Date

8.1 8. a) Let a string of length $n \geq 3$ that contains 3 consecutive 0s such that a string either ends with 1, 10, 100 or 000.

First case: a_{n-1} possibilities

Second case: a_{n-2} possibilities

3^{rd} case: a_{n-3} possibilities

4^{th} case: 2^{n-3} possibilities

Hence the recurrence relation is

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3} \quad \text{for } n \geq 3$$

b) when $n=0, n=1, n=2$

There are no 0 bit strings with 3 consecutive 0s.

When $n=3$, there is 1 bit string with 3 consecutive 0s.

Initial conditions: $a_0 = a_1 = a_2 = 0$.

c) $a_0 = a_1 = a_2 = 0$

$a_3 = 1$

$a_4 = a_3 + \cancel{a_2}^{0} + \cancel{a_1}^{0} + 2^{4-3} = 3$

$a_5 = a_4 + a_3 + a_2 + 2^{5-3} = 3 + 1 + 0 + 4 = 8$

$a_6 = a_5 + a_4 + a_3 + 2^{6-3} = 8 + 3 + 1 + 2^3 = 20$

$a_7 = a_6 + a_5 + a_4 + 2^{7-3} = 20 + 8 + 3 + 2^4 = 47$

$\therefore$ 47 bit strings.

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10. a) Let a_n represent the number of bit strings of length n that contain the string 01.

1^{st} case: bit string begins with 1. Then bit string of $n-1$ has a_{n-1} possible strings that contain the string 01.

2^{nd} case: bit string begins with k zeros followed by a 1. There are 2^{n-k-1} bit strings of length $n-k-1$

3^{rd} case: bit string ending in 01. There are 2^{n-2} bit strings of length $n-2$.

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} 2^{n-k-1}$$

$$= a_{n-1} + \sum_{k=0}^{n-2} 2^k$$

$$= a_{n-1} + 2^{n-1} - 1$$

b) When $n=0$, $n=1$ there are 0 bit strings containing 01.

when $n=2$, there is 1 bit string containing 01.

Initial condition: $a_0 = a_1 = 0$

c) $a_0 = a_1 = 0$

$a_2 = 1$

$a_3 = a_2 + 2^{3-1} - 1 = 4$

$a_4 = a_3 + 2^{4-1} - 1 = 11$

$a_5 = a_4 + 2^{5-1} - 1 = 26$

$a_6 = a_5 + 2^{6-1} - 1 = 57$

$a_7 = a_6 + 2^{7-1} - 1 = 120.$

$\therefore$ 120 bit strings.

Date

26. a) Let a_n represent the number of ways to cover a $2 \times n$ checker board with 1×2 dominoes.

Case 1: The right most domino is placed vertically. We have $2(n-1)$ checkerboard that still needs to be covered. a_{n-1} ways.

Case 2: The right most domino is placed horizontally. We have $2(n-2)$ Checkerboard that still needs to be covered. a_{n-2} ways.

$a_n = a_{n-1} + a_{n-2}$

b) When $n=1$, $a_1 = 1$

when $n=2$, $a_2 = 2$.

c) $a_{17} = a_{16} + a_{15} = 2584$.

$\therefore$ 2584 ways.

29. $S(m,1) = 1$ for $m \geq 1$.

If $m \geq n$, then a function that is not onto from the set with m elements to the set with n elements can be specified by picking the size of the range, which is an integer between 1 and $n-1$ inclusive, picking the elements of the range, which can be done in $C(n,k)$ ways, and picking an onto function onto the range, which can be done in $S(m,k)$ ways. Hence, there are $\sum_{k=1}^{n-1} C(n,k)\,S(m,k)$ functions that are not onto. But there are n^m functions altogether, so $S(m,n) = n^m - \sum_{k=1}^{n-1} C(n,k)S(m,k)$.

Date

32. Tower of Hanoi

a) $H_{k+1} = H_k + 1 + H_k + 1 + H_k$

$= 3H_k + 2$

$H_0 = 0$

0, 2, 8, 26, ...

b) To proof: $H_n = 3^n - 1$

Let $P(n)$ be $H_n = 3^n - 1$

$n = 0$.

