Mathematics

Linear Algebra

Introduction

Elementary row operation:

1. Interchange two rows
2. Multiply (or divide) a row by a (nonzero) constant
3. Add a multiple of one row to another

Rank definition:

Definition 0.0.1 (Rank). *The number of nonzero rows remaining when a matrix has been row reduced is called the rank of the matrix*

Definition 0.0.2 (Rank). *The order of the largest nonzero determinant is the rank of the matrix.*

In a M matrix with m equation (rows) and n unknown (collumn), with and A has one more column (the constants):

1. If (rank M) $<$ (rank A), the equations are inconsistent and there is no solution.
2. f (rank M) $=$ (rank A) $=$ n (number of unknowns), there is one solution.
3. If (rank M) $=$ (rank A) $= R < n$, then R unknowns can be found in terms of the remaining $n - R$ unknowns

Determinant of 2×2 matrix:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Some determinant rule:

$$\begin{aligned} \det(kA) &= k^2 \det A && (2 \times 2) \\ \det(kA) &= k^3 \det A && (3 \times 3) \\ \det(AB) = \det(BA) &= \det(A) \times \det(B) \end{aligned}$$

Cramer's Rule:
the equations

$$\begin{cases} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{cases}$$

has the solution:

$$x = \frac{1}{D} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \text{and} \qquad y = \frac{1}{D} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Scalar Product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

$$= A_x B_x + A_y B_y + A_z B_z$$

The following applies if vector perpendicular:

$$\vec{A} \cdot \vec{B} = 0$$

The following applies if vector parallel:

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

Vector Product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

$$= \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The following applies if vector parallel or antiparalel:

$$\vec{A} \cdot \vec{B} == 0$$

Lines and Plane

Suppose we have vector $\vec{A} = a\hat{i} + b\hat{j}$ and vector $\vec{r} - \vec{r_0} = (x - x_0)\hat{i} + (y - y_0)\hat{j}$, which parallel to $\vec{A}$. We can write:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which is the symmetric equations of a straight line. Note that $\vec{r}$ and $\vec{r_0}$ is not necessarily parallel with $\vec{A}$, but $\vec{r} - \vec{r_0}$ do. The parameter equation is:

$$\vec{r} - \vec{r_0} = \vec{A}t$$

$$\vec{r} = \vec{r_0} + \vec{A}t$$

Now suppose that $N = a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular with $\vec{r} - \vec{r_0} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$. We then have equation of plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Matrix Operation

Matrix AB can be multiplied if thay are conformable, that is if row A = row B. Matrix multiplication in index notation is:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Matrix in general do not commute. We define the commutator of the matrices A and B by

$$[A, B] = AB - BA$$

If a matrix has an inverse we say that it is invertible; if it doesn't have an inverse, it is called singular.

$$M^{-1} = \frac{1}{\det M} C^T \qquad \text{where } C_{ij} \text{ is cofactor of } m_{ij}$$

Linear Algebra

Holy shit! He said it! He said the section title! Truly one of the paper ever. Anyways, a linear combination of $\vec{A}$ and $\vec{B}$ means $a\vec{A}+b\vec{B}$ where a and b are scalars. The vector $r = \hat{\imath}x + \hat{\jmath}y + \hat{k}$z with tail at the origin is a linear combination of the unit basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ A function of a vector, say $f(r)$, is called linear if

$$f(r_1 + r_2) = f(r_1) + f(r_2) \qquad \text{and} \qquad f(ar) = a\, f(r)$$

where a is a scalar. O is a linear operator if

$$O(r_1 + r_2) = O(r_1) + O(r_2) \qquad \text{and} \qquad O(ar) = a\, O(r)$$

Derivative, for example, is linear operator, while square root is not.

Matrix, in context of linear tarnsformation, that preserve the length of vector is said to be orthogonal. Matrix M is orthogonal if

$$M^{-1} = M^T$$

with determinant

$$\det M = \pm 1$$

det M = 1 corresponds geometrically to a rotation, and det M = -1 means that a reflection is involved.

