Appendix: Frobenius' Method

I will demonstrate this technique. Consider the following differential equation.

$$x^2y'' + 4xy' + (x^2 + 2)y = 0$$

The solution will take the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

Substituting this into each terms, we have

$$\begin{aligned}
x^2y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1)a_n x^{n+s} \\
4xy' &= \sum_{n=0}^{\infty} 4(n+s)a_n x^{n+s} \\
x^y &= \sum_{n=0}^{\infty} a_n x^{n+s+2} \\
2y &= \sum_{n=0}^{\infty} 2a_n x^{n+s}
\end{aligned}$$

Then we put them into table.

	x^{n+s}	x^s	x^{s+1}
x^2y''	$(n+s)(n+s-1)a_n$	$s(s-1)a_0$	$s(s+1)a_1$
$4xy'$	$4(n+s)a_n$	$4sa_0$	$4(s+1)a_1$
x^2y	a_{n-2}	$-$	$-$
$2y$	$2a_n$	$2a_0$	$2a_1$

Using the terms on x^s column, we have the following indicial equation.

$$\begin{aligned}
s(s-1)a_0 + 4sa_0 + 2a_0 &= 0 \\
a_0\left[s(s+3) + 2\right] &= 0
\end{aligned}$$

Since a_0 cannot be zero, we write

$$s^2 + 3s + 2 = 0$$

By solving the indicial equation we obtain $s = (-1, -2)$. From the x^{n+s}, we obtain the general formula for a_n in terms of a_{n-2}

$$a_n\left[(n+s)(n+s+3) + 2\right] = -a_{n-2}$$

We also obtain the fact the value of a_1 is zero, proved by the terms in x^{s+1} column

$$\left.\begin{aligned}
&a_1\left[(s+1)(s+4) + 2\right] = 0 \\
&s = (-1, -2)
\end{aligned}\right\} \implies a_0 = 0$$

Since we have two value of s, we first consider the case for $s = -1$. The general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-1)(n+2)+2} = -\frac{a_{n-2}}{n^2+n} = -\frac{a_{n-2}}{n(n+1)}$$

The values of a_n for few n are as follows

$$\begin{aligned}
a_2 &= -\frac{a_0}{3!} \\
a_4 &= -\frac{a_2}{4\cdot 5} = \frac{a_0}{5!} \\
a_6 &= -\frac{a_4}{6\cdot 7} = -\frac{a_0}{7!}
\end{aligned}$$

Thus the solution for this case is

$$\begin{aligned}
y_{-1} = \sum_{n=0}^{\infty} a_n x^{n-1} &= \frac{a_0}{x} - \frac{a_0}{3!}x + \frac{a_0}{5!}x^3 - \frac{a_0}{7!}x^5 + \dots \\
&= \frac{a_0}{x^2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = \frac{a_0}{x^2}\sin x
\end{aligned}$$

For the case of $s = -2$, the general a_n formula evaluated into

$$a_n = -\frac{a_{n-2}}{(n-2)(n+1)+2} = -\frac{a_{n-2}}{n^2 - n} = -\frac{a_{n-2}}{n(n-1)}$$

The values of a_n for few n are as follows

$$\begin{aligned}
a_2 &= -\frac{a_0}{2!} \\
a_4 &= -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!} \\
a_6 &= -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}
\end{aligned}$$

Thus the solution for this case is

$$\begin{aligned}
y_{-2} = \sum_{n=0}^{\infty} a_n x^{n-2} &= \frac{a_0}{x^2} - \frac{a_0}{2!} + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^4 + \dots \\
&= \frac{a_0}{x^2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) = \frac{a_0}{x^2}\cos x
\end{aligned}$$

Hence the complete form of the solution is

$$y = \frac{a_0}{x^2}\left(\cos x + \sin x\right)$$

Appendix: Differential Equation Study Guide

First Order Equations. General Form of ODE

$$\frac{dy}{dx} = f(x,y)$$

Initial Value Problem

$$y' = f(x,y),\ y(x_0) = y_0$$

Linear Equations. General Form:

$$y' + p(x)y = f(x)$$

Integrating Factor

$$\mu(x) = e^{\int p(x)dx}$$
$$\implies \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)f(x)$$

General Solution

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

Homogeneous Equations. General form

$$y' = f(y/x)$$

Substitution

$$y = zx \implies y' = z + xz'$$

The result is always separable in z:

$$\frac{dz}{f(z) - z} = \frac{dx}{x}$$

Bernoulli Equations. General Form

$$y' + p(x)y = q(x)y^n$$

Substitution

$$z = y^{1-n}$$

The result is always linear in z:

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Exact Equations. General Form

$$M(x,y)dx + N(x,y)dy = 0$$

Text for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution

$$\phi = C$$

where

$$M = \frac{\partial \phi}{\partial x} \quad \text{and} \quad N = \frac{\partial \phi}{\partial y}$$

Method for Solving Exact Equations.

