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Calculate the CMB power spectrum: Cosmology II

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May 10, 2023 - GitHub repository link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

m_e - Mass of electron.
$m_e = 9.10938356 \cdot 10^{-31}$ kg.
m_H - Mass of hydrogen atom.
$m_\mathrm{H} = 1.6735575 \cdot 10^{-27}$ kg.
G - Gravitational constant.
$G = 6.67430 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.38064852 \cdot 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.054571817 \cdot 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.99792458 \cdot 10^8$ m s^{-1}.
σ_T - Thomson cross section.
$\sigma_T = 6.6524587158 \cdot 10^{-29}$ m^2.
α - Fine structure constant.
$\alpha = \frac{m_e c}{\hbar}\sqrt{\frac{3\sigma_T}{8\pi}}$

Cosmological parameters

$G_{\mu\nu}$ - Einstein tensor.
$T_{\mu\nu}$ - Stress-energy tensor.
H - Hubble parameter.
$\mathcal{H}$ - Conformal Hubble parameter.
T_CMB0 - Temperature of CMB today.
a - Scale factor.
x - Logarithm of scale factor.
t - Cosmic time.
z - Redshift.
η - Conformal time.
χ - Co-moving distance.
p - Pressure.
ρ - Density.
r - Radial distance.
d_A - Angular diameter distance.
d_L - Luminosity distance.

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n_e - Electron density.
n_b - Baryon density.
X_e - Free electron fraction.
τ - Optical depth.
$\tilde{g}$ - Visibility function.

s - Sound horizon.
r_s - Sound horizon at decoupling.
c_s - Wave propagation speed.

Ψ - Newtonian potential perturbation to the metric.
Φ - Spatial curvature perturbation to the metric.
P^{μ} - Comoving momentum four vector.
$\mathcal{P}_l$ - Legendre polynomial of order l.
k - Fourier mode $k = \mathbf{k}$ of wave vector $\mathbf{k}$.
Θ - Photon perturbation.
Θ_l - l-th order multipole expansion term of Θ.
v_b - Baryon bulk velocity.
v_c - Cold dark matter bulk velocity.
v_γ - Photon bulk velocity.
δ_b - Baryon overdensity.
δ_c - Cold dark matter overdensity.
δ_γ - Photon overdensity.
ϕ - Inflaton field.
ρ_ϕ - Inflaton field density.
p_ϕ - Inflaton field pressure.
$\epsilon_{\rm sr}, \delta_{\rm sr}$ - Slow roll parameters.
$\mathcal{R}$ - Curvature perturbation.
$\mathcal{S}$ - Source function.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

Fiducial cosmology

The fiducial cosmology used throughout this project is based on the observational data obtained by Aghanim et al. (2020):

$$
\begin{aligned}
h &= 0.67, \\
T_{\rm CMB0} &= 2.7255 \text{ K}, \\
N_{\rm eff} &= 3.046, \\
\Omega_{\rm b0} &= 0.05, \\
\Omega_{\rm CDM0} &= 0.267, \\
\Omega_{k0} &= 0, \\
\Omega_{\nu 0} &= N_{\rm eff} \cdot \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \Omega_{\gamma 0}, \\
\Omega_{\Lambda 0} &= 1 - (\Omega_{k0} + \Omega_{\rm b0} + \Omega_{\rm CDM0} + \Omega_{\gamma 0} + \Omega_{\nu 0}), \\
\Omega_{\rm M0} &= \Omega_{b0} + \Omega_{\rm CDM0}, \\
\Omega_{\rm rad} &= \Omega_{\gamma 0} + \Omega_{\nu 0}, \\
n_s &= 0.965, \\
A_s &= 2.1 \cdot 10^{-9}.
\end{aligned}
$$

Introduction

The Cosmic Microwave Background radiation is the leftover radiation from the early universe. It is the most ancient light we can observe, having travelled towards us ever since the Universe became transparent. Therefore, it contains a significant amount of information and it is of great interest to understand why it looks the way it does. Why are there fluctuations in the CMB? In this project we take a closer look at the CMB power spectrum. This is a plot that show the distribution of temperature fluctuations in the CMB across different angular scaler. This is of great significance to us, since the CMB power spectrum is able to reveal information about the cosmological parameters of our universe, such as the various density components and Hubble constant. It is also able to tell us something about the large scale structures of the Universe, and the overall geometry of space itself. Also, which is perhaps the most interesting, it can yield information about the nature of dark energy.

