Astronomy & Astrophysics manuscript no. cosmology2_report
March 5, 2023 ©ESO 2023

Calculate the CMB power spectrum: Cosmology II

Johan Mylius Kroken[1,2]

[1] Institute of Theoretical Astrophysics (ITA), University of Oslo, Norway
[2] Center for Computing in Science Education (CCSE), University of Oslo, Norway

March 5, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

G - Gravitational constant.
$G = 6.6743\times10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.3806 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.0546 \times 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.9979 \times 10^8$ m s^{-1}.

Cosmological parameters

H - Hubble parameter.
H_0 - Hubble constant fill in stuff.
$e^x\mathcal{H}$ - Scaled Hubble parameter.
T_{CMB0} - Temperature of CMB today.
$T_{\mathrm{CMB0}} = 2.7255$ K.
η - Conformal time.
χ - Co-moving distance.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

– Cosmological principle

- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

2. Milestone I - Background Cosmology

Some introduction to milestone 1

2.1. Theory

2.1.1. Fundamentals

If we assume the universe to be homogeneous and isotropic, the line elements ds is given by the FLWR-metric as follows (in polar coordinates)(Weinberg 2008, eq. 1.1.11):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2x}\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right], \tag{2}$$

where we have introduced $x(t) = \ln(a(t))$, the logarithm of the scale factor $a(t)$ include more (about k) as our first measure of time.

We further model all forms of energy in the universe as perfect fluids, only characterised by their rest frame density ρ and isotropic pressure p, and an equation of state relating the two:

$$\omega = \frac{\rho}{p}. \tag{3}$$

By conservation of energy and momentum we must satisfy $\nabla_\mu T^{\mu\nu} = 0$, which results in the following differential equations for the density include more here? of each fluid ρ_i:

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + 3H\rho_i(1+\omega_i) = 0, \tag{4}$$

where we have introduced the Hubble parameter $H \equiv \dot{a}/a = \mathrm{d}x/\mathrm{d}t$. The solution to eq. 4 is of the form:

$$\rho_i \propto e^{-3(1+\omega_i)x}, \tag{5}$$

where $\omega_{\mathrm{M}} = 0$ (matter), $\omega_{\mathrm{rad}} = 1/3$ (radiation), $\omega_\Lambda = -1$ (cosmological constant) and $\omega_k = -1/3$ (curvature).

With these assumptions, the solution to the Einstein equations (eq. 1) are the Friedmann equations, the first of which describes the expansion rate of the universe:

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i - kc^2e^{-2x} \tag{6}$$

and the second describe how this expansion rate changes over time:

$$\frac{\mathrm{d}H}{\mathrm{d}t} + H^2 = -\frac{4\pi G}{3}\sum_i\left(\rho + \frac{3p}{c^2}\right). \tag{7}$$

As of now, we are primarily interested in the first Friedmann equation. By introducing the critical density, $\rho_c \equiv 2H^2/(8\pi G)$, we define the density parameters $\Omega_i = \rho_i/\rho_c$. We further define the density of the curvature $\rho_k \equiv -3kc^2e^{-2x}/(8\pi G)$, which enables us to write eq. 6 as simply:

$$1 = \sum_i \Omega_i, \tag{8}$$

where the curvature density Ω_k is included in the sum. From Eq. (5) we know the evolution of the densities in time, and if we assume the density values today, Ω_{i0}, are known (or are free parameters), then eq. 6 may also be written as:

$$H = H_0\sqrt{\sum_i \Omega_{i0}e^{-3(1+\omega_i)x}}, \tag{9}$$

which is the Hubble equation we will use further. FIXME: references - use cref

2.1.2. Measure of time and space

The main measure of time is usually the scale factor a, or its logarithm x. We then have the *cosmic time* t defined as:

$$t = \int_0^a \frac{\mathrm{d}a}{aH} = \int_{-\infty}^x \frac{\mathrm{d}x}{H}. \tag{10}$$

Another temporal measure is the *conformal time* η defined as $c\mathrm{d}t = e^x\mathrm{d}\eta$ yielding:

$$\eta = \int_0^a \frac{c\mathrm{d}a}{a^2H} = \int_{-\infty}^x \frac{c\mathrm{d}x}{e^xH} \equiv \int_{-\infty}^x \frac{c\mathrm{d}x}{\mathcal{H}}, \tag{11}$$

where $\mathcal{H} = e^xH$ is defined as the *conformal Hubble parameter*. We may also choose to measure time in terms of the *redshift* z, where $1+z = 1/a = e^{-x}$.

