# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Thursday, June 11th, 2015

Time: 09.00 – 13.00

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

*Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.*

**Problem 1 – Background questions (AST5220 and AST9420)**

Answer each question with three or four sentences.

a) Write down the Boltzmann equation on schematic form for a general distribution function $f$. What is the physical interpretation of $f$?

b) Why do we solve the Einstein-Boltzmann equations in Fourier- space instead of real-space?

c) Why is it acceptable to set the curvature potential, $\Phi$, to 1 at early times when solving the Einstein-Boltzmann equations?

d) The Boltzmann equation for two-particle processes reads

$$a^{-3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left( \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right), \tag{1}$$

where $n_i$ is the density of particle species $i = \{1, 2, 3, 4\}$, $n_i^{(0)}$ is the corresponding average density, $\langle \sigma v \rangle$ is a thermally averaged cross-section, and $a$ is the usual cosmological scale factor. What is the corresponding Saha equation, and why does it hold?

e) The Einstein equation reads

$$E_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2}$$

What does the left- and right-hand sides of this equation describe, respectively?

f) What is the main advantage of the line-of-sight integration method for computing the CMB power spectrum?

**Problem 2 – Cosmological parameters (AST5220 and AST9420)**

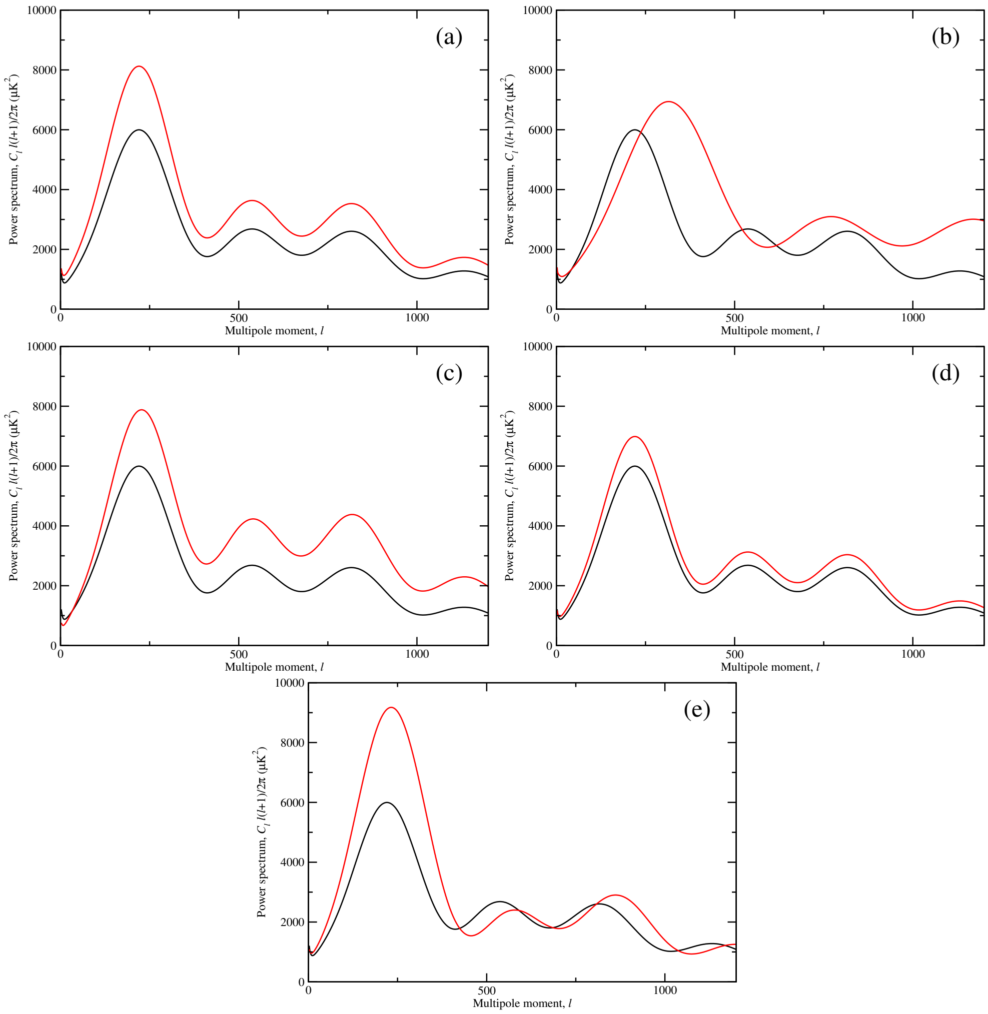


Figure 1: CMB spectra for different cosmological parameters.

