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Calculate the CMB power spectrum: Cosmology II

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April 13, 2023 GitHub repo link: https://github.com/Johanmkr/AST5220/tree/main/project

ABSTRACT

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Nomenclature

Constants of nature

m_e - Mass of electron.
$m_e = 9.10938356 \cdot 10^{-31}$ kg.
m_{H} - Mass of hydrogen atom.
$m_{\mathrm{H}} = 1.6735575 \cdot 10^{-27}$ kg.
G - Gravitational constant.
$G = 6.67430 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}.
k_B - Boltzmann constant.
$k_B = 1.38064852 \cdot 10^{-23}$ m^2 kg s^{-2} K^{-1}.
$\hbar$ - Reduced Planck constant.
$\hbar = 1.054571817 \cdot 10^{-34}$ J s^{-1}.
c - Speed of light in vacuum.
$c = 2.99792458 \cdot 10^8$ m s^{-1}.
σ_T - Thomson cross section.
$\sigma_T = 6.6524587158 \cdot 10^{-29}$ m^2.
α - Fine structure constant.
$\alpha = \frac{m_e c}{\hbar}\sqrt{\frac{3\sigma_T}{8\pi}}$

Cosmological parameters

$G_{\mu\nu}$ - Einstein tensor.
$T_{\mu\nu}$ - Stress-energy tensor.
H - Hubble parameter.
$\mathcal{H}$ - Conformal Hubble parameter.
T_{CMB0} - Temperature of CMB today.
a - Scale factor.
x - Logarithm of scale factor.
t - Cosmic time.
z - Redshift.
η - Conformal time.
χ - Co-moving distance.
p - Pressure.
ρ - Density.
r - Radial distance.
d_A - Angular diameter distance.
d_L - Luminosity distance.

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n_e - Electron density.
n_b - Baryon density.
X_e - Free electron fraction.
τ - Optical depth.
$\tilde{g}$ - Visibility function.
s - Sound horizon.
r_s - Sound horizon at decoupling.
c_s - Wave propagation speed.

Density parameters

Density parameter $\Omega_X = \rho_X/\rho_c$ where ρ_X is the density and $\rho_c = 8\pi G/3H^2$ the critical density. X can take the following values:

- b - Baryons.
- CDM - Cold dark matter.
- γ - Electromagnetic radiation.
- ν - Neutrinos.
- k - Spatial curvature.
- Λ - Cosmological constant.

A 0 in the subscript indicates the present day value.

Fiducial cosmology

The fiducial cosmology used throughout this project is based on the observational data obtained by Aghanim et al. (2020):

$$\begin{aligned}
h &= 0.67, \\
T_{\mathrm{CMB0}} &= 2.7255\ \mathrm{K}, \\
N_{\mathrm{eff}} &= 3.046, \\
\Omega_{\mathrm{b0}} &= 0.05, \\
\Omega_{\mathrm{CDM0}} &= 0.267, \\
\Omega_{k0} &= 0, \\
\Omega_{\nu 0} &= N_{\mathrm{eff}} \cdot \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\Omega_{\gamma 0}, \\
\Omega_{\Lambda 0} &= 1 - (\Omega_{k0} + \Omega_{\mathrm{b0}} + \Omega_{\mathrm{CDM0}} + \Omega_{\gamma 0} + \Omega_{\nu 0}), \\
\Omega_{\mathrm{M0}} &= \Omega_{b0} + \Omega_{\mathrm{CDM0}}, \\
\Omega_{\mathrm{rad}} &= \Omega_{\gamma 0} + \Omega_{\nu 0}, \\
n_s &= 0.965, \\
A_s &= 2.1 \cdot 10^{-9}.
\end{aligned}$$

1. Introduction

Introduce all for Milestones and the overall aim of calculating the CMB power spectrum etc.

TODO: Obviously this introduction will change and amended as more milestones are completed.

2. Milestone III - Perturbations

The aim of this section is to investigate how small fluctuations in the baryon-photon-dark-matter fluid in the early grew into larger structures. This is done by examining the interplay between these fluid fluctuations and the subsequent fluctuations of the space-time geometry. We will model this by perturbing the flat FLRW-metric using the conformal-Newtonian gauge. This will impact how the Boltzmann equations for the different species behaves, from which we are able to construct differential equations for key physical observables, and their initial conditions.

