Astronomy & Astrophysics manuscript no. cosmology2_report
March 28, 2023
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Calculate the CMB power spectrum: Cosmology II

Johan Mylius Kroken[1,2]

[1] Institute of Theoretical Astrophysics (ITA), University of Oslo, Norway
[2] Center for Computing in Science Education (CCSE), University of Oslo, Norway

March 28, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

m_e - Mass of electron.
$m_e = 9.10938356 \cdot 10^{-31}$ kg.
m_H - Mass of hydrogen atom.
$m_\mathrm{H} = 1.6735575 \cdot 10^{-27}$ kg.
G - Gravitational constant.
$G = 6.67430 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.38064852 \cdot 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.054571817 \cdot 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.99792458 \cdot 10^8$ m s^{-1}.
σ_T - Thomson cross section.
$\sigma_T = 6.6524587158 \cdot 10^{-29}$ m^2.
α - Fine structure constant.
$\alpha = \frac{m_e c}{\hbar}\sqrt{\frac{3\sigma_T}{8\pi}}$

Cosmological parameters

$G_{\mu\nu}$ - Einstein tensor.
$T_{\mu\nu}$ - Stress-energy tensor.
H - Hubble parameter.
$\mathcal{H}$ - Conformal Hubble parameter.
T_{CMB0} - Temperature of CMB today.
a - Scale factor.
x - Logarithm of scale factor.
t - Cosmic time.
z - Redshift.
η - Conformal time.
χ - Co-moving distance.
p - Pressure.
ρ - Density.
r - Radial distance.
d_A - Angular diameter distance.
d_L - Luminosity distance.

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n_e - Electron density.
n_b - Baryon density.
X_e - Free electron fraction.
τ - Optical depth.
$\tilde{g}$ - Visibility function.
s - Sound horizon.
r_s - Sound horizon at decoupling.
c_s - Wave propagation speed.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

Fiducial cosmology

The fiducial cosmology used throughout this project is based on the observational data obtained by Aghanim et al. (2020):

$$\begin{aligned}
h &= 0.67, \\
T_{\mathrm{CMB0}} &= 2.7255\ \mathrm{K}, \\
N_{\mathrm{eff}} &= 3.046, \\
\Omega_{\mathrm{b0}} &= 0.05, \\
\Omega_{\mathrm{CDM0}} &= 0.267, \\
\Omega_{k0} &= 0, \\
\Omega_{\nu 0} &= N_{\mathrm{eff}} \cdot \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\Omega_{\gamma 0}, \\
\Omega_{\Lambda 0} &= 1-(\Omega_{k0}+\Omega_{\mathrm{b0}}+\Omega_{\mathrm{CDM0}}+\Omega_{\gamma 0}+\Omega_{\nu 0}), \\
\Omega_{\mathrm{M0}} &= \Omega_{b0}+\Omega_{\mathrm{CDM0}}, \\
\Omega_{\mathrm{rad}} &= \Omega_{\gamma 0}+\Omega_{\nu 0}, \\
n_s &= 0.965, \\
A_s &= 2.1\cdot 10^{-9}.
\end{aligned}$$

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

- Cosmological principle
- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

TODO: Obviously this introduction will change and amended as more milestones are completed.

2. Milestone I - Background Cosmology

The aim of this part is to set up and solve for a uniform background of the Universe. TODO: finish this

2.1. Theory

2.1.1. Fundamentals

If we assume the universe to be homogeneous and isotropic, the line elements $\mathrm{d}s$ is given by the FLWR-metric, here in polar coordinates (Weinberg 2008, eq. 1.1.11):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2x}\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right], \tag{2}$$

where we have introduced $x = \ln(a)$ which will be our primary measure of time.

We further model all forms of energy in the universe as perfect fluids, only characterised by their rest frame density ρ and isotropic pressure p, and an equation of state relating the two:

$$\omega = \frac{\rho}{p}. \tag{3}$$

By conservation of energy and momentum we must satisfy $\nabla_\mu T^{\mu\nu} = 0$, which results in the following differential equations for the density of each fluid ρ_i, from Winther et al. (2023):

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + 3H\rho_i(1+\omega_i) = 0, \tag{4}$$

where we have introduced the Hubble parameter $H \equiv \dot{a}/a = \mathrm{d}x/\mathrm{d}t$. The solution to Eq. (4) is of the form:

$$\rho_i \propto e^{-3(1+\omega_i)x}, \tag{5}$$

where $\omega_{\mathrm{M}} = 0$ (matter), $\omega_{\mathrm{rad}} = 1/3$ (radiation), $\omega_\Lambda = -1$ (cosmological constant) and $\omega_k = -1/3$ (curvature).

With these assumptions, the solution to the Einstein equations (Eq. (1)) are the Friedmann equations, the first of which describes the expansion rate of the universe:

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i - kc^2 e^{-2x} \tag{6}$$

and the second describe how this expansion rate changes over time:

$$\frac{\mathrm{d}H}{\mathrm{d}t} + H^2 = -\frac{4\pi G}{3}\sum_i\left(\rho + \frac{3p}{c^2}\right). \tag{7}$$

As of now, we are primarily interested in the first Friedmann equation. By introducing the critical density, $\rho_c \equiv 2H^2/(8\pi G)$, we define the density parameters $\Omega_i = \rho_i/\rho_c$. We further define the density of the curvature $\rho_k \equiv -3kc^2e^{-2x}/(8\pi G)$, which enables us to write Eq. (6) as simply:

$$1 = \sum_i \Omega_i, \tag{8}$$

where the curvature density Ω_k is included in the sum. From Eq. (5) we know the evolution of the densities in time, and if we assume the density values today, Ω_{i0}, are known (or are free parameters), then Eq. (6) may also be written as:

$$H = H_0\sqrt{\sum_i \Omega_{i0}e^{-3(1+\omega_i)x}}, \tag{9}$$

which is the Hubble equation we will use further.

