In his theory of blackbody radiation, energy density at frequency ν inside cavity with temperature T is given by

$$\mathcal{U}(\nu, T) = \rho u$$

where ρ is the number of radiation modes per unit volume and u is the average energy at said frequency. Both quantity respectively are given by

$$\rho = \frac{8\pi\nu^2}{c^3}, \quad u = \frac{h\nu}{\exp(h\nu/k_B T) - 1}$$

Hence

$$\mathcal{U}(\nu, T) = \frac{\nu^3}{c^3} \frac{8\pi h}{\exp(h\nu/k_B T) - 1}$$

or, we can write it in terms of λ

$$\mathcal{U}(\lambda, T) = \frac{c}{\lambda^5} \frac{8\pi h}{\exp(hc/\lambda k_B T) - 1}$$

Derivation. In his work, Planck assumes that the walls of the cavity act as oscillator with energy of integer multiple ϵ. This has some similarity with Boltzmann discrete model. We denote n_k as the number particle with $k\epsilon$ energy, with maximum energy of $P\epsilon$. Therefore, we have the following constraints.

$$\sum_{k=0}^{P} n_k = N, \quad \sum_{k=0}^{P} k n_k = P$$

Planck he defined entropy as

$$S_P = k_B \ln(W) + C$$

where

$$W = \frac{\mathcal{R}}{\mathcal{J}}$$

is the probability of the N atoms have $p\epsilon$ energy. In other hand, $\mathcal{R}$ denotes the number of said configuration and $\mathcal{J}$ denotes the total configuration. It then may be written as

$$S_P = k_B \ln \mathcal{R}$$

where $\mathcal{R}$ is given by

$$\mathcal{R} = \frac{(N + L - 1)!}{L!(N - 1)!}$$

The logarithm of $\mathcal{R}$ can be evaluated as

$$\begin{aligned} \ln \mathcal{R} = (N + L - 1)[\ln(N + L - 1) - 1] - L[\ln(L) - 1] \\ - (N - 1)[\ln(N - 1) - 1] \end{aligned}$$

and then

$$\ln \mathcal{R} = (N + L - 1)\ln(N + L - 1) - L \ln L - (N - 1)\ln(N - 1)$$

and then

$$\begin{aligned} \ln \mathcal{R} = (N + L)\ln(N + L - 1) - \ln(N + L - 1) - L \ln L \\ - N \ln(N - L) + \ln(N - 1) \end{aligned}$$

and then

$$\ln \mathcal{R} = (N+L)\left[\ln\left(\frac{N+L-1}{N+L}\right) + \ln(N+L)\right] - L\ln L$$
$$- N\left[\ln\left(\frac{N-1}{N}\right) + \ln(N)\right] - \ln\left(\frac{N+L-1}{N+L}\right)$$

then finally we rewrite it as

$$\ln \mathcal{R} = (N+L)\ln(N+L) - L\ln L - N\ln N$$
$$- N\ln\left(\frac{N-1}{N}\right) + (N+L-1)\ln\left(\frac{N+L-1}{N+L}\right)$$

For large N, L; those last two terms will approach zero. Hence,

$$\ln \mathcal{R} = (N+L)\ln(N+L) - L\ln L - N\ln N$$

This equation give the same result for $\ln(\mathcal{P}_{\max})$ in the case of Boltzmann discrete model. Therefore, we write

$$S_P = Nk_B\left[\left(1+\frac{u}{\epsilon}\right)\ln\left(1+\frac{u}{\epsilon}\right) - \frac{u}{\epsilon}\ln\left(\frac{u}{\epsilon}\right)\right]$$

Based on empirical data, Planck concludes that entropy is a function of energy and frequency

$$S_P = f\left(\frac{u}{\nu}\right)$$

On comparing those two equation, it can be seen that energy ϵ is proportional to frequency ν. To show this relationship, we then write $\epsilon = h\nu$. Solving the equation of entropy S_P above for average energy u we have derived the following result.

$$u = \frac{\epsilon}{\exp(\epsilon/k_BT) - 1}$$

Substituting $\epsilon = h\nu$ from our empirical observation, we obtain

$$u = \frac{hv}{\exp(hv/k_BT) - 1} \quad \blacksquare$$