

Solution for Exercize 4

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Abstract

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1 Question 1

1.1 a)

Mean μ of givin points

$$x : \frac{1+2+4+5}{4} = \frac{12}{4} = 3$$

$$y : \frac{-1+1+-1+1}{4} = \frac{0}{4} = 0$$

$$\mu = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

1.2 b)

The Covariancematrix C has the following form:

$$C = \begin{pmatrix} \text{covariance}(x, x) & \text{covariance}(x, y) \\ \text{covariance}(y, x) & \text{covariance}(y, y) \end{pmatrix} = \begin{pmatrix} \text{variance}(x) & \text{covariance}(x, y) \\ \text{covariance}(y, x) & \text{variance}(y) \end{pmatrix}$$

$$\text{covariance}(x, y) = \text{covariance}(y, x) = \frac{\sum_{i=0}^n (x_i - \bar{x}) * (y_i - \bar{y})}{n}$$

$$\begin{aligned} \text{covariance}(x, y) &= \frac{(1-3) * (-1-0) + (2-3) * (1-0) + (4-3) * (-1-0) + (5-3) * (1-0)}{4} \\ &= \frac{2}{4} = 2 \end{aligned}$$

$$\text{variance}(x) = \frac{\sum_{i=0}^n (x_i - \bar{x})^2}{n}$$

$$\text{variance}(x) = \frac{(1-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2}{4-1} = \frac{4+1+1+4}{4} = \frac{10}{4}$$

$$\text{variance}(y) = \frac{(-1-0)^2 + (1-0)^2 + (-1-0)^2 + (1-0)^2}{4-1} = \frac{1+1+1+1}{4} = \frac{4}{4} = 1$$

$$C = \begin{pmatrix} \frac{10}{4} & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix}$$

1.3 c)

$$\det(C - I\lambda) = |C - I\lambda| = 0$$

$$\begin{aligned} \left| \begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \lambda \right| &= 0 \\ \left| \begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| &= 0 \\ \begin{vmatrix} \frac{5}{2} - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} &= 0 \end{aligned}$$

The formula for determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a * d - c * b$$

$$\begin{vmatrix} \frac{5}{2} - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = \left(\frac{5}{2} - \lambda * 1 - \lambda \right) - (2 * 2) = \lambda^2 - \frac{5}{2}\lambda - \lambda + \frac{5}{2} - 4 = \lambda^2 - \frac{7}{2}\lambda - \frac{3}{2}$$

p-q formula

$$\begin{aligned} \lambda_{1,2} &= -\frac{-\frac{7}{2}}{2} \pm \sqrt{\frac{-\frac{7}{2}}{2} - \left(-\frac{3}{2}\right)} = \frac{7}{4} \pm \sqrt{\frac{49}{16} + \frac{24}{16}} = \frac{7}{4} \pm \frac{\sqrt{73}}{4} \\ \lambda_1 &\approx \frac{7}{4} + 2 \approx \frac{15}{4} \\ \lambda_2 &\approx \frac{7}{4} - 2 \approx \frac{-8}{4} \approx -2 \end{aligned}$$

1.4 d)

Insert λ_1 in $(C - I\lambda_1) * x = 0$

$$\begin{aligned} \left(\begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \frac{15}{4} \right) * x &= \left(\begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \frac{15}{4} & 0 \\ 0 & \frac{15}{4} \end{pmatrix} \right) * x \\ &= \begin{pmatrix} \frac{5}{2} - \frac{15}{4} & 2 \\ 2 & 1 - \frac{15}{4} \end{pmatrix} * x = 0 \end{aligned}$$

Now write generell formula

$$\begin{pmatrix} \frac{5}{2} - \frac{15}{4} & 2 \\ 2 & 1 - \frac{15}{4} \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Make a system of linear equations

$$\left(\frac{5}{2} - \frac{15}{4}\right)x_1 + 2x_2 = 0 \quad (1)$$

$$2x_1 + \left(1 - \frac{15}{4}\right)x_2 = 0 \quad (2)$$

$$-\frac{5}{4}x_1 + 2x_2 = 0 \quad (3)$$

$$2x_1 - \frac{11}{4}x_2 = 0 \quad (4)$$

$$x_1 = \frac{11}{8}$$

$$x_2 = 1$$

$$\text{the first eigenvector is } x = \begin{pmatrix} \frac{11}{8} \\ 1 \end{pmatrix}$$

And now the second eigenvector! Insert λ_2 in $(C - I\lambda_2) * x = 0$

$$\begin{aligned} \left(\begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * -2 \right) * x &= \left(\begin{pmatrix} \frac{5}{2} & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \right) * x \\ &= \begin{pmatrix} \frac{5}{2} + 2 & 2 \\ 2 & 1 + 2 \end{pmatrix} * x = 0 \end{aligned}$$

Now write generell formula

$$\begin{pmatrix} \frac{5}{2} + 2 & 2 \\ 2 & 1 + 2 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Make a system of linear equations

$$\left(\frac{5}{2} + 2 + 2\right)x_1 + 2x_2 = 0 \quad (5)$$

$$2x_1 + 1 + 2 = 0 \quad (6)$$

$$\frac{9}{2}x_1 + 2x_2 = 0 \quad (7)$$

$$2x_1 + 3x_2 = 0 \quad (8)$$

$$x_1 = -\frac{5}{8}$$

$$x_2 = 1$$

the second eigenvector is $x = \begin{pmatrix} \frac{5}{8} \\ 1 \end{pmatrix}$

1.5 e)