2.1.2. Measure of time and space

The main measure of time is usually the scale factor a, or its logarithm x. We then have the *cosmic time* t defined as:

$$t = \int_0^a \frac{\mathrm{d}a}{aH} = \int_{-\infty}^x \frac{\mathrm{d}x}{H}. \tag{10}$$

Another temporal measure is the *conformal time* η defined as $c\mathrm{d}t = e^x \mathrm{d}\eta$ yielding:

$$\eta = \int_0^a \frac{c\mathrm{d}a}{a^2 H} = \int_{-\infty}^x \frac{c\mathrm{d}x}{e^x H} \equiv \int_{-\infty}^x \frac{c\mathrm{d}x}{\mathcal{H}}, \tag{11}$$

where $\mathcal{H} = e^x H$ is defined as the *conformal Hubble parameter*. We may also choose to measure time in terms of the *redshift* z, where $1+z = 1/a = e^{-x}$. The comoving distance is defined as follows:

$$\chi = \int_1^a \frac{c\mathrm{d}a}{a^2 H} = \int_0^x \frac{c\mathrm{d}x}{\mathcal{H}} = \eta_0 - \eta \tag{12}$$

The radial distance is given in terms of the comoving distance and the curvature density today Ω_{k0} as:

$$r = \begin{cases} \chi \cdot \frac{\sin\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} < 0 \\ \chi & \Omega_{k0} = 0 \\ \chi \cdot \frac{\sinh\left(\sqrt{|\Omega_{k0}|}H_0\chi/c\right)}{\sqrt{|\Omega_{k0}|}H_0\chi/c} & \Omega_{k0} > 0 \end{cases} \tag{13}$$

It is then straigtforward to define the angular diameter distance:

$$d_A = e^x r, \tag{14}$$

and the luminosity distance:

$$d_L = e^{-x} r, \tag{15}$$

both of which are derived in Appendix A. The temporal quantities η and t have the following evolutions with x:

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}. \tag{16}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{H}. \tag{17}$$

Both differential equations are easy to solve numerically. Their derivation may also be found in Appendix A

2.1.3. ΛCDM-model

In the ΛCDM model, the universe consists of matter in terms of baryonic matter (b) and cold dark matter (CDM), radiation in terms of photons (γ) and neutrinos (ν) and dark energy in terms of a cosmological constant (Λ). In addition, we must allow for some curvature (k). As a result, the parameters of the model will be the present values of the Hubble rate, H_0, the baryon density Ω_{b0}, the cold dark matter density $\Omega_{\mathrm{CDM}0}$, photon density $\Omega_{\gamma0}$, neutrino density $\Omega_{\nu0}$, dark energy density $\Omega_{\Lambda0}$, and the curvature density Ω_{k0}. The present temperature of the cosmic microwave background radiation $T_{\mathrm{CMB}0}$ fixes the radiation density today through:

$$\begin{aligned} \Omega_{\gamma0} &= \frac{16\pi^3 G}{90} \cdot \frac{(k_b T_{\mathrm{CMB}0})^4}{\hbar^3 c^5 H_0^2}, \\ \Omega_{\nu0} &= N_{\mathrm{eff}} \cdot \frac{7}{8} \cdot \left(\frac{4}{3}\right)^{4/3} \cdot \Omega_{\gamma0}. \end{aligned} \tag{18}$$

The total radiation density is $\Omega_{\mathrm{rad}} = \Omega_\gamma + \Omega_\nu$ and the total matter density is $\Omega_{\mathrm{M}} = \Omega_b + \Omega_{\mathrm{CDM}}$.

The Hubble equation from Eq. (9) may be redefined in terms of the conformal Hubble parameter $\mathcal{H}$ as:

$$\begin{aligned} \mathcal{H} &= H_0\sqrt{U} \\ U &\equiv \sum_i \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{19}$$

where we have defined $\alpha_i \equiv (1+3\omega_i)$ and $i \in \{\mathrm{M}, \mathrm{rad}, \Lambda, k\}$. Since we know the values of the various ω_i it follows that:

$$\begin{aligned} \alpha_{\mathrm{M}} &= 1 \\ \alpha_{\mathrm{rad}} &= 2 \\ \alpha_k &= 0 \\ \alpha_\Lambda &= -2 \end{aligned} \tag{20}$$

2.1.4. Equalities and present day values

Given the evolution of the density parameters with time, where the proportionality constant is the present day density, we introduce the *radiation-matter equality*, i.e. the time radiation and matter densities were equal: $\rho_{\mathrm{rad}} = \rho_{\mathrm{M}}$. According to Eq. (5) this can be expressed as:

$$\begin{aligned} \rho_{\mathrm{rad}0} e^{-4x} &= \rho_{\mathrm{M}0} e^{-3x} \\ e^x = \frac{\rho_{\mathrm{rad}0}}{\rho_{\mathrm{M}0}} &\implies x_{\mathrm{rM}} = \ln\left(\frac{\Omega_{\mathrm{rad}0}}{\Omega_{\mathrm{M}0}}\right), \end{aligned} \tag{21}$$

where x_{rM} now denotes the time of radiation-matter equality.

Similarly, the *matter-dark energy equality*, where $\rho_{\mathrm{M}} = \rho_\Lambda$ can be found to be:

$$\begin{aligned} \rho_\Lambda &= \rho_{\mathrm{M}0} e^{-3x} \\ \implies x_{\mathrm{M}\Lambda} &= \frac{1}{3}\ln\left(\frac{\Omega_{\mathrm{M}0}}{\Omega_\Lambda}\right) \end{aligned} \tag{22}$$

The time of matter-dark energy equality coincides with when the universe starts to accelerate, since this acceleration is driven by the dark energy, represented by the cosmological constant. From this time onwards, dark energy dominates the universe, and thus accelerating the expansion.

The age of the universe today, and the conformal time today can both be found by evaluating the solutions to the differential equations of t and η at the present time (where $x = 0$). This is done numerically.

2.1.5. Analytical solutions and sanity checks

There are several ways we may check that both our workings and numerical implementations are indeed correct. The simplest way is to always ensure that the sum of all density parameters add up to 1, for all times: $\sum_i \Omega_i = 1$.

If we only consider the most dominant density parameter, that is $\Omega_i >> \sum_{j\neq i}\Omega_j$, we end up with the following analytical expressions for different temporal regimes:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx -\frac{\alpha_i}{2} = \begin{cases} -1 & \alpha_{\mathrm{rad}} = 2 \\ -\frac{1}{2} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_{\Lambda} = -2 \end{cases} \tag{23}$$

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} \approx \frac{\alpha_i^2}{4} = \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ \frac{1}{4} & \alpha_{\mathrm{M}} = 1 \\ 1 & \alpha_{\Lambda} = -2 \end{cases} \tag{24}$$

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} 1 & \alpha_{\mathrm{rad}} = 2 \\ 2 & \alpha_{\mathrm{M}} = 1 \\ \infty & \alpha_{\Lambda} = -2 \end{cases} \tag{25}$$

These equations will be useful when making sure that the implementations are correct. For a thorough derivation, see Appendix B.

2.2. Methods

2.2.1. Initial equation

We have to consider the time evolution of the density parameters, given some present value, as function of our chosen time parameter, here x. The density evolution is implemented as:

$$\Omega_n = e^{-\alpha_n x}\Omega_{n0}\mathcal{H}_{\mathrm{rat}}^2 \tag{26}$$

where we have defined the ratio $\mathcal{H}_{\mathrm{rat}} \equiv H_0/\mathcal{H}$, and the new index n are all the densitis: $n \in \{b, \mathrm{CDM}, \gamma, \nu, \Lambda, k\}$.