Each of the panels marked (a)-(e) shows the CMB temperature power spectrum for two different cosmological parameter combinations. The black curve shows the standard best-fit ΛCDM spectrum, while the red curve shows a spectrum for which *one* parameter has been changed. **In each case, state which parameter was changed, and describe why it must be that parameter.**

**Problem 3 – The geodesic equation (AST5220 and AST9420)**

In this problem we will consider the geodesic equation.

a) What does Einstein's equivalence principle state regarding an observer in free fall?

b) The equation of motion for a particle in free fall reads

$$\frac{d^2\xi^\mu}{d\tau^2} = 0, \tag{3}$$

where $\xi^\mu$ is the four-position in a coordinate system following the particle, and $\tau$ is the eigentime. Derive the geodesic equation from this by performing a coordinate transformation to an arbitrary coordinate system $x^\mu$. It may be useful to know that the Christoffel symbols may be written as

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \tag{4}$$

c) What are the numerical values of the Christoffel symbols in Euclidean space? What does this tell us about a free particle moving in a flat space?

d) The synchronous gauge is defined through the following line element,

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{5}$$

where

$$h_{ij} = \begin{bmatrix} -2\phi & 0 & 0 \\ 0 & -2\phi & 0 \\ 0 & 0 & h+4\phi \end{bmatrix}, \tag{6}$$

and $\phi = \phi(x,t)$, $h = h(x,t)$ are two general perturbation fields. Consider the special case with $\phi = 0$, and compute $\Gamma^0_{11}$ and $\Gamma^3_{33}$. (PS! The general expression for the Christoffel symbols in terms of the metric is given in the Appendix.)

**Problem 4 – The Boltzmann equation for free photons (AST5220)**

In this problem you will derive a few critical components needed for the Boltzmann equation for free photons, ie., neglecting any collision terms, in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2).$$

a) For a massless photon one has $g_{\mu\nu}P^\mu P^\nu = 0$, where $P^\mu = dx^\mu/d\lambda$ is the four-momentum of the photon. Define $p^2 = g_{ij}P^iP^j$. Show that

$$P^0 = p(1-\Psi)$$
$$P^i = \frac{p}{a}\hat{p}(1-\Phi)$$

to first order in $\Psi$ and $\Phi$, where $\hat{p}$ is the unit vector pointing along the 3-momentum of the photon.

b) Show that

$$\frac{dx^i}{dt} = \frac{\hat{p}}{a}(1-\Phi+\Psi)$$

to first order. Give a physical interpretation of this equation.

c) Write down the Boltzmann equation schematically in term of partial derivatives over each of its dynamic variables, $x^i$, $t$, $p$ and $\hat{p}^i$. Why can the term dependent on $\hat{p}$ be neglected?

d) Starting from the 0-component of the geodesic equation, show that

$$\frac{dp}{dt} = p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

to first order.

e) Inserting both b) and d) into c), adopting a Bose-Einstein distribution to first order in $\Theta$, adopting conformal time, and Fourier transforming the equation eventually yields the following expression:

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b], \tag{7}$$

where $\Theta = \Theta_k(\eta,\mu)$ etc. From this expression, derive the corresponding differential equation for the monopole expansion, $\Theta_0$, knowing that the two lowest-order Legendre polynomials are $P_0(\mu) = 1$ and $P_1(\mu) = \mu$, and that $\Theta_l = \frac{i^l}{2}\int_{-1}^{1}\Theta(\mu)P_l(\mu)d\mu$.

## Problem 5 – Line-of-sight integration (AST9420)

In this problem, we will derive the expression for the transfer function, $\Theta_l(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$
\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
j_l(x) &= \frac{i^l}{2} \int_{-1}^{1} e^{-i\mu x} P_l(\mu) d\mu \\
f_l &= \frac{i^l}{2} \int_{-1}^{1} f(\mu) P_l(\mu) d\mu
\end{aligned}
$$

Here $j_l(x)$ is the spherical Bessel function of order $l$, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and $\mu$ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

b) Assume that $\tilde{S}$ does not depend on $\mu$ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta_0 - \eta)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

c) In reality, $\tilde{S}$ does of course depend on $\mu$, and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and $\mu$ and $k(\eta-\eta_0)$ are therefore Fourier conjugate (just like $k$ and $x$). This allows us to set

$$\mu \to \frac{1}{ik}\frac{d}{d\eta}$$

everywhere $\mu$ appears in $\tilde{S}$, just like we can set $ik \to d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0-\eta)] d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{v_b\dot{\tau}}{k}\right)\right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

# 1 Appendix

## 1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{8}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{9}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{10}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{11}$$

  where $\rho$ is the density of the fluid and $p$ is the pressure.

## 1.2 Background cosmology

- Four "time" variables: $t$ = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, $a$ = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{12}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{13}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{14}$$

## 1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{15}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{16}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{17}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{18}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{19}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{20}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{21}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{22}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{23}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{24}$$

## 1.4 Initial conditions

$$\Phi = 1 \tag{25}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{26}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{27}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{28}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{29}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{30}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{31}$$

## 1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{32}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{33}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{34}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{35}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{36}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left( \frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{37}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[ \beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{38}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{39}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1} \tag{40}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{41}$$

$$n_{1s} = (1 - X_e) n_H \tag{42}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0 / 4T_b} \tag{43}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left( \frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b} \tag{44}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{45}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0 / T_b) \tag{46}$$

## 1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{47}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{48}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{49}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{50}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{51}$$