Astronomy & Astrophysics manuscript no. cosmology2_report
March 21, 2023

Calculate the CMB power spectrum: Cosmology II

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March 21, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

G - Gravitational constant.
$G = 6.6743\times10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.3806\times10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.0546\times10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.9979\times10^{8}$ m s^{-1}.

Cosmological parameters

H - Hubble parameter.
H_0 - Hubble constant fill in stuff.
$e^x\mathcal{H}$ - Scaled Hubble parameter.
$T_{\rm CMB0}$ - Temperature of CMB today.
$T_{\rm CMB0} = 2.7255$ K.
η - Conformal time.
χ - Co-moving distance.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

- Cosmological principle

- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

TODO: Obviously this introduction will change and amended as more milestones are completed.

2. Milestone II

The main goal of this section is to investigate the recombination history of the universe. This can be explained as the point in time when photons decouple from the equilibrium of the opaque, early universe. When this happens, photons scatter for the last time at the *time of last scattering*, and these photons are what we today observe as the CMB. This period of the history of the universe is thus crucial for understanding the CMB.

We will start by calculating the free *electron fraction* X_e, from which we may find the *optical depth* τ. This again enables us to compute the *visibility function*, g, and the *sound horizon*, s. The latter will be of great importance later.

2.1. Theory

Before recombination, the equilibrium between protons, electrons and photons is governed by the following interaction:

$$e^- + p^+ \leftrightharpoons H + \gamma, \tag{2}$$

where a proton and an electron interact to form an excited hydrogen atom, which decays and emits a photon, or a photon excites and split a hydrogen atom into an electron and a proton. This is a reaction of the form $1 + 2 \leftrightharpoons 3 + 4$, and we have from Winther et al. (2023) that the Boltzmann equation for such a reaction is:

$$\frac{1}{n_1 e^{3x}} \frac{\mathrm{d}(n_1 e^{3x})}{\mathrm{d}x} = -\frac{\Gamma_1}{H} \left(1 - \frac{n_3 n_4}{n_1 n_2} \left(\frac{n_1 n_2}{n_3 n_4} \right)_{\mathrm{eq}} \right), \tag{3}$$

where n_i are the number densities of the reactants, Γ_1 is the reaction rate and H the Hubble parameter (expansion rate of the universe). If the reaction rate is much larger than the expansion rate of the universe, $\Gamma_1 \gg H$, then Eq. (2) ensures equilibrium between protons, electron and photons. When Γ_1 drops below H, then the expansion rate becomes dominant and the reaction rate is unable to sustain equilibrium. This happens when the temperature of universe becomes lower than the binding energy of hydrogen, hence stable neutral hydrogen is able to form. As a consequence, the photons *decouple* from the protons and electron. When $\Gamma_1 \ll H$, there are practically no interactions and the number density becomes constant for a comoving volume. Massive particles *freeze out* and their abundance become constant.

2.1.1. Free electron fraction X_e

We express the electron density through the free electron fraction $X_e \equiv n_e/n_\mathrm{H} = n_e/n_b$ where we have assumed that hydrogen make up all the baryons ($n_b = n_\mathrm{H}$). We also ignore the difference between free protons and neutral hydrogen. From Callin (2006) we obtain:

$$n_b = \frac{\rho_b}{m_\mathrm{H}} = \frac{\Omega_b \rho_c}{m_\mathrm{H}} e^{-3x}, \tag{4}$$

where m_H is the mass of the hydrogen atom, and ρ_c the critical density today as defined earlier. Before recombination, no stable neutral hydrogen is formed, thus the electron and baryon density is the same, i.e. there are only free electrons so $X_e \simeq 1$. When in equilibrium, the r.h.s. of Eq. (3) reduces to 0, which is called the *saha approximation*. The solution is in this regime described by the *Saha equation*, which from Dodelson & Schmidt (2020) in physical units is:

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{k_B m_e T_b}{2\pi\hbar^2} \right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{5}$$

where $\epsilon_0 = 13.6$ eV is the ionisation energy of hydrogen. The Saha equation is only a good approximation when $X_e \simeq 1$. Thus for $X_e < (1 - \xi)$,[1] which corresponds to the period during and after recombination, we have to make use of the more accurate *Peebles equation*. From Callin (2006):