The overarching aim is to produce a pipeline that allows us to calculate numerically the CMB power spectrum given some cosmology. The steps will be presented chronologically, and we start by setting up the background cosmology in **??**. Here we solve the evolutionary equations for an isotropic and homogeneous universe using the ΛCDM-model. One particularly important event in the evolution of the CMB is recombination, ultimately leading to photon decoupling. After this, the photons free stream towards us and is what we today see as the CMB. The entire **??** is devoted to this event, and the time right before and right after it. In **??** we take a step away from the isotropic and homogeneous universe and consider perturbations to the metric in conformal Newtonian gauge. The implications these metric perturbations have on the distribution of matter is discussed, and we end up with a set of coupled differential equations that we can solve numerically. The initial condition of these are found by considering the period of inflation in the very early Universe.

TODO: Intro to Milestone 4

1. Milestone IV

Some introduction to milestone 4

1.1. Theory

Some theory

1.2. Methods

some methods

1.3. Results

References

Aghanim, N., Akrami, Y., Ashdown, M., et al. 2020, Astronomy & Astrophysics, 641, A6

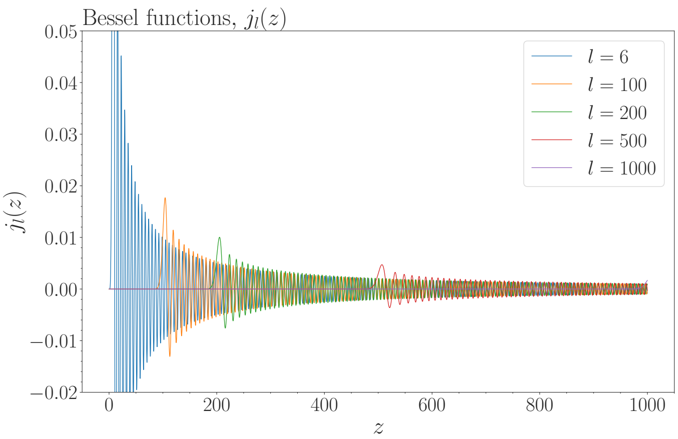
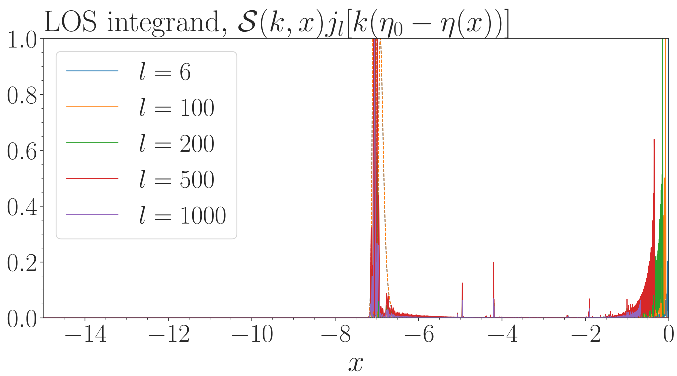


Fig. 1. Some caption



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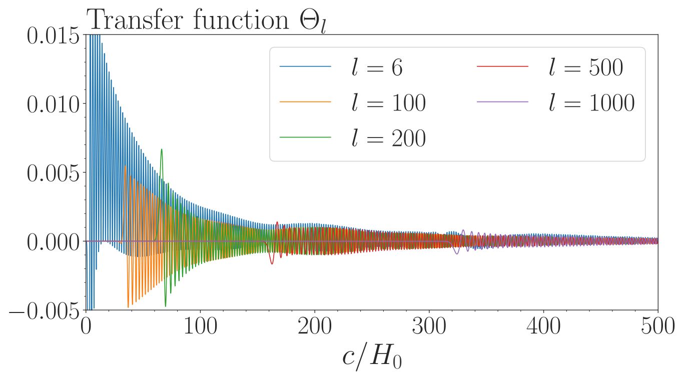

