Physical Mathematics

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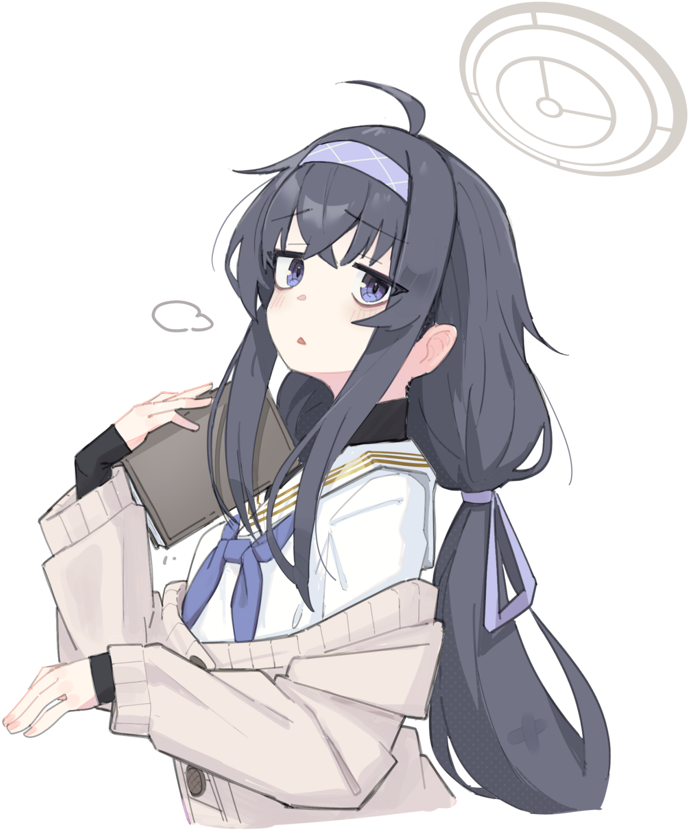

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Mathematics

Partial Derivative

Total Diferential

For a function $f = f(x, y, z, \dots)$, its total derivative is defined as

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

Identity Involving Partial Derivative

The Jacobian of $[u(x, y), v(x, y)]$ with respect to (x, y) is defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Here are some identity relating the Jacobian with partial derivative.

Unity. Unity as in one

$$\frac{\partial(u, v)}{\partial(x, y)} = 1$$

Proof. Trivial

$$\frac{\partial(x, y)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1 \quad \blacksquare$$

Change of order. It can be proved that change of order cost the minus sign

$$\frac{\partial(u, v)}{\partial(x, y)} = -\frac{\partial(v, u)}{\partial(x, y)} = -\frac{\partial(u, v)}{\partial(y, x)}$$

Proof. Those three terms literally have the same value when evaluated

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$-\frac{\partial(v, u)}{\partial(x, y)} = -\begin{vmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{vmatrix} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$\frac{\partial(u,v)}{\partial(y,x)} = -\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \\ \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

See? ■

Jacobian. In terms of Jacobian, partial derivative of u with respect to x can be written as

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{\partial(u,y)}{\partial(x,y)}$$

Proof. Just evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} \qquad \blacksquare$$

Chain rule for partial derivative. The expression is

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)}$$

Proof. The total differential of u and v as function w and z read

$$du = \frac{\partial u}{\partial w}\,dw + \frac{\partial u}{\partial v}\,dz \quad \wedge \quad dv = \frac{\partial v}{\partial w}\,dw + \frac{\partial v}{\partial z}\,dz$$

We can therefore evaluate the Jacobian

$$\frac{\partial(u,y)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial x} & \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial y} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial y} \\ \\ \dfrac{\partial v}{\partial w}\dfrac{\partial w}{\partial x} + \dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial x} & \dfrac{\partial v}{\partial w}\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial y} \end{vmatrix}$$

$$= \left|\begin{pmatrix} \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial z} \\ \\ \dfrac{\partial v}{\partial w} & \dfrac{\partial v}{\partial z} \end{pmatrix}\begin{pmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{pmatrix}\right| = \begin{vmatrix} \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial z} \\ \\ \dfrac{\partial v}{\partial w} & \dfrac{\partial v}{\partial z} \end{vmatrix}\begin{vmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u,y)}{\partial(x,y)} = \frac{\partial(u,y)}{\partial(w,z)}\frac{\partial(w,z)}{\partial(x,y)} \qquad \blacksquare$$

The real chain rule. We have

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y = 1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(x,y)} = \left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial x}\right|_y \qquad \blacksquare$$

Yet another chain rule... Even more chain rule...

$$\left.\frac{\partial x}{\partial y}\right|_w = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Proof. Trivial

$$\left.\frac{\partial x}{\partial y}\right|_w = \frac{\partial(x,w)}{\partial(y,w)} = \frac{\partial(x,w)}{\partial(z,w)}\frac{\partial(z,w)}{\partial(y,w)} = \left.\frac{\partial x}{\partial z}\right|_w \left.\frac{\partial z}{\partial y}\right|_w$$

Cyclic rule. This is chain rule all over again...

$$\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_z = -1$$

Proof. Trivial

$$1 = \frac{\partial(x,y)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(z,y)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(x,y)} = -\frac{\partial(x,y)}{\partial(z,y)}\frac{\partial(y,z)}{\partial(x,z)}\frac{\partial(z,x)}{\partial(y,x)}$$
$$= -\left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial y}{\partial x}\right|_z \left.\frac{\partial z}{\partial y}\right|_x \quad \blacksquare$$

Application in Thermodynamics

Here we will derive some useful intensive parameter used in thermodynamics. We assumed entropy function S has the form of

$$S = S(U, V, N_{i|r})$$

where N is number of chemical potential and $N_{i|r} \equiv N_1, \ldots N_r$. Therefore, its total differential is

$$dS = \left.\frac{\partial S}{\partial U}\right|_{V,N_{i|r}} dU + \left.\frac{\partial S}{\partial V}\right|_{U,N_{i|r}} dV + \sum_{j=1}^{r} \left.\frac{\partial S}{\partial N_j}\right|_{U,V,N_{i\neq r}} dN_j$$

We also assume the following quantities

$$T = \left.\frac{\partial U}{\partial S}\right|_{V,N_i} \quad ; P = -\left.\frac{\partial U}{\partial V}\right|_{S,N_i} \quad ; \mu_j = \left.\frac{\partial U}{\partial N}\right|_{S,V,N_{i\neq j}}$$

First identity. As follows.

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T}$$

Proof. We use chain rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = 1 \implies \left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \left(\left.\frac{\partial U}{\partial S}\right|_{V,N_i}\right)^{-1}$$

Then, from the definition of temperature

$$\left.\frac{\partial S}{\partial U}\right|_{V,N_i} = \frac{1}{T} \quad \blacksquare$$

Second identity. The identity written as

$$\left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T}$$

Proof. We invoke cyclic rule with $x \to U, y \to V, z \to S$; while keeping all the N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial V}\right|_{U,N_i} \left.\frac{\partial V}{\partial U}\right|_{U,N_i}$$

Then, from the first identity and the definition of pressure

$$1 = T \left.\frac{\partial S}{\partial V}\right|_{U,N_i} \frac{1}{P} \implies \left.\frac{\partial S}{\partial V}\right|_{U,N_i} = \frac{P}{T} \quad \blacksquare$$

Third Identity. Expressed as

$$\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{P}{T}$$

Proof. We again invoke cyclic with $x \to U, y \to Nj, z \to S$; while keeping V and all N except N_i constant

$$1 = -\left.\frac{\partial U}{\partial S}\right|_{V,N_i} \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \left.\frac{\partial N_j}{\partial U}\right|_{U,N_{i\neq j}}$$

Then, from the definition of temperature and chemical potential

$$1 = -T\left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} \frac{1}{\mu_j} \implies \left.\frac{\partial S}{\partial N_j}\right|_{U,N_{i\neq j}} = -\frac{\mu_j}{T} \quad \blacksquare$$

Lagrange Multipliers

Let $f(x,y,z)$ be our function that we want to optimize and $\phi(x,y,z) =$ const be our constraint. We then set the total differential of $f(x,y,z)$ and $\phi(x,y,z)$ equal to zero

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial y}dz = 0$$
$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial y}dz = 0$$

Next, we construct the function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

and set its total derivative to zero

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

It follows that, for any value of dx, dy, dz, we choose λ such that

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

Putting it all together, to optimize $f(x,y,z)$ with constraint $\phi(x,y,z)$, we need to optimize $F(x,y,z)$, which obtained by solving three partial derivative equations and constraint equation $\phi(x,y,z) = $ const. The equations in question are

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= 0, & \frac{\partial F}{\partial y} &= 0, \\
\frac{\partial F}{\partial z} &= 0, & \phi &= \text{const.}
\end{aligned}
$$

Multiple constraint. If there are multiple constraints, say ϕ_1 and ϕ_2, we function F we construct instead is

$$F(x, y, z) = f(x, y, z) + \lambda_1 \phi_1(x, y, z) + \lambda_2 \phi_2(x, y, z)$$

As aside, the function that we want to optimize need not to a function of three variable x, y, z. The previous derivation can be justified for any number of variable. Of course, with more variable there are more variables.

Leibniz' rule for Integral

Differentiation under integral sign stated by Leibniz' rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x, t)\, dt = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x, v) \frac{dv}{dx} - f(x, u) \frac{du}{dx}$$

Proof. Suppose we want dI/dx where

$$I = \int_u^v f(t)\, dt$$

By the fundamental theorem of calculus

$$I = F(v) - F(u) = \mathcal{F}(v, u)$$

or I is a function of v and u. Finding dI/dx is then a partial differentiation problem. We can write

$$\frac{dI}{dx} = \frac{\partial I}{\partial v} \frac{dv}{dx} + \frac{\partial I}{\partial u} \frac{du}{dx}$$

By the fundamental theorem of calculus, we have

$$
\begin{aligned}
\frac{d}{dv} \int_a^v f(x)\, dt &= \frac{d}{dv} \big[F(v) - F(a)\big] = f(v) \\
\frac{d}{dv} \int_u^b f(x)\, dt &= \frac{d}{dv} \big[F(b) - F(u)\big] = -f(u)
\end{aligned}
$$

where u and v are a function of x, while a and b are a constant. This is the case when we consider $\partial I/\partial v$ or $\partial I/\partial v$; the other variable is constant. Then

$$\frac{d}{dx} \int_u^v f(t)\, dt = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$$

Under not too restrictive conditions,

$$\frac{d}{dx} \int_a^b f(x, t)\, dt = \int_a^b \frac{\partial f(x, t)}{\partial x}\, dt$$

where, as before, a and b are constant. In other words, we can differentiate under the integral sign. It is convenient to collect these formulas into one formula known as Leibniz' rule:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = \int_u^v \frac{\partial f}{\partial x}\,dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} \qquad \blacksquare$$

Appendix: Lagrange Multipliers

Single constraint. Consider this example.

> Determine the largest volume of parallelepiped–that is, a three-dimensional figure formed by six parallelograms–whose edges parallel with the x, y, z axis inside ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The ellipsoid function above acts as constraints, that left is to determine the function that we want to optimize. This requires some clever thinking. We begin by defining point (x, y, z) be the corner of our parallelepiped. Now, this point is located in the first octant of our parallelepiped. The volume of this octant is

$$v = xyz$$

Since the parallelepiped's sides are parallel the axis, its total volume is

$$V = 8v$$

Hence, the volume of our parallelepiped is

$$V = 8xyz$$

This is the function that we want to maximize. We then construct the function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

The partial derivatives of F read as

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda}{a^2}x, \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda}{b^2}y, \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda}{c^2}z$$

To find the maximum of F, we then must solve the partial derivative equations and constraint equation

$$\begin{aligned} 8yz + \frac{2\lambda}{a^2}x &= 0 \\ 8xz + \frac{2\lambda}{b^2}y &= 0 \\ 8xy + \frac{2\lambda}{c^2}z &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \end{aligned}$$

Multiplying the first equation by x, the second by y, the third by z and adding them all together, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 24xyz + 2\lambda = 0$$

Hence

$$\lambda = -12xyz$$

Substituting this into the partial derivative equation to obtain

$$\begin{aligned}8yz - \frac{24yz}{a^2}x^2 &= 0 \implies x = \frac{\sqrt{3}}{3}a \\ 8xz - \frac{24xz}{b^2}y^2 &= 0 \implies y = \frac{\sqrt{3}}{3}b \\ 8xy - \frac{24xy}{c^2}z^2 &= 0 \implies z = \frac{\sqrt{3}}{3}c\end{aligned}$$

Therefore, the maximum volume of said parallelepiped is

$$V = \frac{24\sqrt{3}}{27}abc$$

Two constraints. Here's an example.