We also implement functions to solve for the luminosity distance (Eq. (15)), angular distance (Eq. (14)), and the conformal distance (Eq. (12)).

2.2.2. ODEs and Splines

The differential equations for η (Eq. (16)) and t (Eq. (17)) are solved numerically as ordinary differential equations with the Runge-Kutta 4 as advancement method. The equations are solved for $x \in (-20, 5)$. As initial condition we would like $\eta(-\infty)$ which is obviously not possible to calculate, so we pick some very early time and use the analytical approximation in the radiation dominated era (Eq. (25)), which yield:

$$\eta(x_0) = \frac{c}{\mathcal{H}(x_0)}. \tag{27}$$

Likewise for t, the initial condition is:

$$t(x_0) = \frac{1}{2H(x_0)}. \tag{28}$$

We then proceed by making splines of both η and t in order to evaluate accurately for any $x \in (-20, 5)$.

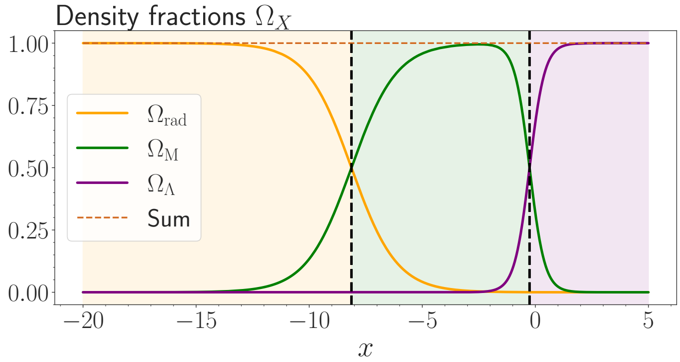

Fig. 1. Density fractions Ω_i as function of x. For low x, radiation dominates, before matter dominates and dark energy has just become the dominant energy density today $x = 0$, and will continue to dominate into the future. The sum of densities sums to one across all times, as required (white dotted line). The black dotted lines are the radiation-matter equality at $x = -8.13$ and the matter-dark energy equality at $x = -0.26$, both stated in Table 1. The domination of each regime is shown as a shaded background with similar colour as its respective graph.

2.2.3. Model evaluation

We evaluate the model by computing the quantities presented in Section 2.1.5 and compare with the analytical solutions in different regimes. This will ensure that the model behave as expected.

Furthermore, we want the model to somewhat resemble reality, we thus use measures of the luminosity distance of supernovas at different redshifts z, acquired by Betoule et al. (2014). This data is compared to the prediction made by our model.

In order to constrain the possible values Ω_{M} and Ω_{Λ} we find the χ^2-error between the luminosity distance of the supernovas and the predictions made by our model. The Ω-s are sampled with Markov-Chain Monte Carlo sampling using the Metropolis-Hastings algorithm. The χ^2-erros is given by:

$$\chi^2(h, \Omega_{m0}, \Omega_{k0}) = \sum_{i=1}^{N} \frac{(d_L(z, \Omega_{m0}, \Omega_{k0}) - d_L^{\mathrm{obs}}(z_i))^2}{\sigma_i^2}. \tag{29}$$

2.3. Results and discussion

2.3.1. Tests

Fig. 1 show the evolution of the density fractions with time. They sum to one across all times which was required. At early times the radiation density dominates (orange line). The intersection between the orange and green lines mark the radiation-matter equality, after which matter is the dominating density. Likewise, the intersection between the green and purple lines mark the matter-dark energy equality, where dark energy (manifested in the cosmological constant) become the dominating density. Time can thus be divided into three regimes; radiation dominated, matter dominated, and dark energy dominated eras.

As explained in section Section 2.1.5, we have analytical solutions for constructions of η and $\mathcal{H}$ in the different regimes. Fig. 2 is the sanity check for η, showing $\eta\mathcal{H}/c$ converging to finite values in the radiation and matter dominated eras (where $\alpha_{\mathrm{rad,M}} > 0$), and diverging towards $+\infty$

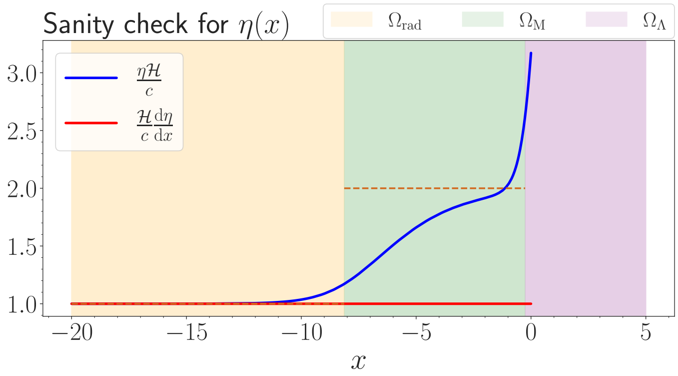

Fig. 2. Sanity check for η. $\eta\mathcal{H}/c$, in blue, is 1 in the radiation regime, 2 in the matter regime and diverging toward $+\infty$ in the dark energy regime, as expected from the analytical approximations in each regime. Remembering that this is strictly correct only in the radiation regime explains the mismatch of the white doted line in the matter regime. $(\mathrm{d}\eta/\mathrm{d}x)\mathcal{H}/c$, in red, is 1 throughout time, as expected from the differential equation for η.

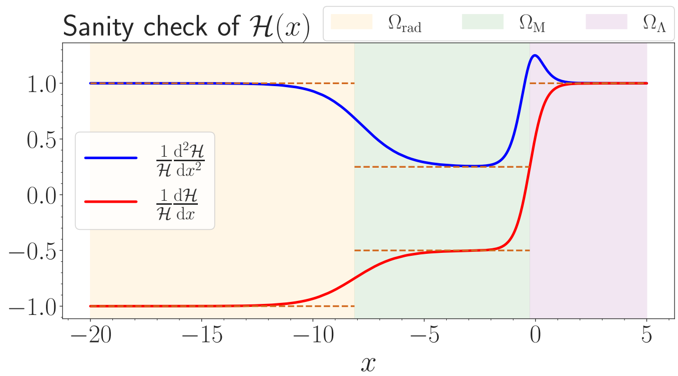

Fig. 3. Sanity check for $\mathcal{H}$, showing that the second derivative (blue) converge to the analytical expressions shown as white dotted lines in the different regimes. The first derivative (red) converge to its analytical values in the same regimes, which again are shown with a shaded colour.

in the dark energy dominated era ($\alpha_\Lambda = -2 < 0$). This is in accordance with the analytical solutions. The different regimes are shown in shaded colour. It is also worth noticing that $(\mathrm{d}\eta/\mathrm{d}x)\mathcal{H}/c$ is one for all regimes, as expected from equation Eq. (16).

