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Calculate the CMB power spectrum: Cosmology II

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March 27, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

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Nomenclature

Constants of nature

G - Gravitational constant.
$G = 6.6743\times10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.3806 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.0546 \times 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.9979 \times 10^8$ m s^{-1}.

Cosmological parameters

H - Hubble parameter.
H_0 - Hubble constant fill in stuff.
$e^x\mathcal{H}$ - Scaled Hubble parameter.
T_{CMB0} - Temperature of CMB today.
$T_{\mathrm{CMB0}} = 2.7255$ K.
η - Conformal time.
χ - Co-moving distance.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

– Cosmological principle

- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

TODO: Obviously this introduction will change and amended as more milestones are completed.

2. Milestone II

The main goal of this section is to investigate the recombination history of the universe. This can be explained as the point in time when photons decouple from the equilibrium of the opaque, early universe. When this happens, photons scatter for the last time at the *time of last scattering*, and these photons are what we today observe as the CMB. This period of the history of the universe is thus crucial for understanding the CMB.

We will start by calculating the free *electron fraction* X_e, from which we may find the *optical depth* τ. This again enables us to compute the *visibility function*, g, and the *sound horizon*, s. The latter will be of great importance later.

2.1. Theory

Before recombination, the equilibrium between protons, electrons and photons is governed by the following interaction:

$$e^- + p^+ \leftrightharpoons H + \gamma, \tag{2}$$

where a proton and an electron interact to form an excited hydrogen atom, which decays and emits a photon, or a photon excites and split a hydrogen atom into an electron and a proton. This is a reaction of the form $1 + 2 \leftrightharpoons 3 + 4$, and we have from Winther et al. (2023) that the Boltzmann equation for such a reaction is:

$$\frac{1}{n_1 e^{3x}} \frac{\mathrm{d}(n_1 e^{3x})}{\mathrm{d}x} = -\frac{\Gamma_1}{H}\left(1 - \frac{n_3 n_4}{n_1 n_2}\left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}}\right), \tag{3}$$

where n_i are the number densities of the reactants, Γ_1 is the reaction rate and H the Hubble parameter (expansion rate of the universe). If the reaction rate is much larger than the expansion rate of the universe, $\Gamma_1 \gg H$, then Eq. (2) ensures equilibrium between protons, electron and photons. When Γ_1 drops below H, then the expansion rate becomes dominant and the reaction rate is unable to sustain equilibrium. This happens when the temperature of universe becomes lower than the binding energy of hydrogen, hence stable neutral hydrogen is able to form. As a consequence, the photons *decouple* from the protons and electron. When $\Gamma_1 \ll H$, there are practically no interactions and the number density becomes constant for a comoving volume. Massive particles *freeze out* and their abundance become constant.

2.1.1. Hydrogen recombination

We express the electron density through the free electron fraction $X_e \equiv n_e/n_{\mathrm{H}} = n_e/n_b$ where we have assumed that hydrogen make up all the baryons ($n_b = n_{\mathrm{H}}$). We also ignore the difference between free protons and neutral hydrogen. From Callin (2006) we obtain:

$$n_b = \frac{\rho_b}{m_{\mathrm{H}}} = \frac{\Omega_b \rho_c}{m_{\mathrm{H}}} e^{-3x}, \tag{4}$$

where m_{H} is the mass of the hydrogen atom, and ρ_c the critical density today as defined earlier. Before recombination, no stable neutral hydrogen is formed, thus the electron and baryon density is the same, i.e. there are only free electrons so $X_e \simeq 1$. When in equilibrium, the r.h.s. of Eq. (3) reduces to 0, which is called the *saha approximation*. The solution is in this regime described by the *Saha equation*, which from Dodelson & Schmidt (2020) in physical units is:

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{5}$$

where $\epsilon_0 = 13.6$ eV is the ionisation energy of hydrogen. The Saha equation is only a good approximation when $X_e \simeq 1$. Thus for $X_e < (1 - \xi)$,[1] which corresponds to the period during and after recombination, we have to make use of the more accurate *Peebles equation*. From Callin (2006):

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_{\mathrm{H}}\alpha^{(2)}(T_b)X_e^2\right], \tag{6}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s-1s} + \Lambda_\alpha}{\Lambda_{2s-1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{6a}$$

$$\Lambda_{2s-1s} = 8.227 \text{ s}^{-1}, \tag{6b}$$

$$\Lambda_\alpha = \frac{1}{(\hbar c)^3} H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \tag{6c}$$

$$n_{1s} = (1 - X_e)n_{\mathrm{H}}, \tag{6d}$$

$$n_{\mathrm{H}} = (1 - Y_p)\frac{3H_0^2\Omega_{b0}}{8\pi G m_{\mathrm{H}}} e^{-3x}, \tag{6e}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4k_B T_b}, \tag{6f}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{6g}$$

$$\alpha^{(2)}(T_b) = \frac{\hbar^2}{c}\frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{k_B T_b}}\phi_2(T_b), \tag{6h}$$

$$\phi_2(T_b) = 0.448 \ln\left(\frac{\epsilon_0}{k_B T_b}\right). \tag{6i}$$

TODO: Add σ_T and α to nomenclature.