The comoving distance is defined as follows:

$$\chi = \int_1^a \frac{c\mathrm{d}a}{a^2H} = \int_0^x \frac{c\mathrm{d}x}{\mathcal{H}} = \eta_0 - \eta \tag{12}$$

The radial distance is given in terms of the comoving distance and the curvature density today Ω_{k0} as:

$$r = \begin{cases} \chi\cdot\frac{\sin\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} < 0 \\ \chi & \Omega_{k0} = 0 \\ \chi\cdot\frac{\sinh\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} > 0 \end{cases} \tag{13}$$

It is then straigtforward to define the angular diameter distance:

$$d_A = e^x r, \tag{14}$$

and the luminosity distance:

$$d_L = e^{-x} r, \tag{15}$$

both of which are derived in Appendix A. The temporal quantities η and t have the following evolutions with x:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}. \tag{16}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{H}. \tag{17}$$

Both differential equations are easy to solve numerically. Their derivation may also be found in Appendix A

2.1.3. ΛCDM-model

In the ΛCDM model, the universe consists of matter in terms of baryonic matter (b) and cold dark matter (CDM), radiation in terms of photons (γ) and neutrinos (ν) and dark energy in terms of a cosmological constant (Λ). In addition, we must allow for some curvature (k). As a result, the parameters of the model will be the present values of the Hubble rate, H_0, the baryon density Ω_{b0}, the cold dark matter density Ω_{CDM0}, photon density $\Omega_{\gamma 0}$, neutrino density $\Omega_{\nu 0}$, dark energy density $\Omega_{\Lambda 0}$, and the curvature density Ω_{k0}. The present temperature of the cosmic microwave background radiation T_{CMB0} fixes the radiation density today through:

$$\begin{aligned}
\Omega_{\gamma 0} &= \frac{16\pi^3 G}{90} \cdot \frac{(k_b T_{\mathrm{CMB0}})^4}{\hbar^3 c^5 H_0^2}, \\
\Omega_{\nu 0} &= N_{\mathrm{eff}} \cdot \frac{7}{8} \cdot \left(\frac{4}{3}\right)^{4/3} \cdot \Omega_{\gamma 0}.
\end{aligned} \tag{18}$$

The total radiation density is $\Omega_{\mathrm{rad}} = \Omega_\gamma + \Omega_\nu$ and the total matter density is $\Omega_{\mathrm{M}} = \Omega_b + \Omega_{\mathrm{CDM}}$. We are thus left with three fixed parameters.

The Hubble equation from Eq. (9) may be redefined in terms of the conformal Hubble parameter $\mathcal{H}$ as:

$$\begin{aligned}
\mathcal{H} &= H_0 \sqrt{U} \\
U &\equiv \sum_i \Omega_{i0} e^{-\alpha_i} x,
\end{aligned} \tag{19}$$

where we have defined $\alpha_i \equiv (1+3\omega_i)$ and $i \in \{\mathrm{M}, \mathrm{rad}, \Lambda, k\}$. Since we know the values of the various ω_i it follows that:

$$\begin{aligned}
\alpha_{\mathrm{M}} &= 1 \\
\alpha_{\mathrm{rad}} &= 2 \\
\alpha_k &= 0 \\
\alpha_\Lambda &= -2
\end{aligned} \tag{20}$$

2.1.4. Equalities and present day values

Given the evolution of the density parameters with time, where the proportionality constant is the present day density, we introduce the *radiation-matter equality*, i.e. the time radiation and matter densities were equal: $\rho_{\mathrm{rad}} = \rho_{\mathrm{M}}$. According to Eq. (5) this can be expressed as:

$$\begin{aligned}
&\rho_{\mathrm{rad0}} e^{-4x} = \rho_{\mathrm{M0}} e^{-3x} \\
&\qquad e^x = \frac{\rho_{\mathrm{rad0}}}{\rho_{\mathrm{M0}}} \implies x_{\mathrm{rM}} = \ln\left(\frac{\Omega_{\mathrm{rad0}}}{\Omega_{\mathrm{M0}}}\right),
\end{aligned} \tag{21}$$

where x_{rM} now denotes the time of radiation-matter equality.

