UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Tuesday, June 12th, 2013

Time: 09.00 – 13.00

The exam set consists of 9 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

What are x^μ, λ and $\Gamma^\mu_{\alpha\beta}$ in this equation, and what does the geodesic equation describe?

b) Why can we adopt the Boltzmann distribution to describe matter in this course?

c) Explain why cosmological perturbations do not grow until after $x \approx -10$, where $x = \ln a$.

d) What is $\theta(x, t, \hat{p})$, and why is this a particularly important quantity in cosmological perturbation theory?

e) When solving the Einstein-Boltzmann equations, we set $\Phi(a = 0) = 1$. Why is this acceptable?

f) The power spectrum source function reads

$$\tilde{S}(k, x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right]$$

Describe each term physically.

Problem 2 – Physical interpretation

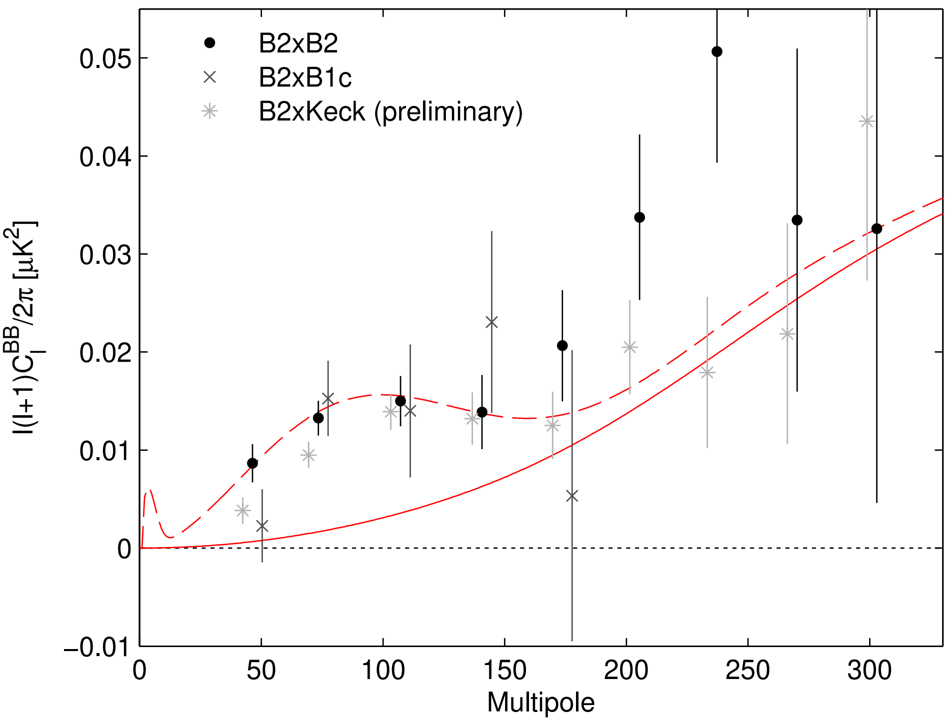

Figure 1: The BICEP2 B-mode power spectrum. The dashed red curve shows the best-fit theoretical spectrum, adopting the parameter values derived by the BICEP2 team.

Figure 1 shows the recently released BICEP2 B-mode polarization power spectrum, which led the BICEP2 team to claim a detection of inflationary gravitational waves with a tensor-to-scalar ratio of $r = 0.2^{+0.07}_{-0.05}$. The best-fit model including all effects are indicated by the dashed red line, and three different features may be seen in this curve, namely 1) a peak at $\ell \approx 10$, 2) a peak at $\ell \approx 100$, and 3) a rise toward high values.

a) Which physical effects cause/create each of the three features?

b) The three most important cosmological parameters for this multipole range are the amplitude of scalar perturbations, A_s, the optical depth of reionization, τ, and the tensor-to-scalar ratio, r. For each parameter, draw a cartoon of the theory spectrum, indicating what happens if you double its value.

c) Explain why the optical depth of reionization, τ, is nearly degenerate with the spectral index of scalar perturbations, n_s.

d) If one only has CMB temperature observations, τ is also almost perfectly degenerate with the amplitude of scalar perturbations, A_s. Explain why. Why do low-multipole polarization observations break this degeneracy?