Matrix that rotate vector $\vec{r} = (x, y)$ into $\vec{R} = (X, Y)$ (in 2D)is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the one that rotate its axis instead

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

in 3D

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $f_1(x), f_2(x), \cdots, f_n(x)$ have derivatives of order $n-1$, and if the Determinant

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix} \neq 0$$

then the functions are linearly independent.

Homogeneous Equations

Definition 0.0.3 (Homogeneous Equations). *Sets of linear equations when the constants on the right hand sides are all zero are called homogeneous equations.*

> Homogeneous equations are never inconsistent; they always have the solution "all unknowns = 0" (often called the "trivial solution"). If the number of independent equations (that is, the rank of the matrix) is the same as the number of unknowns, this is the only solution. If the rank of the matrix is less than the number of unknowns, there are infinitely many solutions.

Consider set of n homogeneous equations in n unknowns. These equations have only the trivial solution unless the rank of the matrix is less than n. This means that at least one row of the row reduced n by n matrix of the coefficients is a zero row. Which mean that the determinant D of the coefficients is zero. This fact will be used in eigenvalue problem.

> A system of n homogeneous equations in n unknowns has solutions other than the trivial solution if and only if the determinant of the coefficients is zero.

Special Matrices

Table of special matrices:

Definition	Condition
Real	$A = \bar{A}$
Symmetric	$A = A^T$
Antisymmetric	$A = -A^T$
Orthogonal	$A^{-1} = A^T$
Pure Imaginary	$A = -\bar{A}$
Hermitian	$A = A^\dagger$
Antihermitian	$A = -A^\dagger$
Unitary	$A^{-1} = A^\dagger$
Normal	$AA^\dagger = A^\dagger A$

Few theorem:

1. $(ABC)^T = C^T B^T A^T$

2. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

3. $Tr(ABC) = Tr(BCA) = Tr(CAB)$. Trace is the sum of main diagonal.It is a theorem that the trace of a product of matrices is not changed by permuting them in cyclic order.

4. If H is a Hermitian matrix, then $U = e^{iH}$ is a unitary matrix.

Vector Analysis

Vector Operation

There are four vector operation: Addition, Multiplication by a scalar, Dot product, and Cross Product. (i) Addition of two vectors. Addition is commutative and associative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

(ii) Multiplication by a scalar. Scalar multiplication is distributive.

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$$

(iii) Dot product of two vectors. The dot product of two vectors is defined

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta$$

where θ is the angle they form. Note that dot product is commutative and distributive.

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$
$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB\sin\theta\ \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of **A** and **B**. The cross product is distributive, but not commutative.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$
$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$$

Few rule for manipulating vector. (i): To add vectors, add like components

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

(ii): To multiply by a scalar, multiply each component.

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}$$

Rule (iii): To calculate the dot product, multiply like components, and add.

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Rule (iv): To calculate the cross product, form the determinant whose first row is unit vector, whose second row is A (in component form), and whose third row is B.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Triple Product

(i) Scalar triple product.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = B \cdot (\mathbf{C} \times A) = C \cdot (\mathbf{A} \times B)$$

They are cyclic and in component form

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector triple product.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

The product is linear combination of vector in parentheses.

Separation Vector

Separation vector defined as vector from the source point $\bar{r'}$ to the field point $\bar{r}$

$$\boldsymbol{\mathscr{r}} \equiv \bar{\mathbf{r}} - \bar{\mathbf{r}'}.$$

Del Operator

Vector operator defined as follows.

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

Operation Involving Del Operator

There are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);

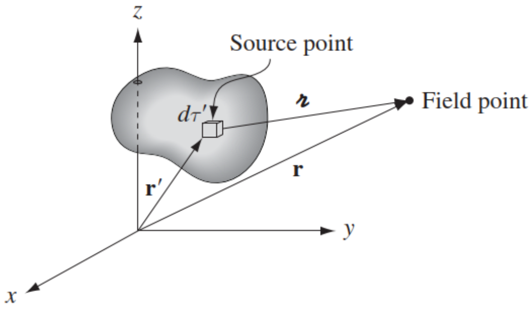

Figure: Separation vector

2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the divergence);

3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the curl).