> Given two equation $z^2 = x^2 + y^2$ and $x + 2z + 3 = 0$, find the shortest and longest distance from the origin and the intersection of those two equations.

Here we want to minimize $f = x^2 + y^2 + z^2$ as usual. We construct auxiliary function

$$F = x^2 + y^2 + z^2 + \lambda_1(z^2 - x^2 - y^2) + \lambda_2(x + 2z)$$

The partial differentials of F read

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2x - 2\lambda_1 x + \lambda_2, \\ \frac{\partial F}{\partial y} &= 2y - 2\lambda_1 y, \\ \frac{\partial F}{\partial z} &= 2z + 2\lambda_1 z + 2\lambda_2\end{aligned}$$

Putting it all together, we have these equations

$$2x - 2\lambda_1 x + \lambda_2 = 0 \tag{1}$$
$$2y - 2\lambda_1 y = 0 \tag{2}$$
$$2z + 2\lambda_1 z + 2\lambda_2 = 0 \tag{3}$$
$$z^2 - x^2 - y^2 = 0 \tag{4}$$
$$x + 2z + 3 = 0 \tag{5}$$

By equation 2, we have two possible cases

$$2y - 2\lambda_1 y = y(1 - \lambda_1) = 0 \implies y = 0 \ \vee \ \lambda_1 = 1$$

First we consider $y = 0$. Equation 4 reads

$$z^2 = x^2 \implies z = \pm x$$

Then in the subcase $y = 0,\ z = x$; equation 5 evaluates into

$$3x + 3 = 0 \implies x = 3$$

In other hand, for subcase $y = 0,\ z = -x$; the same equation evaluates into

$$x = 3$$

Now we consider the case when $\lambda_1 = 1$. Equation 1 reduces into

$$\lambda_2 = 0$$

which means equation 5 turns into

$$4z = 0 \implies z = 0$$

and equation 5

$$x = -3$$

Using this result, equation 4 reads

$$y^2 = -9$$

which is impossible unless we are willing to take a complex value. Suppose we are willing, we have the $y = 3i$. Hence, we have three possibilities that the optimized points might take

$$\{\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}\} = \{(-1, 0, -1), (3, 0, -3), (-3, 3i, 0)\}$$

The distance from origin then evaluated by

$$\begin{aligned} d_1 &= \sqrt{\mathbf{P_1} \cdot \mathbf{P_2}} = \sqrt{2} \\ d_2 &= \sqrt{\mathbf{P_2} \cdot \mathbf{P_2}} = \sqrt{18} \\ d_3 &= \sqrt{\mathbf{P_3} \cdot \overline{\mathbf{P_3}}} = \sqrt{18} \end{aligned}$$

Hence the shortest distance is $d = \sqrt{2}$ and the longest is $d = \sqrt{18}$.

Variation Calculus

The Euler Equation

Any problem in the calculus of variations is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and solving the resulting differential equation. When the function $F = F(r, \theta, \theta')$, the Euler's equation read

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

If $F = F(t, x, \dot{x})$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

Notice that the first derivative in the Euler equation is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative.

Proof. We will try to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

where F is a given function. Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x)$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Differentiating with respect to x, we get

$$Y(x) = y(x)' + \epsilon\eta'(x)$$

Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\, dx$$

Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y} \eta(x) + \frac{\partial F}{\partial Y'} \eta'(x) \right] dx$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$$

Assuming that y'' is continuous, we can integrate the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = - \int_{x_1}^{x_2} \frac{d}{dx} \frac{\partial F}{\partial y'} \eta(x)\, dx + \left. \frac{\partial F}{\partial y'} \eta(x) \right|_{x_1}^{x_2}$$

The first term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \eta(x)\, dx$$

Since $\eta(x)$ is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \blacksquare$$

Notice carefully here that we are not saying that when an integral is zero, the integrand is also zero; this is not true. What we are saying is that the only way $\int f(x)\eta(x)\, dx$ can always be zero for every $\eta(x)$ is for $f(x)$ to be zero.

Several Variables

If there are n dependent variables in the original integral, there are n Euler-Lagrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du} \frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du} \frac{\partial f}{\partial y'}$$

Application: Shortest Between two points

Arbitrary path is given by

$$L = \int_1^2 \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

We factor dx from the integrand in order to make the function we are optimizing not dependent on the y variable and make the evaluation using Euler-Lagrange equation easier

$$f(y, y', x) = \sqrt{1 + y'^2}$$

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial y'}$$

$\partial f/\partial y = 0$ implies simply that $\partial f/\partial y'$ is a constant. Accordingly,

$$\begin{aligned} \frac{\partial f}{\partial y'} &= \frac{y'}{\sqrt{1 + y'^2}} = C \\ y'^2 &= C^2(1 + y^2) \\ y'^2(1 - C^2) &= C^2 \\ y &= \int \frac{C}{\sqrt{1 - C^2}}\, dx = \mathcal{C}x \end{aligned}$$

which is the equation for straight line.

Application: Brachistochrone Given two points 1 and 2, with 1 higher above the ground, in what shape should we build a frictionless roller coaster track so that a car released from point 1 will reach point 2 in the shortest possible time?

The speed at which the coaster descend can be determined by the conservation energy principle

$$mgy = \frac{1}{2}mv^2 v \qquad\qquad = \sqrt{2gy}$$

Thus the time to travel between points

$$t = \int_{t_1}^{t_2} \frac{ds}{v} = \int_{t_1}^{t_2} \sqrt{\frac{dx^2 + dy^2}{2gh}}$$

Since v gives a function of y, we take it as independent variable for the same reason as previously

$$t = \frac{1}{\sqrt{2g}} \int_{t_1}^{t_2} \sqrt{\frac{1 + x'^2}{y}}\, dy$$

Ignoring the constant, the function we want to optimize is

$$f(x, x', y) = \sqrt{\frac{1 + x'^2}{y}}$$

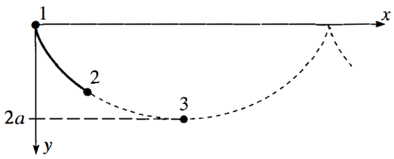


Figure: Brachistochrone problem

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$$

$\partial f/\partial x = 0$ implies simply that $\partial f/\partial x'$ is a constant. Accordingly,

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Here we take the constant as $\sqrt{1/2a}$

$$\begin{aligned}
x'^2 &= \frac{y(1+x'^2)}{2a}\\
x'^2\left(1-\frac{y}{2a}\right) &= \frac{y}{2a}\\
x' &= \sqrt{\frac{y}{2a}\frac{1}{2-y/2a}}\\
x' &= \sqrt{\frac{y}{2a-y}}
\end{aligned}$$

$$x = \int \sqrt{\frac{y}{2a-y}}\ dy$$

To solve this integral, we substitute $y = a(1-\cos\alpha)$ and $dy = a\sin\alpha\ d\theta$

$$\begin{aligned}
x &= \int \left[\frac{a(1-\cos\theta)}{a(1+\cos\theta)}\right]^{1/2} a\sin\theta\ s\theta\\
&= a\int \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{1/2} [(1+\cos\theta)(1-\cos\theta)]^{1/2}\ d\theta\\
&= a\int (1-\cos\theta)\ d\theta\\
x &= a(\theta-\sin\theta)+c
\end{aligned}$$

Therefore the path of the coaster is given by the following parametric equation

$$\begin{cases} x = a(\theta-\sin\theta)+c\\ y = a(1-\cos\theta)\end{cases}$$

Classical Mechanics

Unasorted Classical Mechanics Topics

Newton's Law

First law. In the absence of an external force, when viewed from an inertial frame, an object at rest remains at rest and an object in uniform motion in a straight line maintains that motion.

Second law. Simply put

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Third law. States that if two objects interact, the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1.

Particle Under Constant Acceleration

Here's some kinematics equation for position

$$x(t) = x_i + \frac{1}{2}(v_i + v_f)t$$
$$x(t) = x_i + v_i t + \frac{1}{2}at^2$$

and for velocity

$$v(t) = v_i + at$$
$$v(t)^2 = v_i^2 + 2a(x_f - x_i)$$

Particle in Uniform Circular Motion

If a particle moves in a circular path of radius r with a constant speed v, the magnitude of its centripetal acceleration is given by

$$a_r = \frac{v^2}{r}$$

while its period and angular velocity is

$$T = \frac{2\pi r}{v}, \quad \omega = \frac{2\pi}{T}$$

Applying Newton's second law

$$\sum F = ma_r = m\frac{v^2}{r}$$

Rigid Object Under Constant Angular Acceleration

Analogous to those for translational motion of a particle under constant acceleration

$$\begin{aligned}
\omega(t) &= \omega_i + \alpha t \\
\omega(t)^2 &= \omega_i^2 + 2\alpha(\theta_t - \theta_i) \\
\theta(t) &= \theta_i + \omega t + \frac{1}{2}\alpha t^2 \\
\theta(t) &= \theta_i + \frac{1}{2}(\omega_i + \omega_f)t
\end{aligned}$$

Relation of Linear and Rotational Motion

The following equations show the relation of linear and rotational motion

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha$$

Torque

The torque associated with a force **F** acting on an object

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{d\mathbf{L}}{dt}$$

Moment of Inertia

The moment of inertia of a rigid object is

$$I = \sum mr^2 = \int r^2 dm$$

Parallel Axis Theorem. To calculate the moment inertia from any axis, we use parallel axis theorem

$$I = I_{\mathrm{CM}} + Md^2$$

Terminal velocity

$\boldsymbol{r} \propto \boldsymbol{v}$**.** The velocity as a function of time is

$$v = \frac{mg}{b}\left[1 - \exp\left(-\frac{bt}{m}\right)\right] = v_T\left[1 - \exp\left(-\frac{bt}{m}\right)\right]$$

where b is a resistive constant whose value depends on the properties of the medium.

$\boldsymbol{r \propto v^2}$**.** Given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}$$

where D is a dimensionless empirical quantity called the drag coefficient, ρ is the density of air, and A is the cross-sectional area of the moving object.

Escape velocity. The speed required by an object to escape from any planet orbit is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Work Energy Theorem

It states that if work is done on a system by external forces and the only change in the system is in its speed,

$$W = \Delta T$$

Kinetic Energy

For an object in linear motion, the kinetic energy of said object is

$$T = \frac{1}{2}mv^2$$

whereas for rotational motion

$$T = \frac{1}{2}I\omega^2$$

Hence the total kinetic energy of a rigid object rolling on a rough surface without slipping

$$T = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega_{\text{CM}}^2$$

Potential Energy Function

For conservative energy $\mathbf{F}$, applies

$$V_f - V_i = -\int_{\mathbf{r_i}}^{\mathbf{r_f}} \mathbf{F} \cdot d\mathbf{r}$$

For particle-Earth system, the gravitational potential energy is

$$V = mgy$$

and elastic potential stored in spring

$$V = \frac{1}{2}kx^2$$

Effective potential

Effective potential energy $U_{\text{eff}}(r)$ is the sum of the actual potential energy $U(r)$ and the centrifugal $U_{\text{cf}}(r)$:

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

where l is the angular momentum and μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Momentum Impulse

The linear momentum and impulse are defined as

$$\mathbf{p} = m\mathbf{v}, \quad \mathbf{I} = \int_{t_i}^{t_f} \sum \mathbf{F}\, dt$$

Angular Momentum The angular momentum about an axis through the origin of a particle having linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

The z component of angular momentum of a rigid object rotating about a fixed z axis is

$$L_z = I\omega$$

Center of Mass and Velocity

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{r} = \frac{1}{M} \int \mathbf{r}\, dm$$

where M is the total mass. The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{v} =$$

Collision

Inelastic collision. One for which the total kinetic energy of the system of colliding particles is not conserved.

