Classical Mechanics

Unasorted Classical Mechanics Topics

Newton's Law

First law. in the absence of an external force, when viewed from an inertial frame, an object at rest remains at rest and an object in uniform motion in a straight line maintains that motion.

Second law. Simply put

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Third law. States that if two objects interact, the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1.

Particle Under Constant Acceleration

Here's some kinematics equation for position

$$x(t) = x_i + \frac{1}{2}(v_i + v_f)t$$
$$x(t) = x_i + v_i t + \frac{1}{2}at^2$$

and for velocity

$$v(t) = v_i + at$$
$$v(t)^2 = v_i^2 + 2a(x_f - x_i)$$

Particle in Uniform Circular Motion

If a particle moves in a circular path of radius r with a constant speed v, the magnitude of its centripetal acceleration is given by

$$a_r = \frac{v^2}{r}$$

while its period and angular velocity is

$$T = \frac{2\pi r}{v}, \quad \omega = \frac{2\pi}{T}$$

Applying Newton's second law

$$\sum F = ma_r = m\frac{v^2}{r}$$

Rigid Object Under Constant Angular Acceleration

Analogous to those for translational motion of a particle under constant acceleration

$$\begin{aligned}\omega(t) &= \omega_i + \alpha t \\ \omega(t)^2 &= \omega_i^2 + 2\alpha(\theta_t - \theta_i) \\ \theta(t) &= \theta_i + \omega t + \frac{1}{2}\alpha t^2 \\ \theta(t) &= \theta_i + \frac{1}{2}(\omega_i + \omega_f)t\end{aligned}$$

Relation of Linear and Rotational Motion

The following equations show the relation of linear and rotational motion

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha$$

Torque

The torque associated with a force **F** acting on an object

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{d\mathbf{L}}{dt}$$

Moment of Inertia

The moment of inertia of a rigid object is

$$I = \sum mr^2 = \int r^2 dm$$

Parallel Axis Theorem. To calculate the moment inertia from any axis, we use parallel axis theorem

$$I = I_{\mathrm{CM}} + Md^2$$

Terminal velocity

$\boldsymbol{r} \propto \boldsymbol{v}$. The velocity as a function of time is

$$v = \frac{mg}{b}\left[1 - \exp\left(-\frac{bt}{m}\right)\right] = v_T\left[1 - \exp\left(-\frac{bt}{m}\right)\right]$$

where b is a resistive constant whose value depends on the properties of the medium.

***$r \propto v^2$*.** Given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}$$

where D is a dimensionless empirical quantity called the drag coefficient, ρ is the density of air, and A is the cross-sectional area of the moving object.

Escape velocity. the speed required bu an object to escape from any planet orbit is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Work Energy Theorem

It states that if work is done on a system by external forces and the only change in the system is in its speed,

$$W = \Delta T$$

Kinetic Energy

For an object in linear motion, the kinetic energy of said object is

$$T = \frac{1}{2}mv^2$$

whereas for rotational motion

$$T = \frac{1}{2}I\omega^2$$

Hence the total kinetic energy of a rigid object rolling on a rough surface without slipping

$$T = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega_{\text{CM}}^2$$

Potential Energy Function

For conservative energy $\mathbf{F}$, applies

$$V_f - V_i = -\int_{\mathbf{r_i}}^{\mathbf{r_f}} \mathbf{F} \cdot d\mathbf{r}$$

For particle-Earth system, the gravitational potential energy is

$$V = mgy$$

and elastic potential stored in spring

$$V = \frac{1}{2}kx^2$$

Effective potential

Effective potential energy $U_{\text{eff}}(r)$ is the sum of the actual potential energy $U(r)$ and the centrifugal $U_{\text{cf}}(r)$:

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

where l is the angular momentum and μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Momentum Impulse

The linear momentum and impulse are defined as

$$\mathbf{p} = m\mathbf{v}, \quad \mathbf{I} = \int_{t_i}^{t_f} \sum \mathbf{F}\, dt$$

Angular Momentum The angular momentum about an axis through the origin of a particle having linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

The z component of angular momentum of a rigid object rotating about a fixed z axis is

$$L_z = I\omega$$

Center of Mass and Velocity

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{r} = \frac{1}{M} \int \mathbf{r}\, dm$$

where M is the total mass. The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{v} =$$

Collision

Inelastic collision. One for which the total kinetic energy of the system of colliding particles is not conserved.

