Sample Solutions on HW5 (*13 exercises in total*)

Sec. 3.2 8(c), 26(a), 54, 56

8(c) For large x, this fraction is fairly close to 1. Therefore we can take $n = 0$; that is, this function is $O(x^0) = O(1)$. Formally we can write $f(x) \leqslant 3x^4/x^4 = 3$ for all $x > 1$, so witnesses are $C = 3$ and $k = 1$.

26(a) This is $O(n^3 \cdot \log n + \log n \cdot n^3)$, which is the same as $O(n^3 \cdot \log n)$
(Pick up the most rapidly growing term in each sum and discard the rest, including the multiplicative constant.)

54 For all values of $\boldsymbol{x}$ and $\boldsymbol{y}$ greater than 1, each term of the given expression is greater than $\boldsymbol{x}^3\boldsymbol{y}^3$, so the entire expression is greater than $\boldsymbol{x}^3\boldsymbol{y}^3$. In other words, we take $\boldsymbol{C} = \boldsymbol{k}_1 = \boldsymbol{k}_2 = 1$ in the definition given in Exercise 52.

56 For all positive values of $\boldsymbol{x}$ and $\boldsymbol{y}$, we know that $\lceil xy \rceil \geq xy$ by definition (since the ceiling function value cannot be less than the argument). Thus $\lceil xy \rceil$ is $\boldsymbol{\Omega}(\boldsymbol{xy})$ from the definition, taking $\boldsymbol{C} = 1$ and $\boldsymbol{k}_1 = \boldsymbol{k}_2 = 0$. In fact, $\lceil xy \rceil$ is also $\boldsymbol{O}(\boldsymbol{xy})$ (and therefore $\boldsymbol{\Theta}(\boldsymbol{xy})$); this is easy to see since $\lceil xy \rceil \leq (\boldsymbol{x}+1)(\boldsymbol{y}+1) \leq (2\boldsymbol{x})(2\boldsymbol{y}) = 4\boldsymbol{xy}$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ greater than 1.

Sec. 3.3 7, 10

7 linear search is faster.

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10(a) By the way that $\boldsymbol{S} - 1$ is defined, it is clear that $\boldsymbol{S} \wedge (\boldsymbol{S} - 1)$ is the same as $\boldsymbol{S}$ except that the rightmost 1 bit has been changed to a 0. Thus we add 1 to ***count*** for every one bit (since we stop as soon as $\boldsymbol{S} = 0$, i.e., as soon as $\boldsymbol{S}$ consists of just 0 bits).

10(b) Obviously the number of bitwise AND operations is equal to the final value of ***count***, i.e., the number of one bits in $\boldsymbol{S}$.

Sec. 5.1 46, 47, 48

46 This proof will be similar to the proof in Example 10.
Basis step: since for $\boldsymbol{n} = 3$, the set has exactly one subset containing exactly three elements, and $3(3-1)(3-2)/6 = 1$.
Inductive step: Assume the inductive hypothesis, that a set with n elements has $\boldsymbol{n}(\boldsymbol{n} - 1)(\boldsymbol{n} - 2)/6$ subsets with exactly three elements; we want to prove that a set $\boldsymbol{S}$ with $\boldsymbol{n} + 1$ elements has $(\boldsymbol{n} + 1)\boldsymbol{n}(\boldsymbol{n} - 1)/6$ subsets with exactly three elements. Fix an element $\boldsymbol{a}$ in $\boldsymbol{S}$, and let $\boldsymbol{T}$ be the set of elements of $\boldsymbol{S}$ other than $\boldsymbol{a}$. There are two varieties of subsets of $\boldsymbol{S}$ containing exactly three elements. First there are those that do not contain $\boldsymbol{a}$. These are precisely the three-element subsets of $\boldsymbol{T}$, and by the inductive hypothesis, there are $\boldsymbol{n}(\boldsymbol{n} - 1)(\boldsymbol{n} - 2)/6$ of them. Second, there are those that contain a together with two elements of $\boldsymbol{T}$. Therefore there are just as many of these subsets as there are two-element subsets of $\boldsymbol{T}$. By Exercise 45, there are exactly $\boldsymbol{n}(\boldsymbol{n}-1)/2$ such subsets of $\boldsymbol{T}$; therefore there are also $\boldsymbol{n}(\boldsymbol{n}-1)/2$ three-element subsets of $\boldsymbol{S}$ containing $\boldsymbol{a}$. Thus the total number of subsets of $\boldsymbol{S}$ containing exactly three elements is $(\boldsymbol{n}(\boldsymbol{n} - 1)(\boldsymbol{n} - 2)/6) + \boldsymbol{n}(\boldsymbol{n} - 1)/2$, which simplifies algebraically to $(\boldsymbol{n} + 1)\boldsymbol{n}(\boldsymbol{n} - 1)/6$, as desired.

