UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Monday, June 11th, 2018

Time: 09.00 – 13.00

The exam set consists of 13 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5) Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420) [20 p]

Answer each question with one to four sentences.

a) How is the spatial cold dark matter overdensity, $\delta(\vec{x},\eta) \equiv (\rho(\vec{x},\eta) - \rho^{(0)}(\eta))/\rho^{(0)}(\eta)$, found from the transfer function, $T_\delta(k,\eta)$ (this is the quantity that we called $\delta(k,\eta)$ in the project, and solved for in milestone III), and the initial condition for the potential, $\Phi(\vec{k},\eta_{\rm init})$? [4p]

b) Why is it simpler to solve the first order Einstein-Boltzmann equations in Fourier space rather than in position space? [4p]

c) What is the main effect of photon diffusion on the Cosmic Microwave Bacground (CMB) temperature fluctuations? [4p]

d) Which particle species, and what interactions among them, are relevant for the formation of the CMB? [4p]

e) Explain qualitatively how inflation gave rise to the fluctuations that seeded the subsequent growth of structure in the universe? (Remember, only a short description is needed, max four sentences!) [4p]

Problem 2 – Deriving the Friedmann Equation (AST5220 and AST9420) [20 p]

Consider a spatially flat Friedmann-Robertson-Walker spacetime with metric given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{1}$$

a) Calculate the Christoffel symbols ${\Gamma^0}_{ij}$, ${\Gamma^i}_{0j}$ and ${\Gamma^i}_{j0}$ for this spacetime. All other Christoffel symbols are equal to zero. Show that

$${\Gamma^0}_{ij} = \delta_{ij}a^2 H, \tag{2}$$

$${\Gamma^i}_{0j} = {\Gamma^i}_{j0} = \delta_{ij}H, \tag{3}$$

where $H = (da/dt)/a$ is the Hubble constant. [6p]

b) Show that:

$$R_{00} = -3\frac{d^2a/dt^2}{a}, \tag{4}$$

where $R_{\mu\nu}$ is the Ricci tensor. [6p]

c) We also have (you do not need to show this):

$$R_{ij} = \delta_{ij}\left[2\left(\frac{da}{dt}\right)^2 + a\frac{d^2a}{dt^2}\right]. \tag{5}$$

Use the $_{00}$-component of the Einstein equation to derive the familiar Friedmann equation. You can assume that the matter in the universe can be described as a perfect fluid. [8p]

Problem 3 – Boltzmann Equation for Number Density (AST5220 and AST9420) [30 p]

For this problem, assume that the universe is homogeneous, isotropic and flat.

a) The collisionless Boltzmann equation for the number density of a generic particle species is given by

$$\frac{dn}{dt} + 3Hn = 0, \tag{6}$$

where n is the number density and $H = (da/dt)/a$ is the Hubble constant.

- What is the solution of this equation (it can be smart to solve the equation in terms of a instead of t)? What does the solution mean physically? [6p]

b) If we add a collision term, we can, under certain conditions, write the complete Boltzmann equation for the number density of species 1 on the form:

$$\frac{dn_1}{dt} + 3Hn_1 = n_1^{\rm eq} n_2^{\rm eq} \langle \sigma v \rangle \left[\frac{n_3 n_4}{n_3^{\rm eq} n_4^{\rm eq}} - \frac{n_1 n_2}{n_1^{\rm eq} n_2^{\rm eq}} \right], \tag{7}$$

where we have defined

$$n_i^{\rm eq} \equiv g_i \int \frac{d^3 p}{(2\pi)^3} \exp\left(-E_i/T\right). \tag{8}$$

- What does this collision term represent physically?

- What do the different terms and factors on the right hand side of Eq. 7 mean physically. Do the signs of the two terms make sense?

- Qualitatively, how do the results from a) change when the collision term is introduced? [12p]

c) The Saha equation corresponding to Eq. 7 is given by

$$\frac{n_3 n_4}{n_3^{\rm eq} n_4^{\rm eq}} = \frac{n_1 n_2}{n_1^{\rm eq} n_2^{\rm eq}} \tag{9}$$

- When is the Saha equation a good approximation? Explain! [4p]

d) Derive the Saha equation for recombination (i.e. the process $e^- p^+ \to H\gamma$, where e^- is an electron, p^+ is a proton, H is a neutral hydrogen atom and γ is a photon). Write it in terms of the free electron fraction $X_e \equiv n_e/n_b = n_p/n_b$, where $n_b = n_p + n_H$ is the total amount of protons (we are neglecting Helium and other heavier elements).