Elastic collision. One in which the kinetic energy of the system is conserved.

Perfectly inelastic. A collision which the colliding particles stick together after the collision.

Rocket Propulsion The expression for rocket propulsion is

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$

Power

The rate at which work is done by an external force, called power, is

$$P = \frac{dE}{dt} = Fv = \tau\omega$$

Newton's Law on Gravity

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

For an object at a distance h above the Earth's, the gravitational acceleration is

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$

In general, the gravitational field experienced by mass m is

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

Kepler's Law

First Law. All planets move in elliptical orbits with the Sun at one focus.

Second Law The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

Third Law Simply put

$$T^2 = \frac{4\pi^2 a^3}{GM_S}$$

where a is semimajor axis and M_S is the mass of the sun.

Energy of Gravitational system

Potential energy. The gravitational potential energy associated with a system of two particles is

$$V = -\frac{Gm_1 m_2}{r}$$

Total energy. The total energy of the system is the sum of the kinetic and potential energies

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} = -\frac{GMm}{2r}$$

Central Force

Newton's Second Law in Polar Coordinate

Acceleration in polar coordinate expressed as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}$$

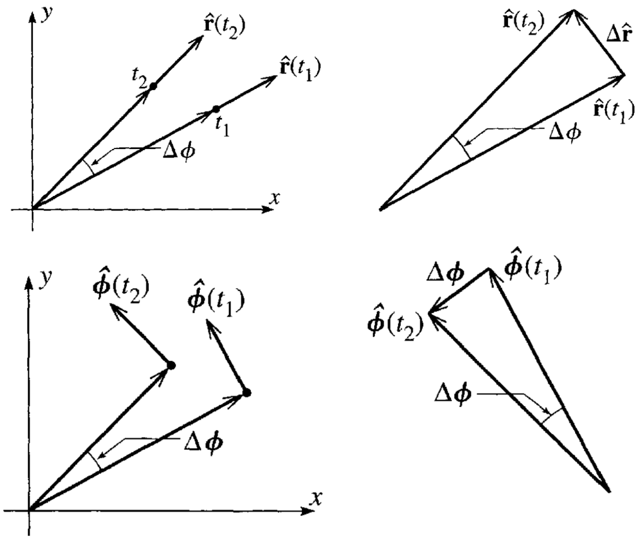
and velocity as

$$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

Hence Newton's law transform into

$$\mathbf{F} = m\mathbf{a} = \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Derivation



The value of $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\phi}}$.

From the figure, we have

$$d\hat{\mathbf{r}} = d\phi\ \hat{\boldsymbol{\phi}}, \quad d\hat{\boldsymbol{\phi}} = -d\phi\ \hat{\mathbf{r}}$$

or equivalently

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\ \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\ \hat{\mathbf{r}}$$

Using these we can now proceed to derive the Newton's law in polar coordinate. In cartesian coordinate, position vector can be writen as

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}}$$

converting it into polar

$$\mathbf{r} = r\ \hat{\mathbf{r}}$$

Next, we determine the velocity as

$$\dot{\mathbf{r}} = \frac{d}{dt} r\ \hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

and acceleration as

$$\begin{aligned}\ddot{r} = \frac{d}{dt}\left(\dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\frac{d}{dt}\left(\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) \\ = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\phi}\hat{\mathbf{r}}\right) \\ = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}\end{aligned}$$

Finally

$$F = F_r\ \hat{\mathbf{r}} + F_\phi\ \hat{\boldsymbol{\phi}} \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Quantum Mechanics

Braket notation

Ket. $|\psi\rangle$ represents quantum state. Written in matrix form as

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

Bra. $\langle\psi|$ is the Hermitian conjugate (complex conjugate transpose) of the ket $|\psi\rangle$

$$\langle\psi| = \begin{pmatrix} \psi_0 & \psi_1 & \dots & \psi_n \end{pmatrix}$$

Inner Product. Written

$$\langle\phi|\psi\rangle = \begin{cases} 0, & \text{if orthogonal} \\ 1, & \text{if orthonormal} \end{cases}$$

Operator

Position Operator. Represents the position of a particle.

$$\hat{x} = x$$

Momentum Operator.

$$\hat{p} = -i\hbar\nabla$$

Energy Operator.