$H_0 = 3^0 - 1$

$= 0$

$P(0)$ is true.

Induction Step.

Lets assume $P(k)$ is true.

$H_k = 3^k - 1$

Prooving H_{k+1}

$H_{k+1} = H_k + 1 + H_k + 1 + H_k$

$= 3H_k + 2$

$= 3(3^k - 1) + 2$

$= 3^{k+1} - 3 + 2$

$= 3^{k+1} - 1$

Thus $P(k+1)$ is true.

c) Randomly assign each of the n disks to a peg. (3 pegs)

$= 3^n$

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d) From (b): $H_n + 1 = 3^n$

Let the 1 represent the initial arrangement, then the moves make $3^n - 1$ different arrangements, resulting in a total of $3^n - 1 + 1 = 3^n$ arrangements.

From (c), there are 3^n possible arrangements and thus every arrangement of the n disks occurs in the solution of this puzzle

8.2 2. a) $a_n = 3a_{n-2}$ Linear, homogeneous, with degree 2.

b) $a_n = 3$ Linear, not homogeneous

c) $a_n = a_{n-1}^2$ Not linear, homogeneous

d) $a_n = a_{n-1} + 2a_{n-3}$ Linear, homogeneous, with degree 3

e) $a_n = a_{n-1}/n$ Not linear, homogeneous

f) $a_n = a_{n-1} + a_{n-2} + n + 3$ Linear, not homogeneous

g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$ Linear, homogeneous, with degree 7.

4g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $\boxed{a_0 = 2,\ a_1 = 8}$

Let $a_{n+2} = r^2$, $a_{n+1} = r$ and $a_n = 1$

$$r^2 = -4r + 5$$
$$r^2 + 4r - 5 = 0$$
$$(r+5)(r-1) = 0$$
$$r = -5 \text{ or } r = 1$$

Recurrence relation $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot -5^n$$
$$= \alpha_1 - 5\alpha_2$$

① $2 = a_0 = \alpha_1 + \alpha_2$

− ② $8 = a_1 = \alpha_1 - 5\alpha_2$

$-6 = 6\alpha_2$

$-1 = \alpha_2$

$$\alpha_1 = 2 - \alpha_2$$
$$= 2 - (-1)$$
$$= 3$$

$\therefore a_n = 3 - (-5)^n$

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20 $a_n = 8a_{n-2} - 16a_{n-4}$

Let $a_n = r^4$, $a_{n-1} = r^3$, $a_{n-2} = r^2$, $a_{n-3} = r^1$, $a_{n-4} = 1$

$r^4 = 8r^2 - 16$

$r^4 - 8r^2 - 16 = 0$

$(r^2 - 4)^2 = 0$

$(r-2)^2 (r+2)^2 = 0$

$r \pm 2 \quad r \pm 2$

$\boxed{a_n = \alpha_1 r_1^n + \alpha_2 n r_2^n + \ldots + \alpha_k n^{k-1} r_1^n}$

$a_n = \alpha_1 (-2)^n + \alpha_2 n(-2)^n + \alpha_3 (2)^n + \alpha_4 n(2)^n$

30. a) $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

Let $a_n = r^2$, $a_{n-1} = r$, $a_{n-2} = 1$

$r^2 = -5r - 6$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2$, $r = -3$

$a_n^{(h)} = \alpha_1 (-2)^n + \alpha_2 (-3)^n$

$F(n) = 42 \cdot 4^n$

$a_n^{(p)} = P_0 4^n$

$a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

$P_0 4^n = -5P_0 4^{n-1} - 6P_0 4^{n-2} + 42 \cdot 4^n$

$16P_0 4^{n-2} = -20P_0 4^{n-2} - 6P_0 4^{n-2} + 672 \cdot 4^{n-2}$

$16P_0 4^{n-2} = (-26P_0 + 672) 4^{n-2}$

$16P_0 = -26P_0 + 672$

$42P_0 = 672$

$P_0 = 16$

$a_n^{(p)} = P_0 4^n = 16 \cdot 4^n$

$= 4^{n+2}$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= \alpha_1 (-2)^n + \alpha_2 (-3)^n + 4^{n+2}$

