Sample Solutions on HW4 (*21 exercises in total*)

Sec. 2.2 17, 48, 57(c)

17.

(a) Suppose $x \in \overline{A \cap B \cap C}$, then $x \notin A \cap B \cap C$. This means

$x \notin A$ or $x \notin B$ or $x \notin C$.

Equivalently, we can say

$x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$

Therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$

Hence $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ (1)

Suppose $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, then $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$

This means $x \notin A$ or $x \notin B$ or $x \notin C$.

So $x \notin A \cap B \cap C$

It follows that $x \in \overline{A \cap B \cap C}$

Hence $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$ (2)

Based on (1) and (2), we can conclude that $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

(b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

Because columns 5 and 9 are identical, we can conclude that $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

48. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

a) $A_n = \{\cdots,-2,-1,0,1,\cdots,n\}$

b) $A_1 = \{\cdots,-2,-1,0,1\}$

57(c)

Assume the universal set U is the set of 26 lower-case English letters, and the ordering of elements of U has the elements in alphabetical order.

Represent sets A, B, C, and D by bit strings:

A 11 1110 0000 0000 0000 0000 0000

B 01 1100 1000 0000 0100 0101 0000

C 00 1010 0010 0000 1000 0010 0111

D 00 0110 0110 0001 1000 0110 0110

$A \cup D$

= 11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110

= 11 1110 0110 0001 1000 0110 0110

$B \cup C$

= 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111

= 01 1110 1010 0000 1100 0111 0111

$(A \cup D) \cap (B \cup C)$

= 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111

= 01 1110 0010 0000 1000 0110 0110, which represents the set

$\{b, c, d, e, i, o, t, u, x, y\}$

Sec. 2.3 22(c), 34, 40(a), 72, 74(c,d)

22(c) This function is a bijection, but not from **R** to **R**. To see that the domain and range are not **R**, note that $\boldsymbol{x} = -2$ is not in the domain, and $\mathbf{x} = 1$ is not in the range. On the other hand, f is a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$, since its inverse is $\mathbf{f}^{-1}(\boldsymbol{x}) = (1 - 2\boldsymbol{x})/(\boldsymbol{x} - 1)$.

34 To clarify the setting, suppose that g: A -> B and f: B -> C, so that $f \circ g$: A -> C. We will prove that if $f \circ g$ is one-to-one, then g is also one-to-one. So not only is the answer to the question "yes," but part of the hypothesis is not even needed.

Suppose that g were not one-to-one. By definition this means that there are distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. Then certainly $f(g(a_1)) = f(g(a_2))$, which is the same statement as $f \circ g(a_1) = f \circ g(a_2)$. By definition this means that f°g is not one-to-one, and our proof by contradiction is complete.

40 (a) This proof has two parts. First suppose that b is in $f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. Either $a \in S$, in which case $b \in f(S)$, or $a \in T$, in which case $b \in f(T)$. Thus in either case, $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, suppose $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$. This means either that $b = f(a)$ for some $a \in S$ or that $b = f(a)$ for some $a \in T$. In either case, $b = f(a)$ for some $a \in S \cup T$, so

$b \in f(S \cup T)$. This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$, and our proof is complete.

72 If ***f*** is 1-1, then every element of ***A*** gets sent to a different element of ***B***. If in addition to the range of ***A*** there were another element in ***B***, then |***B***| would be at least one greater than |***A***|. This cannot happen, so we conclude that ***f*** is onto. Conversely, suppose that ***f*** is onto, so that every element of ***B*** is the image of some element of ***A***. In particular, there is an element of ***A*** for each element of ***B***. If two or more elements of ***A*** were sent to the same element of ***B***, then |***A***| would be at least one greater than |***B***|。 This cannot happen, so we conclude that ***f*** is one-to-one.

74(c) A little trial and error fails to produce a counterexample, so maybe this is true. We look for a proof. Since we are dividing by 4, let us write *x* = 4*n* + *k*, where $0 \le k < 4$. In other words, write *x* in terms of how much it exceeds the largest multiple of 4 not exceeding it. There are three cases. If *k*=0, then *x* is already a multiple of 4, so both sides equal *n*. If $0 < k \le 2$, then $\lceil x/2 \rceil = 2n+1$, so the left-hand side is $\lceil n + \frac{1}{2} \rceil = n+1$. Of course the right-hand side is *n*+1 as well, so again the two sides agree. Finally, suppose that $2 < k < 4$. Then $\lceil x/2 \rceil = 2n+2$, and the left-hand side is $\lceil n+1 \rceil = n+1$; of course the right-hand side is still *n*+1 as well。 Since we proved that the two sides are equal in all cases, the proof is complete.