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Its action on the energy eigenstates is given by:

$$\langle\psi|\hat{E}|\psi\rangle = E_n$$

Hamiltonian Operator.

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

The Hamiltonian can be written in terms of ladder operators as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$H\,|n\rangle = E_n\,|n\rangle$$

where the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Creation operator. Increases the system's energy, thus often said to be raising operator. Defined as

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x - ip)$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Annihilation operator. Decrease the system's energy, thus often said to be lowering operator. Defined as

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x + ip)$$

in matrix representation

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Commutator

Commutator measures how much two physical quantities fail to be simultaneously measurable or well-defined. It is defined as

$$[A, B] = AB - BA$$

If $[A, B] = 0$, then A and B commute and can be simultaneously measured with arbitrary precision. If not, their measurement outcomes interfere with each other.

Expectation value

Braket notation.

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

Matrix notation.

$$\langle \hat{A} \rangle = \psi^\dagger \hat{A} \psi$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \, dx$$

Normalization

Braket notation.

$$\langle \psi | \psi \rangle = 1$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \, dx = 1$$

Normalization Problem

Ex 1. Find the value of A such that the following wavefunction particle inside potential well is normalized.

$$\psi = \frac{1}{\sqrt{10a}} \sin\left(\frac{\pi x}{a}\right) + A\frac{2}{a} \sin\left(\frac{2\pi x}{a}\right) + \frac{3}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right)$$

The wavefunction of said particle is written in the form

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{a}\right)$$

Hence we write the wavefunction as

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3$$

We then normalized the wavefunction by

$$\langle\psi|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \middle| \sqrt{\frac{1}{20}}\psi_1 + A\psi_2 + \sqrt{\frac{9}{10}}\psi_3 \right\rangle$$

Since the wavefunction is orthonormal to itself and orthogonal to another, therefore

$$\langle\psi|\psi\rangle = \frac{1}{20}\langle\psi_1|\psi_1\rangle + A^2\langle\psi_2|\psi_2\rangle + \frac{9}{10}\langle\psi_3|\psi_3\rangle = A^2 + \frac{19}{20}$$

Thus

$$A = \sqrt{\frac{1}{20}}$$

Expectation Value Problem

Ex 1. From the first normalization problem, find the expectation value of the energy. We have the normalized wavefunction

$$\psi = \sqrt{\frac{1}{20}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{1}{20}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{9}{10}} \sin\left(\frac{3\pi x}{a}\right)$$

or simply

$$\psi = \sqrt{\frac{1}{20}}\psi_1 + \sqrt{\frac{1}{20}}\psi_2 + \sqrt{\frac{9}{10}}\psi_2$$

All that left is to do the algebra

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle = \left\langle \sqrt{\frac{1}{20}}\psi_1 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_1 \right\rangle + \left\langle \sqrt{\frac{1}{20}}\psi_2 \middle| \hat{E} \middle| \sqrt{\frac{1}{20}}\psi_2 \right\rangle \\ + \left\langle \sqrt{\frac{9}{10}}\psi_3 \middle| \hat{E} \middle| \sqrt{\frac{9}{10}}\psi_3 \right\rangle\end{aligned}$$

Factoring the constant

$$\begin{aligned}\langle\psi|\hat{E}|\psi\rangle &= \frac{1}{20}\left\langle \psi_1 \middle| \hat{E} \middle| \psi_1 \right\rangle + \frac{1}{20}\left\langle \psi_2 \middle| \hat{E} \middle| \psi_2 \right\rangle + \frac{9}{10}\left\langle \psi_3 \middle| \hat{E} \middle| \psi_3 \right\rangle \\ &= \frac{1}{20}E_1 + \frac{1}{20}E_2 + \frac{9}{10}E_3 = \frac{1}{20}\frac{h^2}{8ma^2} + \frac{1}{20}\frac{4h^2}{8ma^2} + \frac{9}{10}\frac{9h^2}{8ma^2}\end{aligned}$$