Hint 1: You can use the fact that, for a non-relativistic particle species:

$$n_i^{\rm eq} = \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}.$$

Hint 2: For photons $n_\gamma = n_\gamma^{\rm eq}$. [8p]

Problem 4 – Cold Dark Matter (AST5220) [30 p]

In the lectures (and in the book) we derived evolution equation for the cold dark matter (CDM) overdensities, $\delta(\vec{x},t)$, and average velocity, $v^i(\vec{x},t)$, using the Boltzmann equation for CDM. Today, however, you are going to use the fact that CDM can be treated as a perfect fluid to derive the same evolution equations from the conservation equation

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{10}$$

The energy momentum tensor of the CDM fluid is given by

$$T^\mu{}_\nu = \rho\, U^\mu U_\nu, \tag{11}$$

where $\rho(\vec{x},t)$ is the energy density and $U^\mu(\vec{x},t)$ is the four-velocity of the CDM fluid.

It is often useful to separate the energy momentum tensor into a zero'th order part and a perturbed part:

$$T^\mu{}_\nu(\vec{x},t) = T^{(0)\mu}{}_\nu(t) + \delta T^\mu{}_\nu(\vec{x},t). \tag{12}$$

We will be working in the Newtonian gauge, given by:

$$ds^2 = -[1+2\Psi(\vec{x},t)]dt^2 + a^2(t)[1+2\Phi(\vec{x},t)]\delta_{ij}dx^i dx^j. \tag{13}$$

a) The four-momentum is given (to first order) by

$$P^\mu = \left((1-\Psi)E, \frac{(1-\Phi)}{a}p^i\right). \tag{14}$$

Using this, show that for CDM, the four velocity, $U^\mu \equiv dx^\mu/d\tau = P^\mu/m$, is given by

$$U^\mu = \left((1-\Psi), \frac{(1-\Phi)}{a}v^i\right), \tag{15}$$

and that

$$U_\mu = \left(-(1+\Psi), a(1+\Phi)v^i\right). \tag{16}$$

[3p]

b) The four-velocity of the CDM fluid is given by Eqs 15 and 16, only that v^i is then the velocity of the fluid (i.e. the average velocity of the particles), which we assume to be small, and not the velocity of any individual particle. Defining the CDM overdensity

$$\delta(\vec{x},t) \equiv \frac{\rho(\vec{x},t)-\rho^{(0)}(t)}{\rho^{(0)}(t)}, \tag{17}$$

show that, to first order, the perturbed part of the CDM energy momentum tensor is given by

$$\delta T^\mu{}_\nu = \rho^{(0)} \begin{pmatrix} -\delta & av^1 & av^2 & av^3 \\ -v^1/a & 0 & 0 & 0 \\ -v^2/a & 0 & 0 & 0 \\ -v^3/a & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

[6p]

c) Use the time component of the conservation equation

$$\nabla_\mu T^\mu{}_0 = 0 \tag{19}$$

to derive the evolution equations for the CDM density.

You can use the following result (without deriving it):

$$\nabla_\mu T^{(0)\mu}{}_0 = -\frac{d\rho^{(0)}}{dt} - 3\rho^{(0)}\left(H + \frac{\partial \Phi}{\partial t}\right). \tag{20}$$

Remember also that the covariant derivative of $\delta T^\mu{}_0$ is given by:

$$\nabla_\mu \delta T^\mu{}_0 = \frac{\partial \delta T^\mu{}_0}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}\delta T^\alpha{}_0 - \Gamma^\alpha{}_{\mu 0}\delta T^\mu{}_\alpha.$$

Separate the zero'th order equation from the first order one, and show that you get the following

$$\frac{d\rho^{(0)}}{dt} + 3H\rho^{(0)} = 0, \tag{21}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{22}$$

Hint 1: Remember that you can drop any terms second order in the small quantities. Do this as early as possible!