Elastic collision. One in which the kinetic energy of the system is conserved.

Perfectly inelastic. A collision which the colliding particles stick together after the collision.

Rocket Propulsion The expression for rocket propulsion is

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$

Power

The rate at which work is done by an external force, called power, is

$$P = \frac{dE}{dt} = Fv = \tau\omega$$

Newton's Law on Gravity

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

For an object at a distance h above the Earth's, the gravitational acceleration is

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$

In general, the gravitational field experienced by mass m is

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

Kepler's Law

First Law. All planets move in elliptical orbits with the Sun at one focus.

Second Law The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

Third Law Simply put

$$T^2 = \frac{4\pi^2 a^3}{GM_S}$$

where a is semimajor axis and M_S is the mass of the sun.

Energy of Gravitational system

Potential energy. The gravitational potential energy associated with a system of two particles is

$$V = -\frac{Gm_1 m_2}{r}$$

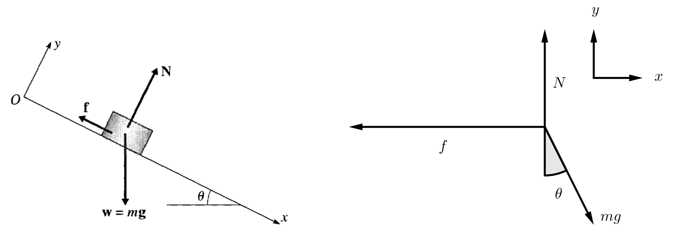
Total energy. The total energy of the system is the sum of the kinetic and potential energies

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} = -\frac{GMm}{2r}$$

Incline Problem

Consider:

> A block of mass m is observed accelerating from rest down an incline that has coefficient of friction μ and is at angle θ from the horizontal. How far will it travel in time t?



Force approach. First we define the direction of displacement as positive x axis and the normal force as positive y axis. The resultant force in y axis written as

$$\sum F_y = N - mg\cos\theta = 0 \implies N = mg\cos\theta$$

and x axis

$$\sum F_x = mg\sin\theta - f = mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$

we then solve for x by

$$\begin{aligned}
\ddot{x} &= g\left(\sin\theta - \mu\cos\theta\right) \\
\dot{x} &= g\left(\sin\theta - \mu\cos\theta\right) t \\
x(t) &= \frac{1}{2} g\left(\sin\theta - \mu\cos\theta\right) t^2
\end{aligned}$$

Since the block stared from rest, its constant of integration is zero.

Energy approach. Here define the zero potential energy at the bottom of the incline. Using the work energy theorem for non-conservative force

$$\begin{aligned}
\Delta T + \Delta U &= W_{\text{fric}} \\
\frac{1}{2}mv^2 - mgh &= -fd \\
\frac{1}{2}mv^2 - mgd\sin\theta &= -\mu mg\cos\theta d \\
v &= \sqrt{2gd(\sin\theta - \mu\cos\theta)}
\end{aligned}$$

Using the kinematics relation $v(t)^2 = v_i^2 + 2a\Delta x$

$$v^2 = 2gd(\sin\theta - \mu\cos\theta) = 2ad \implies a = g(\sin\theta - \mu\cos\theta)$$

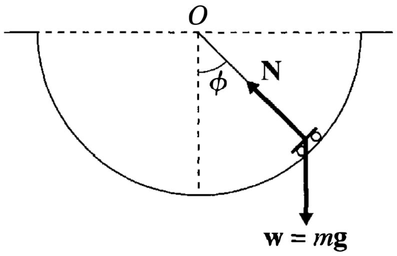
and the relation $v(t) = v_i + at$, we are able to rewrite it as the previous form

$$v = g(\sin\theta - \mu\cos\theta)t$$

Central Force Problem

Consider also:

> A "half-pipe" at a skateboard park consists of a concrete trough with a semicircular cross section of radius $R = 5$m. I hold a frictionless skateboard on the side of the trough pointing down toward the bottom and release it. Find the equation of motion for this system.



In this case r is held constant, thus the expression for resultant force in polar coordinate reads

$$\mathbf{F} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\,\hat{\boldsymbol{\phi}}$$

We also know that the acting force in this system are the normal and the skateboard weigh. Applying this force into equation above

$$(mg\cos\phi - N)\;\hat{\mathbf{r}} - mg\sin\phi\;\hat{\boldsymbol{\phi}} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\,\hat{\boldsymbol{\phi}}$$

We can't do anything with the radial component, we only use the angular component

$$\begin{aligned} mR\ddot{\phi} &= mg\sin\phi \\ \ddot{\phi} &= \frac{g}{R}\sin\phi \end{aligned}$$

This differential equation is solved by

$$\phi(t) = A\sin\sqrt{\frac{g}{R}}t + B\cos\sqrt{\frac{g}{R}}t$$

Since this is released from rest, we have the initial condition of $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$. Apllying the first condition

$$\phi_0 = B$$

and the second

$$\begin{aligned} \dot{\phi}(t) &= A\sqrt{\frac{g}{R}}\cos\sqrt{\frac{g}{R}}t - \phi_0\sqrt{\frac{g}{R}}\sin\sqrt{\frac{g}{R}}t \\ \dot{\phi}(0) &= 0 = A\sqrt{\frac{g}{R}} \end{aligned}$$

Hence the equation of motion reads

$$\phi(t) = \phi_0\cos\sqrt{\frac{g}{R}}t$$

Central Force

Newton's Second Law in Polar Coordinate

Acceleration in polar coordinate expressed as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}$$

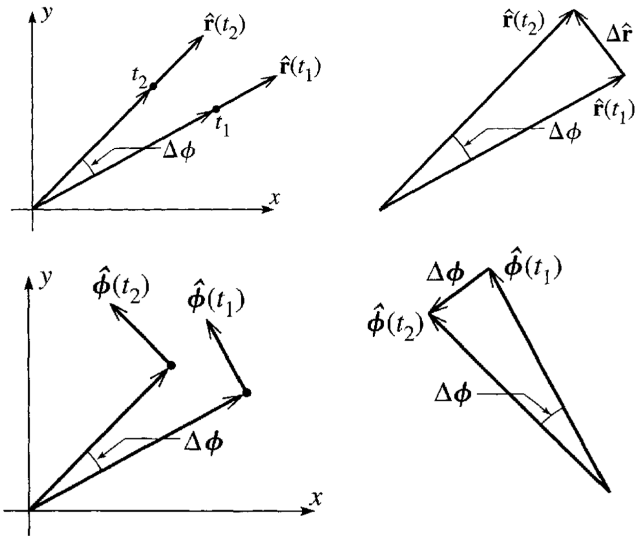
and velocity as

$$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

Hence Newton's law transform into

$$\mathbf{F} = m\mathbf{a} = \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Derivation



The value of $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\phi}}$.

From the figure, we have

$$d\hat{\mathbf{r}} = d\phi\ \hat{\boldsymbol{\phi}}, \quad d\hat{\boldsymbol{\phi}} = -d\phi\ \hat{\mathbf{r}}$$

or equivalently

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\ \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\ \hat{\mathbf{r}}$$

Using these we can now proceed to derive the Newton's law in polar coordinate. In cartesian coordinate, position vector can be writen as

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}}$$

converting it into polar

$$\mathbf{r} = r\ \hat{\mathbf{r}}$$

Next, we determine the velocity as

$$\dot{\mathbf{r}} = \frac{d}{dt} r\ \hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

and acceleration as

$$\begin{aligned}
\ddot{r} = \frac{d}{dt}\left(\dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\frac{d}{dt}\left(\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) \\
= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\phi}\hat{\mathbf{r}}\right) \\
= \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}
\end{aligned}$$

Finally

$$F = F_r\ \hat{\mathbf{r}} + F_\phi\ \hat{\boldsymbol{\phi}} \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Energy

Potential Energy

Gravitational. The potential energy is defined

$$U = mgh$$

where the reference point is chosen to be the ground. This makes it so that

$$F = -\nabla U = -\frac{d}{dr} mgr\,\hat{\mathbf{r}} = -mg\,\hat{\mathbf{r}}$$

the gravitational force is negative, or point downward. Now if we define the downward as positive displacement, the potential energy reads

$$U = -mgh$$

and the gravitational force

$$F = -\nabla U = -\frac{d}{dr}\left(-mgr\,\hat{\mathbf{r}}\right) = mg\,\hat{\mathbf{r}}$$

is positive, or point downward all the same.

Lagrangian Mechanics

The Lagrangian is defined as

$$\mathcal{L} = T - U$$

which mean that the Lagrangian is a function of position and velocity. The path of particle is determined by Hamilton's principle

$$S = \int_{t_1}^{t_2} \mathcal{L} \, dt$$

that is, the particle's path is such that the action integral S is stationary.

We can express the coordinate in other generalized coordinate

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_2)$$

And the same for velocity

$$\mathbf{v} = \mathbf{v}(\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

Now the action integral reads

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) \, dt$$

Therefore we have three Euler-Lagrange equation that must be satisfied by the particle

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

This is the case of one unconstrained particle. For N unconstrained particle, then, we shall have $3N$ Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1, \ldots, 3N$$

The Lagrangian equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$\text{Generalized force} = \text{Rate of change of Generalized momentum}$$

where

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} = i\text{-th component of the Generalized force}$$

and

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = i\text{-th component of the Generalized momentum}$$

Set of generalized coordinates that minimize the number of Euler-Lagrangian equation and be able to uniquely describe said system is said to be natural. This implies that natural coordinate also have minimum degree of freedom, that is the number of coordinate that can be varied independently. In natural coordinate, the generalized coordinate has no time dependence with its Cartesian relative.

System with n degree of freedom that can be described by n generalized coordinate is called holonomic. In general

$$\text{DoF} = \text{No. of Coordinate} - \text{No. Constraint}$$

For example, double pendulum have 4 coordinates and two constraint, thus having two degree of freedom.

The steps to solve problem using Lagrangian formalism are as follows.

1. Write down the Lagrangian $\mathcal{L} = T - U$.
2. Choose generalized n coordinate q_n and $\dot{q}_n$.
3. Rewrite $\mathcal{L}$ in terms of q_n and $\dot{q}_n$.
4. Write n Lagrange equation.

Proof of Lagrange Equation with Constraint

Suppose a particle has two degree of freedom with two kinds of forces act on it: constraint force $\mathbf{F}_{\text{cstr}}$, say interatomic forces that bind rigid body atom together; and the is non constraint conservative forces $\mathbf{F}$, which at minimum must be able to be derived from potential energy $F = -\nabla U(\mathbf{r}, t)$, say gravitational force. The total energy is then

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F}$$

and the Lagrangian

$$\mathcal{L} = T - U$$

where U is the potential energy which can be derived into non constraint conservative force.

The path of the particle can be denoted as

$$\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\epsilon}(t)$$

with $\mathbf{r}(t)$ as the correct path and $\boldsymbol{\epsilon}(t)$ as infinitesimal vector pointing away from the correct path. Then we have two Lagrangians

$$\mathcal{L} = \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t), \qquad \mathcal{L}_0 = \mathcal{L}_0(\mathbf{r}, \dot{\mathbf{r}}, t)$$

and two action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt, \qquad S_0 = \int_{t_1}^{t_2} \mathcal{L}_0\, dt$$

It can be proven that the difference in action integral $\delta S = S - S_0$ is zero, to the first order.