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T_b)}{H} \left[\beta(T_b)(1 - X_e) - n_\mathrm{H} \alpha^{(2)}(T_b) X_e^2 \right], \tag{6}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s-1s} + \Lambda_\alpha}{\Lambda_{2s-1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{6a}$$

$$\Lambda_{2s-1s} = 8.227 \text{ s}^{-1}, \tag{6b}$$

$$\Lambda_\alpha = \frac{1}{(\hbar c)^3} H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \tag{6c}$$

$$n_{1s} = (1 - X_e) n_\mathrm{H}, \tag{6d}$$

$$n_\mathrm{H} = (1 - Y_p) \frac{3 H_0^2 \Omega_{b0}}{8\pi G m_\mathrm{H}} e^{-3x}, \tag{6e}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4k_B T_b}, \tag{6f}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{k_B m_e T_b}{2\pi\hbar^2} \right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{6g}$$

$$\alpha^{(2)}(T_b) = \frac{\hbar^2}{c} \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{k_B T_b}} \phi_2(T_b), \tag{6h}$$

$$\phi_2(T_b) = 0.448 \ln \left(\frac{\epsilon_0}{k_B T_b} \right). \tag{6i}$$

TODO: Add σ_T and α to nomenclature.

TODO: Describe the above equations slightly

We find by X_e by solving Eq. (5) for $X_e > (1 - \xi)$ and Eq. (6) for $X_e < (1 - \xi)$. In theory, it is possible to solve the Peebles equation at very early times, but the equation is very stiff resulting in unstable numerical solutions at early times (high temperatures). TODO: Explain why, difficult to integrate Peebles at early times etc.

[1] Where ξ is some small tolerance, which have to be defined in some numerical model for when to abonden the Saha equation and use the more accurate, but computationally more expensive Peebles equation. This is typically $\xi = 0.001$

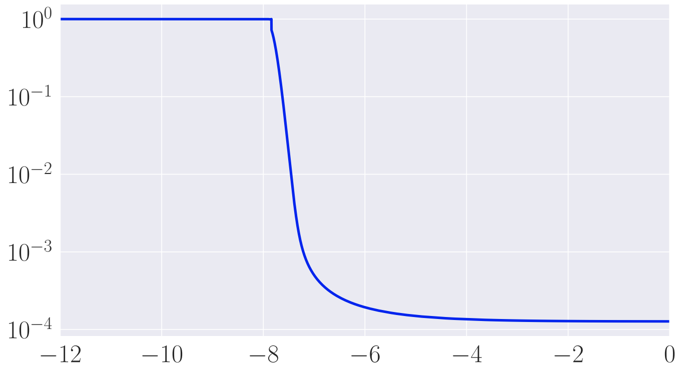

Fig. 1. SOMECAPTION Electron fraction

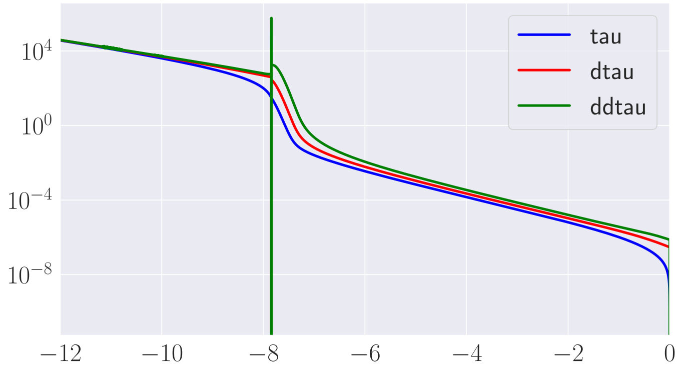

Fig. 2. SOMECAPTION tau of x

2.1.2. Optical depth τ

The optical depth as a function of conformal time is defines as Winther et al. (2023):

$$\tau = \int_{\eta}^{\eta_0} n_e \sigma_{\mathrm{T}} e^{-x} \mathrm{d}\eta', \tag{7}$$

where n_e is the electron density and σ_{T} is the Thompson cross-section. From this we define the visibility function, g:

$$\begin{aligned} g &= -\frac{\mathrm{d}\tau}{\mathrm{d}\eta} e^{-\tau} = -\mathcal{H}\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} \\ \tilde{g} &\equiv -\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} = \frac{g}{\mathcal{H}}, \end{aligned} \tag{8}$$

where $\tilde{g}$ is in terms of the preferred time variable, x. So far, so good, but in order to find $\tilde{g}$ we need τ, which again require n_e, which is not trivial to find, since the electron density changes throughout the evolution of the universe.

