Identity Involving Partial Derivative

The Jacobian of $[u(x,y), v(x,y)]$ with respect to (x,y) is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Here are some identity relating the Jacobian with partial derivative.

Unity. Unity as in one

$$\frac{\partial(u,v)}{\partial(x,y)} = 1$$

Proof. Trivial

$$\frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1 \quad \blacksquare$$

Change of order. It can be proved that change of order cost the minus sign

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(v,u)}{\partial(x,y)} = -\frac{\partial(u,v)}{\partial(y,x)}$$

Proof. Those three terms literally have the same value when evaluated

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$-\frac{\partial(v,u)}{\partial(x,y)} = -\begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{vmatrix} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$\frac{\partial(u,v)}{\partial(y,x)} = -\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

See? $\blacksquare$

Jacobian. In terms of Jacobian, partial derivative of u with respect to x can be written as

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{\partial(u,y)}{\partial(x,y)}$$

Proof. Just evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} \qquad \blacksquare$$

Chain rule for partial derivative. The expression is

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)}$$

Proof. The total differential of u and v as function w and z read

$$du = \frac{\partial u}{\partial w}\,dw + \frac{\partial u}{\partial v}\,dz \quad \wedge \quad dv = \frac{\partial v}{\partial w}\,dw + \frac{\partial v}{\partial z}\,dz$$

We can therefore evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} \\ \frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix}$$

$$= \left| \begin{pmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \right| = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix} \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)} \qquad \blacksquare$$

The real chain rule. We have

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y = 1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(x,y)} = \left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y \qquad \blacksquare$$

Yet another chain rule... Even more chain rule...

$$\left.\frac{\partial x}{\partial y}\right|_w = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Proof. Trivial

$$\left.\frac{\partial x}{\partial y}\right|_w = \frac{\partial(x,w)}{\partial(y,w)} = \frac{\partial(x,w)}{\partial(z,w)}\frac{\partial(z,w)}{\partial(y,w)} = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Cyclic rule. This is chain rule all over again...

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_z = -1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(x,y)} = -\frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(y,z)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(y,x)}$$
$$= -\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial y}{\partial x}\right|_z \left.\frac{\partial z}{\partial y}\right|_x \quad \blacksquare$$

Application in Thermodynamics

Here we will derive some useful intensive parameter used in thermodynamics. We assumed entropy function S has the form of

$$S = S(U, V, N_{i|r})$$

where N is number of chemical potential and $N_{i|r} \equiv N_1, \ldots N_r$. Therefore, its total differential is

$$dS = \left.\frac{\partial S}{\partial U}\right|_{V,N_{i|r}} dU + \left.\frac{\partial S}{\partial V}\right|_{U,N_{i|r}} dV + \sum_{j=1}^{r} \left.\frac{\partial S}{\partial N_j}\right|_{U,V,N_{i\neq r}} dN_j$$

We also assume the following quantities

$$T = \left.\frac{\partial U}{\partial S}\right|_{V,N_i} \quad ; P = -\left.\frac{\partial U}{\partial V}\right|_{S,N_i} \quad ; \mu_j = \left.\frac{\partial U}{\partial N}\right|_{S,V,N_{i\neq j}}$$

First identity. As follows.

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T}$$

Proof. We use chain rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = 1 \implies \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \left(\left.\frac{\partial U}{\partial S}\right|_{V,N_i}\right)^{-1}$$

Then, from the definition of temperature

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T} \quad \blacksquare$$

Second identity. The identity written as

$$\left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T}$$

Proof. We invoke cyclic rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial V}\right|_{U,N_i} \left.\frac{\partial V}{\partial U}\right|_{U,N_i}$$

Then, from the first identity and the definition of pressure

$$1 = T\left.\frac{\partial S}{\partial V}\right|_{U,N_i} \frac{1}{P} \implies \left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T} \quad \blacksquare$$

Third Identity. Expressed as

$$\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{P}{T}$$

Proof. We again invoke cyclic with $x \to U, y \to Nj, z \to S$; while keeping V and all N except N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \left.\frac{\partial N_j}{\partial U}\right|_{U,N_{i\neq j}}$$

Then, from the definition of temperature and chemical potential

$$1 = -T\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \frac{1}{\mu_j} \implies \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{\mu_j}{T} \quad \blacksquare$$

Lagrange Multipliers

Let $f(x, y, z)$ be our function that we want to optimize and $\phi(x, y, z) =$ const be our constraint. We then set the total differential of $f(x, y, z)$ and $\phi(x, y, z)$ equal to zero

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial y}dz = 0$$
$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial y}dz = 0$$

Next, we construct the function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

and set its total derivative to zero

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

It follows that, for any value of dx, dy, dz, we choose λ such that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

Putting it all together, to optimize $f(x, y, z)$ with constraint $\phi(x, y, z)$, we need to optimize $F(x, y, z)$, which obtained by solving three partial derivative equations and constraint equation $\phi(x, y, z) =$ const. The equations in question are

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0, & \frac{\partial F}{\partial y} &= 0, \\ \frac{\partial F}{\partial z} &= 0, & \phi &= \text{const}. \end{aligned}$$

Multiple constraint. If there are multiple constraints, say ϕ_1 and ϕ_2, we function F we construct instead is

$$F(x, y, z) = f(x, y, z) + \lambda_1\phi_1(x, y, z) + \lambda_2\phi_2(x, y, z)$$

As aside, the function that we want to optimize need not to a function of three variable x, y, z. The previous derivation can be justified for any number of variable. Of course, with more variable there are more variables.