Fig. 3 is the sanity check confirming that our constructions of $\mathcal{H}$ and its derivatives converge to the analytical approximation in the different regimes. The second derivative, as shown in blue, takes the value of one in the radiation regime, one half in the matter regime and one in the dark energy regime. Similarly, the first derivative, as shown in green, take the value negative one in the radiation regime, negative one half in the matter regime and one in the dark energy regime. This is well in accordance with the analytical approximations put forth in section Section 2.1.5.

These sanity checks are confirmations that the implementation of the model yields the same result as the analytical approximation in the different regimes for various constructions of η and $\mathcal{H}$ and their derivatives.

Quantity	x	z	t
RM-equality	-8.13	3400	$51{\cdot}10^3$ yr
ML-equality	-0.26	0.29	10.378 Gyr
Accel. start	-0.49	0.63	7.752 Gyr
Age of universe	0.00	0.00	13.858 Gyr
Conformal time	0.00	0.00	46.319 Gyr

Table 1. Important quantities in the evolution of the universe.

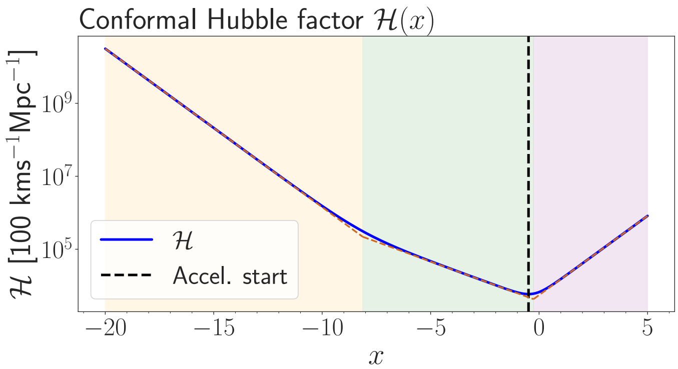

Fig. 4. $\mathcal{H}$ as function of x. It is decreasing in the radiation and matter regimes, and increasing in the dark energy regime, tightly following its analytical approximation in each regime.

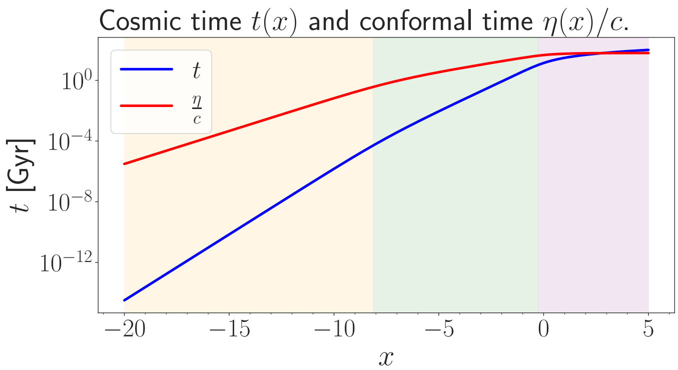

Fig. 5. Cosmic time (in blue) and conformal time (red). TODO: find meaning behind this plot.

2.3.2. Analysis

Section 2.1.4 indicate how we can calculate the radiation-matter equality (RM-equality), matter-dark energy equality (ML-equality), when the acceleration of the universe started, the age of the universe and the conformal time today. The result is shown in Table 1. We note that the equalities is in accordance with the sanity checks, and the age of the universe today (in cosmic time) is about 13.9 Gyr.

The conformal Hubble factor, $\mathcal{H}$, is plotted against time, x, in Fig. 4. It is decreasing in the radiation and matter regimes and increasing in the dark energy regime. Since this is a measure of the expansion of the universe, the acceleration seem to coincide with the matter-dark energy equality.

Fig. 5 show the cosmic time t and the conformal time η/c. We observe that the cosmic time accelerates around matter-dark energy equality, and seems to diverge, whereas the conformal time increases in the matter regime and seem to converge in the dark energy regime.

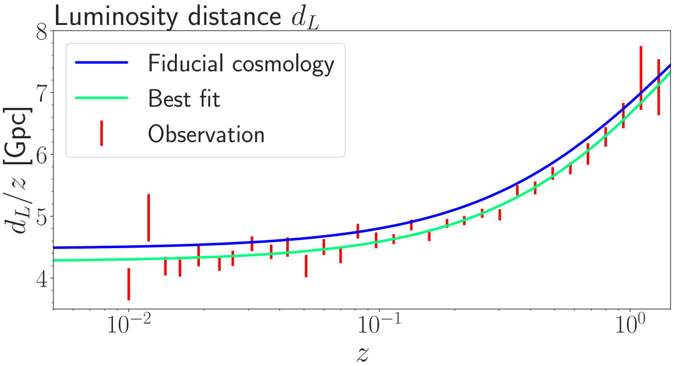

Fig. 6. The luminosity distance predicted using the fiducial cosmology in blue, against observations of actual supernovas in red (or rather the confidence interval of the observations). The green line is found from computing the luminosity distance using a cosmology of the best fit parameters from the supernova fitting; $h = 0.702, \Omega_{\mathrm{b0}} = 0.05, \Omega_{\mathrm{CDM0}} = 0.209, \Omega_{\mathrm{k}} = 0.067, N_{\mathrm{eff}} = 3.046, T_{\mathrm{CMB}} = 2.7255$K. Notice the x-axis is now the redshift $z = e^x - 1$.

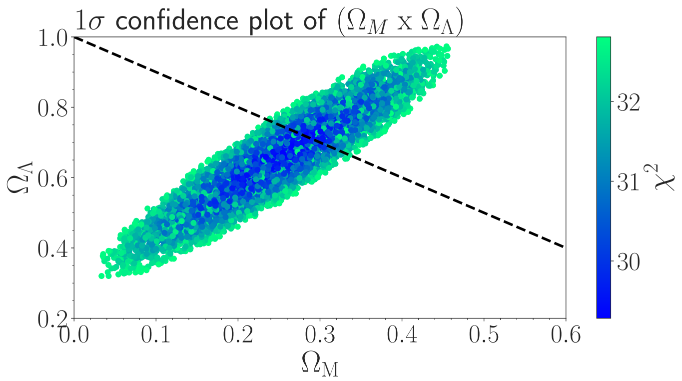

Fig. 7. Scatter plot showing the chi^2-error of the luminosity distance d_L between the observed values and the prediction, as function of Ω_{M} and Ω_Λ. The data shown is within 1σ (standard deviation). The black dotted line signifies a flat universe.

Fig. 6 shows the supernova data as blue error bars, with the theoretical prediction plotted above it. The quantity plotted is the luminosity distance divided by redshift, d_L/z for better comparison. We notice the accordance between the two, and also note that the x-axis in this plot is the redshift z instead of the logarithm of the scale factor. This means that earlier times are to the right in the plot (high redshift), as opposed to the other plots.

