Quantum Statistics

Based on their quantum statistics properties, there are three types of particle, namely

1. **Classics**. This particle obeys Maxwell-Boltzmann (MB) distribution. It assumes particles are distinguishable and can occupy any energy state without restriction.

2. **Boson.** This particle obeys the Bose-Einstein (BE) distribution. It is indistinguishable and can occupy the same quantum state without limit.

3. **Fermion.** This particle obeys Fermi-Dirac (FD) distribution. Fermions are indistinguishable and obey the Pauli exclusion principle, meaning no two fermions can occupy the same quantum state.

Discrete System

Microstate. The number of microstate based on energy level for classical particle is

$$\Omega_{\mathrm{MB}} = N! \prod_k \frac{g_k^{n_k}}{n_k!}$$

for boson

$$\Omega_{\mathrm{BE}} = \prod_k \frac{(n_k + g_k - 1)!}{n_k!(g_k - 1)!}$$

and for fermion

$$\Omega_{\mathrm{FD}} = \prod_k \frac{g_k!}{n_k!(g_k - n_k)!}$$

Example. Consider system with four particle and five unit energy, where each energy level have the following condition.

Table: Random system

n	(g_n, ϵ_n)
4	$(1, 3)$
3	$(3, 2)$
2	$(2, 1)$
1	$(3, 0)$

We have the following macrostate.

$$M_1 = (2, 0, 1, 1)$$
$$M_2 = (1, 2, 0, 1)$$
$$M_3 = (1, 1, 2, 0)$$
$$M_4 = (0, 3, 1, 0)$$

For classical particle, we have also the following number of microstate

$$\begin{aligned}
\Omega_1 &= 4!\frac{3^2}{2!}\frac{2^0}{0!}\frac{3^1}{1!}\frac{1^1}{1!} = 324 \\
\Omega_2 &= 144 \\
\Omega_3 &= 648 \\
\Omega_4 &= 96
\end{aligned}$$

For boson

$$\begin{aligned}
\Omega_1 &= \frac{4!1!3!1!}{2!2!1!2!0!} = 18 \\
\Omega_2 &= 9 \\
\Omega_3 &= 36 \\
\Omega_4 &= 12
\end{aligned}$$

and for fermion

$$\begin{aligned}
\Omega_1 &= \frac{3!}{2!}\frac{2!}{2!}\frac{3!}{2!}\frac{1!}{1!} = 9 \\
\Omega_2 &= 3 \\
\Omega_3 &= 18
\end{aligned}$$

No M_4 for fermion since those three particle cannot have the same energy.

Another example. This time I will do classical particle only. Consider system with $N = E = 7$ unit whose each level of energy obey the following condition.

Table: Random system

n	(g_n, ϵ_n)
7	$(3, 6)$
6	$(3, 5)$
5	$(3, 4)$
4	$(3, 3)$
3	$(3, 2)$
2	$(3, 1)$
1	$(3, 0)$

The corresponding macrostates are as follows.

$$\begin{aligned}
M_1 &= (5,0,0,0,0,0,7) & M_7 &= (2,3,0,1,0,0,0) \\
M_2 &= (4,1,0,0,0,1,0) & M_8 &= (3,0,3,0,0,0,0) \\
M_3 &= (4,0,1,0,1,0,0) & M_9 &= (2,2,2,0,0,0,0) \\
M_4 &= (3,2,0,0,1,0,0) & M_{10} &= (1,4,1,0,0,0,0) \\
M_5 &= (4,0,2,0,0,0,0) & M_{11} &= (0,6,0,0,0,0,0) \\
M_6 &= (3,1,1,1,0,0,0)
\end{aligned}$$

with the following microstate

$$
\begin{array}{ll}
\Omega_1 = 6 \cdot 3^6 & \Omega_7 = 60 \cdot 3^6 \\
\Omega_2 = 30 \cdot 3^6 & \Omega_8 = 20 \cdot 3^6 \\
\Omega_3 = 30 \cdot 3^6 & \Omega_9 = 90 \cdot 3^6 \\
\Omega_4 = 60 \cdot 3^6 & \Omega_{10} = 30 \cdot 3^6 \\
\Omega_5 = 15 \cdot 3^6 & \Omega_{11} = 3^6 \\
\Omega_6 = 120 \cdot 3^6 &
\end{array}
$$