We write the difference in Lagrangian as

$$\begin{aligned}
\delta\mathcal{L} &= \frac{1}{2}m\dot{\mathbf{R}}^2 - U(\mathbf{R},t) - \frac{1}{2}m\dot{\mathbf{r}}^2 + U(\mathbf{r},t) \\
&= \frac{1}{2}m\left[(\dot{\mathbf{r}}^2 + \dot{\boldsymbol{\epsilon}}^2) - \dot{\mathbf{r}}^2\right] - \left[U(\mathbf{r}+\boldsymbol{\epsilon},t) - U(\mathbf{r},t)\right] \\
&= \frac{1}{2}m\left[\dot{\mathbf{r}}^2 + \dot{\boldsymbol{\epsilon}}^2 + 2\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}}^2 - \dot{\mathbf{r}}^2\right] - dU \\
\delta\mathcal{L} &= \frac{1}{2}m\dot{\boldsymbol{\epsilon}}^2 + m\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U
\end{aligned}$$

The difference in action integral, in the first order, is then

$$S = \int_{t_1}^{t_2} \mathcal{L}_0 \; dt = \int_{t_1}^{t_2} \left(m\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U\right) \; dt$$

Using integration by parts

$$\int_a^b f\left(\frac{dg}{dx}\right) \; dx = -\int_a^b g\left(\frac{df}{dx}\right) \; dx + fg\Big|_a^b$$

on the first term

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[m\ddot{\mathbf{r}} + \nabla U\right] \; dt + m\dot{\mathbf{r}}\boldsymbol{\epsilon}\Big|_{t_1}^{t^2}$$

The difference of $\boldsymbol{\epsilon}$ is zero between two end point, so

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[m\ddot{\mathbf{r}} + \nabla U\right] \; dt$$

The path $\mathbf{r}(t)$ satisfies Newton second law, thus $m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}}$. Meanwhile, the gradient of potential energy is the negative of non constraint force

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[\mathbf{F}_{\text{tot}} - \mathbf{F}\right] \; dt = i\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\mathbf{F}_{\text{cstr}} \; dt$$

Note that the constraint force is normal to the particle path and the $\boldsymbol{\epsilon}$, which lies to the same surface of particle path. Therefore, their dot product is zero

$$\delta S = 0$$

and the action integral is stationary.

This justifies the Lagrange equation for system with two degree of freedom where its constraint lie in the same surface as the particle path. In other words, it only applies to particle, or particles in that case, constrained to move in two dimension. Accordingly, the action integral in this case is written as

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t) \; dt$$

which will result in two Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_1}, \qquad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2}$$

Newton Law in 2D Cartesian

The Lagrangian in this case is

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)$$

Here we have two Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

The derivative with respect to position is force

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x, \qquad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial U}{\partial y} = F_y$$

while the derivative with respect to velocity is momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial T}{\partial \dot{y}} = m\dot{y}$$

Substituting this result in the two Euler-Lagrange equation, we have

$$F_x = m\ddot{x}, \qquad F_y = m\ddot{y}$$

which are the two component of Newton second law $\mathbf{F} = m\ddot{\mathbf{r}}$.

Newton Law in Polar Coordinate

The Lagrangian is

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$$

which result into two Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Before moving into evaluating the derivative with respect to r and ϕ, recall the gradient of potential energy in polar coordinate

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} - F_r\hat{\mathbf{r}} - F_\phi\hat{\boldsymbol{\phi}}$$

Evaluating the radial derivative

$$\begin{aligned} mr\dot{\phi} - \frac{\partial U}{\partial r} &= m\ddot{r} \\ F_r &= m(\ddot{r} + r\dot{\phi}) \end{aligned}$$

which is simply the radial component of the Newton's second law in polar coordinate. Evaluating the angular derivative

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}mr^2\dot{\phi} = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

$$-\frac{1}{r}\frac{\partial U}{\partial \phi} = m(2\dot{r}\dot{\phi} + r^2\ddot{\phi})$$
$$F_\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$$

which is, as previously mentioned, the angular component of Newton's second law. It should be noted that the quantity $mr^2\dot{\phi}$ can be recognized as angular momentum L, and the rate of change of it is torque Γ

$$\Gamma = F_\phi r = \frac{dL}{dt} = \frac{d}{dt}mr^2\dot{\phi}$$

Lagrangian with Explicit Constraint Forces Using Lagrange Multipliers

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The modified Euler-Lagrangian that include constraining force can be obtained by Lagrange Multipliers. For two dimension Lagrangian $\mathcal{L}(x, \dot{x}, y, \dot{y}, t)$ with one constraint $f(x, y) = C$, we must solve two modified Lagrange equation plus one for the constraint

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ \frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \\ f(x, y) = \text{Constant}\end{aligned}$$

Lagrange multiplier is not simply mathematical technique, in fact given partial derivatives of the constraint function $f(x, y)$, the Lagrange multiplier $f(x, y)$ gives the corresponding components of the constraint force

$$\lambda\frac{\partial f}{\partial q_i} = F_i^{\text{cstr}}$$

Derivation. We consider the case of two dimension Lagrange. Then deviate the correct path $x(t)$ and $y(t)$ into

$$x(t) \to x(t) + \delta x$$
$$y(t) \to y(t) + \delta y$$

With the Lagrangian of $\mathcal{L}(x, \dot{x}, y, \dot{y})$, the action integral takes the form

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

Assuming the constraint is consisted with the displacement, then, as proved before, the integral is unchanged and $\delta S = 0$. In other words

$$\int \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x} + \frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta \dot{y}\right)\, dt = 0$$

Using integration by part on second and fourth term to move the derivative sign

$$\int \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta x\, dt + \int \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\delta y\, dt = 0$$

For displacement δx and δy, the terms inside parenthesis must be zero for the integral to be zero. Now, this is just the proof of Euler-Lagrange equation. However, we want to explicitly including the constraint, which can be achieved by multiplying the deviation of the constraint equation δf with Lagrange multipliers

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$
$$0 = \lambda\frac{\partial f}{\partial x}\delta x + \lambda\frac{\partial f}{\partial y}\delta y$$

This step is justified because the value is zero anyway. Then we can add it into the integral of δS without changing the integral itself

$$\int \left(\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta x\ dt + \int \left(\frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial \hat{\mathbf{y}}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\delta y\ dt = 0$$

Since the multipliers is arbitrary, we define Lagrange multipliers such that the terms inside parenthesis is zero, thus resulting in two modified Euler-Lagrange equation written previously

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$\frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

Now we add the constraint equation since we need three equation to find three equation in three unknown.

Lagrange multipliers physical meaning. Given that the kinetic energy of the system does not depend on the position and the potential energy does not depend on velocity, the modified Euler-Lagrange equation reads

$$-\frac{\partial U}{\partial q_i} + \lambda\frac{\partial d}{\partial q_i} = m\ddot{q}_i$$
$$\lambda\frac{\partial f}{\partial q_i} = m\ddot{q}_i + \frac{\partial U}{\partial q_i}$$

Recall that the negative gradient of potential energy is the non constraint force, while the product of mass and generalized acceleration is total force. The total force is the sum of non constraint force and constraint force, so

$$\lambda\frac{\partial f}{\partial q_i} = F_i^{\text{cstr}}$$

Comparison with the usual function optimization using Lagrange multipliers. Let us compare the optimization method of Lagrange multipliers when the function we are concerned is $f(x, y)$ instead of the previous case of action integral S. To optimize $f(x, y)$, set its derivative to zero

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0$$

while to optimize S, we use the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

If we include constraint $\phi(x, y)$, we construct function $F = f + \lambda\phi$ such that the multipliers is defined

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

while in the action integral case

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial \phi}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial \phi}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

In any case, we need to solve those two resulting function alongside the constraint equation.

Lagrangian for a Charge in an Electromagnetic Field The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - q(V - \dot{\mathbf{r}} \cdot \mathbf{A})$$

Application: Atwood's Machine

Atwood's machine consist of masses m_1 and m_2 are suspended by an in extensible string (length l) which passes over a massless pulley with frictionless bearings and radius R. The length of the string acts as constraint

$$x + y + 2\pi R = l$$

This implies $y = -x + C$ and $\dot{y} = -\dot{x}$. Thus, the kinetic energy of both mass

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

We define the downward displacement of m_1 and upward displacement of m_2 as positive displacement in our generalized coordinate x. This is the same case of upward acceleration of m_2 being the same as downward acceleration of m_1. In any case, the potential energy is

$$U = -m_1gx - m_2gy = -(m_1 - m_2)gx + C_2$$

We can now write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$$

With only one generalized coordinate, we only have one Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

which on substituting the Lagrangian yield

$$\begin{aligned}(m_1 - m_2)g &= (m_1 + m_2)\ddot{x} \\ \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2}g\end{aligned}$$

Now let's compare it with Newtonian approach, we should obtain the same result. Considering the acceleration direction for both masses, the net forces on both m_1 and m_2 respectively are

$$\begin{aligned}m_1g - F_t &= m_1\ddot{x} \\ F_t - m_2g &= m_2\ddot{x}\end{aligned}$$

Adding both equation

$$\begin{aligned}(m_1 - m_2)g &= (m_1 + m_2)\ddot{x} \\ \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2}g\end{aligned}$$

Now suppose we explicitly include the constraint using Lagrange multipliers. Here, the constraint is the wire and the constraining force is the tension, which takes the form

$$f(x, y) = x + y = \text{Constant}$$

Since we do not need to reduce the number of coordinate, we express the Lagrangian as

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1gx + m_2gy$$

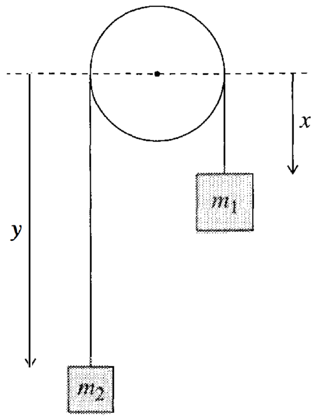

Figure: Atwood's machine configuration

Then the equations that wee need to solve are as follows

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$
$$x + y = \text{Constant}$$

The first two equation yield

$$m_1 g + \lambda = m_1 \ddot{x}, \qquad m_2 g + \lambda = m_2 \ddot{y}$$

Performing second derivative to the third equation with respect to time results

$$\ddot{x} = -\ddot{y}$$

Using this and subtracting the first equation by the second

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$
$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

which is the same result. It can be seen that, by comparing to the Newtonian result, that the Lagrange multipliers gives

$$\lambda = -F_t$$

Application: Particle Constrained on a Cylinder

Consider a particle of mass m constrained to move on a frictionless cylinder of radius R. Besides the force of constraint, the only force on the mass is a force $\mathbf{F} = -k\mathbf{r}$ directed toward the origin. With $\mathbf{r}$ as the position vector of the particle, this force is the three dimension version of Hooke's law.

We shall use cylindrical coordinate to solve this problem. It is known that the radius component is fixed $\rho = R$, so we use (ϕ, z) as our generalized coordinate. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{x}^2)$$

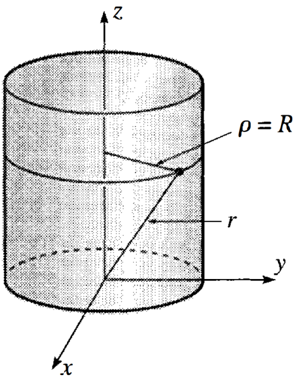

Figure: Particle constrained to move on a cylinder

Recall $\mathbf{F} = -\nabla U$, hence the potential energy

$$-\frac{dU}{dr}\,\hat{\mathbf{r}} = -kr\,\hat{\mathbf{r}}$$
$$U = \frac{1}{2}kr^2$$

The distance of particle from origin is given by $r^2 = R^2 + z^2$, so

$$U = \frac{1}{2}k(z^2 + R^2)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2)$$

We use two generalized coordinates, thus we have two Euler-Lagrangian equation

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \sqcup] \S \sqcup}{\partial \dot{z}}, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

The z equation is

$$-kz = m\ddot{z} \implies z = A\cos(\omega t - \delta)$$

This mean that the mass perform simple harmonic motion in the z direction. Now, the ϕ equation

$$0 = \frac{d}{dt}mR^2\dot{\phi}$$

This mean that the angular momentum $L = mR^2\dot{\phi}$ is conserved and the particle rotate in constant velocity $\dot{\phi}$.

Application: Sliding Block on a Frictionless Wedge

The block (mass m) is free to slide on the wedge, and the wedge (mass M) can slide on the horizontal table, both with negligible friction. The block is released from the top of the wedge, with both initially at rest.

The system has two coordinates and no constraint whatsoever, so it has two degree of freedom. We choose q_1 and q_2 as our generalized coordinate which denote the distance form the block from the top of the wedge and the distance of the wedge from convenient fixed point on the table. We also define positive x displacement to the right and downward as positive y displacement.

