Astronomy & Astrophysics manuscript no. cosmology2_report
March 2, 2023

Calculate the CMB power spectrum: Cosmology II

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March 2, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

G - Gravitational constant.
$G = 6.6743\times10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.3806 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.0546 \times 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.9979 \times 10^8$ m s^{-1}.

Cosmological parameters

H - Hubble parameter.
H_0 - Hubble constant fill in stuff.
$e^x\mathcal{H}$ - Scaled Hubble parameter.
T_{CMB0} - Temperature of CMB today.
$T_{\mathrm{CMB0}} = 2.7255$ K.
η - Conformal time.
χ - Co-moving distance.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

– Cosmological principle

Fig. 1. Penguin making sure that you do all the work necessary!

- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

2. Milestone I - Background Cosmology

Some introduction to milestone 1

2.1. Theory

2.1.1. Fundamentals

If we assume the universe to be homogeneous and isotropic, the line elements $\mathrm{d}s$ is given by the FLWR-metric as follows (in polar coordinates)(Weinberg 2008, eq. 1.1.11):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2x}\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right], \tag{2}$$

where we have introduced $x(t) = \ln(a(t))$, the logarithm of the scale factor $a(t)$ include more (about k) as our first measure of time.

We further model all forms of energy in the universe as perfect fluids, only characterised by their rest frame density ρ and isotropic pressure p, and an equation of state relating the two:

$$\omega = \frac{\rho}{p}. \tag{3}$$

By conservation of energy and momentum we must satisfy $\nabla_\mu T^{\mu\nu} = 0$, which results in the following differential equations for the density include more here? of each fluid ρ_i:

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + 3H\rho_i(1+\omega_i) = 0, \tag{4}$$

where we have introduced the Hubble parameter $H \equiv \dot{a}/a = \mathrm{d}x/\mathrm{d}t$. The solution to eq. 4 is of the form:

$$\rho_i \propto e^{-3(1+\omega_i)x}, \tag{5}$$

where $\omega_\mathrm{M} = 0$ (matter), $\omega_\mathrm{rad} = 1/3$ (radiation), $\omega_\Lambda = -1$ (cosmological constant) and $\omega_k = -1/3$ (curvature).

With these assumptions, the solution to the Einstein equations (eq. 1) are the Friedmann equations, the first of which describes the expansion rate of the universe:

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i - kc^2 e^{-2x} \tag{6}$$

and the second describe how this expansion rate changes over time:

$$\frac{\mathrm{d}H}{\mathrm{d}t} + H^2 = -\frac{4\pi G}{3}\sum_i\left(\rho + \frac{3p}{c^2}\right). \tag{7}$$

As of now, we are primarily interested in the first Friedmann equation. By introducing the critical density, $\rho_c \equiv 2H^2/(8\pi G)$, we define the density parameters $\Omega_i = \rho_i/\rho_c$. We further define the density of the curvature $\rho_k \equiv -3kc^2e^{-2x}/(8\pi G)$, which enables us to write eq. 6 as simply:

$$1 = \sum_i \Omega_i, \tag{8}$$

where the curvature density Ω_k is included in the sum. From Eq. (5) we know the evolution of the densities in time, and if we assume the density values today, Ω_{i0}, are known (or are free parameters), then eq. 6 may also be written as:

$$H = H_0\sqrt{\sum_i \Omega_{i0}e^{-3(1+\omega_i)x}}, \tag{9}$$

which is the Hubble equation we will use further. FIXME: references - use cref

2.1.2. Measure of time and space

The main measure of time is usually the scale factor a, or its logarithm x. We then have the *cosmic time* t defined as:

$$t = \int_0^a \frac{\mathrm{d}a}{aH} = \int_{-\infty}^x \frac{\mathrm{d}x}{H}. \tag{10}$$

Another temporal measure is the *conformal time* η defined as $c\mathrm{d}t = e^x\mathrm{d}\eta$ yielding:

$$\eta = \int_0^a \frac{c\mathrm{d}a}{a^2H} = \int_{-\infty}^x \frac{c\mathrm{d}x}{e^xH} \equiv \int_{-\infty}^x \frac{c\mathrm{d}x}{\mathcal{H}}, \tag{11}$$

where $\mathcal{H} = e^xH$ is defined as the *conformal Hubble parameter*. We may also choose to measure time in terms of the *redshift* z, where $1+z = 1/a = e^{-x}$.