2.2. Methods

some methods

2.3. Results

References

Callin, P. 2006, How to calculate the CMB spectrum
Dodelson, S. & Schmidt, F. 2020, Modern Cosmology (Elsevier Science)
Weinberg, S. 2008, Cosmology, Cosmology (OUP Oxford)

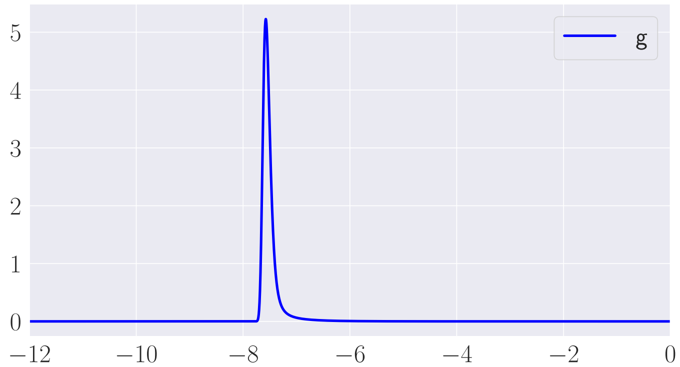

Fig. 3. SOMECAPTION g of x

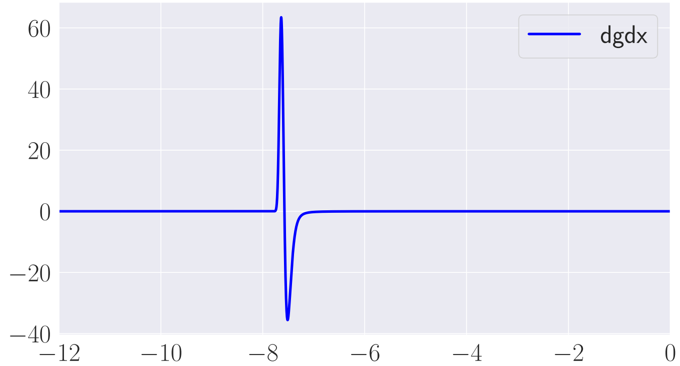

Fig. 4. SOMECAPTION dg of x

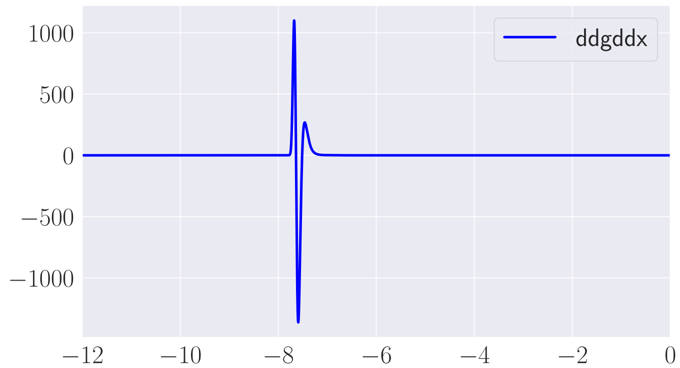

Fig. 5. SOMECAPTION ddg of x

Winther, H. A., Eriksen, H. K., Elgaroy, O., Mota, D. F., & Ihle, H. 2023, Cosmology II, https://cmb.wintherscoming.no/, accessed on March 1, 2023

Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element $\mathrm{d}s$ as given in equation **??**, and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x}r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}H = e^{-x}c \\ \implies \frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{c}{\mathcal{H}}. \end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x}c \\ \implies \frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{e^x}{\mathcal{H}} = \frac{1}{H}. \end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from **??** and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2}U^{-\frac{1}{2}}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right] \end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned} U &\approx \Omega_{i0}e^{-\alpha_i x} \\ \frac{\mathrm{d}^n U}{\mathrm{d}x^n} &\approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4} \end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned} \eta &= \int_{-\infty}^{x}\frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right), \end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} &= -\frac{\alpha_i}{2}\mathcal{H} \\ \implies \mathrm{d}x &= -\frac{2}{\alpha_i\mathcal{H}}\mathrm{d}\mathcal{H}. \end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.