Normalization Problem

Ex 1. Find the value of A such that the following wavefunction particle inside potential well is normalized.

$$\psi = \frac{1}{\sqrt{10a}} \sin\left(\frac{\pi x}{a}\right) + A\frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) + \frac{3}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right)$$

The wavefunction of said particle is written in the form

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{a}\right)$$

Hence we write the wavefunction as

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3$$

We then normalized the wavefunction by

$$\langle\psi|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \middle| \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \right\rangle$$

Since the wavefunction is orthonormal to itself and orthogonal to another, therefore

$$\langle\psi|\psi\rangle = \frac{1}{20}\langle\psi_1|\psi_1\rangle + A^2\langle\psi_2|\psi_2\rangle + \frac{9}{10}\langle\psi_3|\psi_3\rangle = A^2 + \frac{19}{20}$$

Thus

$$A = \sqrt{\frac{1}{20}}$$

Expectation Value Problem

Ex 1. From the first normalization problem, find the expectation value of the energy. We have the normalized wavefunction

$$\psi = \sqrt{\frac{1}{20}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{1}{20}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{9}{10}} \sin\left(\frac{3\pi x}{a}\right)$$

or simply

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + \sqrt{\frac{1}{20}}\psi_2 + \sqrt{\frac{9}{10}}\psi_2$$

All that left is to do the algebra

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_1 \right\rangle + \left\langle \sqrt{\frac{1}{20}}\psi_2 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_2 \right\rangle \\ + \left\langle \sqrt{\frac{9}{10}}\psi_3 \middle| \hat{E} \middle| \sqrt{\frac{9}{10}}\psi_3 \right\rangle\end{aligned}$$

Factoring the constant

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle &= \frac{1}{20}\left\langle \psi_1 \middle| \hat{E} \middle| \psi_1 \right\rangle + \frac{1}{20}\left\langle \psi_2 \middle| \hat{E} \middle| \psi_2 \right\rangle + \frac{9}{10}\left\langle \psi_3 \middle| \hat{E} \middle| \psi_3 \right\rangle \\ &= \frac{1}{20}E_1 + \frac{1}{20}E_2 + \frac{9}{10}E_3 = \frac{1}{20}\frac{h^2}{8ma^2} + \frac{1}{20}\frac{4h^2}{8ma^2} + \frac{9}{10}\frac{9h^2}{8ma^2}\end{aligned}$$