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b) $a_1 = 56$, $a_2 = 278$

From (a): $a_n = \alpha_1(-2)^n + \alpha_2(-3)^n + 4^{n+2}$

$56 = a_1 = -2\alpha_1 - 3\alpha_2 + 64$

$278 = a_2 = -4\alpha_1 + 9\alpha_2 + 256$

$-8 = -2\alpha_1 - 3\alpha_2$ — ① $\longrightarrow$ $\dfrac{-8 + 3\alpha_2}{-2} = \alpha_1$

×2

$22 = -4\alpha_1 + 9\alpha_2$ — ②

$-16 = -4\alpha_1 - 6\alpha_2$ — ③

②+③ $6 = 3\alpha_2$

$2 = \alpha_2$

$\alpha_1 = \dfrac{-8 + 3(2)}{-2}$

$= 1$

$\therefore a_n = \alpha_1(-2)^n + \alpha_2(-3)^n + 4^{n+2}$

$= (-2)^n + 2(-3)^n + 4^{n+2}$

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35. $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$, $a_0 = 1$ and $a_1 = 4$

let $a_n = r^2$, $a_{n-1} = r$, $a_{n-2} = 1$

$r^2 = 4r - 3$

$r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r = 3$ or $r = 1$

$$a_n^{(h)} = \alpha_1(1)^n + \alpha_2(3)^n$$
$$= \alpha_1 + \alpha_2(3)^n$$

$$F(n) = 2^n + n + 3$$
$$= 2^n + (n+3) \cdot 1^n$$

1 is a root of the characteristic eq^n with multiplicity 1 and 2 is not a root.

$$a_n^{(p)} = q2^n + n(p_1 n + p_0)$$
$$= q2^n + p_1 n^2 + p_0 n$$

$$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$$

$$q2^n + p_1 n^2 + p_0 n = 4q2^{n-1} + 4p_1(n-1)^2 + 4p_0(n-1) - 3q2^{n-2} - 3p_1(n-2)^2 + 3p_0(n-2) + 4 \cdot (2^{n-2}) + n + 3$$

$$0 = [-4q + 8q - 3q + 4]2^{n-2} + p_1[-n^2 + 4(n-1)^2 - 3(n-2)^2] + p_0[-n + 4(n-1) - 3(n-2)] + n + 3$$

$$0 = [q+4]2^{n-2} + p_1[4n - 8] + 2p_0 + n + 3$$

$$0 = (q+4)2^{n-2} + (4p_1 + 1)n + (-8p_1 + 2p_0 + 3)$$

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$0 = q + 4$

$0 = 4p_1 + 1$

$0 = 8p_1 + 2p_0 + 3$

$q = -4$

$p_1 = -\frac{1}{4}$

$p_0 = \frac{8p_1 - 3}{2}$, $p_1 = -1/4$

$= -\frac{5}{2}$

$a_n^{(p)} = q2^n + p_1 n^2 + p_0 n$

$= -4(2^n) - \frac{1}{4}n^2 - \frac{5}{2}n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= \alpha_1 + \alpha_2(3^n) - 4(2^n) - \frac{1}{4}n^2 - \frac{5}{2}n$

$1 = a_0 = \alpha_1 + \alpha_2 - 4$

$4 = a_1 = \alpha_1 + 3\alpha_2 - 8 - \frac{1}{4} - \frac{5}{2}$

$5 = \alpha_1 + \alpha_2$ — ① $\rightarrow \alpha_1 = 5 - \alpha_2$

$\frac{59}{4} = \alpha_1 + 3\alpha_2$ — ②

① - ② $\frac{39}{4} = 2\alpha_2$

$\frac{39}{8} = \alpha_2$

$\alpha_1 = 5 - \alpha_2$

$= 5 - \frac{39}{8}$

$= \frac{1}{8}$

Therefore

$a_n = \frac{1}{8} + \frac{39}{8}(3^n) - 4(2^n) - \frac{1}{4}n^2 - \frac{5}{2}n$

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8.4 6 d) $a_n = 1/(n+1)!$ for $n = 0, 1, 2, \cdots$