2.1. Theory

2.1.1. Metric perturbations

The perturbed metric in the conformal-Newtonian gauge is given in Callin (2006) as:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & e^{2x}\delta_{ij}(1+2\Phi) \end{pmatrix} \tag{1}$$

This means that we perturb the FLRW-metric with $\Psi \ll 1$ corresponding to the Newtonian potential governing the motion of non-relativistic particles and $\Phi \ll 1$ governing the perturbation of the spatial curvature. [1] The comoving momentum in this spacetime is:

$$P^\mu = \left[E(1-\Psi), p^i \frac{1-\Phi}{a}\right]. \tag{2}$$

By considering this momentum, and the geodesic equation in this perturbed spacetime we obtain the following (Dodelson & Schmidt 2020, Eqs. 3.62, 3.69, 3.71):

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\hat{p}^i}{a}\frac{p}{E}(1-\Phi+\Psi) \tag{3a}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = -\left(H + \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)p^i - \frac{E}{a}\frac{\partial\Phi}{\partial x^i} - \frac{1}{a}\frac{p^i}{E}p^k\frac{\partial\Phi}{\partial x^k} + \frac{p^2}{aE}\frac{\partial\Phi}{\partial x^i} \tag{3b}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\left(H + \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)p - \frac{E}{a}\hat{p}^i\frac{\partial\Psi}{\partial x^i} \tag{3c}$$

Inserting Eq. (3) into **??**, and for now assuming $C[f] = 0$ yield the *collisionless Bolzmann equations*. Keeping terms to first order only,[2] yield the collisionless Bolzmann equation: (Dodelson & Schmidt 2020, Eq. 3.83):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - \left[H + \frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{E}{ap}\hat{p}^i\frac{\partial\Psi}{\partial x^i}\right]p\frac{\partial f}{\partial p}. \tag{4}$$

Future work consists mainly of evaluating the collision terms for each species and equate it to Eq. (4)

2.1.2. Fourier space and multipole expansion

Consider a function $f(\mathbf{x}, t)$. Its Fourier transform $\mathscr{F}$ and inverse $\mathscr{F}^{-1}$ are defined as:

$$\mathscr{F}[f(\mathbf{x},t)] \equiv \frac{1}{(2\pi)^{3/2}}\int e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x},t)\mathrm{d}^3x = \tilde{f}(\mathbf{k},t), \tag{5}$$

$$\mathscr{F}^{-1}[\tilde{f}(\mathbf{k},t)] \equiv \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k},t)\mathrm{d}^3k = f(\mathbf{x},t). \tag{6}$$

It becomes apparent from these definitions that taking the spatial derivative with respect to $\mathbf{x}$ in real space, is the same as multiplying the function with $i\mathbf{k}$ in Fourier space. This leads to the following property: $\mathscr{F}[\nabla f(\mathbf{x},t)] = i\mathbf{k}\mathscr{F}[f(\mathbf{x},t)]$. This is of major significance when working with partial differential equations (PDEs), where:

$$\begin{aligned}
&\mathscr{F}[\nabla^2 f(\mathbf{x},t)] = i^2\mathbf{k}\cdot\mathbf{k}\mathscr{F}[f(\mathbf{x},t)] = -k^2\mathscr{F}[f(\mathbf{x},t)] \\
&\mathscr{F}\left[\frac{\mathrm{d}^n f(\mathbf{x},t)}{\mathrm{d}t^n}\right] = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathscr{F}[f(\mathbf{x},t)].
\end{aligned} \tag{7}$$

The two equations in Eq. (7) have the ability of reducing PDEs down to a set of decoupled ODEs. This means that

[1] Φ may also be interpreted as a *local perturbation to the scale factor*, Dodelson & Schmidt (2020).

[2] This is justified by the ansatz that deviations away from the equilibrium distribution of radiation in the inhomogeneous universe are of same order as the spacetime perturbations Φ and Ψ, Dodelson & Schmidt (2020).

we are able to solve for each mode $k = |\mathbf{k}|$ independently, which will be of great impact for the equations to come.

We will also work with multipole expansions, which are series written as sums of *Legendre polynomials* expanded in $\mu = \cos\theta \in [-1, 1]$ as:

$$f(\mu) = \sum_{l=0}^{\infty} f_l \mathcal{P}_l(\mu), \tag{8}$$

where $\mathcal{P}_l$ is the l-th Legendre polynomial. These are orthogonal in such a way that they form a complete basis, enabling us to express any $f(\mu)$ as in Eq. (8). The coefficients f_l are the *Legendre multipoles*:

$$f_l = \frac{2l+1}{2} \int_{-1}^{1} f(\mu)\mathcal{P}_l(\mu)\mathrm{d}\mu. \tag{9}$$