The kinetic energy is

$$T = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2$$

The wedge velocity is simply

$$v_M = \dot{q}_2$$

Meanwhile, the block velocity have two component, which are

$$\mathbf{v}_m = (\dot{q}_2 + \dot{q}_1 \cos\alpha)\ \hat{\mathbf{x}} + \dot{q}_2 \sin\alpha\ \hat{\mathbf{y}}$$

In terms of generalized coordinate, the kinetic energy reads

$$\begin{aligned} T &= \frac{1}{2}m\left[(\dot{q}_2 + \dot{q}_1\cos\alpha)^2 + \dot{q}_2 \sin\alpha^2\right] + \frac{1}{2}m\dot{q}_2^2 \\ &= \frac{1}{2}m\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_1\dot{q}_2\cos\alpha +\right] + \frac{1}{2}M\dot{q}_2^2 \\ T &= \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) \end{aligned}$$

In other hand, we defined downward as positive displacement, so the potential energy reads

$$U = -mgy = -mgq_1\sin\alpha$$

The Lagrangian can be evaluated as

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1\sin\alpha$$

With the generalized coordinates we used, we have two Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2} \qquad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2}$$

The q_1 equation yield

$$mg\sin\alpha = \frac{d}{dt}\left[m(\dot{q}_1 + \dot{q}_2\cos\alpha)\right]$$

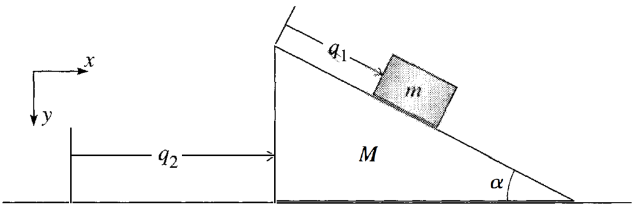

Figure: Block slides on a wedge which is free to move without friction

$$g \sin\alpha = \ddot{q}_1 + \ddot{q}_2 \cos\alpha$$

while the q_2 equation yield

$$\begin{aligned} 0 &= \frac{d}{dt}\left[(M+m)\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1 \cos\alpha)\right] \\ &= (M+m)\ddot{q}_2 + m(\ddot{q}_2 + \ddot{q}_1 \cos\alpha) \\ \ddot{q}_2 &= -\frac{m}{M+m}\ddot{q}_1 \cos\alpha \end{aligned}$$

which is just conservation of momentum in the x direction

$$m\dot{q}_2 + m\left(\dot{q}_2 + \dot{q}_1 \cos\alpha\right) = \text{Constant}$$

Now, combining the q_1 and q_2 result in

$$\begin{aligned} \ddot{q}_1 &= g\sin\alpha + \frac{m}{M+m}\ddot{q}_1 \cos^2\alpha \\ \ddot{q}_1 &= \frac{g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha} \end{aligned}$$

Suppose we want to determine the time it took for the block to reach the bottom of wedge, we can use the kinematic relation $x(t) = x_i + v_i t + at^2/2$ or $t = \sqrt{2l/a}$, with l as the length of the slope, to obtain

$$t = \left(2l\frac{1 - \frac{m}{M+m}\cos^2\alpha}{g\sin\alpha}\right)^{1/2}$$

As a sanity check, consider the case for $\alpha = 90°$. The acceleration $\ddot{q}_1 = g$, which is correct. Another is the case for $M \to \infty$. The acceleration $\ddot{q}_1 = g\sin\alpha$, which is the acceleration for a block on a fixed incline.

Application: Simple Pendulum

A bob of mass m is fixed to a massless rod length $l = \sqrt{x^2 + y^2}$, which is pivoted at O and free to swing without friction in the xy plane. One way to integrate the constraint into the Lagrangian is by expressing both coordinate in terms single generalized coordinate ϕ or by writing one of them in terms other variable, say $y = \sqrt{l^2 - x^2}$.

In terms of ϕ,the kinetic energy is

$$T = \frac{1}{2}ml^2\dot{\phi}^2$$

and the potential energy

$$U = mg(l - l\cos\phi) = mgl(1 - \cos\phi)$$

So, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi)$$

In this case, we only have one Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

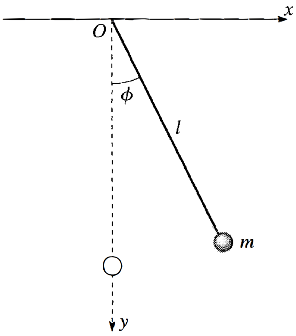

Figure: A simple pendulum

Substituting the Lagrangian

$$-mgl\sin\phi = ml^2\ddot{\phi}$$

$$\sin\phi = -\frac{l}{g}\ddot{\phi}$$

This is the differential equation describing simple pendulum motion, which, on assuming small angle ϕ, has the solution $\phi = A\cos(\omega t + \delta)$. Also, recall $\ddot{\phi}$ is the angular acceleration. This mean that the Euler-Lagrange equation reproduce the formula for torque $\Gamma = I\alpha = ml^2\ddot{\phi}$ or $\Gamma = Fr = -mgl\sin\phi$.

Application: Bead Spinning on a Wire Hoop

A bead of mass m is threaded on a frictionless circular wire hoop of radius R. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with constant angular velocity $\dot{\phi} = \omega$. The bead's position on the hoop is specified by the angle θ measured up from the rotation axis. We shall use θ as our only generalized coordinate.

The kinetic energy is

$$T = \frac{1}{2}m\left(v_\theta^2 + v_\phi^2\right)$$

with v_θ denote the bead tangential velocity with respect to non-rotating hoop, while v_ϕ denote the rotation velocity of the hoop. The tangential velocity is simply $v_\theta = R\dot{\theta}$,

$$v_\theta = R\dot{\theta}$$

with R as the distance of the bead with the axis of rotation–the hoop radius in other words. With the same principle, the hoop velocity is $v_\phi = \rho\dot{\phi}$. From the figure and known quantity, $\rho = R\cos\theta$ and $\dot{\phi} = \omega$, thus

$$v_\phi = R\sin\theta\omega$$

In terms of generalized coordinate, the kinetic energy reads

$$T = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)$$

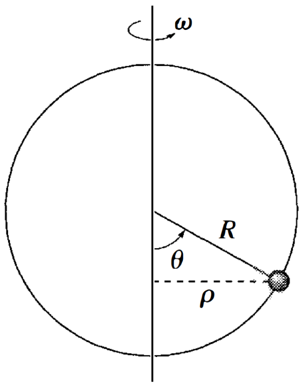

Figure: Bead constrained into moving within wire hoop

The potential energy is

$$U = mg(R - R\cos\theta) = mgR(1-\cos\theta)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mgR(1-\cos\theta)$$

which yield one Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$$

Substituting the Lagrangian

$$mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta = mR^2\ddot{\theta}$$

dividing by mR^2

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$
$$\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$$

This equation can be used to determine the equilibrium point–that is, the point where the position of the system does not change–and the stable point–that is, the position at which the system returns after slightly disturbed–of the system. The requirement of equilibrium point is $\dot{\theta} = 0$, but we can obtain the same result by setting both $\dot{\theta}$ and $\ddot{\theta}$ to zero

$$\left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta = 0$$

If we set $\sin\theta$ to zero, we obtain the following equilibrium points

$$\theta = 0, \pi$$

If we set the term inside parenthesis to zero

$$\cos\theta = \frac{g}{\omega^2 R}$$

or, due to cosine being even function

$$\cos(-\theta) = \frac{g}{\omega^2 R}$$

hence we have the following equilibrium

$$\theta = \pm \arccos \frac{g}{\omega^2 R}$$

Arccos function only has real value at $\theta \in [-1, 1]$, so when

$$\left| \frac{g}{\omega^2 R} \right| > 1 \quad \text{or} \quad \omega^2 < \frac{g}{R}$$

This equation undefined and the stable point disappear. These equilibrium points only appear when $\omega^2 > g/R$ and located on either side of bottom $\theta = 0$.

Out of these four equilibrium points, not all of them are stable.

1. Top $\theta = \pi$ point is unstable point due to not having restorative force, both gravitational and centrifugal push the bead away.

2. Bottom $\theta = 0$ point depends on ω^2. It is stable if $\omega^2 < g/R$, but become unstable if $\omega^2 > g/R$. This can be proven by approximating small θ displacement

   $$\ddot{\theta} = \left(\omega^2 - g/R\right) \theta$$

   If $\omega^2 < g/R$, then

   $$\ddot{\theta} = -\Omega^2 \theta$$

   with

   $$\Omega = \sqrt{\frac{g}{r} - \omega^2}$$

   which mean the bead perform simple harmonic motion about the stable point. If $\omega^2 > g/R$, then

   $$\ddot{\theta} = \Omega^2 \theta$$

   which has the solution $\theta = Ae^{\Omega t} + Be^{-\Omega t}$, so it moves in exponential way and the point is unstable.

3&4 These two point that comes after speeding up the rotation such that $\omega^2 > g/R$ are stable. To proof this, we expand the equation

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right) \sin\theta$$

around stable point θ_0

$$\theta \equiv \theta + \epsilon$$

using Taylor expansion

$$\cos(\theta_0 + \epsilon) \approx \cos(\theta_0) - \epsilon \sin\theta_0, \quad \sin(\theta_0 + \epsilon) \approx \sin\theta_0 + \epsilon \cos\theta_0$$

This result in

$$\ddot{\theta} = \left[\omega^2 \cos\theta_0 - \omega^2 \epsilon \sin\theta_0 - \frac{g}{R}\right] [\sin\theta_0 + \epsilon \cos\theta_0]$$
$$\ddot{\theta} = -\omega^2 \epsilon \sin^2\theta_0 - \omega^2 \epsilon^2 \sin\theta_0 \cos\theta_0$$

Since ϵ is small, we can ignore the second order

$$\ddot{\theta} = -\epsilon\omega^2 \sin^2 \theta_0$$

Since $\ddot{\theta}$ is the same as $\ddot{\epsilon}$

$$\ddot{\epsilon} = -\omega^2 \sin^2 \theta = -\Omega'^2 \epsilon$$

with

$$\Omega = \omega \sin \theta = \sqrt{\omega - \omega \cos^2 \theta_0} = \sqrt{\omega^2 - \left(\frac{g}{\omega R}\right)^2}$$

This mean ϵ oscillates about zero, and the bead itself oscillates about the equilibrium position θ_0 with frequency Ω'.

Hamiltonian Mechanics

The Hamiltonian is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_1 - \mathcal{L}$$

The Hamilton equation derived from this is

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

The conservation of Hamiltonian is stated as follows

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \implies \frac{dH}{dt} = 0$$

In the case of natural coordinate, that is the relation between the generalized coordinates and Cartesian is time-independent, the Hamiltonian takes the simple form $\mathcal{H} = T + U$

$$q_i = q_i(\mathbf{q}) \implies \mathcal{H} = T + U$$

The steps to solve problem using Hamilton formalism are as follows.

1. Choose the generalized coordinate $\mathbf{q}$.
2. Write the T and U in terms of $(\mathbf{q}, \dot{\mathbf{q}})$.
3. Find the generalized momenta $p_i = \partial \mathbf{L} / \partial \dot{q}_i$.
4. Solve for $\dot{\mathbf{q}}$ in terms $(\mathbf{q}, \mathbf{p})$.
5. Write $\mathcal{H}$ in terms $(\mathbf{q}, \mathbf{p})$.
6. Write the Hamilton equation for motion.

Notation

To avoid clutter, we define the following notation. The n-dimension system with n-generalized coordinate is represented by

$$\mathbf{q} \equiv (q_1, \ldots, q_n)$$

while n-generalized velocity

$$\dot{\mathbf{q}} \equiv (\dot{q}_1, \ldots, \dot{q}_n)$$

and generalized momentum

$$\mathbf{p} \equiv (p_1, \ldots, p_n)$$

In this case, $\mathbf{p}$ and $\mathbf{p}$ are n-dimensional vectors in the space of generalized position and generalized momentum.