Gradient of scalar function $T(x, y, z)$

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

can be used to define partial derivative of T

$$\begin{aligned} dT &= \left(\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= \nabla T \cdot \hat{\mathbf{u}} \end{aligned}$$

Note that ∇T is a vector quantity, with three components. The gradient ∇T points in the direction of maximum increase of the function T. Moreover, The magnitude ∇T gives the slope (rate of increase) along this maximal direction.

Divergence of vector function $\mathbf{V}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is a scalar. Divergence is a measure of how much the vector V spreads out (diverges) from the point in question.

Curl of vector function $\mathbf{V}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

The name curl is also well-chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ swirls around the point in question.

Product Rule

There are two ways to construct a scalar as the product of two functions

$$\begin{aligned} fg & \quad \text{(product of two scalar functions)} \\ \mathbf{A} \cdot \mathbf{B} & \quad \text{(dot product of two vector functions)} \end{aligned}$$

and two ways to make a vector

$$\begin{aligned} f\mathbf{A} & \quad \text{(scalar times vector)} \\ \mathbf{A} \times \mathbf{B} & \quad \text{(cross product of two vectors)} \end{aligned}$$

Accordingly, there are six product rule, two for gradients

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

two for divergences

$$\nabla(f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

and two for curls

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivative

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(1) Divergence of gradient: $\nabla \cdot (\nabla T)$. Called Laplacian of T. Notice that the Laplacian of a scalar T is a scalar.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y} + \frac{\partial^2 T}{\partial z}$$

Occasionally, we shall speak of the Laplacian of a vector, $\nabla^2 \mathbf{v}$. By this we mean a vector quantity whose x-component is the Laplacian of V_x, and so on:

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 V_x)\hat{\mathbf{x}} + (\nabla^2 V_y)\hat{y} + (\nabla^2 V_z)\hat{\mathbf{z}}$$

(2) The curl of a gradient: $\nabla \times (\nabla T)$. Always zero.

$$\nabla \cdot (\nabla T) = 0$$

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$. $\nabla(\nabla \cdot \mathbf{v})$ is not the same as the Laplacian of a vector.

$$\nabla(\nabla \cdot \mathbf{v}) \neq \nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v}$$

(4) The divergence of a curl: $\nabla \cdot (\nabla \times \mathbf{v})$. Always zero.

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

(5) Curl of curl:$\nabla \times (\nabla \times \mathbf{v})$. From the definition of ∇,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Fundamental Theorem of Calculus

The fundamental theorem of calculus says the integral of a derivative over some region is given by the value of the function at the end points (boundaries).

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Gradient. The fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (line integral) of a derivative (gradient) is given by the value of the function at the boundaries (a and b).

$$\int_a^b (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Corollary 1: $\int_a^b (\nabla T) \cdot d\mathbf{l}$ is independent of the path.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$ since the beginning and end points are identical.

Divergences . Like the other "fundamental theorems," it says that the integral of a derivative (divergence) over a region (volume V) is equal to thevalue of the function at the boundary (surface S).

$$\int_V (\nabla \cdot \mathbf{v}) \, d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

If v represents the flow of an incompressible fluid, then the flux of **v** is the total amount of fluid passing out through the surface, per unit time. There are two ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. Alternatively,

$$\int \text{(faucets within the volume)} = \oint \text{(flow out through the surface)}$$

Curl. As always, the integral of a derivative (curl) over a region (patch of surface, S) is equal to the value of the function at the boundary (perimeter of the patch, P). Now, the integral of the curl over some surface (flux of the curl) represents the "total amount of swirl," and we can determine that just as well by going around the edge and finding how much the flow is following the boundary.

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Corollary 1. $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line. It doesn't matter which way you go as long as you are consistent. For a closed surface (divergence theorem), $d\mathbf{a}$ points in the direction of the outward normal; but for an open surface is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$.

Corollary 2. $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point.