Fig. 7 shows the χ^2-values found from Eq. (29), to 1σ accuracy. The black dotted line represent a flat universe, where the matter and dark energy are the main constituents of the universe. The supernova data originate in close temporal proximity to us, we thus assume that the contribution from the radiation density is negligible for making constraints on Ω_{M} and Ω_Λ today.

The MCMC method allow us to make a posterior probability distribution of the parameters in question. Fig. 9 the distribution of the Hubble factor H_0 constructed from the sampled values of h. The actual samples are shown in green and the corresponding fitted probability distribution in blue.

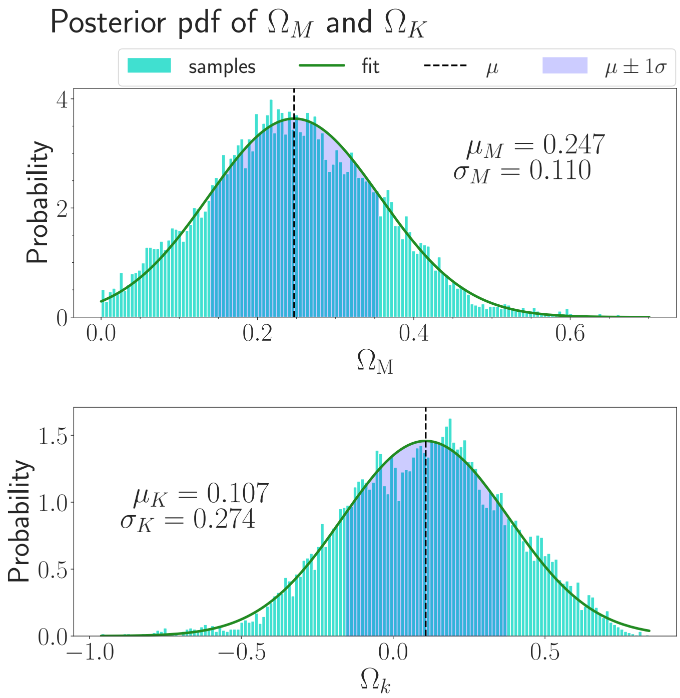

Fig. 8. Posterior probability distributions (pdfs) of $\Omega_mathrmM$ and Ω_{k} as result of the MCMC sampling. The samples as shown in turquoise and the constructed pdfs in green. The mean μ is shown as a black dotted line, with the 1σ confidence interval in shaded blue ($\mu \pm 1\sigma$)

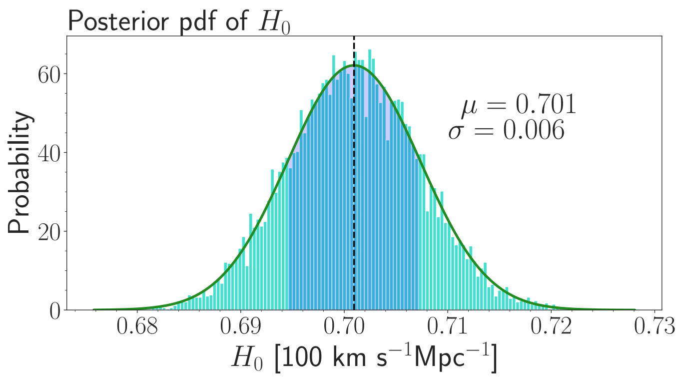

Fig. 9. Posterior probability distribution (pdf) of H_0 as result of the MCMC sampling.

3. Milestone II - Recombination History

The main goal of this section is to investigate the recombination history of the universe. This can be explained as the point in time when photons decouple from the equilibrium of the opaque, early universe. When this happens, photons scatter for the last time at the *time of last scattering*, and these photons are what we today observe as the CMB. This period of the history of the universe is thus crucial for understanding the CMB.

We will start by calculating the free *electron fraction* X_e, from which we may find the *optical depth* τ. This again enables us to compute the *visibility function*, g, and the *sound horizon*, s. The latter will be of great importance later.

Recombination happens because the expansion of the Universe cools it down, making the photons less energetic, which in turn make each interaction in the primordial plasma less energetic. At some point, hydrogen atoms are able to form, reducing the number of free electron, hence reducing photon interactions, until they scatter for the last time. We will determine the time of recombination from the free electron fraction, which indirectly tell us how large portion of the free electron have (re)-combined.[1] Due to the decrease of free electron, photons interact less with them (optical depth is decreased). At some point, photons scatter for the last time, and this information is encapsulated in the visibility function.

3.1. Theory

Before recombination, the equilibrium between protons, electrons and photons is governed by the following interaction, from Weinberg (2008)[2]:

$$e^- + p^+ \leftrightharpoons H^* + \gamma, \tag{30}$$

where a proton and an electron interact to form an excited hydrogen atom, which decays and emits a photon, or a photon excites and split a hydrogen atom into a free electron and a proton through *Compton scattering*.[3] Eq. (30) is a reaction of the form $1+2 \leftrightharpoons 3+4$, and we have from Winther et al. (2023) that the Boltzmann equation for such a reaction is:

$$\frac{1}{n_1 e^{3x}} \frac{\mathrm{d}(n_1 e^{3x})}{\mathrm{d}x} = -\frac{\Gamma}{H}\left(1 - \frac{n_3 n_4}{n_1 n_2}\left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}}\right), \tag{31}$$

where n_i are the number densities of the reactants, Γ is the reaction rate and H the Hubble parameter (expansion rate of the universe). If the reaction rate is much larger than the expansion rate of the universe, $\Gamma \gg H$, then Eq. (30) ensures equilibrium between protons, electron and photons. When Γ drops below H, then the expansion rate becomes dominant and the reaction rate is unable to sustain equilibrium. This happens when the temperature of the Universe becomes lower than the binding energy of hydrogen, hence stable neutral hydrogen is able to form.[4] As a consequence, the photons *decouple* from the protons and electron. When $\Gamma \ll H$, there are practically no interactions and the number density becomes constant for a comoving volume. Massive particles *freeze out* and their abundance become constant.