Buku Pak Rouf, soal nomor 1. Consider

> Tujuh buah partikel klasik didistribusikan pada lima tingkat energi dengan $\epsilon_s = (s-1)$ eV dan $g_s = 3$, dengan indeks s adalah nomor tingkat energi. Jika energi total partikel adalah 8 eV, tentukanlah: Jumlah keadaan makro M dan spesifikasinya; Jumlah keadaan mikro untuk tiap M; Bobot konfigurasi total W_T; Bilangan penempatan rata-rata $\langle n_s \rangle$; dan Peluang untuk mendapatkan partikel dengan energi 2 eV pada pengambilan pertama

Keadaan sistem dapat digambarkan sebagai berikut.

Tabel: Keadaan sistem

s	g_s	$\epsilon_s = (s-1)$ eV
5	3	4
4	3	3
3	3	2
2	3	1
1	3	0

Sistem memiliki 12 keadaan makro dengan spesifikasi sebagai berikut.

$$
\begin{array}{rr}
M_1 = (5,0,0,0,2) & M_7 = (3,1,2,1,0) \\
M_2 = (4,1,0,1,1) & M_8 = (2,3,1,1,0) \\
M_3 = (4,0,2,0,1) & M_9 = (3,0,4,0,0) \\
M_4 = (2,4,0,0,1) & M_{10} = (2,2,3,0,0) \\
M_5 = (4,0,1,2,0) & M_{11} = (1,4,2,0,0) \\
M_6 = (3,2,0,2,0) & M_{12} = (0,6,1,0,0)
\end{array}
$$

Jumlah keadaan mikro untuk setiap keadaan makro adalah sebagai berikut.

$$
\begin{aligned}
W_1 &= 7! \frac{3^5}{5!} \frac{3^0}{0!} \frac{3^0}{0!} \frac{3^0}{0!} \frac{3^2}{2!} = 45\,927 \\
W_2 &= 7! \frac{3^4}{4!} \frac{3^1}{1!} \frac{3^0}{0!} \frac{3^1}{1!} \frac{3^1}{1!} = 459\,720 \\
W_3 &= 7! \frac{3^4}{4!} \frac{3^0}{0!} \frac{3^2}{2!} \frac{3^0}{0!} \frac{3^1}{1!} = 229\,635 \\
W_4 &= 7! \frac{3^2}{2!} \frac{3^4}{4!} \frac{3^0}{0!} \frac{3^0}{0!} \frac{3^1}{1!} = 229\,635 \\
W_5 &= 7! \frac{3^4}{4!} \frac{3^0}{0!} \frac{3^1}{1!} \frac{3^2}{2!} \frac{3^0}{0!} = 229\,635
\end{aligned}
$$

$$W_6 = 7!\frac{3^3}{3!}\frac{3^2}{2!}\frac{3^0}{0!}\frac{3^2}{2!}\frac{3^0}{0!} = 102\ 160$$
$$W_7 = 7!\frac{3^3}{3!}\frac{3^1}{1!}\frac{3^2}{2!}\frac{3^1}{1!}\frac{3^0}{0!} = 302\ 180$$
$$W_8 = 7!\frac{3^2}{2!}\frac{3^3}{3!}\frac{3^1}{1!}\frac{3^1}{1!}\frac{3^0}{0!} = 918\ 540$$
$$W_9 = 7!\frac{3^3}{3!}\frac{3^0}{0!}\frac{3^4}{4!}\frac{3^0}{0!}\frac{3^0}{0!} = 76\ 545$$
$$W_{10} = 7!\frac{3^2}{2!}\frac{3^2}{2!}\frac{3^3}{3!}\frac{3^0}{0!}\frac{3^0}{0!} = 459\ 720$$
$$W_{11} = 7!\frac{3^1}{1!}\frac{3^4}{4!}\frac{3^2}{2!}\frac{3^0}{0!}\frac{3^0}{0!} = 229\ 635$$
$$W_{12} = 7!\frac{3^0}{0!}\frac{3^6}{6!}\frac{3^1}{1!}\frac{3^0}{0!}\frac{3^0}{0!} = 15\ 309$$