Integration by Parts

It applies to the situation in which you are called upon to integrate the product of one function (f) and the derivative of another (g); it says you can transfer the derivative from g to f, at the cost of a minus sign and a boundary term.

$$\int_a^b f\left(\frac{dg}{dx}\right) dx = -\int_a^b g\left(\frac{df}{dx}\right) dx + fg\Big|_a^b$$

Curvilinear Coordinates

I shall use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. Infinitesimal displacement vector can be written

$$d\mathbf{l} = f\ du\ \hat{\mathbf{u}} + g\ dv\ \hat{\mathbf{v}} + h\ dw\ \hat{\mathbf{w}}$$

where f, g, and h are functions of position characteristic of the particular coordinate system. While infinitesimal volume is

$$d\tau = fgh\ du\ dv\ dw$$

Use table 1 for references.

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	r $\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

Table 1

Gradient. The gradient of t is

$$\nabla \mathbf{t} \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$

Divergence. The divergence of $\mathbf{A}$ in curvilinear coordinates:

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$

Curl.

$$\nabla \times \mathbf{A} \equiv \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{\mathbf{u}} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{\mathbf{v}} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}$$

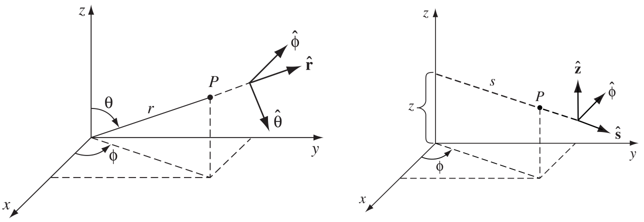
Laplacian.

$$\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]$$

Spherical.

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\ \hat{\mathbf{r}} + \cos\theta\cos\phi\ \hat{\boldsymbol{\theta}} - \sin\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\ \hat{\mathbf{r}} + \cos\theta\sin\phi\ \hat{\boldsymbol{\theta}} + \cos\phi\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\sqrt{x^2+y^2}/z \\ \phi &= \arctan y/z \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\boldsymbol{y}} + \cos\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\ \hat{\mathbf{x}} + \cos\theta\sin\phi\ \hat{\boldsymbol{y}} - \sin\theta\ \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\boldsymbol{y}} \end{cases}$$



Spherical Coordinates and Cylindrical Coordinates

Cylindrical.

$$\begin{cases} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan y/z \\ z &= z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

Dirac Delta

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1. That is to say

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x-a)\;dx = 1$$

It follows that

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Since the product is zero anyway except at $x = a$, we may as well replace $f(x)$ by the value it assumes at the origin. In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\;dx = f(a)$$

It's best to think of the delta function as something that is always intended for use under an integral sign. In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if

$$\int_{-\infty}^{\infty} f(x)D_1(x)\;dx = \int_{-\infty}^{\infty} f(x)D_2(x)\;dx$$

It is easy to generalize the delta function to three dimensions

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

with $\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$, and it's integral

$$\int_{\text{all space}} \delta^3(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\;dxdydz = 1$$

Generalizing Delta function, we get

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r\text{-}a})\, d\tau = f(\mathbf{a})$$

Few Dirac delta function

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

Fourier Transform of a δ function. Using the definition of a Fourier transform, we write

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\alpha x}\, dx = \frac{1}{2\pi}e^{-i\alpha x}$$

and its inverse transform

$$\delta(x-a) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-a)}\, d\alpha$$

The integral however does not converge. If we replace the limits by $-n, n$, we obtain a set of functions which are increasingly peaked around $x = a$ as n increases, but all have area 1.

Derivative of a δ function. Using repeated integrations by parts gives

$$\int_{-\infty}^{\infty} \phi(x)\delta^{(n)}(x-a)\, dx = (-1)^n\phi^{(n)}(a)$$

Few formulas involving δ function. For step function

$$u(x-a) = \begin{cases} 1, x > a \\ 0, x < a \end{cases}$$

$$u'(x-a) = \delta(x-a)$$

It is easy to see how the derivative of step function is equal to delta function.