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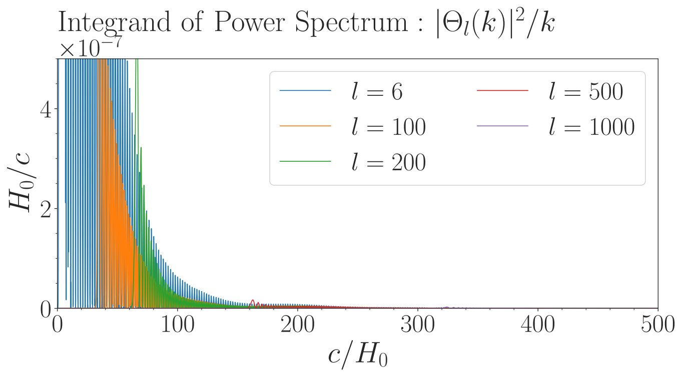

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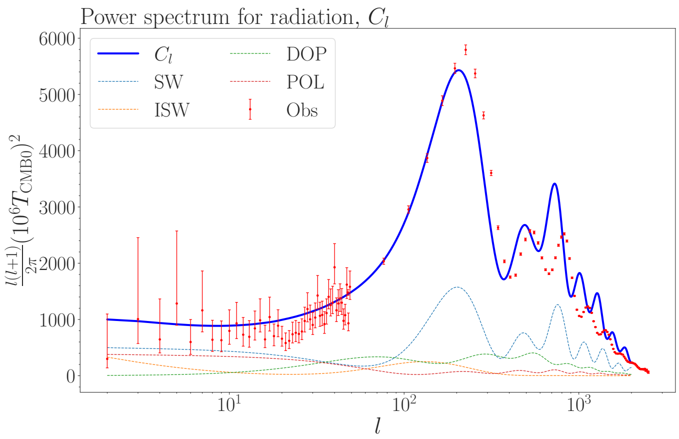

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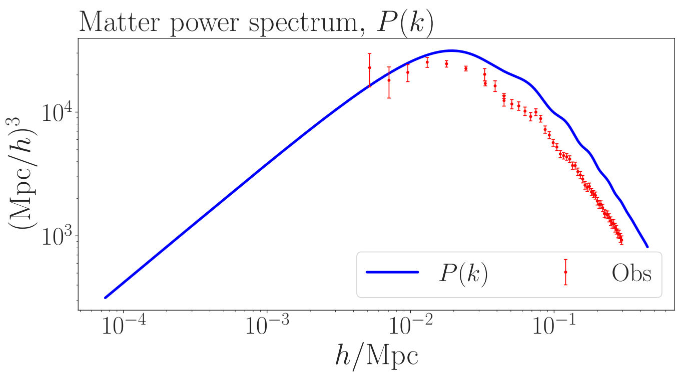

Fig. 6. Some caption

Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element ds as given in equation **??**, and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x}r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x\mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}H = e^{-x}c \\ \implies \frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{c}{\mathcal{H}}. \end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x}c \\ \implies \frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{e^x}{\mathcal{H}} = \frac{1}{H}. \end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from **??** and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2}U^{-\frac{1}{2}}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right] \end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned} U &\approx \Omega_{i0}e^{-\alpha_i x} \\ \frac{\mathrm{d}^n U}{\mathrm{d}x^n} &\approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2\Omega_{i0}e^{-\alpha_i x}}{2\Omega_{i0}e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4} \end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

TODO: fix this In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned} \eta &= \int_{-\infty}^{x}\frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right), \end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} &= -\frac{\alpha_i}{2}\mathcal{H} \\ \implies \mathrm{d}x &= -\frac{2}{\alpha_i\mathcal{H}}\mathrm{d}\mathcal{H}. \end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.