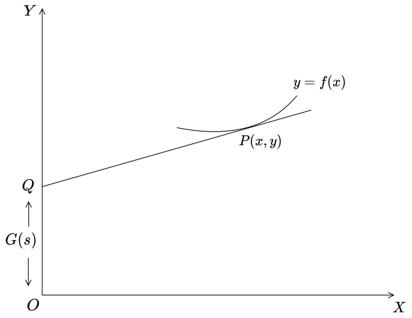

Definition

Consider monotonic function $f(x)$. The slope of $f(x)$ expressed as

$$s(x) = \frac{d}{dx} f(x)$$

Suppose now we want to transform $f(x)$ into function $G(s)$ as function of its slope. We write

$$G(s) = f[x(s)] - sx(s)$$

where $x(s)$ reads as x in terms of its slope s. The function $G(s)$ referred as Legendre transform of $f(x)$. If we want to transform $G(s)$ back into $f(x)$, called inverse transform, we write

$$f(x) = G[s(x)] + xs(x)$$

Derivation

Consider the same monotonic function $f(x)$. Say that its tangent line intercept the y-axis at Q. Another family of the tangent line of the same slope will also intercept the y-axis; they will, however, did it at different point. We define the intercept of origin O and Q as $G(s)$.

To find the actual intercept, note that line passing through Q has the form $y = mx + Q$. Since we define $G(s)$ as the line OG, we have

$$G(s) = f[x(s)] - sx(s)$$

The differential of G is

$$dG(s) = \frac{\partial G}{\partial s} ds = -x(s)\ ds$$

Hence

$$x = -\frac{dG(s)}{ds}$$

In a sense the symmetry of $f(x)$ and $G(s)$ can be traced to equation above. $f(x)$ has the slope of s, where $G(s)$ is x. We can therefore write

$$f(x) = G[s(x)] + xs(x)$$

to transform $G(s)$ back to $f(x)$.

Multivariable Function

Single variable transform. Consider some multivariable function, say $f(x, y, z)$ Suppose we want to construct Legendre transform of $f(x, y, z)$ with respect to x. First we have the slope

$$s = \left.\frac{\partial f}{\partial x}\right|_{y,z}$$

We then write the transform of $f(x)$ as

$$\mathcal{L}_x[f(x, y, z)] = f(x(s), y, z) - sx(s)$$

which is the same for single variable function.

Multivariable transform. Suppose that, with the same function, we want to perform Legendre transform with respect to all variable x, y, z. First we define

$$s = \left.\frac{\partial f}{\partial x}\right|_{y,x} \quad ; t = \left.\frac{\partial f}{\partial y}\right|_{x,z} \quad ; u = \left.\frac{\partial f}{\partial z}\right|_{x,y}$$

We then write

$$\mathcal{L}_{x,y,z}[f(x, y, z)] = f[x(s), y(t), z(u)] - sx(s) - ty(t) - uz(u)$$