Similarly, the *matter-dark energy equality*, where $\rho_{\mathrm{M}} = \rho_\Lambda$ can be found to be:

$$\begin{aligned}
&\rho_\Lambda = \rho_{\mathrm{M0}} e^{-3x} \\
&\implies x_{\mathrm{M}\Lambda} = \frac{1}{3} \ln\left(\frac{\Omega_{\mathrm{M0}}}{\Omega_\Lambda}\right)
\end{aligned} \tag{22}$$

The time of matter-dark energy equality coincides with when the universe starts to accelerate, since this acceleration is driven by the dark energy, represented by the cosmological constant. From this time onwards, dark energy dominates the universe, and thus accelerating the expansion.

The age of the universe today, and the conformal time today can both be found by evaluating the solutions to the differential equations of t and η at the present time (where $x = 0$). This is done numerically.

2.1.5. Analytical solutions and sanity checks

There are several ways we may check that both our workings and numerical implementations are indeed correct. The simplest way is to always ensure that the sum of all density parameters add up to 1, for all times: $\sum_i \Omega_i = 1$.

If we only consider the most dominant density parameter, that is $\Omega_i >> \sum_{j\neq i} \Omega_j$, we end up with the following analytical expressions for different temporal regimes:

$$\frac{1}{\mathcal{H}} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx -\frac{\alpha_i}{2} = \begin{cases} -1 & \alpha_{\mathrm{rad}} = 2 \\ -\frac{1}{2} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_\Lambda = -2 \end{cases} \tag{23}$$

$$\frac{1}{\mathcal{H}} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} \approx \frac{\alpha_i^2}{4} = \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ \frac{1}{4} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_\Lambda = -2 \end{cases} \tag{24}$$

$$\frac{\eta \mathcal{H}}{c} \approx \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ 2 & \alpha_{\mathrm{M}} = 1 \\ \infty & \alpha_\Lambda = -2 \end{cases} \tag{25}$$

These equations will be useful when making sure that the implementations are correct. For a thorough derivation, see Appendix B.

2.2. Methods

2.2.1. Initial equation

We have to consider the time evolution of the density parameters, given some present value, as function of our chosen time parameter, here x. The density evolution is implemented as:

$$\Omega_n = e^{-\alpha_n x} \Omega_{n0} \mathcal{H}_{\mathrm{rat}}^2 \tag{26}$$

where we have defined the ratio $\mathcal{H}_{\rm rat} \equiv H_0/\mathcal{H}$, and the new index n are all the densitis: $n \in \{b, \mathrm{CDM}, \gamma, \nu, \Lambda, k\}$.

We also implement functions to solve for the luminosity distance (Eq. (15)), angular distance (Eq. (14)), and the conformal distance (Eq. (12)).

2.2.2. ODEs and Splines

The differential equations for η (Eq. (16)) and t (Eq. (17)) are solved numerically as ordinary differential equations with the Runge-Kutta 4 as advancement method. The equations are solved for $x \in (-20, 5)$. As initial condition we would like $\eta(-\infty)$ which is obviously not possible to calculate, so we pick some very early time and use the analytical approximation in the radiation dominated era (Eq. (25)), which yield:

$$\eta(x_0) = \frac{c}{\mathcal{H}(x_0)}. \tag{27}$$

Likewise for t, the initial condition is:

$$t(x_0) = \frac{1}{2H(x_0)}. \tag{28}$$

We then proceed by making splines of both η and t in order to evaluate accurately for any $x \in (-20, 5)$.

2.2.3. Model evaluation

We evaluate the model by computing the quantities presented in Section 2.1.5 and compare with the analytical solutions in different regimes. This will ensure that the model behave as expected.

Furthermore, we want the model to somewhat resemble reality, we thus use measures of the luminosity distance of supernovas at different redshifts z, acquired by Betoule et al. (2014). This data is compared to the prediction made by our model.

