Mathematics

Variation Calculus

The Euler Equation

Any problem in the calculus of variations is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and solving the resulting differential equation. When the function $F = F(r, \theta, \theta')$, the Euler's equation read

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

If $F = F(t, x, \dot{x})$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

Notice that the first derivative in the Euler equation is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative.

Proof. We will try to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

where F is a given function. Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x)$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Differentiating with respect to x, we get

$$Y(x) = y(x)' + \epsilon\eta'(x)$$

Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\, dx$$

Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x) \right] dx$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] dx$$

Assuming that y'' is continuous, we can integrate the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)\, dx = -\int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y'}\eta(x)\, dx + \left. \frac{\partial F}{\partial y'}\eta(x) \right|_{x_1}^{x_2}$$

The first term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \eta(x)\, dx$$

Since $\eta(x)$ is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \blacksquare$$

Notice carefully here that we are not saying that when an integral is zero, the integrand is also zero; this is not true. What we are saying is that the only way $\int f(x)\eta(x)\, dx$ can always be zero for every $\eta(x)$ is for $f(x)$ to be zero.

Several Variables

If there are n dependent variables in the original integral, there are n Euler-Lagrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}$$

Application: Shortest Between two points

Arbitrary path is given by

$$L = \int_1^2 \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

We factor dx from the integrand in order to make the function we are optimizing not dependent on the y variable and make the evaluation using Euler-Lagrange equation easier

$$f(y, y', x) = \sqrt{1 + y'^2}$$

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial f}{\partial y'}$$

$\partial f/\partial y = 0$ implies simply that $\partial f/\partial y'$ is a constant. Accordingly,

$$\begin{aligned}
\frac{\partial f}{\partial y'} &= \frac{y'}{\sqrt{1+y'^2}} = C \\
y'^2 &= C^2(1 + y^2) \\
y'^2(1 - C^2) &= C^2 \\
y &= \int \frac{C}{\sqrt{1 - C^2}}\, dx = \mathcal{C}x
\end{aligned}$$

which is the equation for straight line.

Application: Brachistochrone Given two points 1 and 2, with 1 higher above the ground, in what shape should we build a frictionless roller coaster track so that a car released from point 1 will reach point 2 in the shortest possible time?

The speed at which the coaster descend can be determined by the conservation energy principle

$$mgy = \frac{1}{2}mv^2 v \qquad = \sqrt{2gy}$$

Thus the time to travel between points

$$t = \int_{t_1}^{t_2} \frac{ds}{v} = \int_{t_1}^{t_2} \sqrt{\frac{dx^2 + dy^2}{2gh}}$$

Since v gives a function of y, we take it as independent variable for the same reason as previously

$$t = \frac{1}{\sqrt{2g}} \int_{t_1}^{t_2} \sqrt{\frac{1 + x'^2}{y}}\, dy$$

Ignoring the constant, the function we want to optimize is

$$f(x, x', y) = \sqrt{\frac{1 + x'^2}{y}}$$

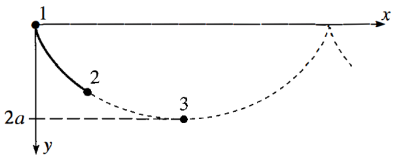


Figure: Brachistochrone problem

Then the Euler-Lagrange equation takes the form of

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}$$

$\partial f/\partial x = 0$ implies simply that $\partial f/\partial x'$ is a constant. Accordingly,

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Here we take the constant as $\sqrt{1/2a}$

$$\begin{aligned}
x'^2 &= \frac{y(1+x'^2)}{2a} \\
x'^2\left(1-\frac{y}{2a}\right) &= \frac{y}{2a} \\
x' &= \sqrt{\frac{y}{2a}\frac{1}{2-y/2a}} \\
x' &= \sqrt{\frac{y}{2a-y}}
\end{aligned}$$

$$x = \int \sqrt{\frac{y}{2a-y}}\; dy$$

To solve this integral, we substitute $y = a(1-\cos\alpha)$ and $dy = a\sin\alpha\; d\theta$

$$\begin{aligned}
x &= \int \left[\frac{a(1-\cos\theta)}{a(1+\cos\theta)}\right]^{1/2} a\sin\theta\; s\theta \\
&= a\int \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{1/2} [(1+\cos\theta)(1-\cos\theta)]^{1/2}\; d\theta \\
&= a\int (1-\cos\theta)\; d\theta \\
x &= a(\theta-\sin\theta)+c
\end{aligned}$$

Therefore the path of the coaster is given by the following parametric equation

$$\begin{cases} x = a(\theta-\sin\theta)+c \\ y = a(1-\cos\theta) \end{cases}$$

Classical Mechanics

Unasorted Classical Mechanics Topics

Newton's Law

First law. In the absence of an external force, when viewed from an inertial frame, an object at rest remains at rest and an object in uniform motion in a straight line maintains that motion.

Second law. Simply put

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Third law. States that if two objects interact, the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1.

Particle Under Constant Acceleration

Here's some kinematics equation for position

$$x(t) = x_i + \frac{1}{2}(v_i + v_f)t$$
$$x(t) = x_i + v_i t + \frac{1}{2}at^2$$

and for velocity

$$v(t) = v_i + at$$
$$v(t)^2 = v_i^2 + 2a(x_f - x_i)$$

Particle in Uniform Circular Motion

If a particle moves in a circular path of radius r with a constant speed v, the magnitude of its centripetal acceleration is given by

$$a_r = \frac{v^2}{r}$$

while its period and angular velocity is

$$T = \frac{2\pi r}{v}, \quad \omega = \frac{2\pi}{T}$$

Applying Newton's second law

$$\sum F = ma_r = m\frac{v^2}{r}$$

Rigid Object Under Constant Angular Acceleration

Analogous to those for translational motion of a particle under constant acceleration

$$\begin{aligned} \omega(t) &= \omega_i + \alpha t \\ \omega(t)^2 &= \omega_i^2 + 2\alpha(\theta_t - \theta_i) \\ \theta(t) &= \theta_i + \omega t + \frac{1}{2}\alpha t^2 \\ \theta(t) &= \theta_i + \frac{1}{2}(\omega_i + \omega_f)t \end{aligned}$$

Relation of Linear and Rotational Motion

The following equations show the relation of linear and rotational motion

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha$$

Torque

The torque associated with a force **F** acting on an object

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{d\mathbf{L}}{dt}$$

Moment of Inertia

The moment of inertia of a rigid object is

$$I = \sum mr^2 = \int r^2 dm$$

Parallel Axis Theorem. To calculate the moment inertia from any axis, we use parallel axis theorem

$$I = I_{\mathrm{CM}} + Md^2$$

Terminal velocity

$\boldsymbol{r} \propto \boldsymbol{v}$**.** The velocity as a function of time is

$$v = \frac{mg}{b}\left[1 - \exp\left(-\frac{bt}{m}\right)\right] = v_T\left[1 - \exp\left(-\frac{bt}{m}\right)\right]$$

where b is a resistive constant whose value depends on the properties of the medium.

$\boldsymbol{r \propto v^2}$. Given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}$$

where D is a dimensionless empirical quantity called the drag coefficient, ρ is the density of air, and A is the cross-sectional area of the moving object.