2.1.3. Einstein-Boltzmann equations

We have two perturbations to the metric, $\Phi(\mathbf{x}, t)$ to the spatial curvature, and $\Psi(\mathbf{x}, t)$ to the Newtonian potential. We seek to find the effect of these perturbations on baryonic matter, dark energy and radiation, as they "live" in a now perturbed spacetime. Let's start by defining the perturbation to the photons, $\Theta(\mathbf{x}, \hat{\mathbf{p}}, t)$, to be the variation of photon temperature around an equilibrium temperature $T^{(0)}$:

$$T(\mathbf{x}, \hat{\mathbf{p}}, t) = T^{(0)} \left[1 + \Theta(\mathbf{x}, \hat{\mathbf{p}}, t)\right]. \tag{10}$$

This is dependent on the location $\mathbf{x}$ and the direction of propagation $\hat{\mathbf{p}}$, thus capturing both inhomogeneities and anisotropies. We assume Θ to be independent of the momentum magnitude.[3] The collision terms for the photons are governed by Compton scattering. We use the form found in (Dodelson & Schmidt 2020, Eq. 5.22)TODO: assumptions: ignore polarisation, and angular dep. of thomson cross sec:

$$C[f(\mathbf{p})] = -p^2 \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{\mathbf{p}}) + \hat{\mathbf{p}} \cdot \mathbf{v}_b] \tag{11}$$

where Θ_0 is the monopole term.[4] $\mathbf{v}_b$ is the bulk velocity of the electrons involved in the process, and is the same as for baryons, hence the subscript. The distribution function for radiation follows the Bose-Einstein distribution function, so we expand f around its zeroth order Bose-Einstein form, (Dodelson & Schmidt 2020, Eq. 5.2-5.9), using the temperature perturbation in Eq. (10)TODO: Include equation 5.9 in Dodelson?. This is then inserted into Eq. (4), which we equate to the collision term in Eq. (11) in order to obtain the following full Boltzmann equation for radiation:[5]

$$\frac{\mathrm{d}\Theta}{\mathrm{dt}} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\mathrm{d}\Phi}{\mathrm{dt}} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0 - \Theta + \hat{\mathbf{p}}\cdot\mathbf{v}_b\right] \tag{12}$$

[3] This follows from the fact that the magnitude of the photon momentum is virtually unchanged by the dominant form of interaction, Compton scattering Dodelson & Schmidt (2020).

[4] This is the integral over the photon perturbation at any given point, over all photon directions. It is given by

$$\Theta_0(\mathbf{x}, t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega' \Theta(\mathbf{x}, \hat{\mathbf{p}}', t)$$

where Ω' is the solid angle spanned by $\hat{\mathbf{p}}'$ Dodelson & Schmidt (2020).

[5] Where of course $m = 0 \iff E = p$.

For massive particles, we start with cold dark matter (CDM). Firstly, we assume cold dark matter to not interact with any other species, nor self-interact. Thus, we do not have any collision terms. Further, we also assume it to behave like a fluid, neglecting any terms not to first order. We consider cold dark matter to be non-relativistic, thus they will only have a sizeable monopole and dipole term, which means that the evolution is fully characterised by the density and velocity, Winther et al. (2023). TODO: how much about moments should I explain? Therefore, we take the first and second moment of Eq. (4) and consider them to first order, in order to retrieve the cosmological generalisation of the continuity equation (Dodelson & Schmidt 2020, Eq. 5.41):

$$\frac{\partial n_c}{\partial t} + \frac{1}{a}\frac{\partial(n_c v_c^i)}{\partial x^i} + 3\left[H + \frac{\partial\Phi}{\partial t}\right] n_c = 0, \tag{13}$$

and the Euler equation (Dodelson & Schmidt 2020, Eq. 5.50):

$$\frac{\partial v_c^i}{\partial t} + H v_c^i + \frac{1}{a}\frac{\partial\Psi}{\partial x^i} = 0. \tag{14}$$

In both Eq. (13) and Eq. (14), n_c is the cold dark matter number density, $\mathbf{v}_c$ its bulk velocity. We then consider the perturbation of n_c to first order:

$$n_c(\mathbf{x}, t) = n_c^{(0)}[1 + \delta_c(\mathbf{x}, t)], \tag{15}$$

and consider the first order perturbation to Eq. (13):

$$\frac{\partial\delta_c}{\partial t} + \frac{1}{a}\frac{\partial v_c^i}{\partial x^i} + 3\frac{\partial\Phi}{\partial t} = 0. \tag{16}$$

Eq. (16) and Eq. (14) now described the evolution of the density perturbation δ_c and bulk velocity $\mathbf{v}_c$ of cold dark matter.