47 Reorder the locations if necessary so that $\mathbf{x}_1 \le \mathbf{x}_2 \le \mathbf{x}_3 \le \cdots \le \mathbf{x_d}$. Place the first tower at position $\mathbf{t}_1 = \mathbf{x}_1+1$. Assume tower **k** has been placed at position $\mathbf{t_k}$. Then place tower $\mathbf{k}+1$ at position $\mathbf{t}_{\mathbf{k}+1} = \mathbf{x}+1$, where x is the smallest $\mathbf{x_i}$ greater than $\mathbf{t_k} + 1$.

48 We will show that any minimum placement of towers can be transformed into the placement produced by the algorithm. Although it does not strictly have the form of a proof by mathematical induction, the spirit is the same. Let $s_1 < s_2 < \cdots < s_k$ be an optimal locations of the towers (i.e., so as to minimize k), and let $t_1 < t_2 < \cdots < t_l$ be the locations produced by the algorithm from Exercise 47. In order to serve the first building, we must have $s_1 \le x_1 +1 = t_1$. If $s_1 \ne t_1$, then we can move the first tower in the optimal solution to position t_1 without losing cell service for any building. Therefore we can assume that $s_1 = t_1$. Let x_j be smallest location of a building out of range of the tower at s_1 ; thus $x_j > s_1 + 1$. In order to serve that building there must be a tower ***si*** such that $s_i \le x_j + 1 = t_2$. If $i > 2$, then towers at positions s_2 through s_{i-1} are not needed, a contradiction. As before, it then follows that we can move the second tower from s_2 to t_2. We continue in this manner for all the towers in the given minimum solution; thus $k = l$. This proves that the algorithm produces a minimum solution.

Sec. 5.2 8, 18, 39

8 Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies. We can achieve the following values of n: 5 = 5, 8 = 8, 10 = 5+5, 13 = 8+5, 15 = 5+5+5, 16 = 8+8, 18 = 8+5+5, 20 = 5+5+5+5+5, 21 = 8+8+5, 23 = 8 + 5 + 5 + 5, 24 = 8 + 8 + 8, 25 = 5 + 5 + 5 + 5 + 5, 26 = 8 + 8 + 5 + 5, 28 = 8 + 5 + 5 + 5 + 5, 29 = 8 + 8 + 8 + 5, 30 = 5 + 5 + 5 + 5 + 5 + 5, 31 = 8 + 8 + 5 + 5 + 5, 32 = 8 + 8 + 8 + 8. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates. (In other words, \$135 is the largest multiple of \$5 that we cannot achieve.)

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all $n \geq 28$. From our work above, we know that $P(n)$ is true for n = 28, 29, 30, 31, 32. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \le j \le k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ is true. Because $k-4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more \$25-dollar certificate, and we have formed $5(k + 1)$ dollars, as desired.

18 We prove something slightly stronger: If a convex n-gon whose vertices are labeled consecutively as $v_m, v_{m+1}, \ldots, v_{m+n-1}$ is triangulated, then the triangles can be numbered from m to $m+n-3$ so that v_i is a vertex of triangle i for $i = m, m+1, \ldots, m+n-3$. (The statement we are asked to prove is the case $m = 1$.) The basis step is $n = 3$, and there is nothing to prove. For the inductive step, assume the inductive hypothesis that the statement is true for polygons with fewer than n vertices, and consider any triangulation of a convex n-gon whose vertices are labeled consecutively as $v_m, v_{m+1}, \ldots, v_{m+n-1}$. One of the diagonals in the triangulation must have either v_{m+n-1} or v_{m+n-2} as an endpoint (otherwise, the region containing v_{m+n-1} would not be a triangle). So there are two cases. If the triangulation uses diagonal $v_k v_{m+n-1}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-1} as v_{k+1} in the polygon that contains v_m. This gives us the desired numbering of the triangles, with numbers v_m through v_{k-1} in the first polygon and numbers v_k

through v_{m+n-3} in the second polygon. If the triangulation uses diagonal $v_k v_{m+n-2}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-2} as v_{k+1} and v_{m+n-1} as v_{k+2} in the polygon that contains v_{m+n-1} , and renumbering v_k as v_{m+n-1} in the other polygon. This gives us the desired numbering of the triangles, with numbers v_m through v_k in the first polygon and numbers v_{k+1} through v_{m+n-3} in the second polygon. Note that we did not need the convexity of our polygons.

39 This is a paradox caused by self-reference. The answer is clearly "no." There are a finite number of English words, so only a finite number of strings of 15 words or fewer; therefore, only a finite number of positive integers can be so described, not all of them.