Escape velocity. The speed required by an object to escape from any planet orbit is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Work Energy Theorem

It states that if work is done on a system by external forces and the only change in the system is in its speed,

$$W = \Delta T$$

Kinetic Energy

For an object in linear motion, the kinetic energy of said object is

$$T = \frac{1}{2}mv^2$$

whereas for rotational motion

$$T = \frac{1}{2}I\omega^2$$

Hence the total kinetic energy of a rigid object rolling on a rough surface without slipping

$$T = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega_{\text{CM}}^2$$

Potential Energy Function

For conservative energy $\mathbf{F}$, applies

$$V_f - V_i = -\int_{\mathbf{r_i}}^{\mathbf{r_f}} \mathbf{F} \cdot d\mathbf{r}$$

For particle-Earth system, the gravitational potential energy is

$$V = mgy$$

and elastic potential stored in spring

$$V = \frac{1}{2}kx^2$$

Effective potential

Effective potential energy $U_{\text{eff}}(r)$ is the sum of the actual potential energy $U(r)$ and the centrifugal $U_{\text{cf}}(r)$:

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

where l is the angular momentum and μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Momentum Impulse

The linear momentum and impulse are defined as

$$\mathbf{p} = m\mathbf{v}, \quad \mathbf{I} = \int_{t_i}^{t_f} \sum \mathbf{F}\, dt$$

Angular Momentum The angular momentum about an axis through the origin of a particle having linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

The z component of angular momentum of a rigid object rotating about a fixed z axis is

$$L_z = I\omega$$

Center of Mass and Velocity

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{r} = \frac{1}{M} \int \mathbf{r}\, dm$$

where M is the total mass. The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{v} =$$

Collision

Inelastic collision. One for which the total kinetic energy of the system of colliding particles is not conserved.

Elastic collision. One in which the kinetic energy of the system is conserved.

Perfectly inelastic. A collision which the colliding particles stick together after the collision.

Rocket Propulsion The expression for rocket propulsion is

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$

Power

The rate at which work is done by an external force, called power, is

$$P = \frac{dE}{dt} = Fv = \tau\omega$$

Newton's Law on Gravity

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

For an object at a distance h above the Earth's, the gravitational acceleration is

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$

In general, the gravitational field experienced by mass m is

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

Kepler's Law

First Law. All planets move in elliptical orbits with the Sun at one focus.

Second Law The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

Third Law Simply put

$$T^2 = \frac{4\pi^2 a^3}{GM_S}$$

where a is semimajor axis and M_S is the mass of the sun.

Energy of Gravitational system

Potential energy. The gravitational potential energy associated with a system of two particles is

$$V = -\frac{Gm_1 m_2}{r}$$

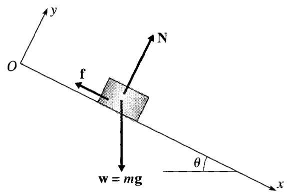
Total energy. The total energy of the system is the sum of the kinetic and potential energies

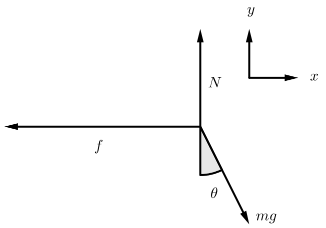
$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} = -\frac{GMm}{2r}$$

Incline Problem

Consider:

> A block of mass m is observed accelerating from rest down an incline that has coefficient of friction μ and is at angle θ from the horizontal. How far will it travel in time t?





Force approach. First we define the direction of displacement as positive x axis and the normal force as positive y axis. The resultant force in y axis written as

$$\sum F_y = N - mg\cos\theta = 0 \implies N = mg\cos\theta$$

and x axis

$$\sum F_x = mg\sin\theta - f = mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$

we then solve for x by

$$\begin{aligned} \ddot{x} &= g\left(\sin\theta - \mu\cos\theta\right) \\ \dot{x} &= g\left(\sin\theta - \mu\cos\theta\right)t \\ x(t) &= \frac{1}{2}g\left(\sin\theta - \mu\cos\theta\right)t^2 \end{aligned}$$

Since the block stared from rest, its constant of integration is zero.

Energy approach. Here define the zero potential energy at the bottom of the incline. Using the work energy theorem for non-conservative force

$$\begin{aligned} \Delta T + \Delta U &= W_{\text{fric}} \\ \frac{1}{2}mv^2 - mgh &= -fd \\ \frac{1}{2}mv^2 - mgd\sin\theta &= -\mu mg\cos\theta d \\ v &= \sqrt{2gd(\sin\theta - \mu\cos\theta)} \end{aligned}$$

Using the kinematics relation $v(t)^2 = v_i^2 + 2a\Delta x$

$$v^2 = 2gd(\sin\theta - \mu\cos\theta) = 2ad \implies a = g(\sin\theta - \mu\cos\theta)$$

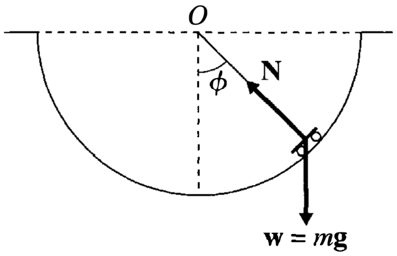
and the relation $v(t) = v_i + at$, we are able to rewrite it as the previous form

$$v = g(\sin\theta - \mu\cos\theta)t$$

Central Force Problem

Consider also:

> A "half-pipe" at a skateboard park consists of a concrete trough with a semicircular cross section of radius $R = 5$m. I hold a frictionless skateboard on the side of the trough pointing down toward the bottom and release it. Find the equation of motion for this system.



In this case r is held constant, thus the expression for resultant force in polar coordinate reads

$$\mathbf{F} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\,\hat{\boldsymbol{\phi}}$$

We also know that the acting force in this system are the normal and the skateboard weigh. Applying this force into equation above

$$(mg\cos\phi - N)\;\hat{\mathbf{r}} - mg\sin\phi\;\hat{\boldsymbol{\phi}} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\;\hat{\boldsymbol{\phi}}$$

We can't do anything with the radial component, we only use the angular component

$$mR\ddot{\phi} = mg\sin\phi$$
$$\ddot{\phi} = \frac{g}{R}\sin\phi$$

This differential equation is solved by

$$\phi(t) = A\sin\sqrt{\frac{g}{R}}t + B\cos\sqrt{\frac{g}{R}}t$$

Since this is released from rest, we have the initial condition of $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$. Apllying the first condition

$$\phi_0 = B$$

and the second

$$\dot{\phi}(t) = A\sqrt{\frac{g}{R}}\cos\sqrt{\frac{g}{R}}t - \phi_0\sqrt{\frac{g}{R}}\sin\sqrt{\frac{g}{R}}t$$
$$\dot{\phi}(0) = 0 = A\sqrt{\frac{g}{R}}$$

Hence the equation of motion reads

$$\phi(t) = \phi_0\cos\sqrt{\frac{g}{R}}t$$

Central Force

Newton's Second Law in Polar Coordinate

Acceleration in polar coordinate expressed as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}$$

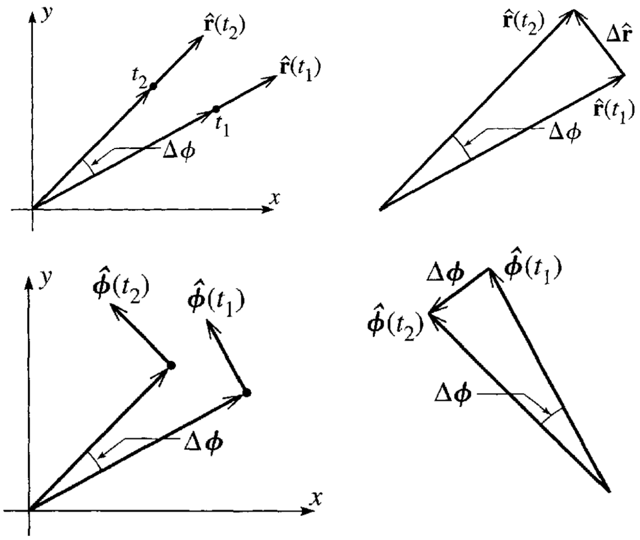
and velocity as

$$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

Hence Newton's law transform into

$$\mathbf{F} = m\mathbf{a} = \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Derivation



The value of $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\phi}}$.