TODO: Describe the above equations slightly

We find X_e by solving Eq. (5) for $X_e > (1 - \xi)$ and Eq. (6) for $X_e < (1 - \xi)$. In theory, it is possible to solve the Peebles equation at very early times, but the equation is very stiff resulting in unstable numerical solutions at early times (high temperatures), hence the Saha approximation.

[1] Where ξ is some small tolerance, which have to be defined in some numerical model for when to abonden the Saha equation and use the more accurate, but computationally more expensive Peebles equation. This is typically $\xi = 0.001$

2.1.2. Visibility

The optical depth as a function of conformal time is defines as Winther et al. (2023):

$$\tau = \int_{\eta}^{\eta_0} n_e \sigma_{\mathrm{T}} e^{-x} \mathrm{d}\eta', \tag{7}$$

where n_e is the electron density and σ_{T} is the Thompson cross-section. In differential form, restoring original units, this is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{c n_e \sigma_T e^x}{\mathcal{H}}. \tag{8}$$

From this we define the visibility function, g:

$$g = -\frac{\mathrm{d}\tau}{\mathrm{d}\eta} e^{-\tau} = -\mathcal{H}\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau}$$
$$\tilde{g} \equiv -\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} = \frac{g}{\mathcal{H}}, \tag{9}$$

where $\tilde{g}$ is in terms of the preferred time variable, x. Notable thing about the visibility function $\tilde{g}$ is that it is a true probability distribution, describing the probability density of some photon to last have scattered at time x. Because of this, we have that $\int_{-\infty}^{0} \tilde{g}(x)\mathrm{d}x = 1$. We also take note of the derivative of the visibility function:

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} = e^{-\tau}\left[\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}^2\tau}{\mathrm{d}x^2}\right] \tag{10}$$

2.1.3. Sound horizon

Let's take a small step back and consider the situation of the early Universe. Before any decoupling, the photons and electrons are coupled through Thompson scattering, and protons and electrons are coupled through coulomb interactions. Because of this, photons interact with baryons and move alongside with them as one fluid, in which wave propagates with a speed c_s, from Dodelson & Schmidt (2020):

$$c_s \equiv c\left[3(1+R)\right]^{-\frac{1}{2}} \quad ; \quad R \equiv \frac{3\Omega_b}{4\Omega_\gamma}, \tag{11}$$

where R is the *baryon-to-photon energy ratio*. By the definition of R, if the baryon density is negligible compared to the radiation density, $R \sim 0$, and we recover the wave propagation speed in a relativistic fluid: $c_s = 3^{-1/2}$ (Dodelson & Schmidt (2020)). The total distance such a wave would have travelled in a time t (since the beginning of the Universe) is called the *sound horizon*, found by simply integrating c_s through time, accounting for the expansion of space itself by including a factor e^{-x}:

$$s = \int_0^t c_s e^{-x}\mathrm{d}t = \int_{-\infty}^{x} \frac{c_s}{\mathcal{H}}\mathrm{d}x, \tag{12}$$

where the variables are changed to x. On differential form:

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{c_s}{\mathcal{H}}, \tag{13}$$

which is a straightforward differential equation to solve given some initial conditions.

2.2. Methods

2.2.1. Computing X_e

First things first, we need to compute the free electron fraction X_e. We are for the most part not interested in things happening in the future here, so the temporal range of choice will be $x \in [-20, 0)$ where $x = 0$ is today, and $x = -20$ is sufficiently long ago, so that the range encapsulated effect studied here. In the early Universe, the energies are so high that all baryonic matter is in the form of free electron, $X_e \simeq 1$, so we will start by solving the Saha equation, Eq. (5). We continue to solve equation Eq. (5) as long as $X_e > 1 - \xi$ where we use $\xi = 0.01$.