For baryons (protons and electrons) we also assume them to behave like a non-relativistic fluid, so taking moments is a similar task as for cold dark matter. The only difference is that baryons interact with each other through Coulomb scattering and Compton scattering. We may ignore Compton scattering between protons and photons due to the small cross-section, but electrons are coupled to both photons and protons. Since the first moment of the Boltzmann equation represents conservations of particle number, and none of the above interactions changes the total baryon particle number, the continuity equation is identical to Eq. (13), but for baryons. We also have a baryon perturbation similar to Eq. (15), which altogether results in the following density perturbation for baryons:

$$\frac{\partial\delta_b}{\partial t} + \frac{1}{a}\frac{\partial v_b^i}{\partial x^i} + 3\frac{\partial\Phi}{\partial t} = 0. \tag{17}$$

$$\dot{\Theta} = -ik\mu(\Theta + \Psi) - \dot{\Phi} - \dot{\tau}\left[\Theta_0 - \Theta + i\mu v_b - \frac{\mathcal{P}_2\Theta_2}{2}\right], \tag{18a}$$

$$\dot{\delta}_{\mathrm{CDM}} = -3\dot{\Phi} + kv_{\mathrm{CDM}} \tag{18b}$$

$$\dot{v}_{\mathrm{CDM}} = -k\Psi - \mathcal{H}v_{\mathrm{CDM}} \tag{18c}$$

$$\tag{18d}$$

Photon temperature multipoles

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \tag{19a}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{19b}$$

$$\Theta_l = \begin{cases} \frac{lck\Theta_{l-1}}{(2l+1)\mathcal{H}} - \frac{(l+1)ck\Theta_{l+1}}{(2l+1)\mathcal{H}} + \tau'\left[\Theta_l - \frac{\Theta_2}{10}\delta_{l,2}\right], & l \geq 2 \\ \frac{ck\Theta_{l-1}}{\mathcal{H}} - c\frac{(l+1)\Theta_l}{\mathcal{H}\eta} + \tau'\Theta_l, & l = l_{\max} \end{cases} \tag{19c}$$

Cold dark matter and baryons

$$\delta_{\mathrm{CDM}}' = \frac{ck}{\mathcal{H}}v_{\mathrm{CDM}} - 3\Phi', \tag{20a}$$

$$v_{\mathrm{CDM}}' = -v_{\mathrm{CDM}} - \frac{ck}{\mathcal{H}}\Psi, \tag{20b}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi', \tag{20c}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}} + \tau' R^{-1}(3\Theta_1 + v_b) \tag{20d}$$

where R is defined in **??**

Metric perturbations

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\mathcal{Y}, \tag{21a}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2}\Omega_\gamma\Theta_2. \tag{21b}$$

where $\mathcal{Y} = \Omega_{\mathrm{CDM}}\delta_{\mathrm{CDM}} + \Omega_b\delta_b + 4\Omega_\gamma\Theta_0$

2.1.4. Tight coupling regime

2.1.5. Inflation

2.1.6. Initial conditions

2.2. Methods

some methods

2.3. Results and discussion

References

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Callin, P. 2006, How to calculate the CMB spectrum
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Appendix A: Useful derivations

A.1. Angular diameter distance

This is related to the physical distance of say, an object, whose extent is small compared to the distance at which we observe is. If the extension of the object is Δs, and we measure an angular size of $\Delta\theta$, then the angular distance to the object is:

$$d_A = \frac{\Delta s}{\Delta\theta} = \frac{\mathrm{d}s}{\mathrm{d}\theta} = \sqrt{e^{2x}r^2} = e^x r, \tag{A.1}$$

where we inserted for the line element $\mathrm{d}s$ as given in equation **??**, and used the fact that $\mathrm{d}t/\mathrm{d}\theta = \mathrm{d}r/\mathrm{d}\theta = \mathrm{d}\phi/\mathrm{d}\theta = 0$ in polar coordinates.