From the figure, we have

$$d\hat{\mathbf{r}} = d\phi\ \hat{\boldsymbol{\phi}}, \quad d\hat{\boldsymbol{\phi}} = -d\phi\ \hat{\mathbf{r}}$$

or equivalently

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\ \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\ \hat{\mathbf{r}}$$

Using these we can now proceed to derive the Newton's law in polar coordinate. In cartesian coordinate, position vector can be writen as

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}}$$

converting it into polar

$$\mathbf{r} = r\ \hat{\mathbf{r}}$$

Next, we determine the velocity as

$$\dot{\mathbf{r}} = \frac{d}{dt} r\ \hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

and acceleration as

$$\begin{aligned}\ddot{r} = \frac{d}{dt}\left(\dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\frac{d}{dt}\left(\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) \\ = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\phi}\hat{\mathbf{r}}\right) \\ = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}\end{aligned}$$

Finally

$$F = F_r\ \hat{\mathbf{r}} + F_\phi\ \hat{\boldsymbol{\phi}} \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\end{cases}$$

Energy

Potential Energy

Gravitational. The potential energy is defined

$$U = mgh$$

where the reference point is chosen to be the ground. This makes it so that

$$F = -\nabla U = -\frac{d}{dr} mgr\,\hat{\mathbf{r}} = -mg\,\hat{\mathbf{r}}$$

the gravitational force is negative, or point downward. Now if we define the downward as positive displacement, the potential energy reads

$$U = -mgh$$

and the gravitational force

$$F = -\nabla U = -\frac{d}{dr}\left(-mgr\,\hat{\mathbf{r}}\right) = mg\,\hat{\mathbf{r}}$$

is positive, or point downward all the same.

Lagrangian Mechanics

The Lagrangian is defined as

$$\mathcal{L} = T - U$$

which mean that the Lagrangian is a function of position and velocity. The path of particle is determined by Hamilton's principle

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

that is, the particle's path is such that the action integral S is stationary.

We can express the coordinate in other generalized coordinate

$$\mathbf{r} = \mathbf{r}(q_1, q_2, q_2)$$

And the same for velocity

$$\mathbf{v} = \mathbf{v}(\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

Now the action integral reads

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)\, dt$$

Therefore we have three Euler-Lagrange equation that must be satisfied by the particle

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

This is the case of one unconstrained particle. For N unconstrained particle, then, we shall have $3N$ Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1, \ldots, 3N$$

The Lagrangian equation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

takes the form

$$\text{Generalized force} = \text{Rate of change of Generalized momentum}$$

where

$$\frac{\partial \mathcal{L}}{\partial q_i} = i\text{-th component of the Generalized force}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = i\text{-th component of the Generalized momentum}$$

Set of generalized coordinates that minimize the number of Euler-Lagrangian equation and be able to uniquely describe said system is said to be natural. This implies that natural coordinate also have minimum degree of freedom, that is the number of coordinate that can be varied independently. System with n degree of freedom that can be described by n generalized coordinate is called holonomic. In general

$$\text{DoF} = \text{No. of Coordinate} - \text{No. Constraint}$$

For example, double pendulum have 4 coordinates and two constraint, thus having two degree of freedom.

The steps to solve problem using Lagrangian formalism are as follows.

1. Write down the Lagrangian $\mathcal{L} = T - U$.
2. Choose generalized n coordinate q_n and $\dot{q}_n$.
3. Rewrite $\mathcal{L}$ in terms of q_n and $\dot{q}_n$.
4. Write n Lagrange equation.

Proof of Lagrange Equation with Constraint

Suppose a particle has two degree of freedom with two kinds of forces act on it: constraint force $\mathbf{F}_{\text{cstr}}$, say interatomic forces that bind rigid body atom together; and the is non constraint conservative forces $\mathbf{F}$, which at minimum must be able to be derived from potential energy $F = -\nabla U(\mathbf{r}, t)$, say gravitational force. The total energy is then

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{cstr}} + \mathbf{F}$$

and the Lagrangian

$$\mathcal{L} = T - U$$

where U is the potential energy which can be derived into non constraint conservative force.

The path of the particle can be denoted as

$$\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\epsilon}(t)$$

with $\mathbf{r}(t)$ as the correct path and $\boldsymbol{\epsilon}(t)$ as infinitesimal vector pointing away from the correct path. Then we have two Lagrangians

$$\mathcal{L} = \mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}, t), \qquad \mathcal{L}_0 = \mathcal{L}_0(\mathbf{r}, \dot{\mathbf{r}}, t)$$

and two action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt, \qquad S_0 = \int_{t_1}^{t_2} \mathcal{L}_0\, dt$$

It can be proven that the difference in action integral $\delta S = S - S_0$ is zero, to the first order.

We write the difference in Lagrangian as

$$\begin{aligned} \delta\mathcal{L} &= \frac{1}{2}m\dot{\mathbf{R}}^2 - U(\mathbf{R}, t) - \frac{1}{2}m\dot{\mathbf{r}}^2 + U(\mathbf{r}, t) \\ &= \frac{1}{2}m\left[(\dot{\mathbf{r}}^2 + \dot{\boldsymbol{\epsilon}}^2) - \dot{\mathbf{r}}^2\right] - \left[U(\mathbf{r} + \boldsymbol{\epsilon}, t) - U(\mathbf{r}, t)\right] \end{aligned}$$

$$= \frac{1}{2}m\left[\dot{\mathbf{r}}^2 + \dot{\boldsymbol{\epsilon}}^2 + 2\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}}^2 - \dot{\mathbf{r}}^2\right] - dU$$

$$\delta\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{\epsilon}}^2 + m\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U$$

The difference in action integral, in the first order, is then

$$S = \int_{t_1}^{t_2} \mathcal{L}_0\ dt = \int_{t_1}^{t_2} \left(m\dot{\mathbf{r}}\cdot\dot{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}\cdot\nabla U\right)\ dt$$

Using integration by parts

$$\int_a^b f\left(\frac{dg}{dx}\right)\ dx = -\int_a^b g\left(\frac{df}{dx}\right)\ dx + fg\Big|_a^b$$

on the first term

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[m\ddot{\mathbf{r}} + \nabla U\right]\ dt + m\dot{\mathbf{r}}\boldsymbol{\epsilon}\Big|_{t_1}^{t^2}$$

The difference of $\boldsymbol{\epsilon}$ is zero between two end point, so

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[m\ddot{\mathbf{r}} + \nabla U\right]\ dt$$

The path $\mathbf{r}(t)$ satisfies Newton second law, thus $m\ddot{\mathbf{r}} = \mathbf{F}_{\text{tot}}$. Meanwhile, the gradient of potential energy is the negative of non constraint force

$$\delta S = -\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\left[\mathbf{F}_{\text{tot}} - \mathbf{F}\right]\ dt = i\int_{t_1}^{t_2} \boldsymbol{\epsilon}\cdot\mathbf{F}_{\text{cstr}}\ dt$$

Note that the constraint force is normal to the particle path and the $\boldsymbol{\epsilon}$, which lies to the same surface of particle path. Therefore, their dot product is zero

$$\delta S = 0$$

and the action integral is stationary.

This justifies the Lagrange equation for system with two degree of freedom where its constraint lie in the same surface as the particle path. In other words, it only applies to particle, or particles in that case, constrained to move in two dimension. Accordingly, the action integral in this case is written as

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2, t)\ dt$$

which will result in two Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_1}, \qquad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2}$$

Newton Law in 2D Cartesian

The Lagrangian in this case is

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y)$$

Here we have two Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}, \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}$$

The derivative with respect to position is force

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x, \qquad \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial U}{\partial y} = F_y$$

while the derivative with respect to velocity is momentum

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial T}{\partial \dot{y}} = m\dot{y}$$

Substituting this result in the two Euler-Lagrange equation, we have

$$F_x = m\ddot{x}, \qquad F_y = m\ddot{y}$$

which are the two component of Newton second law $\mathbf{F} = m\ddot{\mathbf{r}}$.