Nuclear Physics

Introduction

Atomic Species

Characterized by the number of neutron N, number of proton Z, and mass number $A = N + Z$

$$(A, Z) = {}^{A}_{Z}X = {}^{A}_{Z}X_N$$

Nucleon

Defined as bound state of atomic nuclei. The two type are positively charged proton and neutral neutron. Nucleon constitutes three bound fermions called quark: up with charge (2/3) and down with charge (−1/3)

$$\text{proton} = \text{uud}$$
$$\text{neutron} = \text{udd}$$

Both of them are fermion with mass

$$m_e = 939.56 \text{ MeV}/c^2$$
$$m_p = 938.27 \text{ MeV}/c^2$$
$$m_n - m_e = 1.29 \text{ MeV}/c^2$$

The magnetic moment projected by both are

$$\mu_p = 2.792847386\ \mu_N \quad \mu_n = -1.91304275\ \mu_N$$

where μ_N denote nuclear magneton

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15245166\ 10^{-14} \text{ MeV/T}$$

Here are the difference in unit used to describe nucleus compared to atom

Properties	Atom	Nucleus
Radius	Angstrom (10^{-10} m)	Femto (10^{-15} m)
Energy	eV	MeV

Radii. In terms of their mass number A, their radius may be approximated as

$$R = r_0 A^{1/3} \quad \text{with} \quad r_0 = 1.2 \text{ fm}$$

This approximation comes from assuming the radius is proportional to the volume which is also assumed to be spherical. Then $\mathcal{V} = 4\pi R^3/3 \approx A$.

Binding energy. Defined as the difference of the sum of nuclei mass and the nuclear mass

$$B(A, Z) = N m_n c^2 + Z m_p c^2 - m(A, Z)c^2$$

Mass. Three unit most common are atomic mass unit (u), the kilogram (kg), and the electron-volt (eV). The atomic mass unit is defined as the mass of ^{12}C atom divided by 12

$$1\ \text{u} = \frac{m(^{12}C)}{12}$$

electron volt is defined as the kinetic energy of an electron after being accelerated from rest through a potential difference of 1 V.

Nuclear Relative

Isotope. Same number of charge Z, but different number of neutron N. Isotope has identical chemical properties, since they have the same electron, but different nuclear properties. Example are

$$^{238}_{92}\text{U} \quad \text{and} \quad ^{235}_{92}\text{U}$$

Isobar. Same mass A. Frequently have the same nuclear properties due to the same number of nucleon. Example are

$$^{3}\text{He} \quad \text{and} \quad ^{3}\text{H}$$

Isotone. Same number of neutron N, but different number of proton Z. Example are

$$^{14}\text{C}_6 \quad \text{and} \quad ^{16}\text{O}_8$$

Radioactivity

Nuclear Decay

Alpha decay. Occur when parent nucleon decay distribute among daughter nuclei

$$ {}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P} \to {}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D} + {}^{4}_{2}\mathrm{He} $$

The Q value, which is defined as the total released in a given nuclear decay, of alpha decay is

$$ Q = \left[m\left({}^{\mathrm{A}}_{\mathrm{Z}}\mathrm{P}\right) - m({}^{\mathrm{A-4}}_{\mathrm{Z-2}}\mathrm{D}) - m\left({}^{4}_{2}\mathrm{He}\right) \right] c^2 $$

$$ Q = K_D + K_\alpha = \frac{A}{A-4} K_\alpha $$

Q can be determined by applying the energy conservation into given nuclear reaction.

Beta decay. Two example are electron capture and neutrino capture

$$ e^- + p \leftrightarrow \nu_e + n $$

Other interaction can be found by moving particle to different side and changing them to their anti particle, such as

$$ \bar{\nu_e} + p \leftrightarrow e^+ + n $$

where the anti particle of electron e^- and electron neutrino ν_e are positron e^+ and electron antineutrino $\bar{\nu_e}$. Another example is beta negative decay

$$ ({}^{A}_{Z}P) \to {}_{\mathrm{Z+1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \bar{\nu_e} $$

which is the neutron decay inside isotope, and beta positive decay

$$ ({}^{A}_{Z}P) \to {}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D} + e^- + \nu_e $$

which is the proton decay inside isotope. Electron capture inside isotope takes the form of

$$ ({}^{A}_{Z}P) + e^- \to {}_{\mathrm{Z-1}}{}^{\mathrm{A}} + \nu_e $$

The Q value of beta negative is

$$ Q = \left[m\left(({}^{A}_{Z}P) \right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\nu_e} \right] $$

and beta negative decay

$$ Q = \left[m\left(({}^{A}_{Z}P) \right) - m({}_{\mathrm{Z-1}}{}^{\mathrm{A}}\mathrm{D}) + m_e + m_{\bar{\nu_e}} \right] $$

