du + \int_0^{\infty} e^{-u^2} \, du \right)$$
$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\big(x/\sqrt{2}\big) \quad \blacksquare$$

To prove the third relation, we first rewrite the equation as

$$\operatorname{erf}\big(x/\sqrt{2}\big) = 2\Phi(x) - 1$$

then we make the substitution $u = x/\sqrt{2}$

$$\operatorname{erf}(u) = 2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u\sqrt{2}} e^{-t^2/2}\,dt - 1$$
$$\operatorname{erf}(u) = 2\Phi\big(x\sqrt{2}\big) - 1 \quad \blacksquare$$

2.1. Complementary error function.

Defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-t^2}\,dt$$

It relations with the actual error function are as follows.

$$\operatorname{erfc}(x) = 1 - \operatorname{erfc}(x)$$
$$\operatorname{erfc}\left(x/\sqrt{2}\right) = \sqrt{\frac{2}{\pi}}\int_{x}^{\infty} e^{-t^2/2}\,dt$$

Proof. The first relation is quite easy to prove. Consider

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt = 1$$

then

$$\frac{2}{\sqrt{\pi}}\left(\int_{-\infty}^{x} e^{-t^2} + \int_{x}^{\infty} e^{-t^2}\right) = 1$$
$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \quad \blacksquare$$

To proof the second relation, we subtitute the limit of integration from $t = x/\sqrt{2}$ into $x = t\sqrt{2}$

$$\operatorname{erfc}\big(x/\sqrt{2}\big) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} \frac{e^{t^2/2}}{\sqrt{2}}\,dt$$
$$\operatorname{erfc}\big(x/\sqrt{2}\big) = \sqrt{\frac{2}{\pi}}\int_{x}^{\infty} e^{-t^2/2}\,dt \quad \blacksquare$$

2.2. Imaginary error function.

We define

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{t^2}\,dt$$

Here are some relation to the actual error function.

$$\operatorname{erf}(ix) = i\operatorname{erfi}(x)$$
$$\operatorname{erf}\left(\frac{1-i}{\sqrt{2}}x\right) = (1-i)\sqrt{\frac{2}{\pi}}\int_{0}^{x} (\cos^2 u + i\sin^2 u)\,du$$