Definition

The definition Hamiltonian comes from Lagrangian. To see this, consider the change of Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ as the time increase

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial q_1}{\partial t} \cdots + \frac{\partial \mathcal{L}}{\partial q_n}\frac{\partial q_n}{\partial t} + \frac{\partial \mathcal{L}}{\partial \dot{q}_1}\frac{\partial \dot{q}_2}{\partial t} + \cdots + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial \dot{q}_n}{\partial t} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial t} \\
&= \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t} \\
&= \sum_{i=1}^{n}(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} \\
\frac{d\mathcal{L}}{dt} &= \frac{d}{dt}\sum_{i=1}^{n}(p_i\dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t}
\end{aligned}$$

If we assume the Lagrangian does not depend explicitly on time, say in the case of $T = T(\mathbf{q}, \dot{\mathbf{q}})$ and $U = U(\mathbf{q})$, the $\partial\mathcal{L}/\partial t$ term is zero. Thus, the quantity

$$\frac{d}{dt}\left(\sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}\right) = 0$$

which is defined as Hamiltonian, is conserved. In other words,

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \implies \frac{dH}{dt} = 0$$

Special Case of Hamiltonian

The special case we are referring is $\mathcal{H} = T + U$, which occur when the coordinates are natural

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(\mathbf{q})$$

where the α subscript used to denote the coordinate as α-th particle's coordinate. We first begin by expressing the generalized velocity in terms of generalized coordinates, which can be obtained by performing partial derivative with respect to the generalized coordinate

$$\frac{\partial \mathbf{r}_\alpha}{\partial t} = \frac{\partial \mathbf{r}_\alpha}{\partial q_1}\frac{\partial q_1}{\partial t} \cdots + \frac{\partial \mathbf{r}_\alpha}{\partial q_n}\frac{\partial q_n}{\partial t} = \sum_{i=1}^{n}\frac{\partial \mathbf{r}_\alpha}{\partial q}\dot{q}_i$$

Then its square is just the dot product with itself

$$\left(\frac{\partial \mathbf{r}_\alpha}{\partial t}\right)^2 = \sum_{j=1}^{n}\frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i \cdot \sum_{k=1}^{n}\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\dot{q}_j$$

The kinetic energy is then product of triple sum

$$T = \frac{1}{2}\sum_\alpha m_\alpha \mathbf{r}_\alpha = \frac{1}{2}\sum_\alpha m_\alpha \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j}\dot{q}_j \cdot \sum_k \frac{\partial \mathbf{r}_\alpha}{\partial q_k}\dot{q}_k$$

If we define

$$A_{jk} = \sum_\alpha m_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_k}$$

The expression for kinetic energy simplifies into

$$T = \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j\dot{q}_k$$

If we assume kinetic energy $T = T(\mathbf{q}, \dot{\mathbf{q}})$ and potential energy $U = U(\mathbf{q})$, which is what natural coordinates imply anyway, then

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = p_i$$

To evaluate partial derivative, we need the following relation

$$\frac{d}{dv_i}\sum_{j,k} A_{jk}v_jv_k = 2\sum_{j,k} A_{ij}v_j \quad \text{if } A_{jk} = A_{kj}$$

Applying the identity

$$p_i = \frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\sum_{jk} A_{jk}\dot{q}_j\dot{q}_k\right) = \sum_j A_{ij}\dot{q}_j$$

Therefore, the Hamiltonian now reads

$$\begin{aligned}
\mathcal{H} &= \sum_i p_i\dot{q}_i - \mathcal{L} = \sum_i\sum_j A_{ij}\dot{q}_i\dot{q}_j - (T - U) \\
\mathcal{H} &= 2T - T + U = T + U
\end{aligned}$$

Hamilton Equation Derivation

The assumption that must be satisfied first are the constraint must be holonomic–the number of degree of freedom matches the number of generalized coordinates–and that the non constraint force must be derivable from potential energy $\mathbf{F} = -\nabla U$. In principle, to obtain Hamilton equation of motion, we differentiate the Hamiltonian $\mathcal{H}(\mathbf{q}, \mathbf{p})$ with both variable q_i and p_i. To evaluate the derivative, we need to express the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{n} p_i\dot{q}_1 - \mathcal{L}$$

in terms of $\mathbf{q}$ and $\mathbf{p}$

First note that the generalized momentum is a function of generalized velocity and perhaps time, which can be seen from

$$p_i = \frac{\partial}{\partial \dot{q}_i}\mathcal{L}(\mathbf{q}, \dot{q}_i, t) \implies p_i = p_i\left(\dot{q}, t\right)$$

Therefore, in principle, we can solve p_i to express $\dot{q}_i$ in terms of variables $\mathbf{p}$ and $\mathbf{q}$, also time perhaps

$$\dot{q}_i = \dot{q}_i(\mathbf{q}, \mathbf{p}_i, t)$$

Then the next step is to actually perform the differentiation. First we differentiate with respect to q_i

$$\frac{\partial \mathcal{H}}{\partial q_i} = \frac{\partial}{\partial q_i}\sum_{j=1}^{n} p_j\dot{q}_j - \frac{\partial \mathcal{L}}{\partial q_i}$$

The sum differentiation can be evaluated as follows

$$\frac{\partial}{\partial q_i}\sum_j p_j\dot{q}_j = \sum_j\left(\dot{q}_j\frac{\partial p_j}{\partial q_i} + p_j\frac{\partial \dot{q}_j}{\partial q_i}\right) = \sum_j p_j\frac{\partial \dot{q}_j}{\partial q_i}$$

While the Lagrangian term

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_i} &= \frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial q_1}{\partial q_i} + \dots \frac{\partial \mathcal{L}}{\partial q_n}\frac{\partial q_n}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial \dot{q}_i} + \dots + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial \dot{q}_n}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial q_i}\\ &= \sum_j\left(\frac{\partial \mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i}\right)\\ \frac{\partial \mathcal{L}}{\partial q_i} &= \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j p_i\frac{\partial \dot{q}_j}{\partial q_i}\end{aligned}$$

Substituting into the Hamiltonian

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_i\frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\dot{p}_i$$

Next is the derivative with respect to p_i

$$\frac{\partial \mathcal{H}}{\partial p_i} = \frac{\partial}{\partial p_i}\sum_{j=1}^{n} p_i\dot{q}_1 - \frac{\partial \mathcal{L}}{\partial p_i}$$

The sum differentiation can be evaluated as follows

$$\frac{\partial}{\partial p_i}\sum_j p_j\dot{q}_j = \sum_j\left(\dot{q}_j\frac{\partial p_j}{\partial p_i} + p_j\frac{\partial \dot{q}_j}{\partial p_i}\right) = \dot{q}_i + \sum_j p_j\frac{\partial \dot{q}_j}{\partial p_j}$$

While the Lagrangian term

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial p_i} &= \frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial q_1}{\partial p_i} + \dots \frac{\partial \mathcal{L}}{\partial q_n}\frac{\partial q_n}{\partial p_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial p_i} + \dots + \frac{\partial \mathcal{L}}{\partial \dot{q}_n}\frac{\partial \dot{q}_n}{\partial p_i} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial p_i}\\ &= \sum_j\left(\frac{\partial \mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial p_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i}\right)\\ \frac{\partial \mathcal{L}}{\partial p_i} &= \sum_j p_j\frac{\partial \dot{q}_j}{\partial p_i}\end{aligned}$$

Substituting into the Hamiltonian

$$\frac{\partial \mathcal{H}}{\partial q_i} = \dot{q}_i + \sum_j p_j\frac{\partial \dot{q}_j}{\partial p_j} - \sum_j p_j\frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$

Hamiltonian Dependence on Time

Consider the total derivative of the Hamiltonian with respect to time

$$\frac{d\mathcal{H}}{\partial t} = \sum_i\left(\frac{\partial \mathcal{H}}{\partial q_i}\dot{q} + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i\right) + \frac{\partial \mathcal{H}}{\partial t}$$

Then using the Hamilton equation of motion

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

The total derivative $d\mathcal{H}/dt$ denote the actual changes as t moves forward, which will changes both variables $(\mathbf{q}, \mathbf{p})$ also. In other hand, the partial derivative $\partial\mathcal{H}/\partial t$ denote the change of the Hamilton as t moves forward with $(\mathbf{q}, \mathbf{p})$ being held constant. This will be zero if $\mathcal{H}$ does not explicitly depend on t. Thus, if $\mathcal{H}$ does not explicitly depend on t, then $\mathcal{H}$ is conserved.

Newton's Law in 2D

Since we are using 2D Cartesian, the Hamiltonian is simply the system energy

$$\mathcal{H} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + U(x,y) = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + U(x,y)$$

The x-th component reads

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = -\frac{\partial\mathcal{H}}{\partial x} = -\frac{\partial U}{\partial x}$$

while the y-th component

$$\dot{y} = \frac{\partial\mathcal{H}}{\partial p_y} = \frac{p_y}{m} \qquad \dot{p}_y = -\frac{\partial\mathcal{H}}{\partial y} = -\frac{\partial U}{\partial y}$$

Combining both, we obtain the Newton's definition of momentum and force

$$\mathbf{p} = m\mathbf{r} \qquad \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Newton's Law in Polar Coordinate As previously, the Hamiltonian is the system energy

$$\mathcal{H}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

Next, we evaluate the momenta

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}, \qquad p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi}$$

and solve for generalized velocity

$$\dot{r} = \frac{p_r}{m} \qquad \dot{\phi} = \frac{p_\phi}{mr^2}$$

Then rewrite the Hamiltonian in terms of coordinate and momenta

$$\mathcal{H} = \frac{1}{2}m\left(\frac{p_r^2}{m^2} + \frac{p_\phi^2}{m^2r^2}\right) + U(r) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r)$$

The radial components are

$$\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r} = \frac{p_r}{m} \qquad p_r = -\frac{\partial\mathcal{H}}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial U}{\partial r}$$

Combining both, we obtain

$$m\ddot{\mathbf{r}} = \frac{p_\phi^2}{mr^3} - \frac{\partial U}{\partial r}$$

which is the definition of total force $m\ddot{\mathbf{r}}$ as the sum of radial force $-\partial U/\partial r$ and centrifugal force p_ϕ^2/mr^3. Then the angular component

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

which reproduce the definition of angular momentum and conservation of angular momentum.

Application: Mass on a Cone

Consider a mass m which is constrained to move on the frictionless surface of a vertical cone $\rho = cz$.

Here we use cylindrical coordinate. The kinetic energy then is

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[(cz\dot{\phi})^2 + (c^2+1)\dot{z}^2\right]$$

while the potential energy is simply due to gravity

$$U = mgz$$

Thus the Hamiltonian read

$$\mathcal{H} = \frac{1}{2}m\left[(cz\dot{\phi})^2 + (c^2+1)\dot{z}^2\right] + mgz$$

Next we determine the generalized momenta

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = (c^2+1)m\dot{z} \qquad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (c^2z^2)m\dot{\phi}$$

Then we express the Hamiltonian as following

$$\mathcal{H} = \frac{1}{2}\left[\frac{(c^2+1)}{(c^2+1)^2m^2}p_z^2 + + \frac{c^2z^2}{m^2(c^2z^2)^2}p_\phi^2\right] + mgz$$
$$\mathcal{H} = \frac{1}{2m}\left[\frac{p_z^2}{c^2+1} + \frac{p_\phi^2}{c^2z^2}\right] + mgz$$

Finally, we have the equations for z component

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{c^2+1} \qquad \dot{p_z} = -\frac{\partial \mathcal{H}}{\partial z} = \frac{p_\phi^2}{mc^2z^3} - mg$$

and ϕ component

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mc^2z^2} \qquad \dot{p_\phi} = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

Simple 1D Harmonic Oscillator

In this system, the kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2 = \frac{p_x^2}{2m}$$

while the potential energy given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

Thus the Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

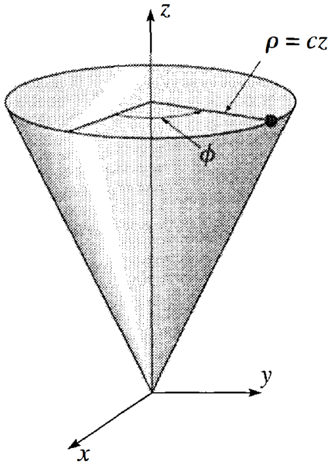

Figure: Particle constrained to move within a cone

This gives the equation

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \qquad \dot{p}_x = \frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x$$

Solving the first equation for $\dot{p}_x$ then equating it with the second

$$\ddot{x} = -\omega^2 x$$

This is the equation for simple harmonic oscillator with the solution

$$x = A\cos(\omega t - \delta)$$

Quantum Mechanics

Braket notation

Ket. $|\psi\rangle$ represents quantum state. Written in matrix form as

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

Bra. $\langle\psi|$ is the Hermitian conjugate (complex conjugate transpose) of the ket $|\psi\rangle$

$$\langle\psi| = \begin{pmatrix} \psi_0 & \psi_1 & \dots & \psi_n \end{pmatrix}$$

Inner Product. Written

$$\langle\phi|\psi\rangle = \begin{cases} 0, & \text{if orthogonal} \\ 1, & \text{if orthonormal} \end{cases}$$

Operator

Position Operator. Represents the position of a particle.

$$\hat{x} = x$$

Momentum Operator.

$$\hat{p} = -i\hbar\nabla$$

Energy Operator.

