Newton's Second Law in Polar Coordinate

Acceleration in polar coordinate expressed as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}$$

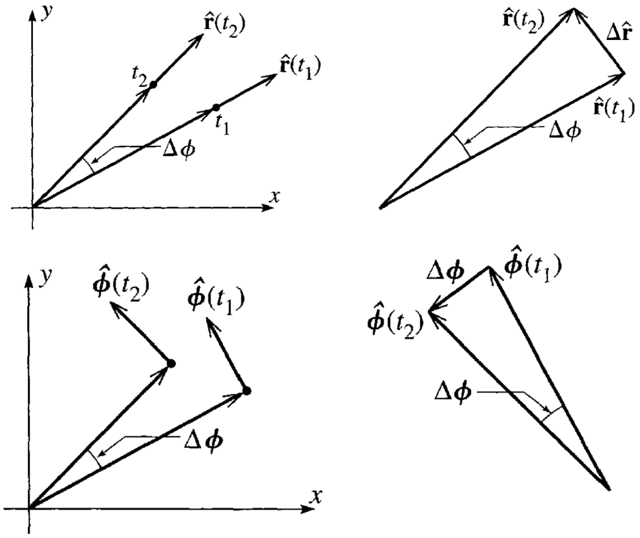
and velocity as

$$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

Hence Newton's law transform into

$$\mathbf{F} = m\mathbf{a} = \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Derivation



The value of $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\phi}}$.

From the figure, we have

$$d\hat{\mathbf{r}} = d\phi\ \hat{\boldsymbol{\phi}}, \quad d\hat{\boldsymbol{\phi}} = -d\phi\ \hat{\mathbf{r}}$$

or equivalently

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\ \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\ \hat{\mathbf{r}}$$

Using these we can now proceed to derive the Newton's law in polar coordinate. In cartesian coordinate, position vector can be writen as

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}}$$

converting it into polar

$$\mathbf{r} = r\ \hat{\mathbf{r}}$$

Next, we determine the velocity as

$$\dot{\mathbf{r}} = \frac{d}{dt} r\ \hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

and acceleration as

$$\begin{aligned}\ddot{r} = \frac{d}{dt}\left(\dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\frac{d}{dt}\left(\dot{\phi}\ \hat{\boldsymbol{\phi}}\right)& \\ = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\phi}\hat{\mathbf{r}}\right)& \\ = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}&\end{aligned}$$

Finally

$$F = F_r\ \hat{\mathbf{r}} + F_\phi\ \hat{\boldsymbol{\phi}} \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\end{cases}$$