Braket notation

Ket. $|\psi\rangle$ represents quantum state. Written in matrix form as

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

Bra. $\langle\psi|$ is the Hermitian conjugate (complex conjugate transpose) of the ket $|\psi\rangle$

$$\langle\psi| = \begin{pmatrix} \psi_0 & \psi_1 & \dots & \psi_n \end{pmatrix}$$

Inner Product. Written

$$\langle\phi|\psi\rangle = \begin{cases} 0, & \text{if orthogonal} \\ 1, & \text{if orthonormal} \end{cases}$$

Operator

Position Operator. Represents the position of a particle.

$$\hat{x} = x$$

Momentum Operator.

$$\hat{p} = -i\hbar\nabla$$

Energy Operator.

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Hamiltonian Operator.

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$$

The Hamiltonian can be written in terms of ladder operators as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$H|n\rangle = E_n|n\rangle$$

where the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Creation operator. Increases the system's energy, thus often said to be raising operator. Defined as

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x - ip)$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Annihilation operator. Decrease the system's energy, thus often said to be lowering operator. Defined as

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega x + ip)$$

in matrix representation

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Its action on the energy eigenstates $|n\rangle$ is given by:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

Commutator

Commutator measures how much two physical quantities fail to be simultaneously measurable or well-defined. It is defined as

$$[A, B] = AB - BA$$

If $[A, B] = 0$, then A and B commute and can be simultaneously measured with arbitrary precision. If not, their measurement outcomes interfere with each other.

Expectation value

Braket notation.

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

Matrix notation.

$$\langle \hat{A} \rangle = \psi^\dagger \hat{A} \psi$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \; dx$$

Normalization

Braket notation.

$$\langle \psi | \psi \rangle = 1$$

Integral notation. If $\psi(x)$ is the wavefunction in the position representation

$$\int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) \; dx = 1$$