The comoving distance is defined as follows:

$$\chi = \int_1^a \frac{c\mathrm{d}a}{a^2 H} = \int_0^x \frac{c\mathrm{d}x}{\mathcal{H}} = \eta_0 - \eta \tag{12}$$

The radial distance is given in terms of the comoving distance and the curvature density today Ω_{k0} as:

$$r = \begin{cases} \chi \cdot \frac{\sin\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} < 0 \\ \chi & \Omega_{k0} = 0 \\ \chi \cdot \frac{\sinh\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} > 0 \end{cases} \tag{13}$$

It is then straigtforward to define the angular diameter distance:

$$d_A = e^x r, \tag{14}$$

and the luminosity distance:

$$d_L = e^{-x} r, \tag{15}$$

both of which are derived in Appendix A. The temporal quantities η and t have the following evolutions with x:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}. \tag{16}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{H}. \tag{17}$$

Both differential equations are easy to solve numerically. Their derivation may also be found in Appendix A

2.1.3. ΛCDM-model

In the ΛCDM model, the universe consists of matter in terms of baryonic matter (b) and cold dark matter (CDM), radiation in terms of photons (γ) and neutrinos (ν) and dark energy in terms of a cosmological constant (Λ). In addition, we must allow for some curvature (k). As a result, the parameters of the model will be the present values of the Hubble rate, H_0, the baryon density Ω_{b0}, the cold dark matter density Ω_{CDM0}, photon density $\Omega_{\gamma 0}$, neutrino density $\Omega_{\nu 0}$, dark energy density $\Omega_{\Lambda 0}$, and the curvature density Ω_{k0}. The present temperature of the cosmic microwave background radiation T_{CMB0} fixes the radiation density today through:

$$\begin{aligned} \Omega_{\gamma 0} &= \frac{16\pi^3 G}{90} \cdot \frac{(k_b T_{\mathrm{CMB0}})^4}{\hbar^3 c^5 H_0^2}, \\ \Omega_{\nu 0} &= N_{\mathrm{eff}} \cdot \frac{7}{8} \cdot \left(\frac{4}{3}\right)^{4/3} \cdot \Omega_{\gamma 0}. \end{aligned} \tag{18}$$

The total radiation density is $\Omega_{\mathrm{rad}} = \Omega_\gamma + \Omega_\nu$ and the total matter density is $\Omega_{\mathrm{M}} = \Omega_b + \Omega_{\mathrm{CDM}}$. We are thus left with three fixed parameters.

The Hubble equation from Eq. (9) may be redefined in terms of the conformal Hubble parameter $\mathcal{H}$ as:

$$\begin{aligned} \mathcal{H} &= H_0\sqrt{U} \\ U &\equiv \sum_i \Omega_{i0} e^{-\alpha_i} x, \end{aligned} \tag{19}$$

where we have defined $\alpha_i \equiv (1+3\omega_i)$ and $i \in \{\mathrm{M}, \mathrm{rad}, \Lambda, k\}$. Since we know the values of the various ω_i it follows that:

$$\begin{aligned} \alpha_{\mathrm{M}} &= 1 \\ \alpha_{\mathrm{rad}} &= 2 \\ \alpha_k &= 0 \\ \alpha_\Lambda &= -2 \end{aligned} \tag{20}$$

2.1.4. Equalities and present day values

TODO: something about the equalities and present day values.