If we define:

$$K \equiv \frac{1}{n_b}\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{14}$$

then equaton Eq. (5) takes the form $X_e^2 + KX_e - K = 0$, which is solved as a normal quadratic equation[2], where $a = 1$, $b = K$ and $c = -K$. Since $0 \leq X_e \leq 1$ we choose the positive solution, given by:

$$X_e = \frac{-K + \sqrt{K^2 + 4K}}{2} = \frac{K}{2}\left(-1 + \sqrt{1 + 4K^{-1}}\right) \tag{15}$$

This solution has the potential to become numerically unstable if the parenthesis is close to zero, i.e. for $K \gg 1$. We then make use of the approximation $\sqrt{1+4K^{-1}} \approx 1 + (2K^{-1})$ for $\left|4K^{-1}\right| \ll 1$, which ensures $X_e \simeq 1$ for very high temperatures (large K).

We continue to solve the Peebles equation as stated in Eq. (6), where the r.h.s. is implemented sequentially as Eq. (6a)- Eq. (6i) in reverse order. The initial condition is the last computed electron fraction above the cut-off: $X_{e0} = \min(X_e > 1 - \xi)$ as found from the Saha equation. It is solved for the x-range not solved by the Saha equation.

Having found X_e for the entire x-range, we compute n_e and spline both results.

2.2.2. Computing τ and $\tilde{g}$

With n_e we are able to solve the optical depth as defined in Eq. (8). The inital condition for this equation is that the optical depth today is zero: $\tau(x = 0) = 0$, meaning we have to solve this backwards in time. This is done by using the negative differential:

$$\frac{\mathrm{d}\tau_{\mathrm{rev}}}{\mathrm{d}x_{\mathrm{rev}}} = -\frac{\mathrm{d}\tau}{\mathrm{d}x} = \frac{c n_e \sigma_T e^x}{\mathcal{H}}, \tag{16}$$

and solving for positive x_{rev}: $x_{\mathrm{rev}} \in [0, 20]$. In order to undo this reversal, we map $\tau = -\tau_{\mathrm{rev}}$ to its corresponding $x = -x_{\mathrm{rev}}$. Having found τ, we find its derivative by solving equation Eq. (8), and further the find the visibility function from Eq. (9) and its derivative from Eq. (10). All of these four quantities are splines, and their derivatives are obtained numerically.

[2] $ay^2 + by + c = 0$ has solutions

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

Phenomenon	x	z	t [Myr]	r_s [Mpc]
Last scattering	-6.9853	1079.67	0.3780	145.31
Recombination	-6.9855	1079.83	0.3779	145.29
Saha	-7.1404	1260.89	0.2909	131.03

Table 1. The times of last scattering and recombination given in terms of x, the redshift z, the cosmic time t and the sound horizon r_s. Also included is the time of recombination found using the Saha approximation only.

In order to solve equation Eq. (12) for the sound horizon, we choose initial conditions $s_i = c_{s,i}/\mathcal{H}_i$ where the subscript i denote a very early time (in our case when $x = -20$). We are then able to solve the differential equation for the sound horizon, Eq. (13), numerically and then spline the result.

2.2.3. Analysis

Having splines for the relevant quantities enables us to compute some important times in the early universe. Firstly, the *last scattering surface*, is the time when most photons scattered for the last time, and decoupled from the plasma. This is not expected to have happened instantly, but recalling that the visibility function $\tilde{g}$ is a probability distribution function for when photons last scattered, we simply use the peak of this function as the definition of the last scattering surface.

Further, we want to find a time for when recombination happened, i.e. when free electron was captured by protons to form hydrogen atoms. Thus, this coincides with the reduction of the free electron fraction, and we will use $X_e = 0.1$ as the definition for when recombination happened. These numbers can also be computed using only the Saha approximation, for comparison. We also compute the sound horizon at these decouplings: $r_s = s(x_{\mathrm{dec}})$.

The last thing we want to compute is the freeze out abundance of free electrons, i.e. the free electron abundance today, which is found by evaluating the spline for X_e at $x = 0$.

2.3. Results and discussion

The relevant times for last scattering, recombination and Saha recombination is obtained as explained in Section 2.2.3, and presented in Table 1. These times are given in terms of x, the redshift z and the cosmic time t (in Myr). The sound horizon is given in units of megaparsecs (Mpc). Last scattering occurred when $x = -6.9853$, at redshift $z = 1079.67$, which is slightly after recombination when $x = -6.9855$ at redshift $z = 1079.83$. If the Saha approximation for valid when the electron fraction dropped, recombination would have happened when $x = -7.1404$ at redshift 1260.89 which is significantly earlier.