Newton Law in Polar Coordinate

The Lagrangian is

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$$

which result into two Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Before moving into evaluating the derivative with respect to r and ϕ, recall the gradient of potential energy in polar coordinate

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} - F_r\hat{\mathbf{r}} - F_\phi\hat{\boldsymbol{\phi}}$$

Evaluating the radial derivative

$$\begin{aligned} mr\dot{\phi} - \frac{\partial U}{\partial r} &= m\ddot{r} \\ F_r &= m(\ddot{r} + r\dot{\phi}) \end{aligned}$$

which is simply the radial component of the Newton's second law in polar coordinate. Evaluating the angular derivative

$$\begin{aligned} -\frac{\partial U}{\partial \phi} &= \frac{d}{dt}mr^2\dot{\phi} = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi}) \\ -\frac{1}{r}\frac{\partial U}{\partial \phi} &= m(2\dot{r}\dot{\phi} + r^2\ddot{\phi}) \\ F_\phi &= m(2\dot{r}\dot{\phi} + r\ddot{\phi}) \end{aligned}$$

which is, as previously mentioned, the angular component of Newton's second law. It should be noted that the quantity $mr^2\dot{\phi}$ can be recognized as angular momentum L, and the rate of change of it is torque Γ

$$\Gamma = F_\phi r = \frac{dL}{dt} = \frac{d}{dt}mr^2\dot{\phi}$$

Application: Atwood's Machine

Atwood's machine consist of masses m_1 and m_2 are suspended by an in extensible string (length l) which passes over a massless pulley with frictionless bearings and radius R. The length of the string acts as constraint

$$x + y + 2\pi R = l$$

This implies $y = -x + C$ and $\dot{y} = -\dot{x}$. Thus, the kinetic energy of both mass

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

We define the downward displacement of m_1 and upward displacement of m_2 as positive displacement in our generalized coordinate x. This is the same case of upward acceleration of m_2 being the same as downward acceleration of m_1. In any case, the potential energy is

$$U = -m_1 gx - m_2 gy = -(m_1 - m_2)gx + C_2$$

We can now write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$$

With only one generalized coordinate, we only have one Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

which on substituting the Lagrangian yield

$$\begin{aligned}(m_1 - m_2)g &= (m_1 + m_2)\ddot{x} \\ \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2}g\end{aligned}$$

Now let's compare it with Newtonian approach, we should obtain the same result. Considering the acceleration direction for both masses, the net forces on both m_1 and m_2 respectively are

$$\begin{aligned}m_1 g - F_t &= m_1\ddot{x} \\ F_t - m_2 g &= m_2\ddot{x}\end{aligned}$$

Adding both equation

$$\begin{aligned}(m_1 - m_2)g &= (m_1 + m_2)\ddot{x} \\ \ddot{x} &= \frac{m_1 - m_2}{m_1 + m_2}g\end{aligned}$$

Application: Particle Constrained on a Cylinder

Consider a particle of mass m constrained to move on a frictionless cylinder of radius R. Besides the force of constraint, the only force on the mass is a force $\mathbf{F} = -k\mathbf{r}$ directed toward the origin. With $\mathbf{r}$ as the position vector of the particle, this force is the three dimension version of Hooke's law.

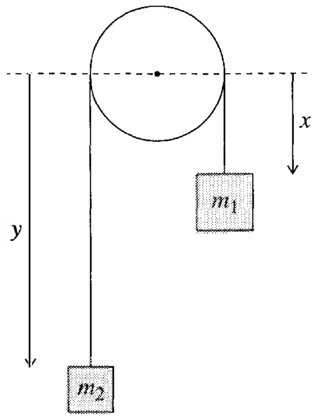


Figure: Atwood's machine configuration

We shall use cylindrical coordinate to solve this problem. It is known that the radius component is fixed $\rho = R$, so we use (ϕ, z) as our generalized coordinate. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{x}^2)$$

Recall $\mathbf{F} = -\nabla U$, hence the potential energy

$$-\frac{dU}{dr}\,\hat{\mathbf{r}} = -kr\,\hat{\mathbf{r}}$$
$$U = \frac{1}{2}kr^2$$

The distance of particle from origin is given by $r^2 = R^2 + z^2$, so

$$U = \frac{1}{2}k(z^2 + R^2)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2)$$

We use two generalized coordinates, thus we have two Euler-Lagrangian equation

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \sqcup]\S\sqcup}{\partial \dot{z}}, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

The z equation is

$$-kz = m\ddot{z} \implies z = A\cos(\omega t - \delta)$$

This mean that the mass perform simple harmonic motion in the z direction. Now, the ϕ equation

$$0 = \frac{d}{dt}mR^2\dot{\phi}$$

This mean that the angular momentum $L = mR^2\dot{\phi}$ is conserved and the particle rotate in constant velocity $\dot{\phi}$.

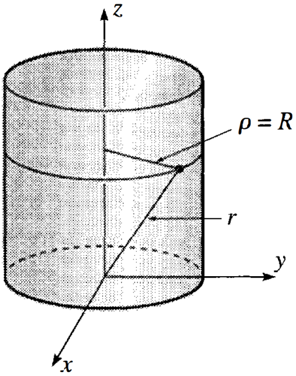


Figure: Particle constrained to move on a cylinder

Application: Sliding Block on a Frictionless Wedge

The block (mass m) is free to slide on the wedge, and the wedge (mass M) can slide on the horizontal table, both with negligible friction. The block is released from the top of the wedge, with both initially at rest.

The system has two coordinates and no constraint whatsoever, so it has two degree of freedom. We choose q_1 and q_2 as our generalized coordinate which denote the distance form the block from the top of the wedge and the distance of the wedge from convenient fixed point on the table. We also define positive x displacement to the right and downward as positive y displacement.

The kinetic energy is

$$T = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2$$

The wedge velocity is simply

$$v_M = \dot{q}_2$$

Meanwhile, the block velocity have two component, which are

$$\mathbf{v}_m = \left(\dot{q}_2 + \dot{q}_1 \cos\alpha\right)\ \hat{\mathbf{x}} + \dot{q}_2 \sin\alpha\ \hat{\mathbf{y}}$$

In terms of generalized coordinate, the kinetic energy reads

$$T = \frac{1}{2}m\left[\left(\dot{q}_2 + \dot{q}_1\cos\alpha\right)^2 + \dot{q}_2 \sin\alpha^2\right] + \frac{1}{2}m\dot{q}_2^2$$

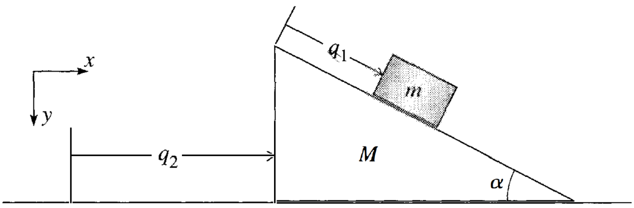


Figure: Block slides on a wedge which is free to move without friction

$$= \frac{1}{2}m\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_1\dot{q}_2\cos\alpha +\right] + \frac{1}{2}M\dot{q}_2^2$$

$$T = \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right)$$

In other hand, we defined downward as positive displacement, so the potential energy reads

$$U = -mgy = -mgq_1\sin\alpha$$

The Lagrangian can be evaluated as

$$\mathcal{L} = \frac{1}{2}(m+M)\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1\sin\alpha$$

With the generalized coordinates we used, we have two Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2} \qquad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_2}$$

The q_1 equation yield

$$mg\sin\alpha = \frac{d}{dt}\left[m(\dot{q}_1 + \dot{q}_2\cos\alpha)\right]$$

$$g\sin\alpha = \ddot{q}_1 + \ddot{q}_2\cos\alpha$$

while the q_2 equation yield

$$0 = \frac{d}{dt}\left[(M+m)\dot{q}_2 + m(\dot{q}_2 + \dot{q}_1\cos\alpha)\right]$$

$$= (M+m)\ddot{q}_2 + m(\ddot{q}_2 + \ddot{q}_1\cos\alpha)$$

$$\ddot{q}_2 = -\frac{m}{M+m}\ddot{q}_1\cos\alpha$$

which is just conservation of momentum in the x direction

$$m\dot{q}_2 + m\left(\dot{q}_2 + \dot{q}_1\cos\alpha\right) = \text{Constant}$$

Now, combining the q_1 and q_2 result in

$$\ddot{q}_1 = g\sin\alpha + \frac{m}{M+m}\ddot{q}_1\cos^2\alpha$$

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha}$$

Suppose we want to determine the time it took for the block to reach the bottom of wedge, we can use the kinematic relation $x(t) = x_i + v_i t + at^2/2$ or $t = \sqrt{2l/a}$, with l as the length of the slope, to obtain

$$t = \left(2l\frac{1 - \frac{m}{M+m}\cos^2\alpha}{g\sin\alpha}\right)^{1/2}$$

As a sanity check, consider the case for $\alpha = 90°$. The acceleration $\ddot{q}_1 = g$, which is correct. Another is the case for $M \to \infty$. The acceleration $\ddot{q}_1 = g\sin\alpha$, which is the acceleration for a block on a fixed incline.