2.1.5. Analytical solutions and sanity checks

There are several ways we may check that both our workings and numerical implementations are indeed correct. The simplest way is to always ensure that the sum of all density parameters add up to 1, for all times: $\sum_i \Omega_i = 1$.

If we only consider the most dominant density parameter, that is $\Omega_i >> \sum_{j\neq i} \Omega_j$, we end up with the following analytical expressions for different temporal regimes:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx -\frac{\alpha_i}{2} = \begin{cases} -1 & \alpha_{\mathrm{rad}} = 2 \\ -\frac{1}{2} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_\Lambda = -2 \end{cases} \tag{21}$$

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} \approx \frac{\alpha_i^2}{4} = \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ \frac{1}{4} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_\Lambda = -2 \end{cases} \tag{22}$$

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ 2 & \alpha_{\mathrm{M}} = 1 \\ \infty & \alpha_\Lambda = -2 \end{cases} \tag{23}$$

These equations will be useful when making sure that the implementations are correct. For a thorough derivation, see Appendix B.

TODO: find place for the below:

$$\chi^2(h, \Omega_{m0}, \Omega_{k0}) = \sum_{i=1}^N \frac{(d_L(z, \Omega_{m0}, \Omega_{k0}) - d_L^{\mathrm{obs}}(z_i))^2}{\sigma_i^2} \tag{24}$$

2.2. Methods

2.2.1. Initial equation

We have to consider the time evolution of the density parameters, given some present value, as function of our chosen time parameter, here x. The density evolution is implemented as:

$$\Omega_n = e^{-\alpha_n x}\Omega_{n0}\mathcal{H}_{\mathrm{rat}}^2 \tag{25}$$

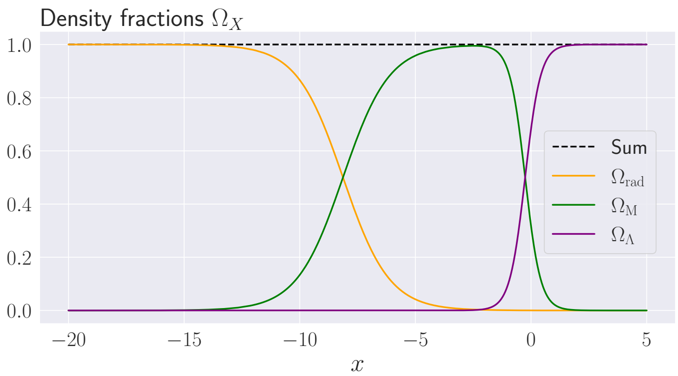

Fig. 2. Omega tests

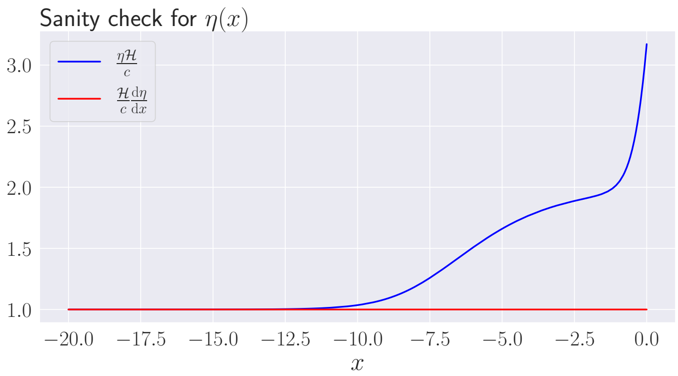

Fig. 3. Eta tests

where we have defined the ratio $\mathcal{H}_{\mathrm{rat}} \equiv H_0/\mathcal{H}$, and the new index n are all the densitis: $n \in \{b, \mathrm{CDM}, \gamma, \nu, \Lambda, k\}$.

We also implement functions to solve for the luminosity distance (Eq. (15)), angular distance (Eq. (14)), and the conformal distance (Eq. (12)).

