Bose-Dirac Distribution

The distribution of particle obeying the Pauli exclusion principle, also called fermion, is governed by Fermi-Dirac distribution, given by

$$n_E = \frac{2\pi}{h^3} \frac{V(2m)^{3/2}\sqrt{E}}{\exp\left[(E-\mu)/k_B T\right]+1}$$

Derivation. The number of configuration in this case is the number of ways to choose N_E particle from P_E cells

$$\mathcal{P}_E = \binom{P_E}{N_E} = \frac{P_E!}{(P_E - N_E)!N_E!}$$

with the number of cells given by

$$P_E = \frac{2\pi}{h^3} V(2m)^{3/2}\sqrt{E}\,dE$$

This is the same value that Bose derived. As for the total number of configuration across all interval of energy

$$\mathcal{P} = \prod_{E=0}^{\infty} \mathcal{P}_E$$

To find the distribution function, we need to determine the entropy of such system. Hence, we need to maximize the logarithm of said configuration. To such end, we write the logarithm as

$$\begin{aligned}\ln\mathcal{P} &= \sum_E \big[P_E\ln(P_E) - P_E - (P_E - N_E)\ln(P_E - N_E) + (P_E - N_E) \\ &\quad - N_E\ln(N_E) + N_E\big] \\ \ln\mathcal{P} &= \sum_E P_E\ln(P_E) - (P_E - N_E)\ln(P_E - N_E) - N_E\ln(N_E)\end{aligned}$$

By Lagrange's method, we consider the following constraints

$$N = \sum_{E=0}^{\infty} N_E, \quad U = \sum_{E=0}^{\infty} E N_E$$

and construct the following function

$$F = \ln\mathcal{P} + \lambda_1 N_E + \lambda_2 E N_E$$

Setting the derivative to zero

$$\ln(P_E - N_E) + 1 - \ln(N_E) - 1\lambda_1 + \lambda_2 E = 0$$

and solving for N_E

$$\begin{aligned}\frac{P_E - N_E}{N_E} &= \exp(\lambda_1 - \lambda_2 E) \\ N_E &= \frac{P_E}{\exp(\lambda_1 - \lambda_2 E)+1}\end{aligned}$$

The expression for entropy then reads

$$S = \sum_E P_E k_B \ln(P_E) - \sum_E (P_E k_B - N_E k_B)$$
$$\left[\ln(P_E) + \ln\left(\frac{\exp(\lambda_1 - \lambda_2 E)}{\exp(\lambda_1 - \lambda_2 E) + 1}\right)\right]$$
$$- \sum_E N_E k_B \left[\ln(P_E) - \ln(\exp(\lambda_1 - \lambda_2 E) + 1)\right]$$

furthermore

$$S = \sum_E (N_E k_B - P_E k_B) \ln\left(\frac{\exp(\lambda_1 - \lambda_2 E)}{\exp(\lambda_1 - \lambda_2 E) + 1}\right)$$
$$+ \ln \sum_E N_E k_B \ln\left(\exp(\lambda_1 - \lambda_2 E) + 1\right)$$

additionally

$$S = N k_B(-\lambda_1 - \lambda_2 E) - \sum_E P_E \ln\left(\frac{\exp(\lambda_1 - \lambda_2 E)}{\exp(\lambda_1 - \lambda_2 E) + 1}\right)$$

By using the thermodynamics relation

$$\frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_{V,N} = -\lambda_2 k_B \implies \lambda_2 = -\frac{1}{k_B T}$$

and

$$-\frac{\mu}{T} = \frac{\partial S}{\partial N} = -k_B \lambda_1 \implies \lambda_1 = \frac{\mu}{k_B T}$$

Therefore

$$N_E = \frac{2\pi}{h^3} \frac{V(2m)^{3/2}\sqrt{E}}{\exp\left[(E-\mu)/k_B T\right] + 1}\, dE$$

and for the distribution function

$$n_E = \frac{2\pi}{h^3} \frac{V(2m)^{3/2}\sqrt{E}}{\exp\left[(E-\mu)/k_B T\right] + 1}$$