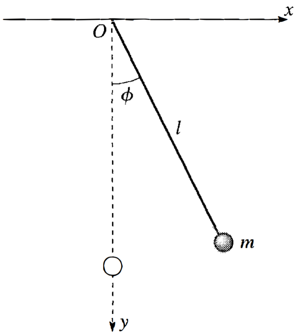


Figure: A simple pendulum

Application: Simple Pendulum

A bob of mass m is fixed to a massless rod length $l = \sqrt{x^2 + y^2}$, which is pivoted at O and free to swing without friction in the xy plane. One way to integrate the constraint into the Lagrangian is by expressing both coordinate in terms single generalized coordinate ϕ or by writing one of them in terms other variable, say $y = \sqrt{l^2 - x^2}$.

In terms of ϕ,the kinetic energy is

$$T = \frac{1}{2}ml^2\dot{\phi}^2$$

and the potential energy

$$U = mg(l - l\cos\phi) = mgl(1 - \cos\phi)$$

So, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi)$$

In this case, we only have one Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Substituting the Lagrangian

$$-mgl\sin\phi = ml^2\ddot{\phi}$$
$$\sin\phi = -\frac{l}{g}\ddot{\phi}$$

This is the differential equation describing simple pendulum motion, which, on assuming small angle ϕ, has the solution $\phi = A\cos(\omega t + \delta)$. Also, recall $\ddot{\phi}$ is the angular acceleration. This mean that the Euler-Lagrange equation reproduce the formula for torque $\Gamma = I\alpha = ml^2\ddot{\phi}$ or $\Gamma = Fr = -mgl\sin\phi$.

Application: Bead Spinning on a Wire Hoop

A bead of mass m is threaded on a frictionless circular wire hoop of radius R. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with constant angular velocity $\dot{\phi} = \omega$. The bead's position on the hoop is specified by the angle θ measured up from the rotation axis. We shall use θ as our only generalized coordinate.

The kinetic energy is

$$T = \frac{1}{2}m\left(v_\theta^2 + v_\phi^2\right)$$

with v_θ denote the bead tangential velocity with respect to non-rotating hoop, while v_ϕ denote the rotation velocity of the hoop. The tangential velocity is simply $v_\theta = R\dot{\theta}$,

$$v_\theta = R\dot{\theta}$$

with R as the distance of the bead with the axis of rotation–the hoop radius in other words. With the same principle, the hoop velocity is $v_\phi = \rho\dot{\phi}$. From the figure and known quantity, $\rho = R\cos\theta$ and $\dot{\phi} = \omega$, thus

$$v_\phi = R\sin\theta\omega$$

In terms of generalized coordinate, the kinetic energy reads

$$T = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)$$

The potential energy is

$$U = mg(R - R\cos\theta) = mgR(1 - \cos\theta)$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mgR(1 - \cos\theta)$$

which yield one Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}$$

Substituting the Lagrangian

$$mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta = mR^2\ddot{\theta}$$

dividing by mR^2

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$
$$\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$$

This equation can be used to determine the equilibrium point–that is, the point where the position of the system does not change–and the stable point–that is, the position at which the system returns after

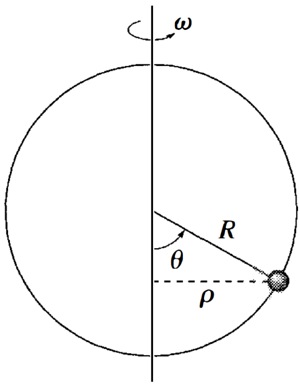


Figure: Bead constrained into moving within wire hoop

slightly disturbed–of the system. The requirement of equilibrium point is $\dot{\theta} = 0$, but we can obtain the same result by setting both $\dot{\theta}$ and $\ddot{\theta}$ to zero

$$\left(\omega^2 \cos\theta - \frac{g}{R}\right) \sin\theta = 0$$

If we set $\sin\theta$ to zero, we obtain the following equilibrium points

$$\theta = 0, \pi$$

If we set the term inside parenthesis to zero

$$\cos\theta = \frac{g}{\omega^2 R}$$

or, due to cosine being even function

$$\cos(-\theta) = \frac{g}{\omega^2 R}$$

hence we have the following equilibrium

$$\theta = \pm \arccos\frac{g}{\omega^2 R}$$

Arccos function only has real value at $\theta \in [-1, 1]$, so when

$$\left|\frac{g}{\omega^2 R}\right| > 1 \quad \text{or} \quad \omega^2 < \frac{g}{R}$$

This equation undefined and the stable point disappear. These equilibrium points only appear when $\omega^2 > g/R$ and located on either side of bottom $\theta = 0$.

Out of these four equilibrium points, not all of them are stable.

1. Top $\theta = \pi$ point is unstable point due to not having restorative force, both gravitational and centrifugal push the bead away.

2. Bottom $\theta = 0$ point depends on ω^2. It is stable if $\omega^2 < g/R$, but become unstable if $\omega^2 > g/R$. This can be proven by approximating small θ displacement

$$\ddot{\theta} = \left(\omega^2 - g/R\right)\theta$$

If $\omega^2 < g/R$, then

$$\ddot{\theta} = -\Omega^2 \theta$$

with

$$\Omega = \sqrt{\frac{g}{r} - \omega^2}$$

which mean the bead perform simple harmonic motion about the stable point. If $\omega^2 > g/R$, then

$$\ddot{\theta} = \Omega^2 \theta$$

which has the solution $\theta = Ae^{\Omega t} + Be^{-\Omega t}$, so it moves in exponential way and the point is unstable.

3&4 These two point that comes after speeding up the rotation such that $\omega^2 > g/R$ are stable. To proof this, we expand the equation

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$

around stable point θ_0

$$\theta \equiv \theta + \epsilon$$

using Taylor expansion

$$\cos(\theta_0 + \epsilon) \approx \cos(\theta_0) - \epsilon \sin\theta_0, \quad \sin(\theta_0 + \epsilon) \approx \sin\theta_0 + \epsilon\cos\theta_0$$

This result in

$$\ddot{\theta} = \left[\omega^2 \cos\theta_0 - \omega^2 \epsilon \sin\theta_0 - \frac{g}{R}\right][\sin\theta_0 + \epsilon\cos\theta_0]$$
$$\ddot{\theta} = -\omega^2 \epsilon \sin^2\theta_0 - \omega^2\epsilon^2 \sin\theta_0 \cos\theta_0$$

Since ϵ is small, we can ignore the second order

$$\ddot{\theta} = -\epsilon\omega^2 \sin^2\theta_0$$

Since $\ddot{\theta}$ is the same as $\ddot{\epsilon}$

$$\ddot{\epsilon} = -\omega^2 \sin^2\theta = -\Omega'^2\epsilon$$

with

$$\Omega = \omega\sin\theta = \sqrt{\omega - \omega\cos^2\theta_0} = \sqrt{\omega^2 - \left(\frac{g}{\omega R}\right)^2}$$

This mean ϵ oscillates about zero, and the bead itself oscillates about the equilibrium position θ_0 with frequency Ω'.

Quantum Mechanics

Braket notation

Ket. $|\psi\rangle$ represents quantum state. Written in matrix form as

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

Bra. $\langle\psi|$ is the Hermitian conjugate (complex conjugate transpose) of the ket $|\psi\rangle$

$$\langle\psi| = \begin{pmatrix} \psi_0 & \psi_1 & \dots & \psi_n \end{pmatrix}$$

Inner Product. Written

$$\langle\phi|\psi\rangle = \begin{cases} 0, & \text{if orthogonal} \\ 1, & \text{if orthonormal} \end{cases}$$

Operator

Position Operator. Represents the position of a particle.

$$\hat{x} = x$$

Momentum Operator.

$$\hat{p} = -i\hbar\nabla$$

Energy Operator.