2.2.2. ODEs and Splines

The differential equations for η (Eq. (16)) and t (Eq. (17)) are solved numerically as ordinary differential equations with the Runge-Kutta 4 as advancement method. The equations are solved for $x \in (-20, 5)$. As initial condition we would like $\eta(-\infty)$ which is obviously not possible to calculate, so we pick some very early time and use the analytical approximation in the radiation dominated era (Eq. (23)), which yield:

$$\eta(x_0) = \frac{c}{\mathcal{H}(x_0)}. \tag{26}$$

Likewise for t, the initial condition is:

$$t(x_0) = \frac{1}{2H(x_0)}. \tag{27}$$

FIXME: WHY divided by 2?

We then proceed by making splines of both η and t.

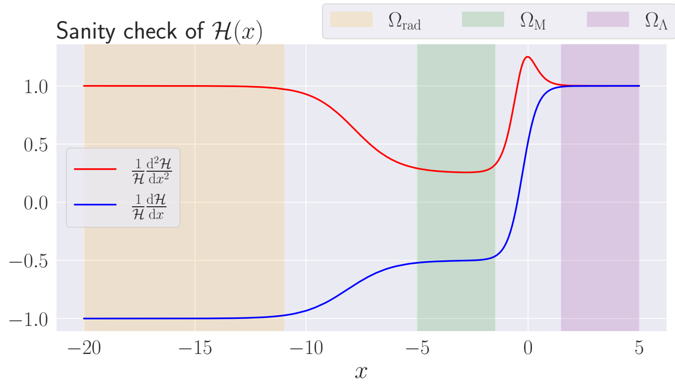

Fig. 4. HP tests

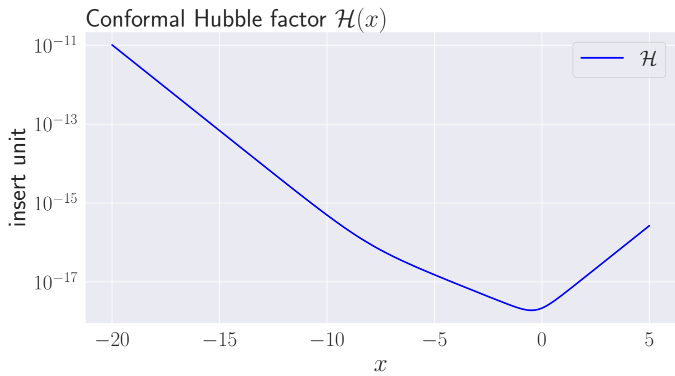

Fig. 5. Conformal Hubble factor.

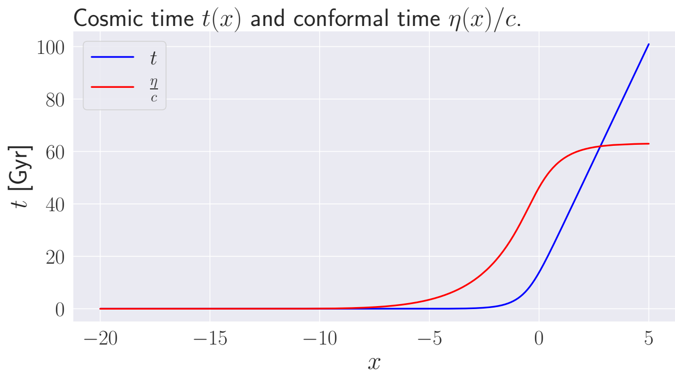

Fig. 6. cosmic time.

