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Calculate the CMB power spectrum: Cosmology II

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March 9, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

SOME ABSTRACT

Contents

Nomenclature

Constants of nature

G - Gravitational constant.
$G = 6.6743\times10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.3806 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.0546 \times 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.9979 \times 10^8$ m s^{-1}.

Cosmological parameters

H - Hubble parameter.
H_0 - Hubble constant fill in stuff.
$e^x\mathcal{H}$ - Scaled Hubble parameter.
$T_{\rm CMB0}$ - Temperature of CMB today.
$T_{\rm CMB0} = 2.7255$ K.
η - Conformal time.
χ - Co-moving distance.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

b - Baryons.
CDM - Cold dark matter.
γ - Electromagnetic radiation.
ν - Neutrinos.
k - Spatial curvature.
Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

1. Introduction

Some citation Dodelson & Schmidt (2020) and Weinberg (2008)

Also write about the following:

– Cosmological principle

- Einstein field equation
- Homogeneity and isotropy
- FLRW metric

In order to explain the connection between spacetime itself and the energy distribution within it we must solve the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the geometry of spacetime, G is the gravitational constant and $T_{\mu\nu}$ is the energy and momentum tensor.

TODO: Obviously this introduction will change and amended as more milestones are completed.

2. Milestone II

The main goal of this section is to investigate the recombination history of the universe, and compute the *visibility function*, g, and the *sound horizon*, s.

Motivate this

2.1. Theory

The optical depth as a function of conformal time is defines as Winther et al. (2023):

$$\tau = \int_{\eta}^{\eta_0} n_e \sigma_{\mathrm{T}} e^{-x} \mathrm{d}\eta', \tag{2}$$

where n_e is the electron density and σ_{T} is the Thompson cross-section. From this we define the visibility function, g:

$$\begin{aligned} g &= -\frac{\mathrm{d}\tau}{\mathrm{d}\eta} e^{-\tau} = -\mathcal{H}\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} \\ \tilde{g} &\equiv -\frac{\mathrm{d}\tau}{\mathrm{d}x} e^{-\tau} = \frac{g}{\mathcal{H}}, \end{aligned} \tag{3}$$

where $\tilde{g}$ is in terms of the preferred time variable, x. So far, so good, but in order to find $\tilde{g}$ we need τ, which again require n_e, which is not trivial to find, since the electron density changes throughout the evolution of the universe.

2.1.1. Finding the free electron fraction X_e

We express the electron density through the free electron fraction $X_e \equiv n_e/n_{\mathrm{H}} = n_e/n_b$ where we have assumed that the hydrogen make up all the baryons ($n_b = n_{\mathrm{H}}$). We also ignore the difference between free protons and neutral hydrogen. From Callin (2006) we obtain:

$$n_b = \frac{\rho_b}{m_{\mathrm{H}}} = \frac{\Omega_b \rho_c}{m_{\mathrm{H}}} e^{-3x}, \tag{4}$$

where m_{H} is the mass of the hydrogen atom, and ρ_c the critical density today as defined earlier (FIXME: cite this?). At early times, before recombination, $X_e \simeq 1$ (FIXME: why?), and is in this regime described by the *Saha equation*, from Dodelson & Schmidt (2020):

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{5}$$

where $\epsilon_0 = 13.6$ eV is the ionisation energy of hydrogen. The Saha equation is only a good approximation when $X_e \simeq 1$, thus for $X_e < (1 - \xi)$, (where we have to define ξ) which corresponds to the period during and after recombination, we have to make use of the more accurate *Peebles equation*. From Callin (2006):

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_{\mathrm{H}}\alpha^{(2)}(T_b)X_e^2\right], \tag{6}$$

where

$$\begin{aligned} C_r(T_b) &= \frac{\Lambda_{2s-1s} + \Lambda_\alpha}{\Lambda_{2s-1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \\ \Lambda_{2s-1s} &= 8.227\ \mathrm{s}^{-1}, \\ \Lambda_\alpha &= H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}}, \\ n_{1s} &= (1 - X_e)n_{\mathrm{H}}, \\ n_{\mathrm{H}} &= (1 - Y_p)\frac{3H_0^2\Omega_{b0}}{8\pi G m_{\mathrm{H}}} e^{-3x}, \\ \beta^{(2)}(T_b) &= \beta(T_b)e^{3\epsilon_0/4T_b}, \\ \beta(T_b) &= \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \\ \alpha^{(2)}(T_b) &= \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b), \\ \phi_2(T_b) &= 0.448\ln\left(\frac{\epsilon_0}{T_b}\right). \end{aligned} \tag{7}$$

TODO: Add σ_T and α to nomenclature.

TODO: Describe the above equations slightly

We find by X_e by solving Eq. (5) for $X_e > (1 - \xi)$ and Eq. (6) for $X_e < (1 - \xi)$. TODO: Explain why, difficult to integrate Peebles at early times etc.

2.2. Methods

some methods

2.3. Results

3. Milestone III

Some introduction to milestone 3

3.1. Theory

Some theory

3.2. Methods

some methods

3.3. Results

4. Milestone IV

Some introduction to milestone 4

4.1. Theory

Some theory

4.2. Methods

some methods

4.3. Results

5. Conclusion

Some overall conclusion

References

Callin, P. 2006, How to calculate the CMB spectrum

Dodelson, S. & Schmidt, F. 2020, Modern Cosmology (Elsevier Science)

Weinberg, S. 2008, Cosmology, Cosmology (OUP Oxford)

Winther, H. A., Eriksen, H. K., Elgaroy, O., Mota, D. F., & Ihle, H. 2023, Cosmology II, https://cmb.wintherscoming.no/, accessed on March 1, 2023

Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta \theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element ds as given in equation **??**, and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x} r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned}
&\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x} H = e^{-x} c \\
\implies &\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{c}{\mathcal{H}}.
\end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned}
&\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x} c \\
\implies &\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{e^x}{\mathcal{H}} = \frac{1}{H}.
\end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from **??** and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2} U^{-\frac{1}{2}} \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\
&= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\
&= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right]
\end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned}
\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\
&= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2
\end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned}
U &\approx \Omega_{i0} e^{-\alpha_i x} \\
\frac{\mathrm{d}^n U}{\mathrm{d}x^n} &\approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x},
\end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned}
\frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2 \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\
&= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4}
\end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned}
\eta &= \int_{-\infty}^{x} \frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\
&= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right),
\end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = -\frac{\alpha_i}{2}\mathcal{H} \\
\implies &\mathrm{d}x = -\frac{2}{\alpha_i \mathcal{H}}\mathrm{d}\mathcal{H}.
\end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.