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Its action on the energy eigenstates is given by:

$$\langle\psi|\hat{E}|\psi\rangle = E_n$$

Hamiltonian Operator.

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

The Hamiltonian can be written in terms of ladder operators as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$H|n\rangle = E_n|n\rangle$$

where the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Creation operator. Increases the system's energy, thus often said to be raising operator. Defined as

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x - ip\right)$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Annihilation operator. Decrease the system's energy, thus often said to be lowering operator. Defined as

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x + ip\right)$$

in matrix representation

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Commutator

Commutator measures how much two physical quantities fail to be simultaneously measurable or well-defined. It is defined as

$$[A, B] = AB - BA$$

If $[A, B] = 0$, then A and B commute and can be simultaneously measured with arbitrary precision. If not, their measurement outcomes interfere with each other.

Expectation value

Braket notation.

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

Matrix notation.

$$\langle \hat{A} \rangle = \psi^\dagger \hat{A} \psi$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \, dx$$

Normalization

Braket notation.

$$\langle \psi | \psi \rangle = 1$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \, dx = 1$$

Normalization Problem

Ex 1. Find the value of A such that the following wavefunction particle inside potential well is normalized.

$$\psi = \frac{1}{\sqrt{10a}} \sin\left(\frac{\pi x}{a}\right) + A\frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) + \frac{3}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right)$$

The wavefunction of said particle is written in the form

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{a}\right)$$

Hence we write the wavefunction as

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3$$

We then normalized the wavefunction by

$$\langle\psi|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \middle| \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \right\rangle$$

Since the wavefunction is orthonormal to itself and orthogonal to another, therefore

$$\langle\psi|\psi\rangle = \frac{1}{20}\langle\psi_1|\psi_1\rangle + A^2\langle\psi_2|\psi_2\rangle + \frac{9}{10}\langle\psi_3|\psi_3\rangle = A^2 + \frac{19}{20}$$

Thus

$$A = \sqrt{\frac{1}{20}}$$

Expectation Value Problem

Ex 1. From the first normalization problem, find the expectation value of the energy. We have the normalized wavefunction

$$\psi = \sqrt{\frac{1}{20}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{1}{20}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{9}{10}} \sin\left(\frac{3\pi x}{a}\right)$$

or simply

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + \sqrt{\frac{1}{20}}\psi_2 + \sqrt{\frac{9}{10}}\psi_2$$

All that left is to do the algebra

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_1 \right\rangle + \left\langle \sqrt{\frac{1}{20}}\psi_2 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_2 \right\rangle \\ + \left\langle \sqrt{\frac{9}{10}}\psi_3 \middle| \hat{E} \middle| \sqrt{\frac{9}{10}}\psi_3 \right\rangle\end{aligned}$$

Factoring the constant

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle &= \frac{1}{20}\left\langle \psi_1 \middle| \hat{E} \middle| \psi_1 \right\rangle + \frac{1}{20}\left\langle \psi_2 \middle| \hat{E} \middle| \psi_2 \right\rangle + \frac{9}{10}\left\langle \psi_3 \middle| \hat{E} \middle| \psi_3 \right\rangle \\ &= \frac{1}{20}E_1 + \frac{1}{20}E_2 + \frac{9}{10}E_3 = \frac{1}{20}\frac{h^2}{8ma^2} + \frac{1}{20}\frac{4h^2}{8ma^2} + \frac{9}{10}\frac{9h^2}{8ma^2}\end{aligned}$$

Nuclear Physics

Introduction

Atomic Species

Characterized by the number of neutron N, number of proton Z, and mass number $A = N + Z$

$$(A, Z) = {}^{A}_{Z}X = {}^{A}_{Z}X_N$$

Nucleon

Defined as bound state of atomic nuclei. The two type are positively charged proton and neutral neutron. Nucleon constitutes three bound fermions called quark: up with charge (2/3) and down with charge (−1/3)

$$\text{proton} = \text{uud}$$
$$\text{neutron} = \text{udd}$$

Both of them are fermion with mass

$$m_e = 939.56 \text{ MeV}/c^2$$
$$m_p = 938.27 \text{ MeV}/c^2$$
$$m_n - m_e = 1.29 \text{ MeV}/c^2$$

The magnetic moment projected by both are

$$\mu_p = 2.792847386\ \mu_N \quad \mu_n = -1.91304275\ \mu_N$$

where μ_N denote nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15245166\ 10^{-14} \text{ MeV/T}$$

Here are the difference in unit used to describe nucleus compared to atom

Properties	Atom	Nucleus
Radius	Angstrom (10^{-10} m)	Femto (10^{-15} m)
Energy	eV	MeV

Radii. In terms of their mass number A, their radius may be approximated as

$$R = r_0 A^{1/3} \quad \text{with} \quad r_0 = 1.2 \text{ fm}$$

This approximation comes from assuming the radius is proportional to the volume which is also assumed to be spherical. Then $\mathcal{V} = 4\pi R^3/3 \approx A$.

Binding energy. Defined as the difference of the sum of nuclei mass and the nuclear mass

$$B(A, Z) = Nm_nc^2 + Zm_pc^2 - m(A, Z)c^2$$

Mass. Three unit most common are atomic mass unit (u), the kilogram (kg), and the electron-volt (eV). The atomic mass unit is defined as the mass of ^{12}C atom divided by 12

$$1\ \mathrm{u} = \frac{m(^{12}C)}{12}$$

electron volt is defined as the kinetic energy of an electron after being accelerated from rest through a potential difference of 1 V.

Nuclear Relative

Isotope. Same number of charge Z, but different number of neutron N. Isotope has identical chemical properties, since they have the same electron, but different nuclear properties. Example are

$$^{238}_{92}\mathrm{U} \quad \text{and} \quad ^{235}_{92}\mathrm{U}$$

Isobar. Same mass A. Frequently have the same nuclear properties due to the same number of nucleon. Example are

$$^{3}\mathrm{He} \quad \text{and} \quad ^{3}\mathrm{H}$$

Isotone. Same number of neutron N, but different number of proton Z. Example are

$$^{14}\mathrm{C}_6 \quad \text{and} \quad ^{16}\mathrm{O}_8$$

Radioactivity

Nuclear Decay

Alpha decay. Occur when parent nucleon decay distribute among daughter nuclei

$$ {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P} \rightarrow {}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D} + {}^{4}_{2}\mathrm{He} $$

The Q value, which is defined as the total released in a given nuclear decay, of alpha decay is

$$ Q = \left[m\left({}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P}\right) - m({}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D}) - m\left({}^{4}_{2}\mathrm{He}\right)\right] c^2 $$
$$ Q = K_D + K_\alpha = \frac{A}{A-4} K_\alpha $$

Q can be determined by applying the energy conservation into given nuclear reaction.

Beta decay. Two example are electron capture and neutrino capture

$$ e^- + p \leftrightarrow \nu_e + n $$

Other interaction can be found by moving particle to different side and changing them to their anti particle, such as

$$ \bar{\nu_e} + p \leftrightarrow e^+ + n $$

where the anti particle of electron e^- and electron neutrino ν_e are positron e^+ and electron antineutrino $\bar{\nu_e}$. Another example is beta negative decay

$$ ({}^{A}_{Z}P) \rightarrow {}_{\mathrm{Z+1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \bar{\nu_e} $$

which is the neutron decay inside isotope, and beta positive decay

$$ ({}^{A}_{Z}P) \rightarrow {}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \nu_e $$

which is the proton decay inside isotope. Electron capture inside isotope takes the form of

$$ ({}^{A}_{Z}P) + e^- \rightarrow {}_{\mathrm{Z-1}}{}^{\mathrm{A}} + \nu_e $$

The Q value of beta negative is

$$ Q = \left[m\left(({}^{A}_{Z}P)\right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\nu_e}\right] $$

and beta negative decay

$$ Q = \left[m\left(({}^{A}_{Z}P)\right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\bar{\nu_e}}\right] $$

Radiation

Radioactive decay. The number of decay events over a time inteval is proportional to the number of particle

$$-\frac{dN}{dt} = \lambda N$$

This first order ODE is solved by integration

$$-\int_{N_0}^{N} \frac{1}{N}\, dN = \int_0^t \lambda t\, dt$$
$$\ln \frac{N}{N_0} = -\lambda t$$
$$N(t) = N_0 e^{-\lambda t}$$

Now consider the case of chain radiation, that is the particle undergoes decay $N_A \to N_B \to N_C$. For the first decay of particles N_A into N_B

$$\frac{dN_A}{dt} = -\lambda_A N_A \implies N_A = N_{A0} e^{-\lambda_A t}$$

Now the total rate of creation for the second particle N_B is the sum of the decay of said particle and the decay of the first particle into the second particle.

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A$$
$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A0} e^{-\lambda_A t}$$

This ODE is solved by integrating factor method

$$I = \int \lambda_B\, dt = \lambda_B t$$

Then

$$N_B(t) = e^{-\lambda_B t} \int \lambda_A N_{A0} e^{-\lambda_A t}\, dt + Ce^{-\lambda_B t}$$
$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} e^{-\lambda_A t} + Ce^{-\lambda_B t}$$

If we assume at $t = 0$ we have zero second particle, then

$$N_B(0) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} + C = 0 \implies C = -\frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1}$$

Thus, the complete solution is

$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} \left(e^{\lambda_A t} - e^{\lambda_B t}\right)$$

In practice, often we are not given the number of particle, but rather the mol n or mass m (in gram) of said particle. Those quantities are related by

$$n = \frac{m}{M_r} \quad \text{or} \quad n = \frac{N}{N_A}$$

where $M_r \approx A$ is the molecular mass (gram/mol) and $N_A = 6.02 \cdot 10^{23}$ (mol^{-1}) is the Avogadro constant.

Halflife. Defined as the time required for particles to reduce to half of its initial value. The value of halflife can be determined by considering $N(t)$ at $t_{1/2}$, which by definition

$$\frac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}$$
$$t_{1/2} = \frac{\ln 2}{\lambda}$$

This equation can also be used to determine the decay constant.

Activity. Defined as the number of radioactive transformations per second

$$A \equiv -\frac{dN}{dt} = \lambda N$$

Using the solution for N, we can write

$$A = \lambda N_0 e^{-\lambda t} = A_0 e^{-\lambda t}$$

Specific Activity. Quantity related to activity; specific activity is the activity per unit mass

$$a \equiv \frac{A}{m}$$

Using the relation of mass with mol and halflife relation

$$a = \frac{\lambda N}{\frac{N}{N_A} M_r} = \frac{N_A \lambda}{M_r} = \frac{\ln 2 N_A}{t_{1/2} M_r}$$

On evaluating the numerator constant

$$a = \frac{1.32 \cdot 10^{16}}{t_{1/2} M}$$

Nuclear Stability

Valley of stability. Consist of long-lived isotope that do not simultaneously decay.

Below the valley of stability. Consist of isotopes with more N than those of the valley of stability, thus they decay either by beta negative, more likely, or neutron decay, less likely.

Above the valley of stability. Consist of isotope with more Z than those of the valley of stability, thus it decays either by beta positive or electron capture.

Beyond the valley of stability. Consist of heavy isotope with $Z > 83$, $N > 126$, and $A > 209$. They decay with alpha radiation.

