UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Tuesday, June 4th, 2013

Time: 09.00 – 13.00

The exam set consists of 13 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) Why are tensor equations important in General Relativity?

b) What do the Fermi-Dirac and Bose-Einstein distributions describe? What does the Maxwell distribution describe?

c) The Einstein equation for tensor perturbations reads

$$\ddot{h} + 2\frac{\dot{a}}{a}h + k^2 h = 0. \tag{1}$$

What sort of an equation is this, and how can it be quantized?

d) Why is it acceptable to set the curvature potential, Φ, 1 at early times when solving the Einstein-Boltzmann equations?

e) When deriving the Boltzmann equation for photons, one finds that

$$\frac{dx^i}{dt} \approx \frac{\hat{p}^i}{a}(1 - \Phi + \Psi), \tag{2}$$

where (t, x^i) is the photon position four-vector, $\hat{p}$ is the photon direction, a is the scale factor, and Φ and Ψ are the curvature and Newtonian potentials, respectively. What is the physical interpretation of this equation?

f) What is the main advantage of the CMB power spectrum line-of-sight integration method?

Problem 2 – Physical interpretation

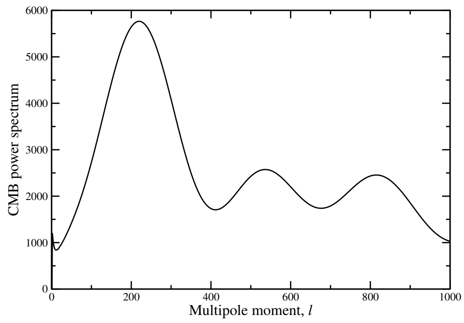

Figure 1: The best-fit LCDM WMAP temperature power spectrum.

Figure 1 shows the best-fit LCDM WMAP power spectrum as computed from the 7-year data release. In the following we will consider how this compares to the more recent Planck release in terms of cosmological parameters:

a) The WMAP best-fit value of the spectral index of scalar perturbations, $n_{\rm s}$, is 0.972 ± 0.013, while Planck finds 0.9616 ± 0.0094. How does this difference manifest itself in the power spectrum? Draw a cartoon illustrating the difference. What physical mechanism determines the value of n_s?

b) The best-fit WMAP value of the baryon density is $\Omega_b = 0.0463$, while Planck finds $\Omega_b = 0.049$. How is this seen in the observed power spectrum? Draw a cartoon to illustrate this. Explain this effect physically.

c) How does the optical depth, τ, affect the power spectrum at low and high multipoles? Explain why there is a strong degeneracy between the power spectrum amplitude, A, and the optical depth.

d) As the first experiment ever, Planck has sufficient angular resolution and sensitivity to exploit gravitational lensing for parameter estimation. This effect is due to CMB photons being bent by the presence of dark matter fluctuations between the last-scattering surface and ourselves, such that the photon appear to us to come from a slightly different direction than its true point of origin. How is the high-l power spectrum changed by these distortions, and how does this effect break the A-τ degeneracy?

Problem 3 – The Saha equation

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 >> kT$, ie., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{3}$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u}e^{-u}du = \frac{\sqrt{\pi}}{2}.$$

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here?

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times?

Problem 4 – Initial conditions

In order to solve the Boltzmann-Einstein equations, one needs initial conditions, and in this problem you will derive the appropriate expressions for Ψ, δ, δ_b, v, v_b and Θ_0, relating all to Φ. You should neglect neutrinos and polarization in this problem, and you need only consider adiabatic initial conditions.

a) Starting from the full set of perturbation equations listed in the Appendix, write down a simplified set that is valid for very early times. Why can terms containing factors of $k/\mathcal{H}$ be neglected?

b) Derive a second-order differential equation for Φ as a function of a. Show that this equation has two solutions. Which solution survives and is therefore cosmologically relevant?

c) Find the appropriate initial conditions for Θ_0, δ and δ_b expressed relative to Φ.

d) Why is $\Theta_3 << \Theta_2 << \Theta_1 << \Theta_0$ at early times?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{4}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{5}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{6}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{7}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{8}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{9}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{10}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{11}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{12}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{13}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{14}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{15}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{16}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{17}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{18}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{19}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{20}$$

1.4 Initial conditions

$$\Phi = 1 \tag{21}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{22}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{23}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{24}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{25}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{26}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{27}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a d\eta' \tag{28}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{29}$$

- Visibility function:

$$g(\eta) = -\dot{\tau}e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{30}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{31}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^0 \tilde{g}(x) dx = 1. \tag{32}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{33}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{34}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{35}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{36}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{37}$$

$$n_{1s} = (1-X_e)n_H \tag{38}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{39}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{40}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{41}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{42}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{43}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{44}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{45}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\propto}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{46}$$

3. The CMB spectrum:

$$C_l = \int_0^{\propto} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{47}$$