UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Friday, June 11th, 2010

Time: 09.00 – 12.00

The exam set consists of 16 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420)

Answer each question with three or four sentences.

a) The metric for the conformal Newtonian gauge is

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2).$$

What is the physical interpretation of Ψ and Φ? At late times, how is Ψ numerically related to Φ?

b) The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

What are x^μ, λ and $\Gamma^\mu_{\alpha\beta}$ in this equation? What does the geodesic equation describe?

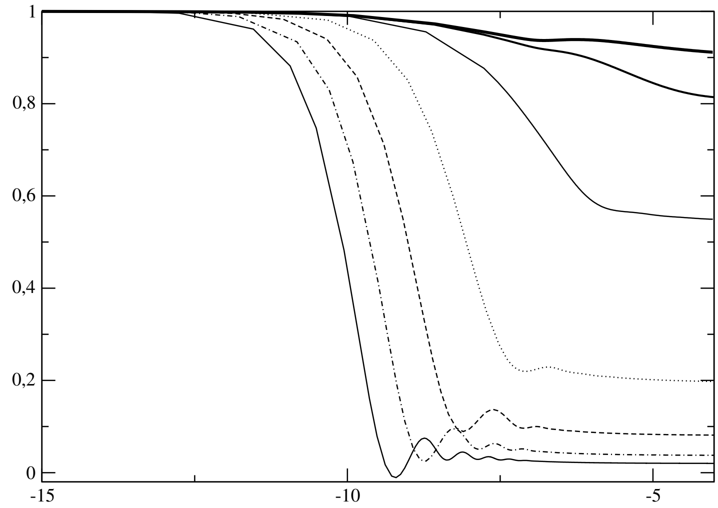
c) Write down the Boltzmann equation on schematic form. Which physical effects must be included in the collision term for 1) cold dark matter and 2) baryonic matter, respectively?

d) What is the optical depth, τ? Why is $\tau \approx 1$ a special value?

e) What is the main advantage of solving the Boltzmann-Einstein in Fourier space instead of in real-space?

f) Briefly list the (up to four) main motivations for introducing inflation in cosmology. (No more than two sentences per item.)

Problem 2 – Physical interpretation (AST5220 and AST9420)



Figur 1: The function Φ is plotted for a few different values of the wavenumber k, as function of time, x.

Figure 1 shows the curvature potential, Φ, which enters into the solution of the Einstein-Boltzmann equations, plotted for a few different values of k (between 0.1 and 1000 H_0/c) and as a function of time, x, where $x = \ln a$.

a) Which main phases can you see here?

b) Explain physically the general behaviour of these functions. Why are the functions flat after $x \approx -6$? In the standard ΛCDM model (ie., $\Omega_b \approx 0.05$, $\Omega_m \approx 0.25$ and $\Omega_\Lambda \approx 0.7$), will they remain flat to $x = 0$? If not, why?

c) Would the late-time constant plateau for small-scale modes shift to higher or lower values (up or down) if we double Ω_m? Why?

Problem 3 – The Saha equation (AST5220 and AST9420)

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 >> kT$, ie., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{1}$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u} e^{-u} du = \frac{\sqrt{\pi}}{2}.$$

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here?

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times?

Problem 4 – The Boltzmann equation for free photons (AST5220)

In this problem you will derive the Boltzmann equation for free photons, ie., neglecting any collision terms, in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2).$$

a) For a massless photon one has $g_{\mu\nu}P^\mu P^\nu = 0$, where $P^\mu = dx^\mu/d\lambda = (E, \vec{p})$ is the four-momentum of the photon. Define $p^2 = g_{ij}P^iP^j$. Show that

$$\begin{aligned} P^0 &= p(1-\Psi) \\ P^i &= \frac{p}{a}\hat{p}(1-\Phi) \end{aligned}$$

to first order in Ψ and Φ, where $\hat{p}$ is the unit vector pointing along the 3-momentum of the photon.

b) Show that

$$\frac{dx^i}{dt} = \frac{\hat{p}}{a}(1-\Phi-\Psi)$$

to first order. Give a physical interpretation of this equation.

c) Write down the Boltzmann equation expanded into its dynamic variables, x^i, t, p and $\hat{p}^i$. Why can the term dependent on $\hat{p}$ be neglected?

d) Starting from the 0-component of the geodesic equation, show that

$$\frac{dp}{dt} = p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

to first order.

e) The expression derived in d) can be written out in full, and one then obtains

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} - H - \frac{\partial \Phi}{\partial t}. \tag{2}$$

What does this equation tell us?

f) Using the above information, write down the Boltzmann equation for the distribution of photons. Which assumption/approximation is needed to get from this equation to an equation for $\Theta(x,t)$, where $\Theta = \frac{T(x)-T_0}{T_0}$ is the photon temperature perturbation, $T(x)$ is the physical temperature at position x, and T_0 is the average background temperature?

Problem 5 – Line-of-sight integration (AST9420)

A crucial part in speeding up a Boltzmann code is to change from direct integration of the Boltzmann-Einstein equations to a smarter approach, called “line-of-sight” integration. In this problem, we will derive the expression for the transfer function $\Theta_l(k)$ in terms of the source function.

Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_l(\mu) = (-1)^l P_l(-\mu)$$
$$\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu = \delta_{ll'} \frac{2}{2l+1}$$
$$j_l(x) = \frac{1}{2i^l} \int_{-1}^{1} e^{i\mu x} P_l(\mu) d\mu$$
$$f_l = \frac{i^l}{2} \int_{-1}^{1} f(\mu) P_l(\mu) d\mu$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) First, from an coding point of view, why does one obtain such a large speed-up with the line-of-sight integration approach compared to direct integration?

b) In a few sentences, explain what the main physical difference is between the line-of-sight integration and the direct solution approaches.

c) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

d) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta - \eta_0)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

e) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta - \eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik} \frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k, \eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{v_b\dot{\tau}}{k}\right)\right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

f) Introducing the visibility function, $g(x) \equiv -\dot{\tau}e^{-\tau}$, the source function can be rewritten into the form

$$S(k,\eta) = g[\Theta_0 + \Psi] + \frac{d}{d\eta}\left(\frac{gv_b}{k}\right) + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right].$$

What is the physical interpretation of each of these terms?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{3}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{4}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{5}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{6}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{7}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{8}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{9}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{10}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{11}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{12}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{13}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{14}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{15}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{16}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{17}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{18}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{19}$$

1.4 Initial conditions

$$\Phi = 1 \tag{20}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{21}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{22}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{23}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{24}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{25}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{26}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a d\eta' \tag{27}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{28}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{29}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{30}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^0 \tilde{g}(x) dx = 1. \tag{31}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{32}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{33}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{34}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{35}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{36}$$

$$n_{1s} = (1-X_e)n_H \tag{37}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{38}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{39}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{40}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{41}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{42}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{43}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{44}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\propto}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{45}$$

3. The CMB spectrum:

$$C_l = \int_0^{\propto} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{46}$$