Trajectory Simulation

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1 Introduction

Simulation is a powerful tool, which can be used to help understand or evaluate processes working within complex systems. One example for such a process is the drop of a ping pong ball onto a surface. One might want to simulate this process, to find both the speed with which the ball reaches the surface as well as the height of the subsequent bounces. This can be accomplished by using mathematical models to calculate the trajectory of the falling ball based on its previous position and the forces impacting it. Depending on the needed accuracy of the simulation the Problem can be simplified in order to make computation both less complex and thus less time intensive. In the following I will describe a simplified version of the problem and the mathematical models used to simulate it.

2 Description of the Problem

The problem discussed in the following is dropping a ping pong ball onto a flat surface. To initially lower the complexity of the simulation not every possible parameter is used. The following assumptions are made before the simulation. First, the ball is perfectly round and will fall onto a perfectly flat surface. Second, the ball falls in a perfectly straight line towards the ground. Third, the ball is dropped on earth, however the air resistance will not be a factor. This means the simulation will take place in a single dimension, which will be representing the height during the fall and bounces of the ball. This also means that sound won't be part of the simulation.

3 Possible Mathematical Model

When analyzing motion along a straight line a few base concepts need to be introduced. When looking at the motion of objects in space the central metric is their velocity. The velocity of an Object describes its direction in combination with its speed. Velocities are usually represented through vectors. In the simplified version of the problem the ball is dropped in a single dimension thus there are only two directions for the ball to travel in. An objects velocity can then be used to calculate its acceleration. Acceleration is the change in speed and/or direction of an objects velocity. An objects velocity can be calculated with the equation:

$$a = \frac{v - u}{t}$$

With a being the acceleration of the object, u being the initial velocity, v being the final velocity and t being the time between u and v. Since air-resistance is not measured into the calculation in the simplified version of the problem, the change in speed of the ball is coming from its free fall acceleration. The free fall acceleration is independent of an objects properties and lies at $9.8m/s^2$ at sea level on earth. This means when dropped the ping pong ball will accelerate at a constant speed of $9.8m/s^2$. With constant acceleration the current velocity of the ball can be calculated using the following equation:

$$v = v_0 + at$$

With v_0 being the starting position. To calculate the distance traveled from its initial starting point the following equation can be used:

$$x - x_0 = v_0 t + \frac{1}{2} a t^2$$

With x_0 being the starting point and x being the current position at t. Through solving this quadratic equation for t the time the ball takes to reach the ground can be determined. With the equations mentioned above one is now able to find the velocity with which the balls is impacting the ground. When reaching the ground the ball is met with normal force $\vec{F_N}$ from the surface pressing against the ball. The normal force can be calculated through Newtons second law $\vec{F_n} et = m\vec{a}$, which states that the net force on a body is equal to the product of the body's mass and its acceleration. This means that if the ball is perfectly elastic it will return upward with the same velocity it had just before reaching the ground. However due to different materials of the ball as well as the surface not all of the kinetic energy exerted on impact stays as kinetic energy. Thus the upwards velocity on the rebound of the ball is lower than on impact. When calculating the upwards movement one has to remember that the gravitation acceleration now works opposite to the velocity of the ball.

4 Euler-Integration

4.1 Introduction

When looking at real problems one always wants to calculate the exact result. Actually doing so requires an equally exact mathematical model, usually containing various forces with differing effects on the problem at hand. However once this model is finished it gets apparent just how many calculations are necessary to compute results. More often than not this means a model has to exclude or approximate certain factors during calculation to fit the needed efficiency of its use case. The best approach is then to start stripping or approximating less impactful factors first in order to keep the results of the model as exact as possible. Still one problem remains, key to many physical problems, including bouncing ping pong balls, is the differential equation or ordinary differential equation (ODE in short). Working with modern computers trying to solve problems like these analytically is either highly inefficient or downright impossible. This results in the need for a method of approximating ODE's through numerical means. While in many regards out shined by newer methods the Euler Integration introduced this core idea.

4.2 Deriving Euler's Method

Euler-Integration or also known as Euler's method is based on the definition of the derivative

$$f(y(t)) = \frac{dy}{dt} \tag{1}$$

Euler makes use of a simple numerical approximation of the above function

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} \cong f(y(t)) \tag{2}$$

Which says $f(x(t))$ is approximately y at time t plus a chosen interval Δt minus y at time t divided by Δt. This can then be further rewritten to the following

$$y(t + \Delta t) = y(t) + \Delta t f(y(t)) \tag{3}$$

Which gives us the approximate y value at Δt time into the future from the current time t. To verify the above being an approximate version of $F(y(t))$ one can look at y one Δt into the future

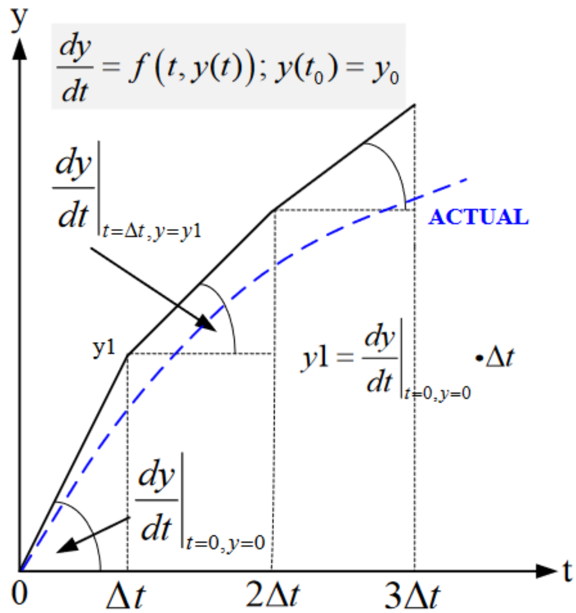
$$y(t_k + 1) = F(y(t_k)) \tag{4}$$

With k being an arbitrary number of Δt time intervals the above can be rewritten to

$$y((k + 1)\Delta t) = y(k\Delta t + \Delta t) = y(t_k + \Delta t) \tag{5}$$

4.3 Which inaccuracies result out of the Euler-Integration?

While this approximation gives a solution to the inefficiencies of computing ODE's analytically Euler's method naturally doesn't come without flaws. It is important to mention that the accuracy of this approximation strongly depends on the size of delta t. The closer the value gets to 0, the more accurate the approximation gets. Furthermore since Euler-Integration uses previous results when calculating multiple steps into the future errors from earlier steps increase following ones. This can be seen in the following graphic.



4.4 What happens at $y(t) = 0$?

When reaching $y(t_k) = 0$ at time t_k during the process of integrating with Δt time steps the next results at time $t_k + 1$ is calculated only through the derivative $f(y(t_k))$ times Δt. If the derivative is still unequal to zero the next value $y(t_k + 1)$ will also be unequal to zero. If however the derivative is zero every subsequent value following $y(t)$ will be zero as well.

References

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