Helmholtz Theorem

Suppose we are told that the divergence of a vector function F(r) is a specified scalar function D(r):

$$\nabla \cdot \mathbf{F} = D$$

and the curl of F(r) is a specified vector function C(r):

$$\nabla \times \mathbf{F} = \mathbf{C}$$

For consistency, C must be divergenceless $\nabla \cdot \mathbf{C} = 0$. Helmholtz theorem state if the divergence D(r) and the curl C(r) of a vector function F(r) are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$ and if F(r) goes to zero as $r \to \infty$, then F is given uniquely by

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$$

Potential Theorem

Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, F satisfies one if and only if it satisfies all the others):

- $\nabla \times \mathbf{F} = 0$ everywhere.
- $\int_a^b \mathbf{F} \cdot \, d\mathbf{l}$ is independent of path, for any given end points.
- $\oint \mathbf{F} \cdot \, d\mathbf{l} = 0$ for any closed loop.
- **F** is the gradient of some scalar function: $F = -\nabla V$.

Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

- $\nabla \cdot \mathbf{F} = 0$ everywhere.
- $\int \mathbf{F} \cdot \, d\mathbf{a}$ is independent of surface, for any given boundary line.
- $\oint \mathbf{F} \cdot \, d\mathbf{a} = 0$ for any closed surface.
- **F** is the curl of some scalar function: $\mathbf{F} = -\nabla \mathbf{A}$.

Variation Calculus

The Euler Equation

Any problem in the calculus of variations is solved by setting up the integral which is to be stationary, writing what the function F is, substituting it into the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and solving the resulting differential equation. When the function $F = F(r, \theta, \theta')$, the Euler's equation read

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

If $F = F(t, x, \dot{x})$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0$$

Notice that the first derivative in the Euler equation is with respect to the integration variable in the integral. The partial derivatives are with respect to the other variable and its derivative.

Proof. We will try to find the y which will make stationary the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

where F is a given function. Let $\eta(x)$ represent a function of x which is zero at x_1 and x_2, and has a continuous second derivative in the interval x_1 to x_2, but is otherwise completely arbitrary. We define the function $Y(x)$ by the equation

$$Y(x) = y(x) + \epsilon\eta(x)$$

where $y(x)$ is the desired extremal and ϵ is a parameter. Differentiating with respect to x, we get

$$Y(x) = y(x)' + \epsilon\eta'(x)$$

Then we have

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')\, dx$$

Now I is a function of the parameter ϵ; when $\epsilon = 0$, $Y = y(x)$, the desired extremal. Our problem then is to make $I(\epsilon)$ take its minimum value when $\epsilon = 0$. In other words, we want

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

Remembering that Y and Y' are functions of ϵ, and differentiating under the integral sign with respect to ϵ

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\epsilon}\right) dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y} \eta(x) + \frac{\partial F}{\partial Y'} \eta'(x) \right] dx$$

We want $dI/d\epsilon = 0$ at $\epsilon = 0$

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$$

Assuming that y'' is continuous, we can integrate the second term by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'(x)\, dx = - \int_{x_1}^{x_2} \frac{d}{dx} \frac{\partial F}{\partial y'} \eta(x)\, dx + \left. \frac{\partial F}{\partial y'} \eta(x) \right|_{x_1}^{x_2}$$

The first term is zero as before because $\eta(x)$ is zero at x_1 and x_2. Then we have

$$\left. \frac{dI}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \eta(x)\, dx$$

Since $\eta(x)$ is arbitrary, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \blacksquare$$

Notice carefully here that we are not saying that when an integral is zero, the integrand is also zero; this is not true. What we are saying is that the only way $\int f(x)\eta(x)\, dx$ can always be zero for every $\eta(x)$ is for $f(x)$ to be zero.

Several Variable

If there are n dependent variables in the original integral, there are n Euler-Lagrange equations. For instance, an integral of the form

$$S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du$$

with two dependent variables [$x(u)$ and $y(u)$], is stationary with respect to variations of $x(u)$ and $y(u)$ if and only if these two functions satisfy the two equations

$$\frac{\partial f}{\partial x} = \frac{d}{du} \frac{\partial f}{\partial x'} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{d}{du} \frac{\partial f}{\partial y'}$$

Classical Mechanics

Unasorted Classical Mechanics Topics

Newton's Law

First law. in the absence of an external force, when viewed from an inertial frame, an object at rest remains at rest and an object in uniform motion in a straight line maintains that motion.

Second law. Simply put

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Third law. States that if two objects interact, the force exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force exerted by object 2 on object 1.

