UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Tuesday, June 14th, 2011

Time: 09.00 – 13.00

The exam set consists of 13 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420)

Answer each question with three or four sentences.

a) The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

What are x^μ, λ and $\Gamma^\mu_{\alpha\beta}$ in this equation? What does the geodesic equation describe?

b) The Boltzmann equation on schematic form read

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C[f] \tag{1}$$

What is the physical interpretation of f?

c) Why can we describe cosmological electrons by the Maxwell distribution instead of the full Fermi-Dirac distribution?

d) When solving the Saha and Peebles equations for the ionization fraction, X_e, one finds that X_e does not fall to zero at late times, but rather flattens out. Why?

e) What is the physical interpretation of $\Theta_l(k,\eta)$? What is the difference between T_0 and $\Theta_0(k,\eta)$?

f) Why do we solve the Einstein-Boltzmann equations in Fourier-space instead of real-space?

Problem 2 – Physical interpretation (AST5220 and AST9420)

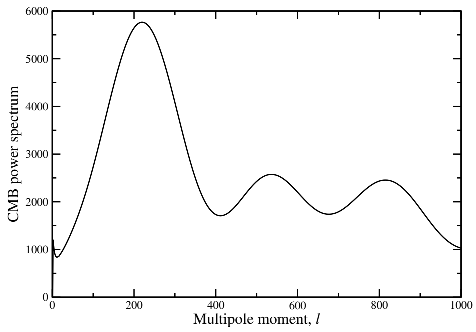

Figure 1: The best-fit LCDM WMAP7 temperature power spectrum.

Figure 1 shows the best-fit LCDM WMAP power spectrum as computed from the 7-year data release. In the following we will consider the behaviour of this curve as a function of cosmological parameters:

a) What happens if we multiply the amplitude A by a factor of two? What physical process determines the value of A?

b) What happens if we *decrease* the spectral index of scalar perturbations, n_s? What physical process determines the value of n_s?

c) What is the main distinguishing change if we *increase* the baryon density, Ω_b? How do we physically explain the observed effect?

d) What happens if we *increase* the total density Ω_0?

e) (**AST9420 only**) What happens if we *increase* the optical depth of reionization, τ? How do we explain this effect?

Problem 3 – The Boltzmann equation for CDM (AST5220 and AST9420)

To get the perturbations in an expanding universe right, one of the most central equations is the Boltzmann equation for cold dark matter (CDM). In this problem we will review some of the most important steps in relevant derivation, and we will work in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2), \qquad (2)$$

where Φ is the Newtonian potential and Φ is the curvature potential.

a) First, the symbolic Boltzmann equation is written in terms of a total time derivative, while it is computationally more convenient to work with partial derivatives in t, x, E and $\hat{p}$. Therefore, we first rewrite the Boltzmann equation in terms of partial derivatives:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} = 0 \qquad (3)$$

Why can we neglect the last term, depending on the direction of the momentum?

b) The energy of a massive particle is given by $E = \sqrt{p^2+m^2}$, and one also one knows from special relativity that $g_{\mu\nu}P^\mu P^\nu = -m^2$, where $P^\mu = (E, p^i)$ is the four-momentum of the particle. Show that $P^0 \approx E(1-\Psi)$.

c) Show that $P^i = \frac{p}{a}\hat{p}^i(1-\Phi)$.

d) Derive an expression for $\frac{\mathrm{d}x^i}{\mathrm{d}t}$ in terms of $\hat{p}$, p, E, a, Φ and Ψ. Physically, what does this equation tell us?

e) Finally, one has to compute $\frac{\mathrm{d}E}{\mathrm{d}t}$, which one can obtain from the geodesic equation – but, fortunately, we won't do that here. Instead, we simply write down the answer, namely

$$\frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\frac{p\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} + \frac{p^2}{E}H + \frac{p^2}{E}\frac{\partial \Phi}{\partial t}\right] = 0 \quad (4)$$

Then, we recall the definitions of the particle density and the mean velocity,

$$n = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f \quad (5)$$

$$v^i = \frac{1}{n}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f \frac{p\hat{p}^i}{E}, \quad (6)$$

$$(7)$$

where $n = n^{(0)}(1+\delta)$. From this,derive the Boltzmann equation for the density of cold dark matter,

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \quad (8)$$

f) To close the system, we need actually two Boltzmann equations for CDM, because there are two unknown, δ and v. Outline schematically how one can obtain this second equation from equation 4.