In order to constrain the possible values $\Omega_{\rm M}$ and Ω_Λ we find the χ^2-error between the luminosity distance of the supernovas and the predictions made by our model. The Ω-s are sampled with Markov-Chain Monte Carlo sampling using the Metropolis-Hastings algorithm. The χ^2-erros is given by:

$$\chi^2(h, \Omega_{m0}, \Omega_{k0}) = \sum_{i=1}^{N} \frac{(d_L(z, \Omega_{m0}, \Omega_{k0}) - d_L^{\rm obs}(z_i))^2}{\sigma_i^2}. \tag{29}$$

2.3. Results

2.3.1. Tests

Fig. 1 show the evolution of the density fractions with time. They sum to one across all times which was required. At early times the radiation density dominates (orange line). The intersection between the orange and green lines mark the radiation-matter equality, after which matter is the dominating density. Likewise, the intersection between the green and purple lines mark the matter-dark energy equality, where dark energy (manifested in the cosmological constant) become the dominating density. Time can thus be divided into three regimes; radiation dominated, matter dominated, and dark energy dominated eras.

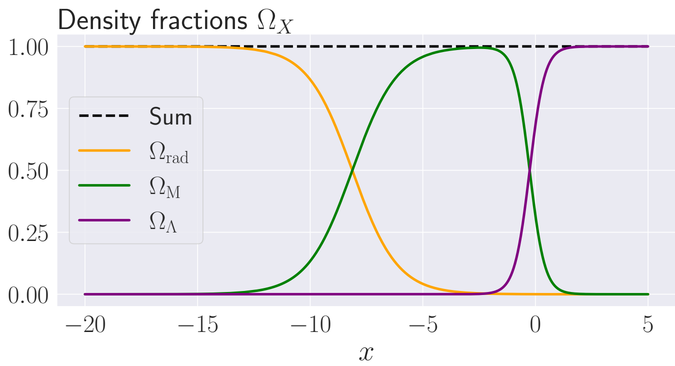

Fig. 1. Density fractions Ω_i as function of x. For low x, radiation dominates, before matter dominates and dark has just become the dominant energy density today $x = 0$, and will continue to dominate into the future. The sum of densities sums to one across all times, as required.

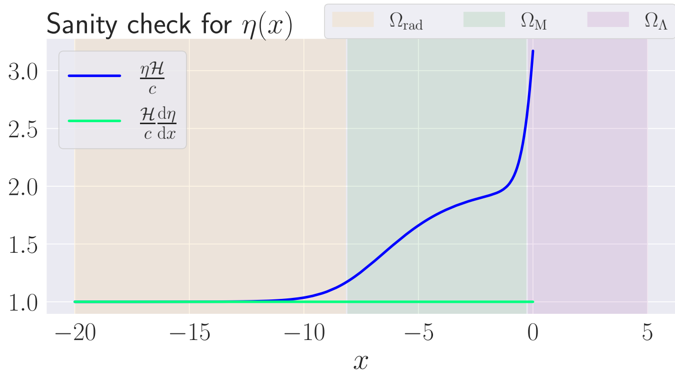

Fig. 2. Sanity check for η. $\eta\mathcal{H}/c$, in blue, is one in the radiation regime, two in the matter regime and diverging toward $+\infty$ in the dark energy regime, as expected from the analytical approximations in each regime. $(\mathrm{d}\eta/\mathrm{d}x)\mathcal{H}/c$, in green, is one throughout time, as expected from the differential equation for η.

As explained in section Section 2.1.5, we have analytical solutions for constructions of η and $\mathcal{H}$ in the different regimes. Fig. 2 is the sanity check for η, showing $\eta\mathcal{H}/c$ converging to finite values in the radiation and matter dominated eras (where $\alpha_{\rm rad,M} > 0$), and diverging towards $+\infty$ in the dark energy dominated era ($\alpha_\Lambda = -2 < 0$). This is in accordance with the analytical solutions. The different regimes are shown in shaded colour. It is also worth noticing that $(\mathrm{d}\eta/\mathrm{d}x)\mathcal{H}/c$ is one for all regimes, as expected from equation Eq. (16).

Fig. 3 is the sanity check confirming that our constructions of $\mathcal{H}$ and its derivatives converge to the analytical approximation in the different regimes. The second derivative, as shown in blue, takes the value of one in the radiation regime, one half in the matter regime and one in the dark energy regime. Similarly, the first derivative, as shown in green, take the value negative one in the radiation regime, negative one half in the matter regime and one in the dark energy regime. This is well in accordance with the analytical approximations put forth in section Section 2.1.5.

These sanity checks are confirmations that the implementation of the model yields the same result as the analytical approximation in the different regimes for various constructions of η and $\mathcal{H}$ and their derivatives.