Fig. 1 shows the free electron fraction X_e as a function of x found using both the Saha and Peebles equation, as explained in Section 2.2.1, in blue. Also shown is the results found from the Saha equation only, which tends to zero a lot faster. This is used for comparison only, as we have already stated that the Saha approximation is only valid for $X_e \simeq 1$. The time of recombination is shown for both cases, which for the Saha approximation happens significantly earlier than what is the actual case. The Peebles solution falls off gradually, and converges towards a constant value, which is the present day abundance of free electrons (freeze out abundance). This is found to be $X_e(x = 0) = 0.0002$, shown as a brown dashed line in Fig. 1.

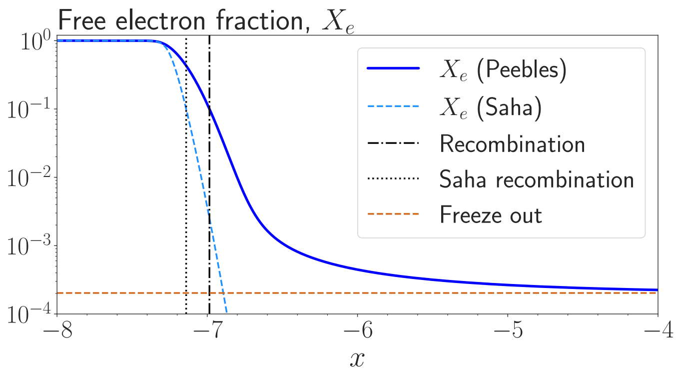

Fig. 1. The free electron fraction X_e as function of x, found from the Saha and Peebles equation (blue). The result using only the Saha equation is shown in dashed light blue. The time of recombination is shown as a dashed black line. Likewise, recombination in the Saha approximation is shown as a dotted black line, appearing earlier. The freeze out abundance of hydrogen (the present value) is shown as a brown dashed line.

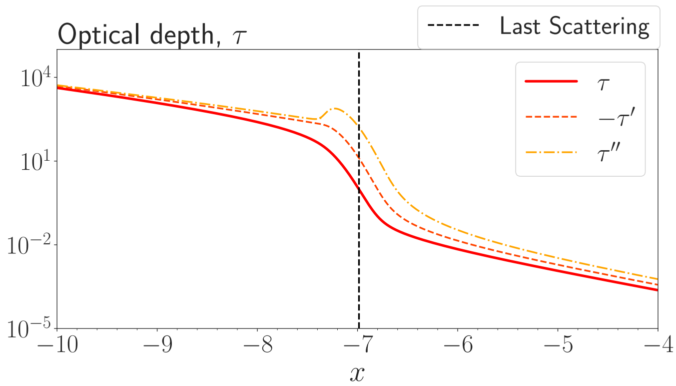

Fig. 2. The optical depth τ and its first and second derivatives as functions of x. The time of last scattering is shown as a dashed black line, before which the Universe was optically thick. Right before last scattering can e find a peak in the

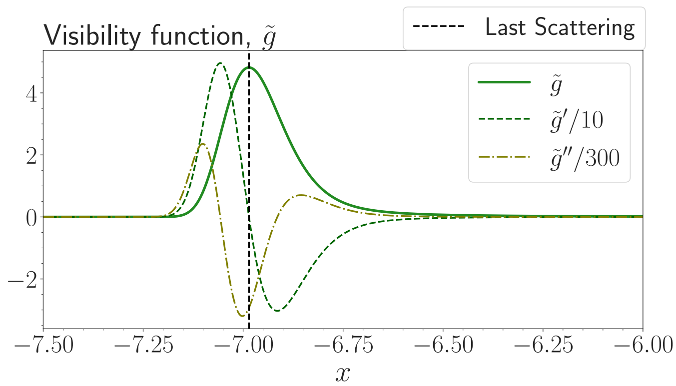

Fig. 3. Somecaption

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Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element $\mathrm{d}s$ as given in equation **??**, and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x} r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x} H = e^{-x} c \\ \implies \frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{c}{\mathcal{H}}. \end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x} c \\ \implies \frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{e^x}{\mathcal{H}} = \frac{1}{H}. \end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from **??** and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2} U^{-\frac{1}{2}} \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right] \end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned} U &\approx \Omega_{i0} e^{-\alpha_i x} \\ \frac{\mathrm{d}^n U}{\mathrm{d}x^n} &\approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2 \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4} \end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned} \eta &= \int_{-\infty}^{x} \frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right), \end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} &= -\frac{\alpha_i}{2}\mathcal{H} \\ \implies \mathrm{d}x &= -\frac{2}{\alpha_i \mathcal{H}}\mathrm{d}\mathcal{H}. \end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.