$$\boxed{G(x) = \sum_{k=0}^{\infty} a_k x^k}$$

$$G(x) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} x^k$$

$$= \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}$$

let $k+1 = m$

$$= \frac{1}{x} \sum_{m=1}^{\infty} \frac{x^m}{m!}$$

$$= \frac{1}{x}\left(\sum_{m=0}^{\infty} \frac{x^m}{m!} - 1\right) \quad , \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$= \frac{1}{x}(e^x - 1)$$

$$= \frac{e^x - 1}{x}$$

f) $a = \binom{10}{n+1}$ for all $n = 0, 1, 2, \cdots$

$$G(x) = \sum_{k=0}^{\infty} \binom{10}{k+1} x^k$$

$$= \sum_{k=0}^{9} \binom{10}{k+1} x^k \quad , \binom{10}{k+1} = 0 \text{ when } k+1 > 10$$

$$= \frac{1}{x} \sum_{k=0}^{9} \binom{10}{k+1} x^{k+1} \quad , \text{ let } m = k+1$$

$$= \frac{1}{x} \sum_{m=1}^{10} \binom{10}{m} x^m$$

$$= \frac{1}{x}\left(\sum_{m=0}^{10} \binom{10}{m} x^m - 1\right) \quad , \quad \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

$$= \frac{1}{x}\left((1+x)^{10} - 1\right)$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k \qquad (1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

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10. c) $(x^3+x^5+x^6)(x^3+x^4)(x+x^2+x^3+x^4+\dots)$

$= x^3(1+x^2+x^3)\ x^3(1+x)\ x(1+x+x^2+x^3+\dots)$

$= x^7(1+x^2+x^3)(1+x)\cdot \sum_{k=0}^{\infty} x^k$

$= x^7(1+x^2+x^3+x+x^3+x^4)\cdot \sum_{k=0}^{\infty} x^k$

$= x^7(1+x+x^2+2x^3+x^4)\cdot \sum_{k=0}^{\infty} x^k$

$= x^7 \cdot \sum_{m=0}^{4} a_m x^m \cdot \sum_{k=0}^{\infty} b_k x^k$

$a_0 = a_1 = a_2 = a_4 = 1$, $a_3 = 2$, $b_k = 1$ for all k.

x^9 can be obtained if $7+m+k=9$.

Either $m=0, k=2$

or $m=1, k=1$

or $m=2, k=0$

$a_9 = a_0b_2 + a_1b_1 + a_2b_0$

Sum of coeff. $= 1+1+1$

$= 3$

d) $(x+x^4+x^7+x^{10}+\dots)(x^2+x^4+x^6+x^8+\dots)$

$= x(1+x^3+x^6+x^9+\dots)\ x^2(1+x^2+x^4+x^6+\dots)$

$= x^3 \cdot \sum_{m=0}^{\infty} x^{3m} \cdot \sum_{k=0}^{\infty} x^{2k}$

x^9 can be obtained if $3+3m+2k=9$

Either $m=2, k=0$

or $m=0, k=3$

coeff of $x^9 = 2$.

c) $(1+x+x^2)^3$

$= (1+x+x^2)(1+x+x^2+x+x^2+x^3+x^2+x^3+x^4)$

$= (1+x+x^2)(1+2x+3x^2+2x^3+x^4)$

$= (1+2x+3x^2+2x^3+x^4+x+2x^2+3x^3+2x^4+x^5+x^2+2x^3+3x^4+2x^5+x^6)$

$= 1+3x+6x^2+7x^3+6x^4+3x^5+x^6$

Power series does not contain x^9 thus, coeff. $x^9 = 0$.

