Definition

Consider monotonic function $f(x)$. The slope of $f(x)$ expressed as

$$s(x) = \frac{d}{dx} f(x)$$

Suppose now we want to transform $f(x)$ into function $G(s)$ as function of its slope. We write

$$G(s) = f[x(s)] - sx(s)$$

where $x(s)$ reads as x in terms of its slope s. The function $G(s)$ referred as Legendre transform of $f(x)$. If we want to transform $G(s)$ back into $f(x)$, called inverse transform, we write

$$f(x) = G[s(x)] + xs(x)$$

Derivation

Consider the same monotonic function $f(x)$. Say that its tangent line intercept the y-axis at Q. Another family of the tangent line of the same slope will also intercept the y-axis; they will, however, did it at different point. We define the intercept of origin O and Q as $G(s)$.

To find the actual intercept, note that line passing through Q has the form $y = mx + Q$. Since we define $G(s)$ as the line OG, we have

$$G(s) = f[x(s)] - sx(s)$$

The differential of G is

$$dG(s) = \frac{\partial G}{\partial s} ds = -x(s)\ ds$$

Hence

$$x = -\frac{dG(s)}{ds}$$

In a sense the symmetry of $f(x)$ and $G(s)$ can be traced to equation above. $f(x)$ has the slope of s, where $G(s)$ is x. We can therefore write

$$f(x) = G[s(x)] + xs(x)$$

to transform $G(s)$ back to $f(x)$.

Multivariable Function

Single variable transform. Consider some multivariable function, say $f(x, y, z)$ Suppose we want to construct Legendre transform of $f(x, y, z)$ with respect to x. First we have the slope

$$s = \left.\frac{\partial f}{\partial x}\right|_{y,z}$$

We then write the transform of $f(x)$ as

$$\mathcal{L}_x[f(x, y, z)] = f(x(s), y, z) - sx(s)$$

which is the same for single variable function.

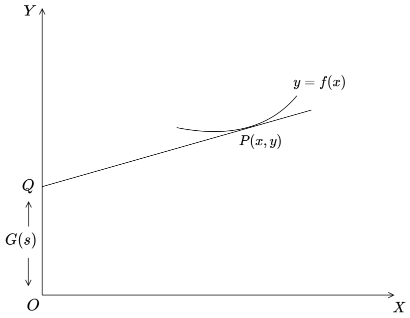

Figure: Geometric interpertation of Legendre transformation

Multivariable transform. Suppose that, with the same function, we want to perform Legendre transforms with respect to all variable x, y, z. First we define

$$s = \left.\frac{\partial f}{\partial x}\right|_{y,x} \quad ; t = \left.\frac{\partial f}{\partial y}\right|_{x,z} \quad ; u = \left.\frac{\partial f}{\partial z}\right|_{x,y}$$

We then write

$$\mathcal{L}_{x,y,z}[f(x,y,z)] = f[x(s), y(t), z(u)] - sx(s) - ty(t) - uz(u)$$