Radiation

Radioactive decay. The number of decay events over a time inteval is proportional to the number of particle

$$-\frac{dN}{dt} = \lambda N$$

This first order ODE is solved by integration

$$-\int_{N_0}^{N} \frac{1}{N}\, dN = \int_0^t \lambda t\, dt$$
$$\ln \frac{N}{N_0} = -\lambda t$$
$$N(t) = N_0 e^{-\lambda t}$$

Now consider the case of chain radiation, that is the particle undergoes decay $N_A \to N_B \to N_C$. For the first decay of particles N_A into N_B

$$\frac{dN_A}{dt} = -\lambda_A N_A \implies N_A = N_{A0} e^{-\lambda_A t}$$

Now the total rate of creation for the second particle N_B is the sum of the decay of said particle and the decay of the first particle into the second particle.

$$\frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A$$
$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{A0} e^{-\lambda_A t}$$

This ODE is solved by integrating factor method

$$I = \int \lambda_B\, dt = \lambda_B t$$

Then

$$N_B(t) = e^{-\lambda_B t} \int \lambda_A N_{A0} e^{-\lambda_A t}\, dt + C e^{-\lambda_B t}$$
$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} e^{-\lambda_A t} + C e^{-\lambda_B t}$$

If we assume at $t = 0$ we have zero second particle, then

$$N_B(0) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} + C = 0 \implies C = -\frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1}$$

Thus, the complete solution is

$$N_B(t) = \frac{\lambda_A N_{A0}}{\lambda_B - \lambda_1} \left(e^{\lambda_A t} - e^{\lambda_B t} \right)$$

In practice, often we are not given the number of particle, but rather the mol n or mass m (in gram) of said particle. Those quantities are related by

$$n = \frac{m}{M_r} \quad \text{or} \quad n = \frac{N}{N_A}$$

where $M_r \approx A$ is the molecular mass (gram/mol) and $N_A = 6.02 \cdot 10^{23}$ (mol^{-1}) is the Avogadro constant.

Halflife. Defined as the time required for particles to reduce to half of its initial value. The value of halflife can be determined by considering $N(t)$ at $t_{1/2}$, which by definition

$$\frac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}$$
$$t_{1/2} = \frac{\ln 2}{\lambda}$$

This equation can also be used to determine the decay constant.

Activity. Defined as the number of radioactive transformations per second

$$A \equiv -\frac{dN}{dt} = \lambda N$$

Using the solution for N, we can write

$$A = \lambda N_0 e^{-\lambda t} = A_0 e^{-\lambda t}$$

Specific Activity. Quantity related to activity; specific activity is the activity per unit mass

$$a \equiv \frac{A}{m}$$

Using the relation of mass with mol and halflife relation

$$a = \frac{\lambda N}{\frac{N}{N_A} M_r} = \frac{N_A \lambda}{M_r} = \frac{\ln 2 N_A}{t_{1/2} M_r}$$

On evaluating the numerator constant

$$a = \frac{1.32 \cdot 10^{16}}{t_{1/2} M}$$

Nuclear Stability

Valley of stability. Consist of long-lived isotope that do not simultaneously decay.

Below the valley of stability. Consist of isotopes with more N than those of the valley of stability, thus they decay either by beta negative, more likely, or neutron decay, less likely.

Above the valley of stability. Consist of isotope with more Z than those of the valley of stability, thus it decays either by beta positive or electron capture.

Beyond the valley of stability. Consist of heavy isotope with $Z > 83$, $N > 126$, and $A > 209$. They decay with alpha radiation.

