UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Wednesday, June 14th, 2012

Time: 09.00 – 13.00

The exam set consists of 15 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) Write down the Boltzmann equation on schematic form, and expand the left-hand side into partial derivatives using x, p, $\hat{p}$ and t as free variables. Why can we neglect the term depending on $\hat{p}$?

b) What is the physical interpretation of the conformal time, η?

c) What is the main difference between dark matter and baryons?

d) How does inflation solve the so-called isotropy problem?

e) During tight coupling we neglect all photon moments except Θ_0, Θ_1 and Θ_2. Why can't we neglect also Θ_1 and Θ_2?

f) Why is the position of the first peak in the CMB power spectrum a sensitive probe of the total density of the universe? (Explain with a drawing if you think that is useful.)

Problem 2 – Recombination and the electron fraction

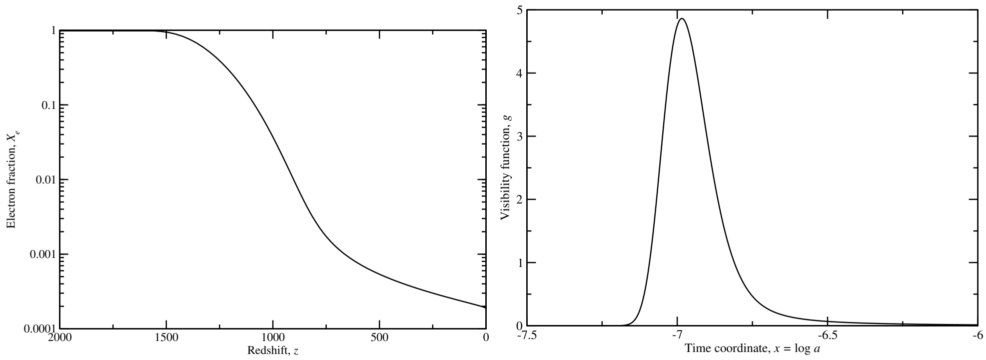

Figure 1: Left: The electron fraction as a function of redshift. Right: The visibility function, g.

The CMB field we observe today is to a very large extent created during the recombination epoch, when electrons and protons combined into neutral hydrogen. In order to understand this period quantitatively, it is necessary to know the electron fraction as a function of time, and we have in fact computed this during the course work, and the result is shown in the left panel of Figure 1. The right panel shows the corresponding visibility function, g.

a) As seen in the plot of X_e above, it is possible to define more or less three redshift ranges for the electron fraction, namely $z > 1500$, $1500 > z > 750$ and $z < 750$. Explain this behaviour.

b) As long as $X_e > 0.99$ we used the Saha equation

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \tag{1}$$

to solve for X_e, which basically is the Boltzmann equation applied to the process $e+p \leftrightarrow H+\gamma$. But it does not apply always. What requirement(s) must be fulfilled in order for the Saha equations to hold?

c) At some stage one has to switch from the Saha equation the more accurate Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right]. \quad (2)$$

This describes recombination to the first excited hydrogen state ($n=2$), not to the ground state ($n=1$). Why is recombination to the ground state not relevant in our case?

d) What is the physical interpretation of the visibility function? Why is it zero at $x < -8$? Why is it zero at $x > -5$?

Problem 3 – The Einstein equations

One of the most central parts of AST5220 is to derive and solve the linear Boltzmann-Einstein equations. In this problem we will therefore derive the first Einstein equation, precisely like we did in the lectures.

Before we start the real work, we have to choose a gauge, and we decided to adopt the conformal Newtonian gauge for our studies,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2). \quad (3)$$

Here Φ is the Newtonian potential and Φ is the curvature potential.

a) Given this metric, the first step is to derive the Christoffel symbols. The only non-zero Christoffel symbols are

$$\Gamma^0_{00} = \Psi_{,0} \tag{4}$$
$$\Gamma^0_{0i} = \Gamma^0_{i0} = ik_i\Psi \tag{5}$$
$$\Gamma^0_{ij} = \delta_{ij}a^2[H + 2H(\Phi - \Psi) + \Phi_{,0}] \tag{6}$$
$$\Gamma^i_{00} = \frac{ik^i}{a^2}\Psi \tag{7}$$
$$\Gamma^i_{j0} = \Gamma^i_{0j} = \delta_{ij}(H + \Phi_{,0}) \tag{8}$$
$$\Gamma^i_{jk} = i\Phi[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i] \tag{9}$$