Particle Under Constant Acceleration

Here's some kinematics equation for position

$$x(t) = x_i + \frac{1}{2}(v_i + v_f)t$$
$$x(t) = x_i + v_i t + \frac{1}{2}at^2$$

and for velocity

$$v(t) = v_i + at$$
$$v(t)^2 = v_i^2 + 2a(x_f - x_i)$$

Particle in Uniform Circular Motion

If a particle moves in a circular path of radius r with a constant speed v, the magnitude of its centripetal acceleration is given by

$$a_r = \frac{v^2}{r}$$

while its period and angular velocity is

$$T = \frac{2\pi r}{v}, \quad \omega = \frac{2\pi}{T}$$

Applying Newton's second law

$$\sum F = ma_r = m\frac{v^2}{r}$$

Rigid Object Under Constant Angular Acceleration

Analogous to those for translational motion of a particle under constant acceleration

$$\begin{aligned} \omega(t) &= \omega_i + \alpha t \\ \omega(t)^2 &= \omega_i^2 + 2\alpha(\theta_t - \theta_i) \\ \theta(t) &= \theta_i + \omega t + \frac{1}{2}\alpha t^2 \\ \theta(t) &= \theta_i + \frac{1}{2}(\omega_i + \omega_f)t \end{aligned}$$

Relation of Linear and Rotational Motion

The following equations show the relation of linear and rotational motion

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha$$

Torque

The torque associated with a force $\mathbf{F}$ acting on an object

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = I\alpha = \frac{d\mathbf{L}}{dt}$$

Moment of Inertia

The moment of inertia of a rigid object is

$$I = \sum mr^2 = \int r^2 dm$$

Parallel Axis Theorem. To calculate the moment inertia from any axis, we use parallel axis theorem

$$I = I_{\mathrm{CM}} + Md^2$$

Terminal velocity

$\boldsymbol{r} \propto \boldsymbol{v}$**.** The velocity as a function of time is

$$v = \frac{mg}{b}\left[1 - \exp\left(-\frac{bt}{m}\right)\right] = v_T\left[1 - \exp\left(-\frac{bt}{m}\right)\right]$$

where b is a resistive constant whose value depends on the properties of the medium.

$\boldsymbol{r \propto v^2}$**.** Given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}$$

where D is a dimensionless empirical quantity called the drag coefficient, ρ is the density of air, and A is the cross-sectional area of the moving object.

Escape velocity. the speed required bu an object to escape from any planet orbit is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Work Energy Theorem

It states that if work is done on a system by external forces and the only change in the system is in its speed,

$$W = \Delta T$$

Kinetic Energy

For an object in linear motion, the kinetic energy of said object is

$$T = \frac{1}{2}mv^2$$

whereas for rotational motion

$$T = \frac{1}{2}I\omega^2$$

Hence the total kinetic energy of a rigid object rolling on a rough surface without slipping

$$T = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I\omega_{\text{CM}}^2$$

Potential Energy Function

For conservative energy $\mathbf{F}$, applies

$$V_f - V_i = -\int_{\mathbf{r_i}}^{\mathbf{r_f}} \mathbf{F} \cdot d\mathbf{r}$$

For particle-Earth system, the gravitational potential energy is

$$V = mgy$$

and elastic potential stored in spring

$$V = \frac{1}{2}kx^2$$

Effective potential

Effective potential energy $U_{\text{eff}}(r)$ is the sum of the actual potential energy $U(r)$ and the centrifugal $U_{\text{cf}}(r)$:

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

where l is the angular momentum and μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Momentum Impulse

The linear momentum and impulse are defined as

$$\mathbf{p} = m\mathbf{v}, \quad \mathbf{I} = \int_{t_i}^{t_f} \sum \mathbf{F}\, dt$$

Angular Momentum The angular momentum about an axis through the origin of a particle having linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

The z component of angular momentum of a rigid object rotating about a fixed z axis is

$$L_z = I\omega$$

Center of Mass and Velocity

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{r} = \frac{1}{M} \int \mathbf{r}\, dm$$

where M is the total mass. The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{1}{M} \sum m\mathbf{v} =$$

Collision

Inelastic collision. One for which the total kinetic energy of the system of colliding particles is not conserved.