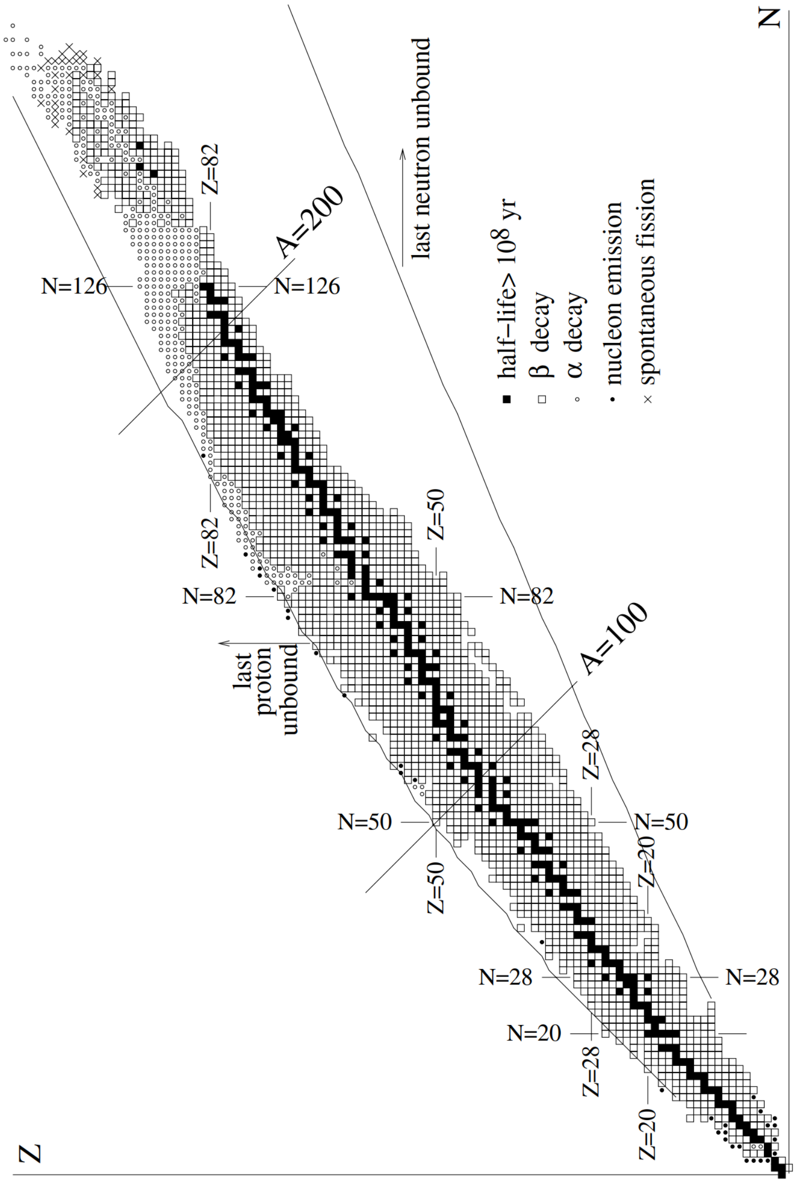


Figure: Valley of stability in ZN graph.

Particle Physics

Elementary Particle

Particle can be categorized by their spin, mass, and the type of interaction.

Boson. A pseudo-particle with interger spin that mediate the interaction. Two types of boson are gauge, or force carrier, and scalar, whose function is to give particle mass. Also, $W^{\pm}$ and Z^0 boson both belong to the same isospin triplet with $I = 1$

Table: Boson properties

Gauge Boson	
Family	Function/Force
Photon γ	Electromagnetic
Gluon g	Strong: binds the quark
W Boson $W^{\pm}$	Weak:radioactive decay
Z Boson Z^0	Weak: same as above
Scalar Boson	
Family	Function/Force
Higgs Boson H^0	Gives particle mass

Fermion. Consist of quark and lepton. All has half interger spin. Bound state of quark is called hadron and quark is able to experience all four fundamental forces.

Lepton experiences the weak force, however for charged lepton is also able to experience electromagnetic forces. This is why neutral lepton, that is neutrino, is hard to detect.

Hadron. Two types of hadron are baryon, which consist of three quarks, and meson, which consist of two quarks and one antiquark. Since baryon is made up of three spin-1/2, its spin is always half interger, opposite with meson; thus, both are also able to experience all four fundamental forces. Anti baryon consist of antiquark with the same configuration, for example is anti proton $\bar{p} = \bar{u}\bar{u}\bar{d}$.