A.2. Luminosity distance

If the intrinsic luminosity, L of an object is known, we can calculate the flux as: $F = L/(4\pi d_L^2)$, where d_L is the luminosity distance. It is a measure of how much the light has dimmed when travelling from the source to the observer. For further analysis we observe that the luminosity of objects moving away from us is changing by a factor a^{-4} due to the energy loss of electromagnetic radiation, and the observed flux is changed by a factor $1/(4\pi d_A^2)$. From this we draw the conclusion that the luminosity distance may be written as:

$$d_L = \sqrt{\frac{L}{4\pi F}} = \sqrt{\frac{d_A^2}{a^4}} = e^{-x} r \tag{A.2}$$

A.3. Differential equations

From the definition of $e^x \mathrm{d}\eta = c\mathrm{d}t$ we have the following:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}x} H = e^{-x} c \\ \implies \frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{c}{\mathcal{H}}. \end{aligned} \tag{A.3}$$

Likewise, for t we have:

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\mathrm{d}\eta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{c}{\mathcal{H}} = e^{-x} c \\ \implies \frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{e^x}{\mathcal{H}} = \frac{1}{H}. \end{aligned} \tag{A.4}$$

Appendix B: Sanity checks

B.1. For $\mathcal{H}$

We start with the Hubble equation from **??** and realize that we may write any derivative of U as

$$\frac{\mathrm{d}^n U}{\mathrm{d}x^n} = \sum_i (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}. \tag{B.1}$$

We further have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{H_0}{2} U^{-\frac{1}{2}} \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.2}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \\ &= \frac{H_0}{2}\left[\frac{\mathrm{d}U}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}U^{-\frac{1}{2}}\right) + U^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}U}{\mathrm{d}x}\right)\right] \\ &= H_0\left[\frac{1}{2U^{\frac{1}{2}}}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^{\frac{3}{2}}}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2\right] \end{aligned} \tag{B.3}$$

Multiplying both equations with $\mathcal{H}^{-1} = 1/(H_0 U^{\frac{1}{2}})$ yield the following:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} = \frac{1}{2U}\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{B.4}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \frac{1}{4U^2}\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \\ &= \frac{1}{2U}\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 \end{aligned} \tag{B.5}$$

We now make the assumption that one of the density parameters dominate $\Omega_i >> \sum_{j\neq i}\Omega_i$, enabling the following approximation:

$$\begin{aligned} U &\approx \Omega_{i0} e^{-\alpha_i x} \\ \frac{\mathrm{d}^n U}{\mathrm{d}x^n} &\approx (-\alpha_i)^n \Omega_{i0} e^{-\alpha_i x}, \end{aligned} \tag{B.6}$$

from which we are able to construct:

$$\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} \approx \frac{-\alpha_i \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} = -\frac{\alpha_i}{2}, \tag{B.7}$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}}\frac{\mathrm{d}^2\mathcal{H}}{\mathrm{d}x^2} &\approx \frac{\alpha_i^2 \Omega_{i0} e^{-\alpha_i x}}{2\Omega_{i0} e^{-\alpha_i x}} - \left(\frac{\alpha_i}{2}\right)^2 \\ &= \frac{\alpha_i^2}{2} - \frac{\alpha_i^2}{4} = \frac{\alpha_i^2}{4} \end{aligned} \tag{B.8}$$

which are quantities which should be constant in different regimes and we can easily check if our implementation of $\mathcal{H}$ is correct, which is exactly what we sought.

B.2. For η

TODO: fix this In order to test η we consider the definition, solve the integral and consider the same regimes as above, where one density parameter dominates:

$$\begin{aligned} \eta &= \int_{-\infty}^{x} \frac{c\mathrm{d}x}{\mathcal{H}} = \frac{-2c}{\alpha_i}\int_{x=-\infty}^{x=x}\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} \\ &= \frac{2c}{\alpha_i}\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right), \end{aligned} \tag{B.9}$$

where we have used that:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}x} &= -\frac{\alpha_i}{2}\mathcal{H} \\ \implies \mathrm{d}x &= -\frac{2}{\alpha_i \mathcal{H}}\mathrm{d}\mathcal{H}. \end{aligned} \tag{B.10}$$

Since we consider regimes where one density parameter dominates, we have that $\mathcal{H}(x) \propto \sqrt{e^{-\alpha_i x}}$, meaning that we have:

$$\left(\frac{1}{\mathcal{H}(x)} - \frac{1}{\mathcal{H}(-\infty)}\right) \approx \begin{cases} \frac{1}{\mathcal{H}} & \alpha_i > 0 \\ -\infty & \alpha_i < 0. \end{cases} \tag{B.11}$$

Combining the above yields:

$$\frac{\eta\mathcal{H}}{c} \approx \begin{cases} \frac{2}{\alpha_i} & \alpha_i > 0 \\ \infty & \alpha_i < 0. \end{cases} \tag{B.12}$$

Notice the positive sign before ∞. This is due to α_i now being negative.