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors. Case 1. If $P(x,y)$ depends only on x, where

$$P(x,y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$$

then

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact.

Case 2. If $Q(x,y)$ depends only on y, where

$$Q(x,y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$$

Then

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact.

Second Order Linear Equations General Form of the Equation

$$a(t)y'' + b(t)y' + c(t)y = g(t) \tag{1}$$

Homogeneous

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{2}$$

Standard Form

$$y'' + p(t)y' + q(t)y = f(t) \tag{3}$$

General Solution. The general solution of (1) or (3) is

$$y = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \tag{4}$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (2).

Linear Independence and The Wronskian. Two functions $f(x)$ and $g(x)$ are linearly dependent if there exist numbers a and b, not both zero, such that $af(x)+bg(x)=0$ for all x. If y_1 and y_2 are two solutions of (2), then Wronskian

$$W(t)=y_1(t)y_2'(t)-y_1'(t)y_2(t)$$

and Abel's Formula

$$W(t)=Ce^{-\int p(t)dt}$$

and the following are all equivalent:

1. $\{y_1,y_2\}$ are linearly independent.
2. $\{y_1,y_2\}$ are a fundamental set of solutions.
3. $W(y_1,y_2)(t_0)\neq 0$ at some point t_0.
4. $W(y_1,y_2)(t)\neq 0$ for all t.

Initial Value Problem. The initial value problem includes two initial conditions at the same point in time, one condition on $y(t)$ and one condition on $y'(t)$.

$$\begin{cases} y''+p(t)y'+q(t)y=0 \\ y(t_0)=y_0 \\ y'(t_0)=y_1 \end{cases}$$

The initial conditions are applied to the entire solution $y=y_h+y_p$.

Linear Equation With Constant Coefficients. The general form of the homogeneous equation is

$$ay''+by'+cy=0 \tag{5}$$

Non-homogeneous

$$ay''+by'+cy=g(t) \tag{6}$$

Characteristic Equation

$$ar^2+br+c=0$$

Quadratic Roots

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{7}$$

The solution of (5) of Real Roots ($r_1\neq r_2$)

$$y_h=C_1e^{r_1t}+C_2e^{r_2t} \tag{8}$$

Repeated ($r_1=r_2$)

$$y_h=(C_1+C_2t)e^{r_1t} \tag{9}$$

Complex ($r=\alpha\pm i\beta$)

$$y_H=e^{\alpha t}(C_1\cos\beta t+C_2\sin\beta t) \tag{10}$$

The solution of (6) is $y=y_p+y_h$ where y_h is given by (8) through (10) and y_p is found by undetermined coefficients or reduction of order.

Heuristics for Undetermined Coefficients. Also called Trial and Error

If $f(t) =$	then guess that a particular solution $y_p =$.
$P_n(t)$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{at}$
$P_n(t)e^{at}\sin bt$ or $P_n(t)e^{at}\cos bt$	$t^s e^{at}[(A_0 + A_1 t + \cdots + A_n t^n)\cos bt$ $+(A_0 + A_1 t + \cdots + A_n t^n)\sin bt]$

Method of Reduction of Order. When solving (2), given y_1, then y_2 can be found by solving

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}dx}{y_1(x)^2} \tag{11}$$

Method of Variation of Parameters. If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (2) then a particular solution to (3) is

$$y_P(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}dt \tag{12}$$

Cauchy-Euler Equation. For ODE

$$ax^2y'' + bxy' + cy = 0 \tag{13}$$

with auxiliary Equation

$$ar(r-1) + br + c = 0 \tag{14}$$

The solutions of (13) depend on the roots $r_{1,2}$ of (14). For Real Roots

$$y = C_1 x^{r_1} + C_2 x^{r_2}$$

Repeated Root

$$y = C_1 x^r + C_2 x^r \ln x$$

Complex

$$y = x^\alpha[C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)] \tag{15}$$

In (15) $r_{1,2} = \alpha \pm i\beta$, where $\alpha,\beta \in \mathbb{R}$

Series Solutions.

$$(x-x_0)^2y'' + (x-x_0)p(x)y' + q(x)y = 0 \tag{16}$$

If x_0 is a regular point of (16) then

$$y_1(t) = (x-x_0)^n \sum_{k=0}^{\infty} a_k(x-x_k)^k$$

At a Regular Singular Point x_0, the indicial Equation

$$r^2 + (p(0)-1)r + q(0) = 0 \tag{17}$$

First Solution

$$y_1 = (x-x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x-x_k)^k$$

Where r_1 is the larger real root if both roots of (17) are real or either root if the solutions are complex.