One difference is that, unlike baryon, meson do not follow Pauli exclusion principle, since its total spin is interger. Most mesons are not their own antiparticle: charged mesons like K^+ with K^-; and neutral meson like K^0 with $\bar{K^0}$.

Quantum Number. Summarized as follows. First we have the fundamental particles.

Table: fundamental particle properties

Boson

Name	q	L	B	S	I	I_z
γ	0	0	0	0	0	0
g	0	0	0	0	0	0
W^+	+1	0	0	0	1	+1
Z^0	0	0	0	0	1	0
W^-	-1	0	0	0	1	-1
H^0	0	0	0	0	0	0

Quark

Name	q	L	B	S	I	I_z
u	+2/3	0	+1/3	0	1/2	+1/2
d	-1/3	0	+1/3	0	1/2	-1/2
s	-1/3	0	+1/3	-1	0	0
c	+2/3	0	+1/3	0	0	0
b	-1/3	0	+1/3	0	0	0
t	+2/3	0	+1/3	0	0	0

Lepton

Name	q	L	B	S	I	I_z
e	-1	$L_e = 1$	0	0	1/2	+1/2
μ	-1	$L_\mu = 1$	0	0	1/2	-1/2
τ	-1	$L_\tau = 1$	0	0	0	0
ν_e	0	$L_e = 1$	0	0	0	0
ν_μ	0	$L_\mu = 1$	0	0	0	0
ν_τ	0	$L_\tau = 1$	0	0	0	0

Then the Hadron.

Table: Hadron properties

Baryon

Name	Content	q	L	B	S	I	I_z
p	uud	+1	0	+1	0	1/2	+1/2
n	udd	0	0	+1	0	1/2	-1/2
Λ^0	uds	0	0	+1	-1	I	0
Σ^+	uus	+1	0	+1	-1	1	+1
Σ^0	uds	0	0	+1	-1	1	0
Σ^-	dds	-1	0	+1	-1	1	-1
Ξ^0	uss	0	0	+1	-2	1/2	+1/2
Ξ^-	dss	-1	0	+1	-2	1/2	-1/2
Ω^-	sss	-1	0	+1	-3	0	0

Meson

Name	Content	q	L	B	S	I	I_z
π^+	$u\bar{d}$	+1	0	0		1	+1
π^0	$u\bar{u}, d\bar{d}$	0	0	0	0	1	0
π^-	$d\bar{u}$	-1	0	0	0	1	-1
K^+	$d\bar{s}$	+1	0	0	+1	1/2	+1/2
K^0	$d\bar{s}$	0	0	0	+1	1/2	-1/2

$\bar{K^0}$	$s\bar{d}$	0	0	0	-1	1/2	+1/2
K^-	$s\bar{u}$	-1	0	0	-1	1/2	-1/2
η	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0
η'	$u\bar{u}, d\bar{d}, s\bar{s}$	0	0	0	0	0	0

Conservation Laws

Energy. The total energy $E = K + mc^2$ must be conserved in all types of nuclear reaction.

Momentum. Same as energy conservation law.

Mass number. Same as energy conservation law.

Charge. Same as energy conservation law.

Lepton number. We assign lepton a lepton number $L = +1$, anti lepton $L = -1$, and non lepton $L = 0$. Each family of lepton–such as electron e, muon μ, tau τ and their neutrino sibling–has separate conserved neutrino number–L_e, L_μ, L_τ repectively.

This quantum number almost conserved in all reaction, exception exist in neutrino oscillations; however only family lepton number is violated, while the total lepton number is conserved.

Baryon number. Like before, we assign baryon a baryon number $L = +1$, anti baryon $L = -1$, and non baryon $L = 0$. As an aside, baryon is the defined as the bound state of three quarke and that strong force works on all of them.

Strangeness number. Defined as the negative of the number of strange quarks in it, in particular strange quark s has $S = -1$ and antistrange quark $\bar{s}$ has $S = +1$.

Isospin (Isotropic Spin). We define isospin I such that $2I + 1$ is equal to the multiplet type of the baryon. Recall that the strong force does not differentiate between proton and nucleon, hence both particle can be defined as the different state of the same particle and thus can be categorized as doublet. To differentiate them, then, we define proton with $I_z = 1/2$ and neutron with $I_z = -1/2$. In general,

$$I_z = I, I - 1, \ldots, -I$$

Isospin is conserved in strong force interaction, but may not in other interaction.