16. dozen bagels, 3 varieties - egg, salty, plain. At least 2 bagels of each kind, no more than 3 salty bagels.

egg & plain variety $= (x^2+x^3+x^4+\dots)^2$

salty variety $= x^2+x^3$

Therefore: $(x^2+x^3)(x^2+x^3+x^4+\dots)^2$

$= x^2(1+x^2)\, x^{2^2}(1+x+x^2+\dots)^2$

$= x^6(1+x^2)(1+x+x^2+\dots)^2$

$= x^6\left(\sum_{k=0}^{1} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right)^2$

$= x^6\left(\frac{1-x^2}{1-x}\right)\left(\frac{1}{1-x}\right)^2$, $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$= x^6(1-x^2)(1-x)^{-3}$

$= x^6(1-x^2)\left(\sum_{k=0}^{\infty}\binom{-3}{k}(-x)^k\right)$

$= x^6 \cdot \sum_{m=0}^{\infty}\binom{-3}{m}(-1)^m x^m - x^8\left(\sum_{k=0}^{\infty}\binom{-3}{k}(-1)^k x^k\right)$

let $b_m = \binom{-3}{m}(-1)^m$ and $C_k = \binom{-3}{k}(-1)^k$

$= x^6 \cdot \sum_{m=0}^{\infty} b_m x^m - x^8 \cdot \sum_{k=0}^{\infty} C_k x^k$

If $m=6$, $k=4$, coeff x^{12}: $a_{12} = b_6 - C_4$

$= \binom{-3}{6}(-1)^6 - \binom{-3}{4}(-1)^4$

$= 28 - 15 = 13$.

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24 a) Generating function for $\{a_k\}$

$a_k : x_1 + x_2 + x_3 + x_4 = k$, $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, $x_4 \geq 1$

$x_1 \geq 3 : x^3 + x^4 + x^5 + \ldots$

$1 \leq x_2 \leq 5 : x + x^2 + x^3 + x^4 + x^5$

$0 \leq x_3 \leq 4 : 1 + x + x^2 + x^3 + x^4$

$x_4 \geq 1 : x + x^2 + x^3 + \ldots$

$$(1+x+x^2+x^3+x^4)(x+x^2+x^3+x^4+x^5)(x^3+x^4+x^5+\ldots)(x+x^2+x^3+\ldots)$$
$$= x(1+x+x^2+x^3+x^4)^2 \cdot x(1+x+x^2+\ldots)^2 \cdot x^3$$
$$= x^5(1+x+x^2+x^3+x^4)^2(1+x+x^2+\ldots)^2$$
$$= x^5(1+x+x^2+x^3+x^4)^2\left(\sum_{k=0}^{\infty} x^k\right)^2$$
$$= x^5(1+x+x^2+x^3+x^4)^2\left(\frac{1}{1-x}\right)^2 \quad ; \quad \sum_{k=0}^{\infty} x^k = \left(\frac{1}{1-x}\right)$$

30. a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$

$$G_a(x) = 2a_0 + 2a_1x + 2a_2x^2 + \ldots$$
$$= 2(a_0 + a_1x + a_2x^2 + \ldots)$$
$$= 2\sum_{k=0}^{\infty} a_k x^k$$
$$= 2G(x)$$

b) $0, a_0, a_1, a_2, a_3, \ldots$

$$G_a(x) = 0 + a_0 + a_1 + a_2 + a_3 + \ldots$$
$$= a_0x + a_1x^2 + a_2x^3 + \ldots$$
$$= x(a_0 + a_1x + a_2x^2 + \ldots)$$
$$= x\sum_{k=0}^{\infty} a_k x^k$$
$$= xG(x)$$

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c) $0, 0, 0, 0, a_2, a_3, \ldots$

$G_a(x) = 0 + 0x + 0x^2 + 0x^3 + a_2x^4 + a_3x^5 + \ldots$

$= a_2x^4 + a_3x^5 + a_4x^6 + \ldots$

$= -a_0x^2 - a_1x^3 + a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + a_4x^6 + \ldots$

$= -a_0x^2 - a_1x^3 + x^2(a_0 + a_1x + a_2x^2 + \ldots)$

$= -a_0x^2 - a_1x^3 + x^2 \sum_{k=0}^{\infty} a_kx^k$

$= -a_0x^2 - a_1x^3 + x^2 G(x)$

$= x^2(G(x) - a_0 - a_1x)$

d) $a_2, a_3, a_4, \ldots$

$G_a(x) = a_2 + a_3x + a_4x^2 + \ldots$

$= \frac{1}{x^2}(a_2x^2 + a_3x^3 + a_4x^3 + \ldots)$

$= \frac{1}{x^2}(-a_0 - a_1x + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots)$