Derive the expressions for Γ^0_{0i} and Γ^i_{jk}.

b) The next step is to compute the Ricci tensor, which in general reads

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}. \tag{10}$$

The $0i$-component of this tensor is zero, while the ij-component is

$$\begin{aligned} R_{ij} = \delta_{ij}[&(2a^2H^2 + a\ddot{a})(1 + 2\Phi - 2\Psi) + \\ &a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi] + k_ik_j(\Phi + \Psi). \end{aligned}$$

But what is R_{00}? As a help on your way, I will let you know that

$$\Gamma^i_{00}, i = \frac{-k^2}{a^2}\Psi, \qquad \Gamma^i_{i\beta}\Gamma^\beta_{00} = \Gamma^i_{i0}\Gamma^0_{00} = 3H\Psi_{,0}. \tag{11}$$

You do not have to show this.

c) Third, we have to compute the Ricci scalar, $\mathcal{R} \equiv R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij}$. This is rather ugly, since there are quite a lot of terms involved, and we won't spend time on it here. Instead, we just write down the final expression, including only first-order terms,

$$\delta\mathcal{R} = -12\Psi(H^2 + \frac{\ddot{a}}{a}) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}$$

Using this expression and the results derived above, show that the first-order contribution to the Einstein tensor is

$$\delta G^0_0 = -6H\Phi, 0 + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2} \tag{12}$$

d) To complete the Einstein equation,

$$\delta G^0_0 = 8\pi\delta T^0_0$$

we need an energy-momentum tensor on the right-hand side. You will not be asked to derive this, but only describe qualitatively what goes into it: Which components are the bare minimum we need to include in order to obtain a physically relevant power spectrum, ie., one that looks qualitatively similar to the current ΛCDM spectrum? Which of these components was most important at very early times, at $x = \log a \sim -10$, and why?

Problem 4 – Line-of-sight integration

A crucial part in speeding up a Boltzmann code is to change from direct integration of the Boltzmann-Einstein equations to a smarter approach, called "line-of-sight" integration. In this problem, we will derive the expression for the transfer function $\Theta_l(k)$ in terms of the source function.

Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
j_l(x) &= \frac{1}{2i^l} \int_{-1}^{1} e^{i\mu x} P_l(\mu) d\mu \\
f_l &= \frac{i^l}{2} \int_{-1}^{1} f(\mu) P_l(\mu) d\mu
\end{aligned}$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) First, from an coding point of view, why does one obtain such a large speed-up with the line-of-sight integration approach compared to direct integration?

b) In a few sentences, explain what the main physical difference is between the line-of-sight integration and the direct solution approaches.

c) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 + \mu v_b],$$

and show that equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

d) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta - \eta_0)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

e) (*Hint: This sub-problem is the toughest in the exam set. Don't spend all your time on getting this right, but rather do it after finishing up the other problems.*) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta - \eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik}\frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{v_b\dot{\tau}}{k}\right)\right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

f) Introducing the visibility function, $g(x) \equiv -\dot{\tau}e^{-\tau}$, the source function can be rewritten into the form

$$S(k,\eta) = g[\Theta_0 + \Psi] + \frac{d}{d\eta}\left(\frac{gv_b}{k}\right) + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right].$$

What is the physical interpretation of each of these terms?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.

- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{13}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{14}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{15}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{16}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \qquad (17)$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \qquad (18)$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \qquad (19)$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{20}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{21}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{22}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{23}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{24}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{25}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{26}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{27}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{28}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{29}$$

1.4 Initial conditions

$$\Phi = 1 \tag{30}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{31}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{32}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{33}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{34}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{35}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{36}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{37}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{38}$$

- Visibility function:

$$g(\eta) = -\dot{\tau}e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{39}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{40}$$

$$\int_0^{\eta_0} g(\eta)d\eta = \int_{-\infty}^{0} \tilde{g}(x)dx = 1. \tag{41}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \tag{42}$$

  where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{43}$$

  where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{44}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1} \tag{45}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{46}$$

$$n_{1s} = (1 - X_e) n_H \tag{47}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{48}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{49}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{50}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{51}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{52}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{53}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{54}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{55}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{56}$$