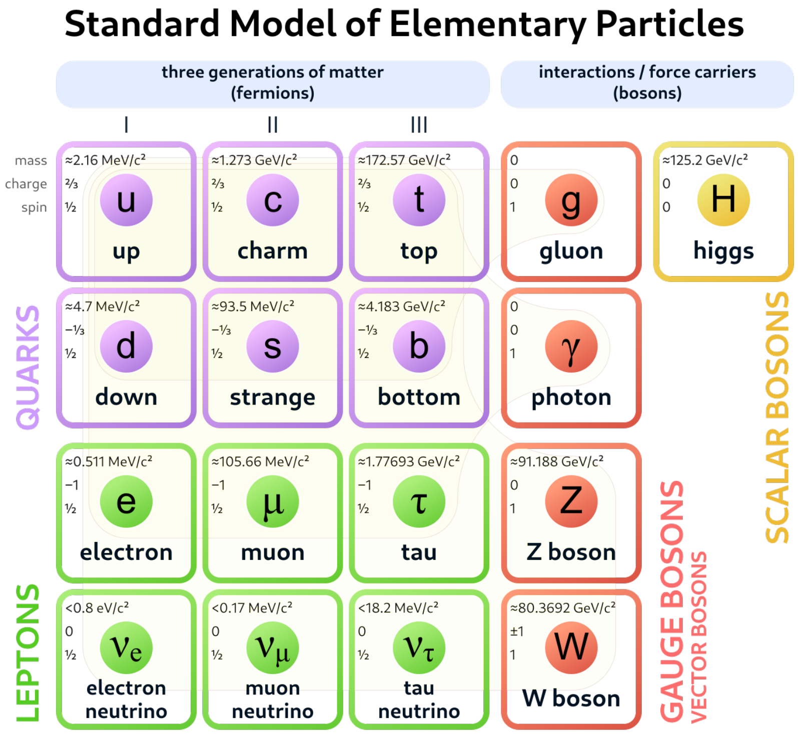


Figure: Standard model.

Nuclear Model

Liquid Drop Model

The binding energy by this model is given by

$$B(A,Z) = a_V A - a_s A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta(A)$$

where

$$\begin{aligned} a_V &= 15.753 \\ a_s &= 17.804 \\ a_C &= 0.7103 \\ a_a &= 23.69 \\ \delta(A) &= \begin{cases} 33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are even} \\ -33.6A^{-3/4} & \text{if } N \text{ and } Z \text{ are odd} \\ 0 & \text{if } A = N + Z \text{ is odd} \end{cases} \end{aligned}$$

Volume term. Recall that the volume of nucleon is proportional to A; on using this we have obtained $r = r_0 A^{1/3}$, which means that nuclei have constant density, like a drop of water.

Experiment shows that the binding energy per nucleon is rougly constant, $B/A \approx 8$ MeV. The overshoot of the volem term, then, require corection that will lower the value of the binding energy.

Surface term. Like water molecule, internal nucleon experience isotropic force, while surface nucleon only from inside. This resulting the force, and consequentl the energy, to be proportional to area $4\pi r^2 \approx a_s A^{1/3}$, with $r = r_0 A^{1/3}$.

Columb term. The binding energy due to charged particle is proportional to $Q^2/R \approx Z^2/A^{1/3}$.

Asymmetry term. By The Pauli exclusion principle, the configuration with different N and Z will have more energy than that with the same due to the different nucleon will simply occupy the higher energy state. This is the basis of $N - Z$ term.

Quantum pairing term. This term captures the effect of spin coupling. Odd-odd nuclei tend to undergo beta decay to an adjacent even-even nucleus by changing an N to a Z or vice versa.

Stable nuclei. By setting the derivative of B with respect to Z to zero, we have the maximum binding energy, that is the most state nuclei. If we ignore the quantum pairing term, or we are considering the case of odd A,we have

$$Z(A) = \frac{A}{2 + a_c A^{2/3}/2a_a} \approx \frac{A/2}{1 + 0.0075\ A^{2/3}}$$