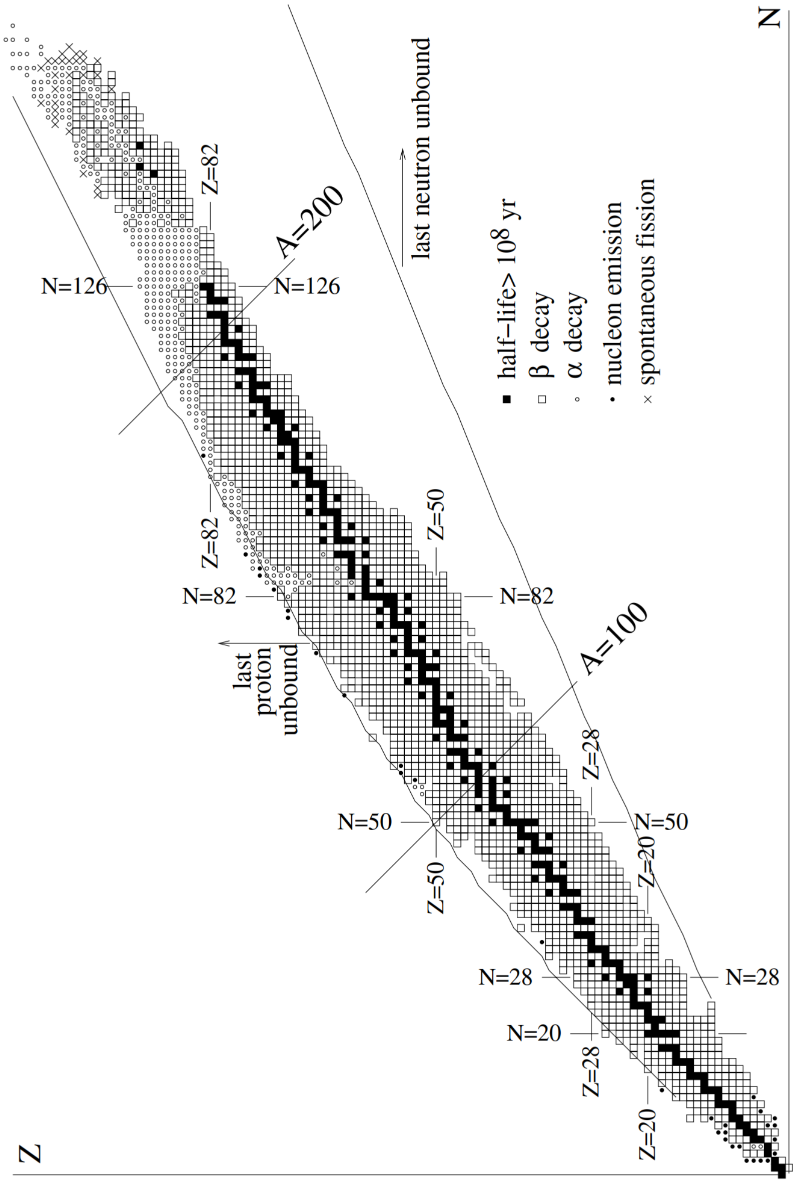

Figure: Valley of stability in ZN graph.

Particle Physics

Elementary Particle

Particle can be categorized by their spin, mass, and the type of interaction.

Boson. A pseudo-particle with interger spin that mediate the interaction. Two types of boson are gauge, or force carrier, and scalar, whose function is to give particle mass. Also, $W^{\pm}$ and Z^0 boson both belong to the same isospin triplet with $I = 1$

Table: Boson properties

Gauge Boson	
Family	Function/Force
Photon γ	Electromagnetic
Gluon g	Strong: binds the quark
W Boson $W^{\pm}$	Weak:radioactive decay
Z Boson Z^0	Weak: same as above
Scalar Boson	
Family	Function/Force
Higgs Boson H^0	Gives particle mass

Fermion. Consist of quark and lepton. All has half interger spin. Bound state of quark is called hadron and quark is able to experience all four fundamental forces.

Lepton experiences the weak force, however for charged lepton is also able to experience electromagnetic forces. This is why neutral lepton, that is neutrino, is hard to detect.

Hadron. Two types of hadron are baryon, which consist of three quarks, and meson, which consist of two quarks and one antiquark. Since baryon is made up of three spin-1/2, its spin is always half interger, opposite with meson; thus, both are also able to experience all four fundamental forces. Anti baryon consist of antiquark with the same configuration, for example is anti proton $\bar{p} = \bar{u}\bar{u}\bar{d}$.

One difference is that, unlike baryon, meson do not follow Pauli exclusion principle, since its total spin is interger. Most mesons are not their own antiparticle: charged mesons like K^+ with K^-; and neutral meson like K^0 with $\bar{K^0}$.

Quantum Number. Summarized as follows. First we have the fundamental particles.

Table: fundamental particle properties

Boson

Name	q	L	B	S	I	I_z
γ	0	0	0	0	0	0
g	0	0	0	0	0	0
W^+	+1	0	0	0	1	+1
Z^0	0	0	0	0	1	0
W^-	-1	0	0	0	1	-1
H^0	0	0	0	0	0	0

Quark

Name	q	L	B	S	I	I_z
u	+2/3	0	+1/3	0	1/2	+1/2
d	-1/3	0	+1/3	0	1/2	-1/2
s	-1/3	0	+1/3	-1	0	0
c	+2/3	0	+1/3	0	0	0
b	-1/3	0	+1/3	0	0	0
t	+2/3	0	+1/3	0	0	0

Lepton

Name	q	L	B	S	I	I_z
e	-1	$L_e = 1$	0	0	1/2	+1/2
μ	-1	$L_\mu = 1$	0	0	1/2	-1/2
τ	-1	$L_\tau = 1$	0	0	0	0
ν_e	0	$L_e = 1$	0	0	0	0
ν_μ	0	$L_\mu = 1$	0	0	0	0
ν_τ	0	$L_\tau = 1$	0	0	0	0

Then the Hadron.

Table: Hadron properties

Baryon

Name	Content	q	L	B	S	I	I_z
p	uud	+1	0	+1	0	1/2	+1/2
n	udd	0	0	+1	0	1/2	-1/2
Λ^0	uds	0	0	+1	-1	I	0
Σ^+	uus	+1	0	+1	-1	1	+1
Σ^0	uds	0	0	+1	-1	1	0
Σ^-	dds	-1	0	+1	-1	1	-1
Ξ^0	uss	0	0	+1	-2	1/2	+1/2
Ξ^-	dss	-1	0	+1	-2	1/2	-1/2
Ω^-	sss	-1	0	+1	-3	0	0

Meson

Name	Content	q	L	B	S	I	I_z
π^+	$u\bar{d}$	+1	0	0		1	+1
π^0	$u\bar{u}, d\bar{d}$	0	0	0	0	1	0
π^-	$d\bar{u}$	-1	0	0	0	1	-1
K^+	$d\bar{s}$	+1	0	0	+1	1/2	+1/2
K^0	$d\bar{s}$	0	0	0	+1	1/2	-1/2

$\bar{K^0}$	$s\bar{d}$	0	0	0	-1	1/2	+1/2
K^-	$s\bar{u}$	-1	0	0	-1	1/2	-1/2
η	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0
η'	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0

Conservation Laws

Energy. The total energy $E = K + mc^2$ must be conserved in all types of nuclear reaction.

Momentum. Same as energy conservation law.

Mass number. Same as energy conservation law.

Charge. Same as energy conservation law.

Lepton number. We assign lepton a lepton number $L = +1$, anti lepton $L = -1$, and non lepton $L = 0$. Each family of lepton–such as electron e, muon μ, tau τ and their neutrino sibling–has separate conserved neutrino number–L_e, L_μ, L_τ repectively.

This quantum number almost conserved in all reaction, exception exist in neutrino oscillations; however only family lepton number is violated, while the total lepton number is conserved.

Baryon number. Like before, we assign baryon a baryon number $L = +1$, anti baryon $L = -1$, and non baryon $L = 0$. As an aside, baryon is the defined as the bound state of three quarke and that strong force works on all of them.

Strangeness number. Defined as the negative of the number of strange quarks in it, in particular strange quark s has $S = -1$ and antistrange quark $\bar{s}$ has $S = +1$.

Isospin (Isotropic Spin). We define isospin I such that $2I + 1$ is equal to the multiplet type of the baryon. Recall that the strong force does not differentiate between proton and nucleon, hence both particle can be defined as the different state of the same particle and thus can be categorized as doublet. To differentiate them, then, we define proton with $I_z = 1/2$ and neutron with $I_z = -1/2$. In general,

$$I_z = I, I - 1, \ldots, -I$$

Isospin is conserved in strong force interaction, but may not in other interaction.

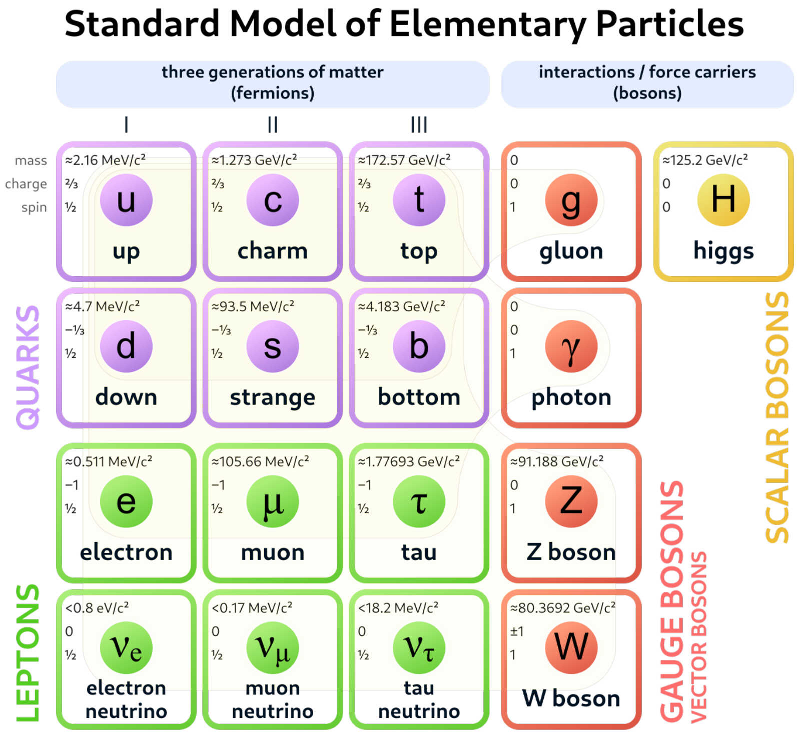

Figure: Standard model.

Nuclear Model

Liquid Drop Model

The binding energy by this model is given by

$$B(A,Z) = a_V A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta(A)$$

where

$$\begin{aligned} a_V &= 15.753 \\ a_s &= 17.804 \\ a_C &= 0.7103 \\ a_a &= 23.69 \\ \delta(A) &= \begin{cases} 33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are even} \\ -33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are odd} \\ 0 & \text{if } A = N+Z \text{ is odd} \end{cases} \end{aligned}$$

Volume term. Recall that the volume of nucleon is proportional to A; on using this we have obtained $r = r_0 A^{1/3}$, which means that nuclei have constant density, like a drop of water.

Experiment shows that the binding energy per nucleon is rougly constant, $B/A \approx 8$ MeV. The overshoot of the volem term, then, require corection that will lower the value of the binding energy.

Surface term. Like water molecule, internal nucleon experience isotropic force, while surface nucleon only from inside. This resulting the force, and consequentl the energy, to be proportional to area $4\pi r^2 \approx a_s A^{1/3}$, with $r = r_0 A^{1/3}$.

Columb term. The binding energy due to charged particle is proportional to $Q^2/R \approx Z^2/A^{1/3}$.

Asymmetry term. By The Pauli exclusion principle, the configuration with different N and Z will have more energy than that with the same due to the different nucleon will simply occupy the higher energy state. This is the basis of $N - Z$ term.

Quantum pairing term. This term captures the effect of spin coupling. Odd-odd nuclei tend to undergo beta decay to an adjacent even-even nucleus by changing an N to a Z or vice versa.

Stable nuclei. By setting the derivative of B with respect to Z to zero, we have the maximum binding energy, that is the most state nuclei. If we ignore the quantum pairing term, or we are considering the case of odd A,we have

$$Z(A) = \frac{A}{2 + a_c A^{2/3}/2a_a} \approx \frac{A/2}{1 + 0.0075\ A^{2/3}}$$