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Its action on the energy eigenstates is given by:

$$\langle\psi|\hat{E}|\psi\rangle = E_n$$

Hamiltonian Operator.

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

The Hamiltonian can be written in terms of ladder operators as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$H|n\rangle = E_n|n\rangle$$

where the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Creation operator. Increases the system's energy, thus often said to be raising operator. Defined as

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x - ip)$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Annihilation operator. Decrease the system's energy, thus often said to be lowering operator. Defined as

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x + ip)$$

in matrix representation

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Commutator

Commutator measures how much two physical quantities fail to be simultaneously measurable or well-defined. It is defined as

$$[A, B] = AB - BA$$

If $[A, B] = 0$, then A and B commute and can be simultaneously measured with arbitrary precision. If not, their measurement outcomes interfere with each other.

Expectation value

Braket notation.

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

Matrix notation.

$$\langle \hat{A} \rangle = \psi^\dagger \hat{A} \psi$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x)\, dx$$

Normalization

Braket notation.

$$\langle \psi | \psi \rangle = 1$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x)\, dx = 1$$

Normalization Problem

Ex 1. Find the value of A such that the following wavefunction particle inside potential well is normalized.

$$\psi = \frac{1}{\sqrt{10a}} \sin\left(\frac{\pi x}{a}\right) + A\frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) + \frac{3}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right)$$

The wavefunction of said particle is written in the form

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{a}\right)$$

Hence we write the wavefunction as

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3$$

We then normalized the wavefunction by

$$\langle\psi|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \middle| \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \right\rangle$$

Since the wavefunction is orthonormal to itself and orthogonal to another, therefore

$$\langle\psi|\psi\rangle = \frac{1}{20}\langle\psi_1|\psi_1\rangle + A^2\langle\psi_2|\psi_2\rangle + \frac{9}{10}\langle\psi_3|\psi_3\rangle = A^2 + \frac{19}{20}$$

Thus

$$A = \sqrt{\frac{1}{20}}$$

Expectation Value Problem

Ex 1. From the first normalization problem, find the expectation value of the energy. We have the normalized wavefunction

$$\psi = \sqrt{\frac{1}{20}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{1}{20}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{9}{10}} \sin\left(\frac{3\pi x}{a}\right)$$

or simply

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + \sqrt{\frac{1}{20}}\psi_2 + \sqrt{\frac{9}{10}}\psi_2$$

All that left is to do the algebra

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle = &\left\langle \sqrt{\frac{1}{20}}\psi_1 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_1 \right\rangle + \left\langle \sqrt{\frac{1}{20}}\psi_2 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_2 \right\rangle \\ &+ \left\langle \sqrt{\frac{9}{10}}\psi_3 \middle| \hat{E} \middle| \sqrt{\frac{9}{10}}\psi_3 \right\rangle\end{aligned}$$

Factoring the constant

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle &= \frac{1}{20}\left\langle \psi_1 \middle| \hat{E} \middle| \psi_1 \right\rangle + \frac{1}{20}\left\langle \psi_2 \middle| \hat{E} \middle| \psi_2 \right\rangle + \frac{9}{10}\left\langle \psi_3 \middle| \hat{E} \middle| \psi_3 \right\rangle \\ &= \frac{1}{20}E_1 + \frac{1}{20}E_2 + \frac{9}{10}E_3 = \frac{1}{20}\frac{h^2}{8ma^2} + \frac{1}{20}\frac{4h^2}{8ma^2} + \frac{9}{10}\frac{9h^2}{8ma^2}\end{aligned}$$

Nuclear Physics

Introduction

Atomic Species

Characterized by the number of neutron N, number of proton Z, and mass number $A = N + Z$

$$(A, Z) = {}^A_Z X = {}^A_Z X_N$$

Nucleon

Defined as bound state of atomic nuclei. The two type are positively charged proton and neutral neutron. Nucleon constitutes three bound fermions called quark: up with charge $(2/3)$ and down with charge $(-1/3)$

$$\text{proton} = \text{uud}$$
$$\text{neutron} = \text{udd}$$

Both of them are fermion with mass

$$m_e = 939.56 \text{ MeV}/c^2$$
$$m_p = 938.27 \text{ MeV}/c^2$$
$$m_n - m_e = 1.29 \text{ MeV}/c^2$$

The magnetic moment projected by both are

$$\mu_p = 2.792847386\ \mu_N \quad \mu_n = -1.91304275\ \mu_N$$

where μ_N denote nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15245166\ 10^{-14} \text{ MeV/T}$$

Here are the difference in unit used to describe nucleus compared to atom

Properties	Atom	Nucleus
Radius	Angstrom (10^{-10} m)	Femto (10^{-15} m)
Energy	eV	MeV

Radii. In terms of their mass number A, their radius may be approximated as

$$R = r_0 A^{1/3} \quad \text{with} \quad r_0 = 1.2 \text{ fm}$$

This approximation comes from assuming the radius is proportional to the volume which is also assumed to be spherical. Then $\mathcal{V} = 4\pi R^3/3 \approx A$.

Binding energy. Defined as the difference of the sum of nuclei mass and the nuclear mass

$$B(A, Z) = N m_n c^2 + Z m_p c^2 - m(A, Z)c^2$$

Mass. Three unit most common are atomic mass unit (u), the kilogram (kg), and the electron-volt (eV). The atomic mass unit is defined as the mass of ^{12}C atom divided by 12

$$1\ \mathrm{u} = \frac{m(^{12}C)}{12}$$

electron volt is defined as the kinetic energy of an electron after being accelerated from rest through a potential difference of 1 V.

Nuclear Relative

Isotope. Same number of charge Z, but different number of neutron N. Isotope has identical chemical properties, since they have the same electron, but different nuclear properties. Example are

$$^{238}_{92}\mathrm{U} \quad \text{and} \quad ^{235}_{92}\mathrm{U}$$

Isobar. Same mass A. Frequently have the same nuclear properties due to the same number of nucleon. Example are

$$^{3}\mathrm{He} \quad \text{and} \quad ^{3}\mathrm{H}$$

Isotone. Same number of neutron N, but different number of proton Z. Example are

$$^{14}\mathrm{C}_6 \quad \text{and} \quad ^{16}\mathrm{O}_8$$

Radioactivity

Nuclear Decay

Alpha decay. Occur when parent nucleon decay distribute among daughter nuclei

$$ {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P} \to {}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D} + {}^{4}_{2}\mathrm{He} $$

The Q value, which is defined as the total released in a given nuclear decay, of alpha decay is

$$ Q = \left[m\left({}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P}\right) - m({}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D}) - m\left({}^{4}_{2}\mathrm{He}\right)\right]c^2 $$
$$ Q = K_D + K_\alpha = \frac{A}{A-4}K_\alpha $$

Q can be determined by applying the energy conservation into given nuclear reaction.

Beta decay. Two example are electron capture and neutrino capture

$$ e^- + p \leftrightarrow \nu_e + n $$

Other interaction can be found by moving particle to different side and changing them to their anti particle, such as

$$ \bar{\nu_e} + p \leftrightarrow e^+ + n $$

where the anti particle of electron e^- and electron neutrino ν_e are positron e^+ and electron antineutrino $\bar{\nu_e}$. Another example is beta negative decay

$$ ({}^{A}_{Z}P) \to {}_{\mathrm{Z+1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \bar{\nu_e} $$

which is the neutron decay inside isotope, and beta positive decay

$$ ({}^{A}_{Z}P) \to {}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \nu_e $$

which is the proton decay inside isotope. Electron capture inside isotope takes the form of

$$ ({}^{A}_{Z}P) + e^- \to {}_{\mathrm{Z-1}}{}^{\mathrm{A}} + \nu_e $$

The Q value of beta negative is

$$ Q = \left[m\left(({}^{A}_{Z}P)\right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\nu_e}\right] $$

and beta negative decay

$$ Q = \left[m\left(({}^{A}_{Z}P)\right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\bar{\nu_e}}\right] $$

Radiation

Radioactive decay. The number of decay events over a time inteval is proportional to the number of particle

$$-\frac{dN}{dt} = \lambda N$$

This first order ODE is solved by integration

$$-\int_{N_0}^{N} \frac{1}{N}\, dN = \int_0^t \lambda t\, dt$$
$$\ln \frac{N}{N_0} = -\lambda t$$
$$N(t) = N_0 e^{-\lambda t}$$

Now consider the case of chain radiation, that is the particle undergoes decay $N_A \to N_B \to N_C$. For the first decay of particles N_A into N_B

$$\frac{dN_A}{dt} = -\lambda_A N_A \implies N_A = N_{A0} e^{-\lambda_A t}$$

Now the total rate of creation for the second particle N_B is the sum of the decay of said particle and the decay of the first particle into the second particle.