Elastic collision. One in which the kinetic energy of the system is conserved.

Perfectly inelastic. A collision which the colliding particles stick together after the collision.

Rocket Propulsion The expression for rocket propulsion is

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$

Power

The rate at which work is done by an external force, called power, is

$$P = \frac{dE}{dt} = Fv = \tau\omega$$

Newton's Law on Gravity

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

For an object at a distance h above the Earth's, the gravitational acceleration is

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2}$$

In general, the gravitational field experienced by mass m is

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

Kepler's Law

First Law. All planets move in elliptical orbits with the Sun at one focus.

Second Law The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

Third Law Simply put

$$T^2 = \frac{4\pi^2 a^3}{GM_S}$$

where a is semimajor axis and M_S is the mass of the sun.

Energy of Gravitational system

Potential energy. The gravitational potential energy associated with a system of two particles is

$$V = -\frac{Gm_1 m_2}{r}$$

Total energy. The total energy of the system is the sum of the kinetic and potential energies

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r} = -\frac{GMm}{2r}$$

Central Force

Newton's Second Law in Polar Coordinate

Acceleration in polar coordinate expressed as

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}$$

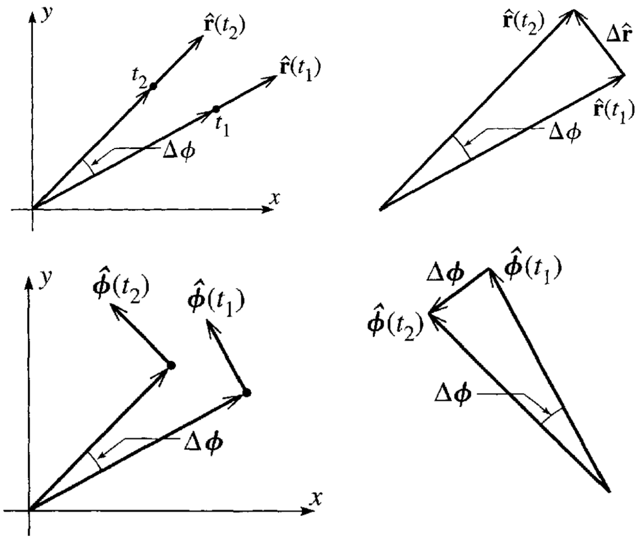
and velocity as

$$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

Hence Newton's law transform into

$$\mathbf{F} = m\mathbf{a} = \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right) \end{cases}$$

Derivation



The value of $d\hat{\mathbf{r}}$ and $d\hat{\boldsymbol{\phi}}$.

From the figure, we have

$$d\hat{\mathbf{r}} = d\phi\ \hat{\boldsymbol{\phi}}, \quad d\hat{\boldsymbol{\phi}} = -d\phi\ \hat{\mathbf{r}}$$

or equivalently

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\ \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\ \hat{\mathbf{r}}$$

Using these we can now proceed to derive the Newton's law in polar coordinate. In cartesian coordinate, position vector can be writen as

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}}$$

converting it into polar

$$\mathbf{r} = r\ \hat{\mathbf{r}}$$

Next, we determine the velocity as

$$\dot{\mathbf{r}} = \frac{d}{dt} r\ \hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}$$

and acceleration as

$$\begin{aligned}\ddot{r} = \frac{d}{dt}\left(\dot{r}\ \hat{\mathbf{r}} + r\dot{\phi}\ \hat{\boldsymbol{\phi}}\right) = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\frac{d}{dt}\left(\dot{\phi}\ \hat{\boldsymbol{\phi}}\right)& \\ = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\left(\ddot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\phi}\hat{\mathbf{r}}\right)& \\ = \left(\ddot{r} - r\dot{\phi}^2\right)\ \hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\ \hat{\boldsymbol{\phi}}&\end{aligned}$$

Finally

$$F = F_r\ \hat{\mathbf{r}} + F_\phi\ \hat{\boldsymbol{\phi}} \begin{cases} F_r & = m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi & = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\end{cases}$$