3.1.1. Hydrogen recombination

We express the electron density through the free electron fraction $X_e \equiv n_e/n_{\mathrm{H}} = n_e/n_b$ where we have assumed that hydrogen make up all the baryons ($n_b = n_{\mathrm{H}}$). We also ignore the difference between free protons and neutral hydrogen. From Callin (2006) we obtain:

$$n_b = \frac{\rho_b}{m_{\mathrm{H}}} = \frac{\Omega_b \rho_c}{m_{\mathrm{H}}} e^{-3x}, \tag{32}$$

where m_{H} is the mass of the hydrogen atom, and ρ_c the critical density today as defined earlier. Before recombination, no stable neutral hydrogen is formed, thus the electron and baryon density is the same, i.e. there are only free electrons so $X_e \simeq 1$. When in equilibrium, the r.h.s. of Eq. (31) reduces to 0, which is called the *Saha approximation*. The solution is in this regime described by the *Saha equation*, which from Dodelson & Schmidt (2020) in physical units is:

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{33}$$

where $\epsilon_0 = 13.6$ eV is the ionisation energy of hydrogen. The Saha equation is only a good approximation when $X_e \simeq 1$. Thus for $X_e < (1-\xi)$,[5] which corresponds to the period during and after recombination, we have to make use of the more accurate *Peebles equation*. From Callin (2006):

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1-X_e) - n_{\mathrm{H}}\alpha^{(2)}(T_b)X_e^2\right], \tag{34}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s-1s} + \Lambda_\alpha}{\Lambda_{2s-1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{34a}$$

$$\Lambda_{2s-1s} = 8.227 \text{ s}^{-1}, \tag{34b}$$

$$\Lambda_\alpha = \frac{1}{(\hbar c)^3} H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \tag{34c}$$

$$n_{1s} = (1-X_e)n_{\mathrm{H}}, \tag{34d}$$

$$n_{\mathrm{H}} = (1-Y_p)\frac{3H_0^2\Omega_{b0}}{8\pi G m_{\mathrm{H}}} e^{-3x}, \tag{34e}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4k_B T_b}, \tag{34f}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{34g}$$

$$\alpha^{(2)}(T_b) = \frac{\hbar^2}{c}\frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{k_B T_b}}\phi_2(T_b), \tag{34h}$$

$$\phi_2(T_b) = 0.448 \ln\left(\frac{\epsilon_0}{k_B T_b}\right). \tag{34i}$$

TODO: Add σ_T and α to nomenclature.

The Peebles equation takes into account that the energy (excitation) of hydrogen formed through Eq. (30) vary, and

[1] As with any good article on the subject, we ought to say that recombination is a funny wording, as this is the first time in the history of the Universe that protons and electrons combine to form hydrogen.

[2] Where H^* denotes excited states of hydrogen which will decay into neutral hydrogen.

[3] Elastic scattering of photons is technically Thomson scattering, but Compton scattering is a more general term and will be used (Dodelson & Schmidt (2020)). This is also why we later use the Thomson cross section σ_T. The reaction is when a photon scatters of an electron, and possibly energises it enough to break out of the hydrogen atom, if already bound:

$$\gamma + e^- \leftrightharpoons \gamma + e^-.$$

[4] Well, it is really not as simple, as neutral hydrogen is obtained from excited hydrogen and how this process go about is non-trivial. As we ignore re-ionisation, I will not delve into this. However, both (Weinberg 2008, p. 113-129), (Dodelson & Schmidt 2020, p. 95-99) and Winther et al. (2023) elaborate further on this.

[5] Where ξ is some small tolerance, which have to be defined in some numerical model for when to abonden the Saha equation and use the more accurate, but computationally more expensive Peebles equation. This is typically $\xi = 0.001$

that decays take place until we reach the $n = 2$ level (first excited state), denoted by $^{(2)}$ in Eq. (34a)- Eq. (34i). Recombination to the ground state is not relevant, as this leads to an ionised photon which immediately ionises a neutral hydrogen atom (Dodelson & Schmidt 2020, p. 97). The C_r is the probability that singly ionised hydrogen is reionised further, where $\beta^{(2)}$ and β are the collisional ionisations from the first ionised state and ground state respectively. $\alpha^{(2)}$ is the recombination rate to excited states. For more detailed description of these terms, see Ma & Bertschinger (1995).[6]

We find X_e by solving Eq. (33) for $X_e > (1-\xi)$ and Eq. (34) for $X_e < (1-\xi)$. In theory, it is possible to solve the Peebles equation at very early times, but the equation is very stiff resulting in unstable numerical solutions at early times (high temperatures), hence the Saha approximation.

3.1.2. Visibility

Visibility is a concept tied to the optical depth and mean free path of a medium. The two latter are inversely proportional to each other. The mean free path is the average distance a photon travels before its direction is changed (often by scattering). Thus, a small mean free path gives results in a lot of collision across short distances, which occurs in optically thick media. The optical depth as a function of conformal time is defined as Winther et al. (2023):

$$\tau = \int_{\eta}^{\eta_0} n_e \sigma_{\mathrm{T}} e^{-x} \mathrm{d}\eta', \tag{35}$$

where n_e is the electron density and σ_{T} is the Thompson cross-section. In differential form, restoring original units, this is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = -\frac{c n_e \sigma_T e^x}{\mathcal{H}}. \tag{36}$$

From this we define the visibility function, g:

$$\begin{aligned} g &= -\frac{\mathrm{d}\tau}{\mathrm{d}\eta} e^{-\tau} = -\mathcal{H}\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} \\ \tilde{g} &\equiv -\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} = \frac{g}{\mathcal{H}}, \end{aligned} \tag{37}$$

where $\tilde{g}$ is in terms of the preferred time variable, x. Notable thing about the visibility function $\tilde{g}$ is that it is a true probability distribution, describing the probability density of some photon to last have scattered at time x. Because of this, we have that $\int_{-\infty}^{0} \tilde{g}(x)\mathrm{d}x = 1$. We also take note of the derivative of the visibility function:

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} = e^{-\tau}\left[\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}^2\tau}{\mathrm{d}x^2}\right] \tag{38}$$

3.1.3. Sound horizon

Let's take a small step back and consider the situation of the early Universe. Before any decoupling, the photons and electrons are coupled through Thompson scattering, and protons and electrons are coupled through coulomb interactions. Because of this, photons interact with baryons and move alongside with them as one fluid, in which wave propagates with a speed c_s, from Dodelson & Schmidt (2020):

$$c_s \equiv c\left[3(1+R)\right]^{-\frac{1}{2}} \quad ; \quad R \equiv \frac{3\Omega_b}{4\Omega_\gamma}, \tag{39}$$

where R is the *baryon-to-photon energy ratio*. By the definition of R, if the baryon density is negligible compared to the radiation density, $R \sim 0$, and we recover the wave propagation speed in a relativistic fluid: $c_s = 3^{-1/2}$ (Dodelson & Schmidt (2020)). The total distance such a wave would have travelled in a time t (since the beginning of the Universe) is called the *sound horizon*, found by simply integrating c_s through time, accounting for the expansion of space itself by including a factor e^{-x}:

$$s = \int_0^t c_s e^{-x} \mathrm{d}t = \int_{-\infty}^{x} \frac{c_s}{\mathcal{H}} \mathrm{d}x, \tag{40}$$

where the variables are changed to x. On differential form:

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{c_s}{\mathcal{H}}, \tag{41}$$

which is a straightforward differential equation to solve given some initial conditions.