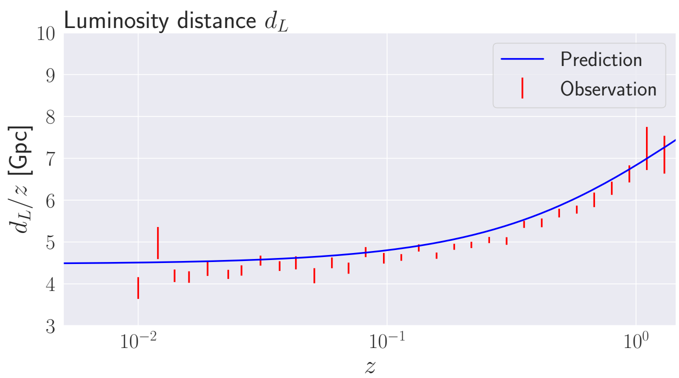

Fig. 7. Supernova data fitted

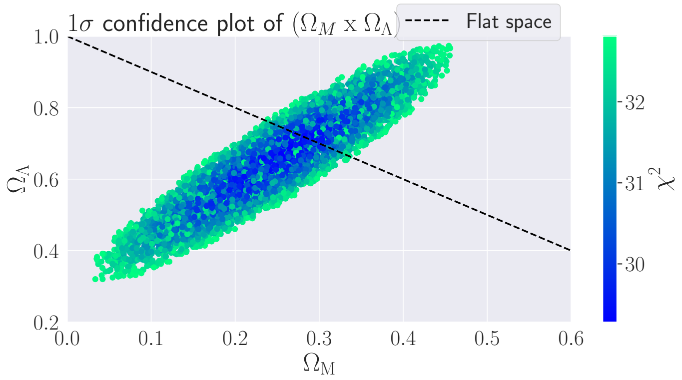

Fig. 8. one sigma confidence plot

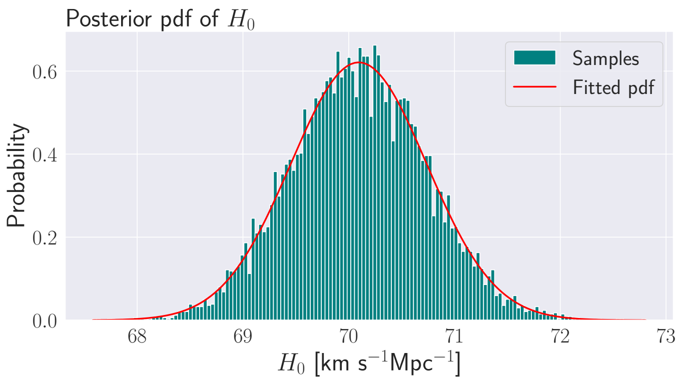

Fig. 9. posterior pdf.

2.3. Results

2.3.1. Tests

2.3.2. Analysis

3. Milestone II

Some introduction to milestone 2

3.1. Theory

Some theory

3.2. Methods

some methods

3.3. Results

4. Milestone III

Some introduction to milestone 3

4.1. Theory

Some theory

4.2. Methods

some methods

4.3. Results

5. Milestone IV

Some introduction to milestone 4

5.1. Theory

Some theory

5.2. Methods

some methods

5.3. Results

6. Conclusion

Some overall conclusion

References

Dodelson, S. & Schmidt, F. 2020, Modern Cosmology (Elsevier Science)

Weinberg, S. 2008, Cosmology, Cosmology (OUP Oxford)

Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta \theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element ds as given in equation Eq. (2), and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x} r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned}
&\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x} H = e^{-x} c \\
\implies &\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}.
\end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned}
&\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x} c \\
\implies &\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{e^x}{\mathcal{H}} = \frac{1}{H}.
\end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from Eq. (19) and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2} U^{-\frac{1}{2}} \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\
&= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\
&= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right]
\end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned}
\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\
&= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2
\end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned}
&U \approx \Omega_{i0} e^{-\alpha_i x} \\
&\frac{\mathrm{d}^n U}{\mathrm{d}x^n} \approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x},
\end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned}
\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha^2 \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\
&= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4}
\end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned}
\eta &= \int_{-\infty}^{x} \frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\
&= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right),
\end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -\frac{\alpha_i}{2}\mathcal{H} \\
\implies &\mathrm{d}x = -\frac{2}{\alpha_i \mathcal{H}}\mathrm{d}\mathcal{H}.
\end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.