Continuous Energy Levels

Assume continuous energy E within interval $(0, \infty)$, with an interval of ϵ. The function $f(E)$ denote the number of atoms per unit energy. The density function $f(E)$ for continuous energy levels at equilibrium is given by

$$f(E) = \frac{N}{u} e^{-E/u}$$

Permutability measure is defined as

$$\Omega = -\int_0^\infty f(E) \ln\left[f(E)\right] \, dE$$

which, on using the given expression for density function evaluates into

$$\Omega = N(1 + \ln u - \ln N)$$

Derivation. Let n_k be the number of atoms whose energy lies between $(k\epsilon, k\epsilon + \epsilon)$. For any positive integer k,

$$n_k = \epsilon f(k\epsilon)$$

By taking the limit of $\epsilon \to 0$, n_k now denote the number of atoms whose energy lies between $(E, E + dE)$

$$\lim_{\epsilon \to 0} n_k = f(E) \, dE$$

As it the case with discrete model, we have the following restriction

$$\sum_{k=0}^{P} n_k = N, \quad \epsilon \sum_{k=0}^{P} k n_k = Nu$$

By using the expression for n_k when $\epsilon \to 0$ these restrictions now read as

$$\int_0^\infty f(E) \, dE = N, \quad \int_0^\infty E f(E) \, dE = Nu$$

Now we consider the expression for number of configuration in the limit $D \to \mathcal{P}$. The expression for the logarithm of D written as

$$\ln D = N \ln N - N - \sum_{k=0}^{\infty} (n_k \ln n_k - n_k)$$

By taking the limit, we have

$$\begin{aligned}
\ln \mathcal{P} &= N \ln N - N - \int_0^\infty \left[f(E) \ln(n_k) - f(E)\right] \, dE \\
&= N \ln N - N - \int_0^\infty f(E) \ln[f(E)] \, dE - \lim_{\epsilon \to 0} \int_0^\infty f(E) \ln(\epsilon) \, dE \\
&\quad + \int_0^\infty f(E) \, dE
\end{aligned}$$

$$\ln \mathcal{P} = N \ln N - \int_0^\infty f(E) \ln[f(E)] \, dE - \lim_{\epsilon \to 0} N \ln(\epsilon)$$

Recall that equilibrium state correspond to the largest number of configuration. Although the equation above, due to the last term, diverges; it can be ignored since maximization does not concern constant. In essence, we what to maximize the logarithm of $\mathcal{P}$ by varying the expression for $f(E)$. Therefore, we maximize the quantity of

$$\Omega \equiv -\int_0^\infty f(E)\ln[f(E)]\,dE$$

which is defined as permutability measure, with N and Nu constraint. By the Lagrange multiplier method, we have the following auxiliary function

$$F(f) = \int_0^\infty [f\ln(f) + \lambda_1 f + \lambda_2 E f]\,dE$$

Then, we set its derivative to zero

$$\frac{dF}{df} = \int_0^\infty [\ln f + 1\lambda_1 + \lambda_2 E]\,dE = 0$$

One possible way for an integral to be zero is that the integrand is zero, hence we have

$$\ln f + 1 + \lambda_1 + \lambda_2 E = 0 \implies \begin{aligned} f(E) &= \exp(-1-\lambda_1-\lambda_2 E) \\ f(E) &= Ce^{-\lambda_2 E} \end{aligned}$$

On Using this expression, N constraint now may be evaluated as

$$N = \int_0^\infty Ce^{-\lambda_2 E}\,dE = \left.\frac{Ce^{\lambda_2 E}}{\lambda_2}\right|_\infty^0 = \frac{C}{\lambda_2}$$

As for Nu constraint

$$\begin{aligned} Nu &= \int_0^\infty ECe^{-\lambda_2 E}\,dE = Ce^{-\lambda_2 E}\left.\left(\frac{E}{-\lambda_2} - \frac{1}{\lambda_2^2}\right)\right|_0^\infty \\ &= \left.\frac{CEe^{\lambda_2 E}}{\lambda_2}\right|_\infty^0 + \left.\frac{Ce^{-\lambda_2 E}}{\lambda_2^2}\right|_\infty^0 = \frac{C}{\lambda_2^2} = \frac{N}{\lambda_2} \end{aligned}$$

We have

$$C = \frac{N}{u} \quad \wedge \quad \lambda_2 = \frac{1}{u}$$

The equilibrium distribution now may be written as

$$f(E) = \frac{N}{u}e^{-E/u} \quad \blacksquare$$

Now we evaluate the permutability measure

$$\begin{aligned} \Omega &= -\int_0^\infty \frac{N}{u}e^{-E/u}\ln\left[\frac{N}{u}e^{-E/u}\right]\,dE \\ &= -\int_0^\infty \frac{N}{u}e^{-E/u}\left[\ln\left(\frac{N}{u}\right) - \frac{E}{u}\right]\,dE \\ &= Ne^{E/u}\ln\left(\frac{N}{u}\right)\Bigg|_0^\infty + \frac{N}{u^2}\left(-Eu-u^2\right)e^{E/u}\Bigg|_0^\infty \\ \Omega &= -N\ln\left(\frac{N}{u}\right) + N = N(1+\ln U - \ln N) \quad \blacksquare \end{aligned}$$