3.2. Methods

3.2.1. Computing X_e

First things first, we need to compute the free electron fraction X_e. We are for the most part not interested in things happening in the future here, so the temporal range of choice will be $x \in [-20, 0)$ where $x = 0$ is today, and $x = -20$ is sufficiently long ago, so that the range encapsulated effect studied here. In the early Universe, the energies are so high that all baryonic matter is in the form of free electron, $X_e \simeq 1$, so we will start by solving the Saha equation, Eq. (33). We continue to solve equation Eq. (33) as long as $X_e > 1 - \xi$ where we use $\xi = 0.01$.

If we define:

$$K \equiv \frac{1}{n_b}\left(\frac{k_B m_e T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\epsilon_0/k_B T_b}, \tag{42}$$

then equaton Eq. (33) takes the form $X_e^2 + KX_e - K = 0$, which is solved as a normal quadratic equation[7], where $a = 1$, $b = K$ and $c = -K$. Since $0 \leq X_e \leq 1$ we choose the positive solution, given by:

$$X_e = \frac{-K + \sqrt{K^2 + 4K}}{2} = \frac{K}{2}\left(-1 + \sqrt{1 + 4K^{-1}}\right) \tag{43}$$

This solution has the potential to become numerically unstable if the parenthesis is close to zero, i.e. for $K \gg 1$. We then make use of the approximation $\sqrt{1 + 4K^{-1}} \approx$

[6] Because of this non-trivial path into the ground state, and the large photon to baryon number ratio, recombination happens later than when the temperature of the universe correspond to exactly the binding energy of neutral hydrogen (Callin (2006))

[7] $ay^2 + by + c = 0$ has solutions

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}.$$

$1 + (2K^{-1})$ for $|4K^{-1}| \ll 1$, which ensures $X_e \simeq 1$ for very high temperatures (large K).

We continue to solve the Peebles equation as stated in Eq. (34), where the r.h.s. is implemented sequentially as Eq. (34a)- Eq. (34i) in reverse order. The initial condition is the last computed electron fraction above the cut-off: $X_{e0} = \min(X_e > 1 - \xi)$ as found from the Saha equation. It is solved for the x-range not solved by the Saha equation. On thing to notice is that for late time, T_b becomes small, meaning that $e^{\epsilon_0/k_B T_b}$ becomes enormous. This term is found in Eq. (34f), and we solve it by setting $\beta^{(2)}(T_b) = 2$ if $\epsilon/k_B T_b > 200$, in order to avoid overflow.

Having found X_e for the entire x-range, we compute n_e and spline both results.

3.2.2. Computing τ and $\tilde{g}$

With n_e we are able to solve the optical depth as defined in Eq. (36). The inital condition for this equation is that the optical depth today is zero: $\tau(x = 0) = 0$, meaning we have to solve this backwards in time. This is done by using the negative differential:

$$\frac{\mathrm{d}\tau_{\text{rev}}}{\mathrm{d}x_{\text{rev}}} = -\frac{\mathrm{d}\tau}{\mathrm{d}x} = \frac{c n_e \sigma_T e^x}{\mathcal{H}}, \tag{44}$$

and solving for positive x_{rev}: $x_{\text{rev}} \in [0, 20]$. In order to undo this reversal, we map $\tau = -\tau_{\text{rev}}$ to its corresponding $x = -x_{\text{rev}}$. Having found τ, we find its derivative by solving equation Eq. (36), and further the find the visibility function from Eq. (37) and its derivative from Eq. (38). We ensure that $\int_{-\infty}^{0} \tilde{g} \mathrm{d}x = 1$ as a sanity check. All of these four quantities are splined, and their derivatives are obtained numerically.

In order to solve equation Eq. (40) for the sound horizon, we choose initial conditions $s_i = c_{s,i}/\mathcal{H}_i$ where the subscript i denote a very early time (in our case when $x = -20$). We are then able to solve the differential equation for the sound horizon, Eq. (41), numerically and then spline the result.

3.2.3. Analysis

Having splines for the relevant quantities enables us to compute some important times in the early universe. Firstly, the *last scattering surface*, is the time when most photons scattered for the last time, and decoupled from the plasma. This is not expected to have happened instantly, but recalling that the visibility function $\tilde{g}$ is a probability distribution function for when photons last scattered, we simply use the peak of this function as the definition of the last scattering surface.

Further, we want to find a time for when recombination happened, i.e. when free electron was captured by protons to form hydrogen atoms. Thus, this coincides with the reduction of the free electron fraction, and we will use $X_e = 0.1$ as the definition for when recombination happened. These numbers can also be computed using only the Saha approximation, for comparison. We also compute the sound horizon at these decouplings: $r_s = s(x_{\text{dec}})$.

The last thing we want to compute is the freeze out abundance of free electrons, i.e. the free electron abundance today, which is found by evaluating the spline for X_e at $x = 0$.

Phenomenon	x	z	t [Myr]	r_s [Mpc]
Last scattering	-6.9853	1079.67	0.3780	145.31
Recombination	-6.9855	1079.83	0.3779	145.29
Saha	-7.1404	1260.89	0.2909	131.03

Table 2. The times of last scattering and recombination given in terms of x, the redshift z, the cosmic time t and the sound horizon r_s. Also included is the time of recombination found using the Saha approximation only.

3.3. Results and discussion

3.3.1. Times and sound horizon

The relevant times for last scattering, recombination and Saha recombination are obtained as explained in Section 3.2.3, and presented in Section 3.3.1. These times are given in terms of x, the redshift z and the cosmic time t (in Myr). The sound horizon is given in units of megaparsecs (Mpc). Last scattering occurred when $x = -6.9853$, at redshift $z = 1079.67$, which is slightly after recombination when $x = -6.9855$ at redshift $z = 1079.83$. If the Saha approximation was valid when the electron fraction dropped, recombination would have happened when $x = -7.1404$ at redshift 1260.89 which is significantly earlier. However, this is not the case since photons drop out of equilibrium with the primordial plasma as soon as hydrogen begin to form, and the free electron fraction is reduced. Thus, this number may only be used for comparison. Another thing worth noting is the validity of these numbers.

3.3.2. Free electron fraction

Fig. 10 shows the free electron fraction X_e as a function of x found using both the Saha and Peebles equation, as explained in Section 3.2.1, in blue. Also shown is the results found from the Saha equation only, which tends to zero a lot faster. This is used for comparison only, as we have already stated that the Saha approximation is only valid for $X_e \simeq 1$. The time of recombination is shown for both cases, which for the Saha approximation happens significantly earlier than what is the actual case. The Peebles solution falls off gradually, and converges towards a constant value, which is the present day abundance of free electrons (freeze out abundance). This is found to be $X_e(x = 0) = 0.0002$, shown as a brown dashed line in Fig. 10.