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A$$
$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A0} e^{-\lambda_A t}$$

This ODE is solved by integrating factor method

$$I = \int \lambda_B\, dt = \lambda_B t$$

Then

$$N_B(t) = e^{-\lambda_B t} \int \lambda_A N_{A0} e^{-\lambda_A t}\, dt + C e^{-\lambda_B t}$$
$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} e^{-\lambda_A t} + C e^{-\lambda_B t}$$

If we assume at $t = 0$ we have zero second particle, then

$$N_B(0) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} + C = 0 \implies C = -\frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1}$$

Thus, the complete solution is

$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} \left(e^{\lambda_A t} - e^{\lambda_B t}\right)$$

In practice, often we are not given the number of particle, but rather the mol n or mass m (in gram) of said particle. Those quantities are related by

$$n = \frac{m}{M_r} \quad \text{or} \quad n = \frac{N}{N_A}$$

where $M_r \approx A$ is the molecular mass (gram/mol) and $N_A = 6.02 \cdot 10^{23}$ (mol^{-1}) is the Avogadro constant.

Halflife. Defined as the time required for particles to reduce to half of its initial value. The value of halflife can be determined by considering $N(t)$ at $t_{1/2}$, which by definition

$$\frac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}$$
$$t_{1/2} = \frac{\ln 2}{\lambda}$$

This equation can also be used to determine the decay constant.

Activity. Defined as the number of radioactive transformations per second

$$A \equiv -\frac{dN}{dt} = \lambda N$$

Using the solution for N, we can write

$$A = \lambda N_0 e^{-\lambda t} = A_0 e^{-\lambda t}$$

Specific Activity. Quantity related to activity; specific activity is the activity per unit mass

$$a \equiv \frac{A}{m}$$

Using the relation of mass with mol and halflife relation

$$a = \frac{\lambda N}{\frac{N}{N_A} M_r} = \frac{N_A \lambda}{M_r} = \frac{\ln 2 N_A}{t_{1/2} M_r}$$

On evaluating the numerator constant

$$a = \frac{1.32 \cdot 10^{16}}{t_{1/2} M}$$

Nuclear Stability

Valley of stability. Consist of long-lived isotope that do not simultaneously decay.

Below the valley of stability. Consist of isotopes with more N than those of the valley of stability, thus they decay either by beta negative, more likely, or neutron decay, less likely.

Above the valley of stability. Consist of isotope with more Z than those of the valley of stability, thus it decays either by beta positive or electron capture.

Beyond the valley of stability. Consist of heavy isotope with $Z > 83$, $N > 126$, and $A > 209$. They decay with alpha radiation.

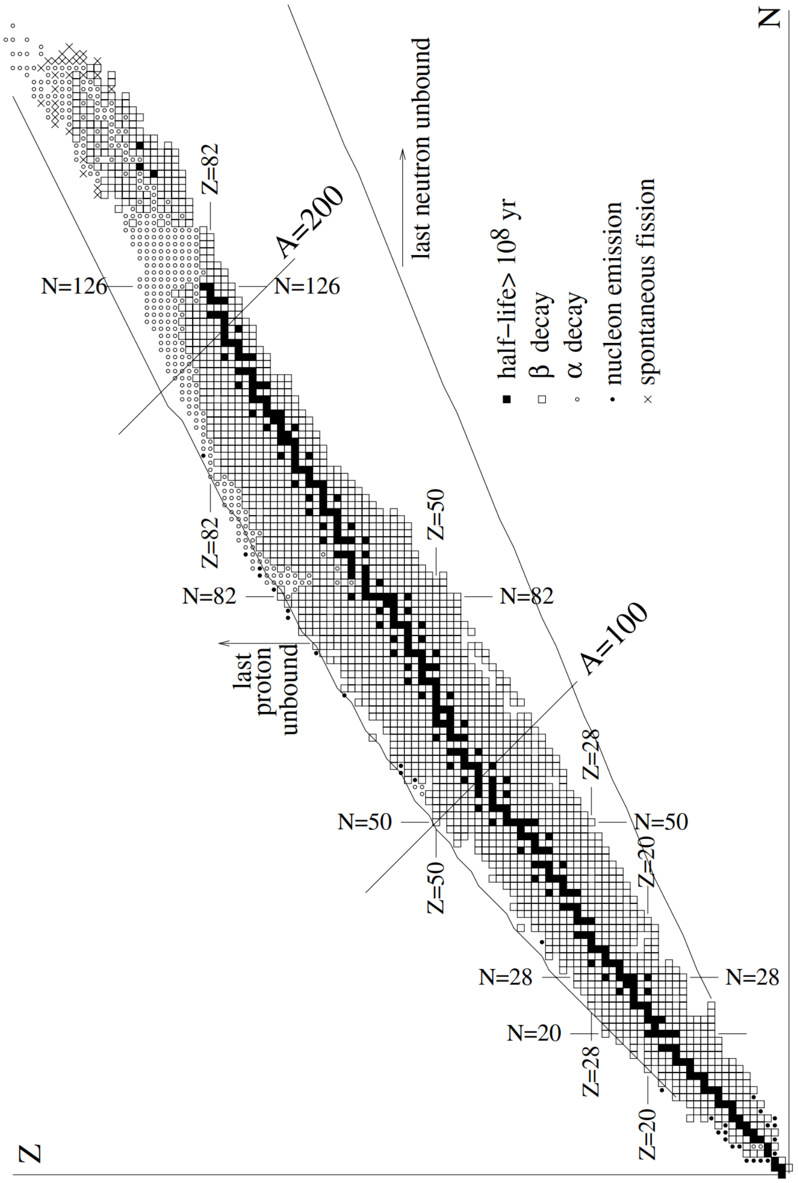


Figure: Valley of stability in ZN graph.

Particle Physics

Elementary Particle

Particle can be categorized by their spin, mass, and the type of interaction.

Boson. A pseudo-particle with interger spin that mediate the interaction. Two types of boson are gauge, or force carrier, and scalar, whose function is to give particle mass. Also, $W^{\pm}$ and Z^0 boson both belong to the same isospin triplet with $I = 1$

Table: Boson properties

Gauge Boson	
Family	Function/Force
Photon γ	Electromagnetic
Gluon g	Strong: binds the quark
W Boson $W^{\pm}$	Weak:radioactive decay
Z Boson Z^0	Weak: same as above
Scalar Boson	
Family	Function/Force
Higgs Boson H^0	Gives particle mass

Fermion. Consist of quark and lepton. All has half interger spin. Bound state of quark is called hadron and quark is able to experience all four fundamental forces.

Lepton experiences the weak force, however for charged lepton is also able to experience electromagnetic forces. This is why neutral lepton, that is neutrino, is hard to detect.

Hadron. Two types of hadron are baryon, which consist of three quarks, and meson, which consist of two quarks and one antiquark. Since baryon is made up of three spin-1/2, its spin is always half interger, opposite with meson; thus, both are also able to experience all four fundamental forces. Anti baryon consist of antiquark with the same configuration, for example is anti proton $\bar{p} = \bar{u}\bar{u}\bar{d}$.