Untuk menghitung bilangan penempatan rata, diperlukan jumlah keadaan mikro total:

$$\begin{aligned} W_T = \sum_{k=1}^{12} W_k &= 45\ 927 + 459\ 270 + 229\ 635 \\ &+ 229\ 635 + 229\ 635 + 102\ 060 + 306\ 180 + 918\ 540 \\ &+ 76\ 545 + 459\ 270 + 229\ 635 + 15\ 309 = 3\ 301\ 641 \end{aligned}$$

Jumlah partkel rata-rata untuk tingkat energi pertama $s = 1$ adalah

$$\begin{aligned} \langle n_1 \rangle = \frac{\sum_k n_1 W_k}{W_T} &= \frac{1}{3\ 301\ 641}(5 \cdot 45927 + 4 \cdot 459270 + 4 \cdot 229635 \cdot \\ &+ 2 \cdot 229635 \cdot +4 \cdot 229635 \cdot +3 \cdot 102060 + 3 \cdot 306180 + 2 \cdot 918540 \\ &+ 3 \cdot 76545 + 2 \cdot 459270 + 1 \cdot 229635 + 0 \cdot 15309) \approx 2.666 \end{aligned}$$

Untuk tingkat energi kedua

$$\begin{aligned} \langle n_2 \rangle = \frac{\sum_k n_2 W_k}{W_T} &= \frac{1}{3\ 301\ 641}(0 \cdot 45927 + 1 \cdot 459270 + 0 \cdot 229635 \\ &+ 4 \cdot 229635 + 0 \cdot 229635 + 2 \cdot 102060 + 1 \cdot 306180 + 3 \cdot 918540 \\ &+ 0 \cdot 76545 + 2 \cdot 459270 + 4 \cdot 229635 + 6 \cdot 15309) \approx 1.991 \end{aligned}$$

Untuk tingkat energi ketiga

$$\begin{aligned} \langle n_3 \rangle = \frac{\sum_k n_3 W_k}{W_T} &= \frac{1}{3\ 301\ 641}(0 \cdot 45927 + 0 \cdot 459270 + 2 \cdot 229635 \\ &+ 0 \cdot 229635 + 1 \cdot 229635 + 0 \cdot 102060 + 2 \cdot 306180 + 1 \cdot 918540 \\ &+ 4 \cdot 76545 + 3 \cdot 459270 + 2 \cdot 229635 + 1 \cdot 15309) \approx 1.326 \end{aligned}$$

Untuk tingkat energi keempat

$$\begin{aligned} \langle n_4 \rangle = \frac{\sum_k n_4 W_k}{W_T} &= \frac{1}{3\ 301\ 641}(0 \cdot 45927 + 1 \cdot 459270 + 0 \cdot 229635 \\ &+ 0 \cdot 229635 + 2 \cdot 229635 + 2 \cdot 102060 + 1 \cdot 306180 + 1 \cdot 918540 \\ &+ 0 \cdot 76545 + 0 \cdot 459270 + 0 \cdot 229635 + 0 \cdot 15309) \approx 0.7109 \end{aligned}$$

Dan untuk tingkat kelima

$$\langle n_4 \rangle = \frac{\sum_k n_4 W_k}{W_T} = \frac{1}{3\ 301\ 641}(2 \cdot 45927 + 1 \cdot 459270 + 1 \cdot 229635$$
$$+ 1 \cdot 229635 + 0 \cdot 229635 + 0 \cdot 102060 + 0 \cdot 306180 + 0 \cdot 918540$$
$$+ 0 \cdot 76545 + 0 \cdot 459270 + 0 \cdot 229635 + 0 \cdot 15309) \approx 0.3060$$

Sebagai pembuktian,

$$N = \sum_s^5 n_s = 2.666 + 1.991 + 1.326 + 0.7109 + 0.3060 = 7$$

sesuai keadaan sistem.

Kemungkinan mendapatkan partikel dengan energi 2 eV, atau dengan tingkat energi ketiga, adalah

$$P(\epsilon_3) = \frac{\langle n_3 \rangle}{N} = \frac{1,326}{7} \approx 19\%$$