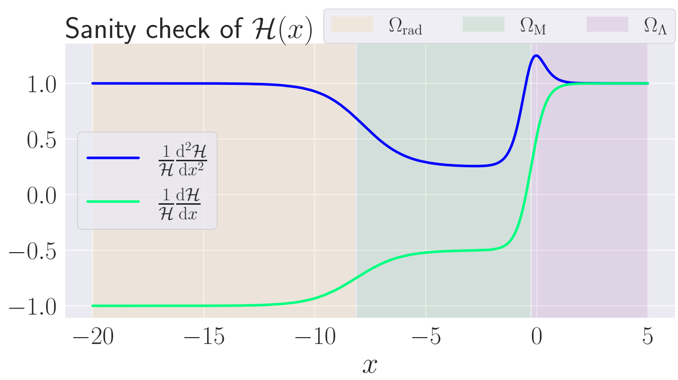

Fig. 3. Sanity check for $\mathcal{H}$, showing that the second derivative (blue) converge to one, one half, and one in the radiation, matter and dark energy regimes respectively. The first derivative (green) converge to negative one, negative one half and one in the same regimes, which are shown as a shaded background.

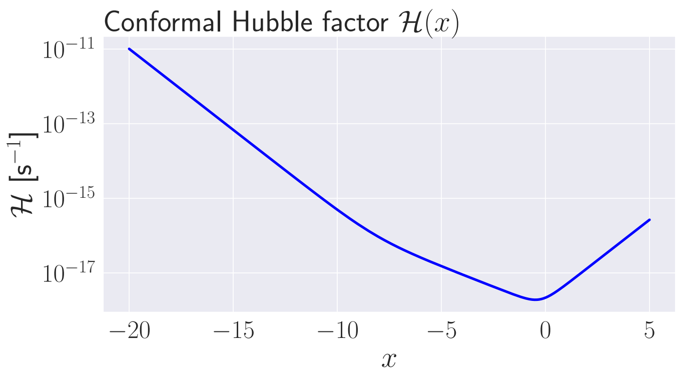

Fig. 4. $\mathcal{H}$ as function of x. It is decreasing in the radiation and matter regimes, and increasing in the dark energy regime.

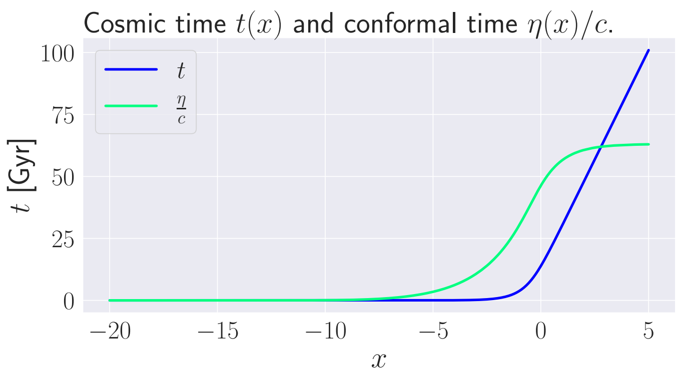

Fig. 5. Cosmic time (in blue) and conformal time (green). Cosmic times increase drastically in the dark energy regime showing a seemingly divergent behaviour. Conformal time increased in the matter regime and seem to converge in the dark energy regime.

2.3.2. Analysis

The conformal Hubble factor, $\mathcal{H}$, is plotted against time, x, in Fig. 4. It is decreasing in the radiation and matter regimes and increasing in the dark energy regime. Since this is a measure of the expansion of the universe, the acceleration seem to coincide with the matter-dark energy equality.

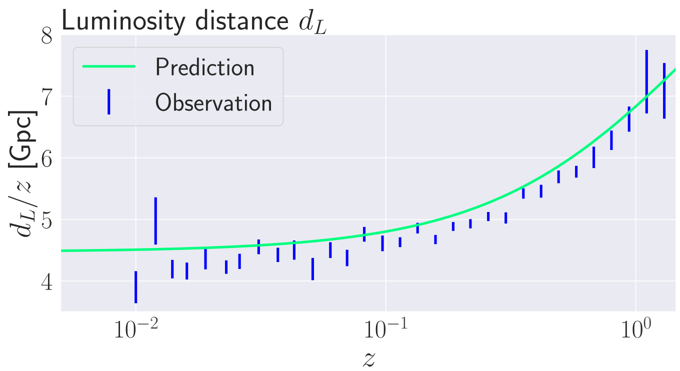

Fig. 6. Luminosity distance from supernova data shown in blue, and the prediction from the model in green. Notice the x-axis is now the redshift $z = e^x - 1$.

