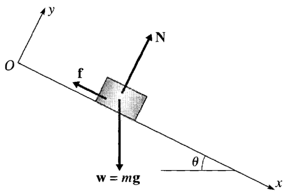
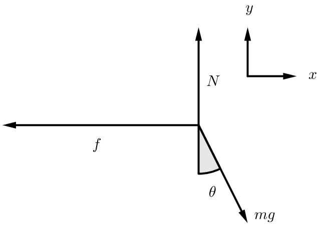
Incline Problem

Consider:

> A block of mass m is observed accelerating from rest down an incline that has coefficient of friction μ and is at angle θ from the horizontal. How far will it travel in time t?





First we define the direction of displacement as positive x axis and the normal force as positive y axis. The resultant force in y axis written as

$$\sum F_y = N - mg\cos\theta = 0 \implies N = mg\cos\theta$$

and x axis

$$\sum F_x = mg\sin\theta - f = mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$

we then solve for x by

$$\begin{aligned}
\ddot{x} &= g\left(\sin\theta - \mu\cos\theta\right) \\
\dot{x} &= g\left(\sin\theta - \mu\cos\theta\right)t \\
x(t) &= \frac{1}{2}g\left(\sin\theta - \mu\cos\theta\right)t^2
\end{aligned}$$

Since the block stared from rest, its constant of integration is zero.

Central Force Problem

Consider also:

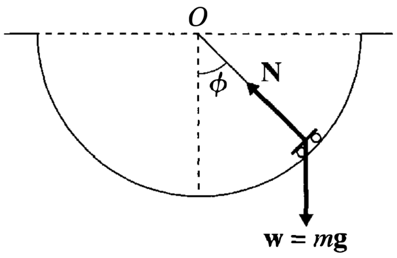
> A "half-pipe" at a skateboard park consists of a concrete trough with a semicircular cross section of radius $R = 5$m. I hold a frictionless skateboard on the side of the trough pointing down toward the bottom and release it. Find the equation of motion for this system.

In this case r is held constant, thus the expression for resultant force in polar coordinate reads

$$\mathbf{F} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\,\hat{\boldsymbol{\phi}}$$

We also know that the acting force in this system are the normal and the skateboard weigh. Applying this force into equation above

$$\left(mg\cos\phi - N\right)\,\hat{\mathbf{r}} - mg\sin\phi\,\hat{\boldsymbol{\phi}} = -m\dot{\phi}^2 R\,\hat{\mathbf{r}} + mR\ddot{\phi}\,\hat{\boldsymbol{\phi}}$$



We can't do anything with the radial component, we only use the angular component

$$\begin{aligned} mR\ddot{\phi} &= mg\sin\phi \\ \ddot{\phi} &= \frac{g}{R}\sin\phi \end{aligned}$$

This differential equation is solved by

$$\phi(t) = A\sin\sqrt{\frac{g}{R}}t + B\cos\sqrt{\frac{g}{R}}t$$

Since this is released from rest, we have the initial condition of $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$. Apllying the first condition

$$\phi_0 = B$$

and the second

$$\begin{aligned} &\dot{\phi}(t) = A\sqrt{\frac{g}{R}}\cos\sqrt{\frac{g}{R}}t - \phi_0\sqrt{\frac{g}{R}}\sin\sqrt{\frac{g}{R}}t \\ &\dot{\phi}(0) = 0 = A\sqrt{\frac{g}{R}} \end{aligned}$$

Hence the equation of motion reads

$$\phi(t) = \phi_0\cos\sqrt{\frac{g}{R}}t$$