$= \frac{1}{x^2}\left(\sum_{k=0}^{\infty} a_kx^k - a_0 - a_1x\right)$

$= \frac{1}{x^2}(G(x) - a_0 - a_1x)$

e) $a_1, 2a_2, 3a_3, 4a_4, \ldots$

$G(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k + \ldots$

$G'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + ka_kx^{k-1}$

$\therefore G'(x)$

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f) $a_0^2,\ 2a_0a_1,\ a_1^2+2a_0a_2,\ 2a_0a_3+2a_1a_2,\ 2a_0a_4+2a_1a_3+a_2^2+\dots$

From theorem 1: let $f(x)=\sum_{k=0}^{\infty} a_k x^k$

$$g(x)=\sum_{k=0}^{\infty} b_k x^k$$

$$f(x)\cdot g(x)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

$$\therefore\ G(x)\cdot G(x)=G^2(x)$$

34. $a_k=3a_{k-1}+4^{k-1}\quad ,\ a_0=1$

let $G(x)=\sum_{k=0}^{\infty} a_k x^k$

$$\begin{aligned}G(x)-a_0&=\sum_{k=1}^{\infty}(3a_{k-1}+4^{k-1})x^k\\&=3\sum_{k=1}^{\infty}a_{k-1}x^k+\sum_{k=1}^{\infty}4^{k-1}x^k\\&=3x\sum_{k=1}^{\infty}a_{k-1}x^{k-1}+x\sum_{k=1}^{\infty}4^{k-1}x^{k-1}\\&=3x\sum_{m=0}^{\infty}a_m x^m+x\sum_{m=0}^{\infty}(4x)^m\quad ,\ \text{let } m=k-1\\&=3xG(x)+\frac{x}{1-4x}\quad ,\ \sum_{k=1}^{\infty}x^k=\frac{1}{1-x}\end{aligned}$$

$a_0=1$

$$G(x)-a_0=3xG(x)+\frac{x}{1-4x}$$

$$G(x)-1=3xG(x)+\frac{x}{1-4x}$$

$$G(x)-3xG(x)=\frac{x}{1-4x}+1$$

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$(1-3x)G(x) = \frac{x}{1-4x} + 1$

$(1-3x)\,G(x) = \frac{1-3x}{1-4x}$

$G(x) = \frac{1}{1-4x}$, $\sum_{k=1}^{\infty} x^k = \frac{1}{1-x}$

$G(x) = \sum_{k=1}^{\infty} (4x)^k$

$= \sum_{k=1}^{\infty} 4^k x^k$

$a_k = 4^k$

43. To prove $(1+x)^{m+n} = \sum_{k=0}^{r} C(m, r-k)\, C(n, k)$

Bino theorem: $(1+x)^{m+n} = \sum_{r=0}^{\infty} \binom{m+n}{r} x^r = \sum_{r=0}^{\infty} C(m+n, r) x^r$

$(1+x)^m (1+x)^n = \left(\sum_{r=0}^{\infty} \binom{m}{r} x^r\right)\left(\sum_{r=0}^{\infty} \binom{n}{r} x^r\right)$

$= \left(\sum_{r=0}^{\infty} \binom{n}{r} x^r\right)\left(\sum_{r=0}^{\infty} \binom{m}{r} x^r\right)$

$= \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k}\right) x^k$

$= \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} C(n,k)\, C(m, r-k)\right) x^k$

Since $(1+x)^{m+n} = (1+x)^n (1+x)^n$

$\sum_{r=0}^{\infty} C(m+n, r) x^r = \sum_{k=0}^{\infty} \left(\sum_{k=0}^{r} C(m, r-k)\, C(n, k)\right) x^k$

$\therefore$ The coeff must be equal:

$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)\, C(n, k)$