Problem 4 – The second Einstein equation (AST5220)

In the lectures we derived only one Einstein equation,

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left[\rho_m\delta + \rho_b\delta_b + 4\rho_r\Theta_0\right], \tag{9}$$

and left the second Einstein equation,

$$k^2(\Phi + \Psi) = -32\pi G a^2\rho_r\Theta_2, \tag{10}$$

as an exercise. Now the time has come for doing that exercise.

Recall the expression for the spatial part of the Ricci tensor in the conformal Newtonian gauge,

$$R_{ij} = \delta_{ij}\Bigg[\left(2a^2H^2 + a\frac{\mathrm{d}^2a}{\mathrm{d}t^2}\right)(1 + 2\Phi - 2\Psi) + \tag{11}$$

$$+ a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi\Bigg] + k_ik_j(\Phi + \Psi), \tag{12}$$

and the first-order part of the Ricci scalar,

$$\delta R = -12\Psi(H^2 + \Psi\frac{\mathrm{d}^2a/\mathrm{d}t^2}{a}) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00} \tag{13}$$

$$- 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}. \tag{14}$$

Also, remember that the Einstein equation reads

$$E^\mu_\nu = 8\pi G T^\mu_\nu \tag{15}$$

with $T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$, and

$$\rho = \sum_{\text{species}} g_i\int\frac{\mathrm{d}^3p_i}{(2\pi)^3}f_iE_i \tag{16}$$

$$P = \sum_{\text{species}} g_i\int\frac{\mathrm{d}^3p_i}{(2\pi)^3}f_i\frac{p_i^2}{3E_i}. \tag{17}$$

Your task is straightforward: Derive Equation 10.
Hints:

- Consider the spatial part of the Einstein equation, and apply the projection operator $(\hat{k}_i\hat{k}^j - 1/3\delta_i^j)$ to both sides of the equation. What is
- The second Legendre polynomial is $P_2(\mu) = \mu^2 - \frac{1}{3}$.
- Dark matter and baryons do not have a quadrupole moment, only monopole and dipole moments; radiation does have a quadrupole moment, namely Θ_2.

Problem 5 – Initial conditions (AST9420)

In order to solve the Boltzmann-Einstein equations, one obviously need initial conditions. Therefore: Derive the appropriate initial conditions for Ψ, δ, δ_b, v, v_b and Θ_0, relating all to Φ. You can neglect neutrinos and polarization in this problem, and you need only consider adiabatic initial conditions.

Hint: Start with the perturbation equations listed in the Appendix; choose some particular time for when you want to set your initial conditions; simplify as much as possible using physical arguments; derive an ordinary differential equation for Φ, and solve this; relate other quantities to Φ.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.

- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{18}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{19}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{20}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{21}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{22}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{23}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{24}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{25}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{26}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{27}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{28}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{29}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{30}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{31}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{32}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{33}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{34}$$

1.4 Initial conditions

$$\Phi = 1 \tag{35}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{36}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{37}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{38}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{39}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{40}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{41}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a d\eta' \tag{42}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{43}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{44}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{45}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^0 \tilde{g}(x) dx = 1. \tag{46}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{47}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{48}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{49}$$

$$\Lambda_{2s\to 1s} = 8.227\text{s}^{-1} \tag{50}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{51}$$

$$n_{1s} = (1-X_e)n_H \tag{52}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{53}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{54}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{55}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{56}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{57}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{58}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{59}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{60}$$

3. The CMB spectrum:

$$C_l = \int_0^{\infty} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{61}$$