Hint 2: If you use the results from Problem 2 a), you should not have to calculate any new Christoffel symbols. [7p]

d) Using the spatial components of the conservation equation, we can also derive an equation for the CDM velocity (you do not have to do this!). Give a physical/intuitive explanation of the different terms in each of the three equations:

$$\frac{d\rho^{(0)}}{dt} + 3H\rho^{(0)} = 0, \tag{23}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0, \tag{24}$$

$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0. \tag{25}$$

[8p]

e) Figure 1 shows the angular power spectrum for temperature fluctuations in the CMB.

- How would this curve change (roughly) if you changed the ratio of CDM to baryons (i.e. replacing some of the CDM by baryons or vise versa, keeping everything else fixed)? Please draw a sketch and explain your reasoning. [6p]

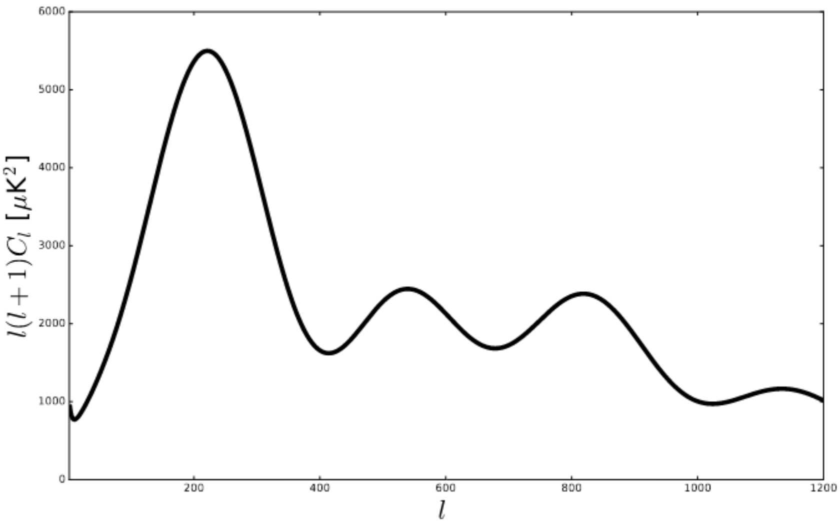

Figure 1: Angular power spectrum of temperature fluctuations in the CMB.

Problem 5 – Line-of-sight integration (AST9420) [30 p]

In this problem, we will derive the expression for the transfer function, $\Theta_l(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
j_l(x) &= \frac{(-i)^l}{2} \int_{-1}^{1} e^{i\mu x} P_l(\mu) d\mu \\
f_l &= \frac{i^l}{2} \int_{-1}^{1} f(\mu) P_l(\mu) d\mu
\end{aligned}$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and $\mu \equiv \hat{k} \cdot \hat{p}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta - \eta_0) - \tau} d\eta.$$

Hint: Note that

$$\dot{\Theta} + (ik\mu - \dot{\tau})\Theta = e^{-ik\mu\eta + \tau} \frac{d}{d\eta} \left[\Theta e^{ik\mu\eta - \tau}\right].$$

[6p]

b) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta - \eta_0)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$. [4p]

c) In reality, $\tilde{S}$ does of course depend on μ, and this has to be taken into account. The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta-\eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \to \frac{1}{ik}\frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik \to d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0-\eta)]d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right] \tag{26}$$

Hint: You may need your old knowledge about integration-by-parts to get this right. [8p]

d) Eq. 26 can be rewritten in the following form

$$S(k,\eta) = g(\eta)\left[\Theta_0 + \Psi\right] + \frac{d}{d\eta}\left(\frac{iv_b g(\eta)}{k}\right) + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right]. \tag{27}$$

- What is $g(\eta)$ here? (Explain physically!)

- Explain what each of these terms means physically (You can draw a sketch if you want to). [12p]

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{28}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{29}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{30}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor (in the rest frame of the fluid) is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{31}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{32}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{33}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{34}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{35}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{36}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{37}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{38}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{39}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{40}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{41}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{42}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{43}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{44}$$

1.4 Initial conditions

$$\Phi = 1 \tag{45}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{46}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{47}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{48}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{49}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{50}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{51}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{52}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{53}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{54}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{55}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{56}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{57}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{58}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{59}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{60}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{61}$$

$$n_{1s} = (1 - X_e) n_H \tag{62}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{63}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{64}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{65}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{66}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{67}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{68}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{69}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{70}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{71}$$