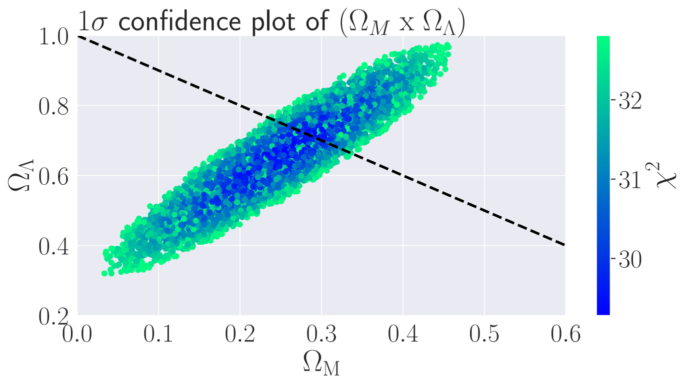

Fig. 7. Scatter plot showing the chi^2-error of the luminosity distance d_L between the observed values and the prediction, as function of $\Omega_{\rm M}$ and Ω_Λ. The data shown is within 1σ (standard deviation). The black dotted line signifies a flat universe.

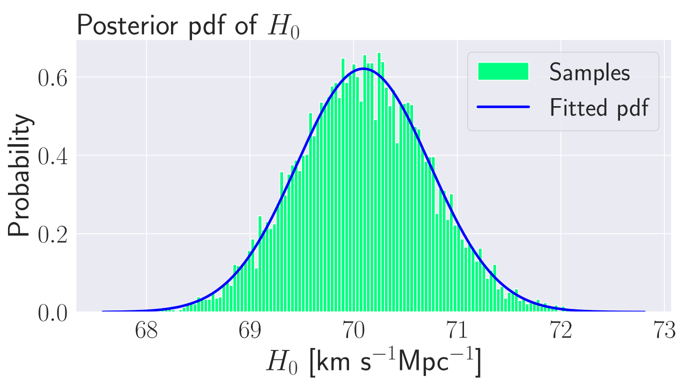

Fig. 8. Posterior probability distribution (pdf) of H_0 as result of the MCMC sampling. The samples are shown in green, and the construction pdf in blue.

Quantity	x	z	t [Gyr]
RM-equality	-8.13	3400.33	0.000051
ML-equality	-0.26	0.29	10.378200
Accel. start	-0.26	0.29	10.378200
Age of universe	0.00	0.00	13.857700
Conformal time	0.00	0.00	46.318700

3. Milestone II

Some introduction to milestone 2

3.1. Theory

Some theory

3.2. Methods

some methods

3.3. Results

4. Milestone III

Some introduction to milestone 3

4.1. Theory

Some theory

4.2. Methods

some methods

4.3. Results

5. Milestone IV

Some introduction to milestone 4

5.1. Theory

Some theory

5.2. Methods

some methods

5.3. Results

6. Conclusion

Some overall conclusion

References

Betoule, M., Kessler, R., Guy, J., et al. 2014, Astronomy & Astrophysics, 568, A22

Dodelson, S. & Schmidt, F. 2020, Modern Cosmology (Elsevier Science)

Weinberg, S. 2008, Cosmology, Cosmology (OUP Oxford)

Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element ds as given in equation Eq. (2), and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x} r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned} &\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x} H = e^{-x} c \\ \implies &\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}. \end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned} &\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x} c \\ \implies &\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{e^x}{\mathcal{H}} = \frac{1}{H}. \end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from Eq. (19) and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2} U^{-\frac{1}{2}} \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right] \end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned} &U \approx \Omega_{i0} e^{-\alpha_i x} \\ &\frac{\mathrm{d}^n U}{\mathrm{d}x^n} \approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2 \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4} \end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned} \eta &= \int_{-\infty}^{x} \frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right), \end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned} &\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -\frac{\alpha_i}{2}\mathcal{H} \\ \implies &\mathrm{d}x = -\frac{2}{\alpha_i \mathcal{H}}\mathrm{d}\mathcal{H}. \end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.