Since the Peebles equation is a solution of the Boltzmann equation, it takes into account the particle interaction with changing abundance, after the photons decouple from the primordial plasma. It is thus expected that this will result in a much more gradual fall off of the free electron fraction, just as we observe in Fig. 10.

3.3.3. Visibility

Fig. 11 shows the optical depth and its first two derivatives as functions of x. The surface of last scattering is shown with a black dashed line, before which the primordial plasma is optically thick, meaning the photons have a short mean free path. The decrease of the optical depth means that the photons gradually travel longer distances before interacting with free electrons. There are two processes going on here; the expansion of space itself, and the formation of

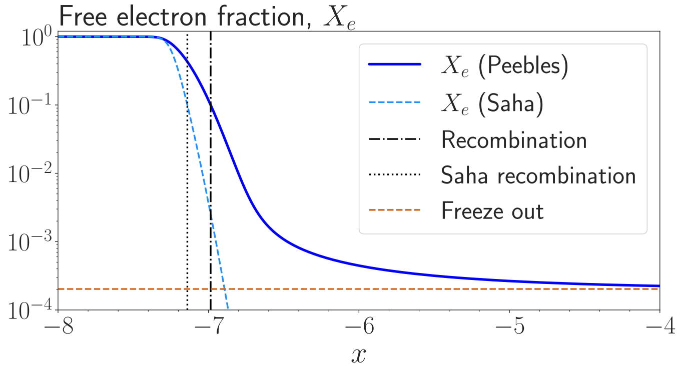

Fig. 10. The free electron fraction X_e as function of x, found from the Saha and Peebles equation (blue). The result using only the Saha equation is shown in dashed light blue. The time of recombination is shown as a dashed black line. Likewise, recombination in the Saha approximation is shown as a dotted black line, appearing earlier. The freeze out abundance of hydrogen (the present value) is shown as a brown dashed line.

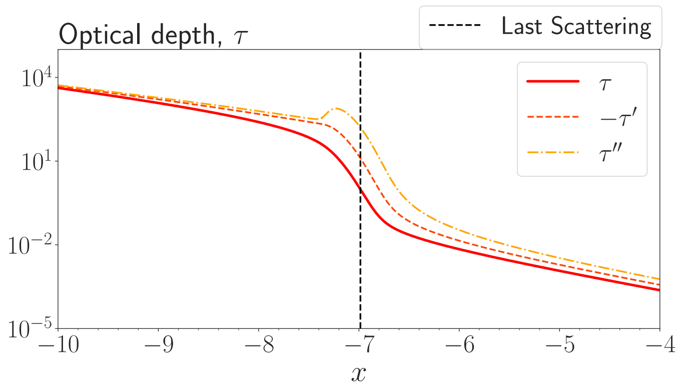

Fig. 11. The optical depth τ and its first and second derivatives as functions of x. The time of last scattering is shown as a dashed black line, before which the Universe was optically thick.

neutral hydrogen. Both of which contribute to the increased mean free path of the photons. The contribution from the expansion of space is slow compared to the seemingly rapid change in the free electron fraction once neutral hydrogen is able to form. Thus, the rapid decrease of free electrons, as seen is Fig. 10 makes the mean free paths of photon to increase beyond the horizon. This effectively enable them to travel through space without interacting with matter, and this is what we observe as the CMB today - the Universe becomes transparent. This sudden decrease of optical depth is clearly seen in Fig. 11, both in τ itself, but also in its derivatives.

Another way of arriving at similar conclusions is by considering the visibility function in Fig. 12. Here, $\tilde{g}$ is shown in green along with its derivatives. The scaling follow that of Callin (2006), in order to fit the graphs into the same figure. $\tilde{g}$ describes the probability that a photon reaching us today scattered at time x. The peak of this function indicates the time were *the most* photons scattered for the last time, and is thus used as a definition of the last scattering surface. The visibility function is skewed forward in time. TODO: why?

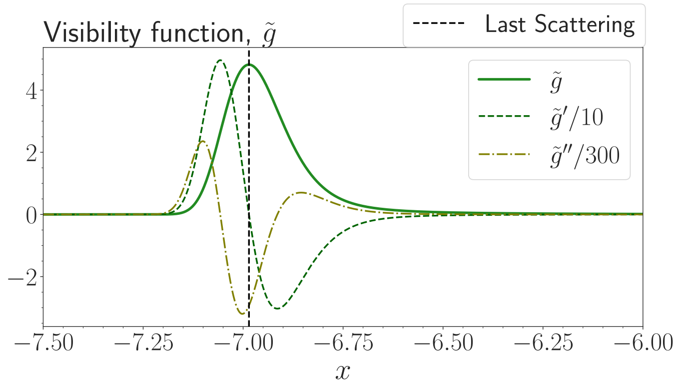

Fig. 12. The visibility function $\tilde{g}$ and its first and second derivatives as functions of x. The time of last scattering is shown as a dashed black line, which by definition coincides with the peak of the visibility function

.

3.3.4. General discussion

One key thing to keep in mind is that recombination did not happen instantaneously, but rather over a relatively short period in which neutral hydrogen formed rapidly. This caused a rapid decrease of the free electron fraction, which again caused the optical depth to decrease by several magnitudes. In the same period, we see a that the visibility function is non-zero, meaning the probability of last scattering is (relatively) very high in this period. The times quoted in Section 3.3.1 are times that arise from our quite rigid, but fair definition of last scattering and recombination. However, these times do not encapsulate the duration of the abovementioned period. One could also define the last scattering surface as the time when $\tau = 1$ which is the transition between optically thick and optically thin media (when the photon travels exactly one mean free path before scattering). However, the visibility function is arguably a better choice since this is a proper probability distribution, so its peak represents the *actual* time when the probability of last scattering was the highest. Nevertheless, if we change these definitions we ought to expect different times as a result.

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Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element $\mathrm{d}s$ as given in equation Eq. (2), and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x}r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x\mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned}\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}H = e^{-x}c \\ \implies \frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{c}{\mathcal{H}}.\end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned}\frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x}c \\ \implies \frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{e^x}{\mathcal{H}} = \frac{1}{H}.\end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from Eq. (19) and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^nU}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2}U^{-\frac{1}{2}}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right]\end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned}\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned}U &\approx \Omega_{i0}e^{-\alpha_i x} \\ \frac{\mathrm{d}^nU}{\mathrm{d}x^n} &\approx (-\alpha_i)^n\Omega_{i0}e^{-\alpha_i x},\end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned}\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4}\end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned}\eta &= \int_{-\infty}^{x}\frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right),\end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} &= -\frac{\alpha_i}{2}\mathcal{H} \\ \implies \mathrm{d}x &= -\frac{2}{\alpha_i\mathcal{H}}\mathrm{d}\mathcal{H}.\end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.