Problem 3 – The Boltzmann equation for CDM

To get the perturbations in an expanding universe right, one of the most central equations is the Boltzmann equation for cold dark matter (CDM). In this problem we will review some of the most important steps in relevant derivation, and we will work in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2), \tag{1}$$

where Φ is the Newtonian potential and Φ is the curvature potential.

a) First, the symbolic Boltzmann equation is written in terms of a total time derivative, while it is computationally more convenient to work with partial derivatives in t, x, E and $\hat{p}$. Therefore, we first rewrite the Boltzmann equation in terms of partial derivatives:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} = 0 \tag{2}$$

Why can we neglect the last term, depending on the direction of the momentum?

b) The energy of a massive particle is given by $E = \sqrt{p^2+m^2}$, and one also one knows from special relativity that $g_{\mu\nu}P^\mu P^\nu = -m^2$, where $P^\mu = (E, p^i)$ is the four-momentum of the particle. Show that $P^0 \approx E(1-\Psi)$.

c) Show that $P^i = \frac{p}{a}\hat{p}^i(1-\Phi)$.

d) Derive an expression for $\frac{\mathrm{d}x^i}{\mathrm{d}t}$ in terms of $\hat{p}$, p, E, a, Φ and Ψ. Physically, what does this equation tell us?

e) Finally, one has to compute $\frac{\mathrm{d}E}{\mathrm{d}t}$, which one can obtain from the geodesic equation – but, fortunately, we won't do that here. Instead, we simply write down the answer, namely

$$\frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\frac{p\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} + \frac{p^2}{E}H + \frac{p^2}{E}\frac{\partial \Phi}{\partial t}\right] = 0 \tag{3}$$

Then, we recall the definitions of the particle density and the mean velocity,

$$n = \int \frac{\mathrm{d}^3p}{(2\pi)^3} f \tag{4}$$

$$v^i = \frac{1}{n}\int \frac{\mathrm{d}^3p}{(2\pi)^3} f \frac{p\hat{p}^i}{E}, \tag{5}$$

$$\tag{6}$$

where $n = n^{(0)}(1+\delta)$. From this, derive the Boltzmann equation for the density of cold dark matter,

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \tag{7}$$

f) To close the system, we need actually two Boltzmann equations for CDM, because there are two unknown, δ and v. Outline schematically how one can obtain this second equation from equation 3.

Problem 4 – Christoffel symbols in the synchronous gauge

The synchronous gauge is defined through the following line element,

$$ds^2 = -dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{8}$$

where

$$h_{ij} = \begin{bmatrix} -2\phi & 0 & 0 \\ 0 & -2\phi & 0 \\ 0 & 0 & h + 4\phi \end{bmatrix}, \tag{9}$$

and $\phi = \phi(x, t)$, $h = h(x, t)$ are two perturbation fields. Compute Γ^0_{00}, Γ^0_{03}, Γ^0_{11}, Γ^0_{23}, Γ^0_{33}, Γ^1_{11}, Γ^2_{10}, Γ^3_{22}, and Γ^3_{33}. (PS! The general expression for the Christoffel symbols is given in the Appendix. PPS! For those of you who thinks this is a lot of work – the first version of this problem requested calculation of all 64 symbols. Consider yourself lucky!)

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{10}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{11}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{12}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{13}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{14}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{15}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{16}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{17}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{18}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{19}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{20}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{21}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{22}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{23}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{24}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{25}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{26}$$

1.4 Initial conditions

$$\Phi = 1 \tag{27}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{28}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{29}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{30}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{31}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{32}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{33}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{34}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{35}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{36}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{37}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{38}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{39}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{40}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{41}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1} \tag{42}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{43}$$

$$n_{1s} = (1 - X_e) n_H \tag{44}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{45}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{46}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{47}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{48}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{49}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{50}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{51}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{52}$$

3. The CMB spectrum:

$$C_l = \int_0^{\infty} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{53}$$