One difference is that, unlike baryon, meson do not follow Pauli exclusion principle, since its total spin is interger. Most mesons are not their own antiparticle: charged mesons like K^+ with K^-; and neutral meson like K^0 with $\bar{K^0}$.

Quantum Number. Summarized as follows. First we have the fundamental particles.

Table: fundamental particle properties

Boson

Name	q	L	B	S	I	I_z
γ	0	0	0	0	0	0
g	0	0	0	0	0	0
W^+	+1	0	0	0	1	+1
Z^0	0	0	0	0	1	0
W^-	-1	0	0	0	1	-1
H^0	0	0	0	0	0	0

Quark

Name	q	L	B	S	I	I_z
u	+2/3	0	+1/3	0	1/2	+1/2
d	-1/3	0	+1/3	0	1/2	-1/2
s	-1/3	0	+1/3	-1	0	0
c	+2/3	0	+1/3	0	0	0
b	-1/3	0	+1/3	0	0	0
t	+2/3	0	+1/3	0	0	0

Lepton

Name	q	L	B	S	I	I_z
e	-1	$L_e = 1$	0	0	1/2	+1/2
μ	-1	$L_\mu = 1$	0	0	1/2	-1/2
τ	-1	$L_\tau = 1$	0	0	0	0
ν_e	0	$L_e = 1$	0	0	0	0
ν_μ	0	$L_\mu = 1$	0	0	0	0
ν_τ	0	$L_\tau = 1$	0	0	0	0

Then the Hadron.

Table: Hadron properties

Baryon

Name	Content	q	L	B	S	I	I_z
p	uud	+1	0	+1	0	1/2	+1/2
n	udd	0	0	+1	0	1/2	-1/2
Λ^0	uds	0	0	+1	-1	I	0
Σ^+	uus	+1	0	+1	-1	1	+1
Σ^0	uds	0	0	+1	-1	1	0
Σ^-	dds	-1	0	+1	-1	1	-1
Ξ^0	uss	0	0	+1	-2	1/2	+1/2
Ξ^-	dss	-1	0	+1	-2	1/2	-1/2
Ω^-	sss	-1	0	+1	-3	0	0

Meson

Name	Content	q	L	B	S	I	I_z
π^+	$u\bar{d}$	+1	0	0		1	+1
π^0	$u\bar{u}, d\bar{d}$	0	0	0	0	1	0
π^-	$d\bar{u}$	-1	0	0	0	1	-1
K^+	$d\bar{s}$	+1	0	0	+1	1/2	+1/2
K^0	$d\bar{s}$	0	0	0	+1	1/2	-1/2

$\bar{K^0}$	$s\bar{d}$	0	0	0	-1	1/2	+1/2
K^-	$s\bar{u}$	-1	0	0	-1	1/2	-1/2
η	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0
η'	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0

Conservation Laws

Energy. The total energy $E = K + mc^2$ must be conserved in all types of nuclear reaction.

Momentum. Same as energy conservation law.

Mass number. Same as energy conservation law.

Charge. Same as energy conservation law.

Lepton number. We assign lepton a lepton number $L = +1$, anti lepton $L = -1$, and non lepton $L = 0$. Each family of lepton–such as electron e, muon μ, tau τ and their neutrino sibling–has separate conserved neutrino number–L_e, L_μ, L_τ repectively.

This quantum number almost conserved in all reaction, exception exist in neutrino oscillations; however only family lepton number is violated, while the total lepton number is conserved.

Baryon number. Like before, we assign baryon a baryon number $L = +1$, anti baryon $L = -1$, and non baryon $L = 0$. As an aside, baryon is the defined as the bound state of three quarke and that strong force works on all of them.

Strangeness number. Defined as the negative of the number of strange quarks in it, in particular strange quark s has $S = -1$ and antistrange quark $\bar{s}$ has $S = +1$.

Isospin (Isotropic Spin). We define isospin I such that $2I + 1$ is equal to the multiplet type of the baryon. Recall that the strong force does not differentiate between proton and nucleon, hence both particle can be defined as the different state of the same particle and thus can be categorized as doublet. To differentiate them, then, we define proton with $I_z = 1/2$ and neutron with $I_z = -1/2$. In general,

$$I_z = I, I - 1, \ldots, -I$$

Isospin is conserved in strong force interaction, but may not in other interaction.

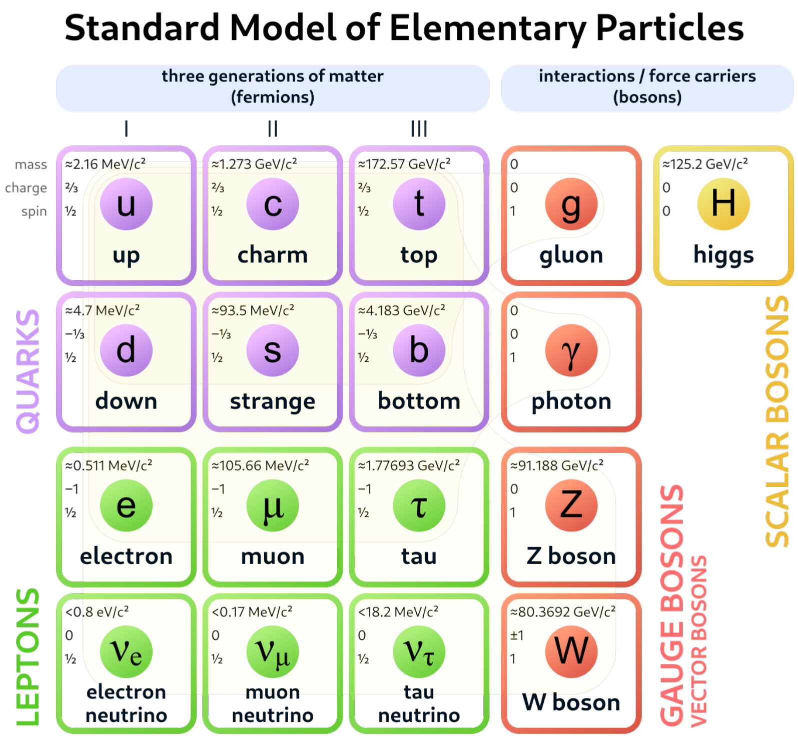


Figure: Standard model.

Nuclear Model

Liquid Drop Model

The binding energy by this model is given by

$$B(A, Z) = a_V A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta(A)$$

where

$$\begin{aligned} a_V &= 15.753 \\ a_s &= 17.804 \\ a_C &= 0.7103 \\ a_a &= 23.69 \\ \delta(A) &= \begin{cases} 33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are even} \\ -33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are odd} \\ 0 & \text{if } A = N + Z \text{ is odd} \end{cases} \end{aligned}$$

Volume term. Recall that the volume of nucleon is proportional to A; on using this we have obtained $r = r_0 A^{1/3}$, which means that nuclei have constant density, like a drop of water.

Experiment shows that the binding energy per nucleon is rougly constant, $B/A \approx 8$ MeV. The overshoot of the volem term, then, require corection that will lower the value of the binding energy.

Surface term. Like water molecule, internal nucleon experience isotropic force, while surface nucleon only from inside. This resulting the force, and consequentl the energy, to be proportional to area $4\pi r^2 \approx a_s A^{1/3}$, with $r = r_0 A^{1/3}$.

Columb term. The binding energy due to charged particle is proportional to $Q^2/R \approx Z^2/A^{1/3}$.

Asymmetry term. By The Pauli exclusion principle, the configuration with different N and Z will have more energy than that with the same due to the different nucleon will simply occupy the higher energy state. This is the basis of $N - Z$ term.

Quantum pairing term. This term captures the effect of spin coupling. Odd-odd nuclei tend to undergo beta decay to an adjacent even-even nucleus by changing an N to a Z or vice versa.

Stable nuclei. By setting the derivative of B with respect to Z to zero, we have the maximum binding energy, that is the most state nuclei. If we ignore the quantum pairing term, or we are considering the case of odd A,we have

$$Z(A) = \frac{A}{2 + a_c A^{2/3}/2a_a} \approx \frac{A/2}{1 + 0.0075\ A^{2/3}}$$