74(d) For x=8.5, the left-hand side is 3, whereas the right-hand side is 2.

Sec. 2.5 4(c,d), 28 , 36, 38

4(c) (The set described here is a bit unclear. It may or may not include real numbers whose absolute values are less than 1. Negative ones may also be included. But in any case, the answer is the same: the set is countable. The following gives a solution for one of possible cases. All other cases can be proved similarly.)

This set is countable. We can arrange the number in a 2-dimensional table as follows:

$.\overline{1}$	.1	.11	.111	.1111	.11111	.111111	$\ldots$
$1.\overline{1}$	1	1.1	1.11	1.111	1.1111	1.11111	$\ldots$
$11.\overline{1}$	11	11.1	11.11	11.111	11.1111	11.11111	$\ldots$
$111.\overline{1}$	111	111.1	111.11	111.111	111.1111	111.11111	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in the proof of countability of the set of positive rational numbers.

4(d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable. All we need to do is choose $d_i=1$ when $d_{ii}=9$ and choose

d_i=9 when d_{ii}=1 or d_{ii} is blank (if the decimal expansion is finite).

28 We can think of $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$ as the countable union of countable sets, where the $\boldsymbol{i}^{\text{th}}$ set in the collection, for $\boldsymbol{i} \in \boldsymbol{Z}^+$, is $\{(\boldsymbol{i}, \boldsymbol{n}) \mid \boldsymbol{n} \in \boldsymbol{Z}^+\}$.

36 We can encode subsets of the set of positive integers as strings of, say, 5's and 6's, where the $\boldsymbol{i}^{\text{th}}$ symbol is a 5 if $\boldsymbol{i}$ is in the subset and a 6 otherwise. If we interpret this string as a real number by putting a 0

and a decimal point in front, then we have constructed a one-to-one function from $\mathbf{P}(\boldsymbol{Z}^+)$ to (0, 1). Also, we can construct a one-to-one function from (0, 1) to $\mathbf{P}(\boldsymbol{Z}^+)$ by sending the number whose binary expansion is $0.d_1d_2d_3 \ldots$ to the set $\{i \mid d_i = 1\}$. Therefore by the Schr¨oder-Bernstein theorem we have $|\mathbf{P}(\boldsymbol{Z}^+)| = |(0, 1)|$. We have known from the lecture that $|(0, 1)| = |\mathbf{R}|$, so we have shown that $|\mathbf{P}(\boldsymbol{Z}^+)| = |\mathbf{R}|$. (We already know from Cantor's diagonal argument that $\aleph_0 < |\mathbf{R}|$.) There is one technical point here. In order for our function from (0, 1) to $\mathbf{P}(\boldsymbol{Z}^+)$ to be well-defined, we must choose which of two equivalent expressions to represent numbers that have terminating binary expansions to use (for example, 0.100101111… versus 0.100110000…); we can decide to always use the terminating form, i.e., the one ending in all 0's.)

38 We know that the set of real numbers between 0 and 1 is uncountable. Let us associate each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set {0,1,2,3,4,5,6,7,8,9} as follows: If x is a real number whose decimal representation is $0.d_1d_2d_3d_4\ldots$(with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associated to x the function whose rule is given by $f(n) = d_n$. Clearly this is a bijection from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set {0,1,2,3,4,5,6,7,8,9}. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as 0.2399999…) Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable.

Sec. 3.1 2, 4

2

a) This procedure is not finite, since execution of the while loop continues forever.

b) This procedure is not effective, because the step $m := 1/n$ cannot be performed when $n = 0$, which will eventually be the case.

c) This procedure lacks definiteness, since the value of i is never set.

d) This procedure lacks definiteness, since the statement does not tell whether x is to be set equal to a or to b.

4 Set the answer to be $-\infty$. For i going from 1 through $n - 1$, compute the value of the $(i + 1)^{st}$ element in the list minus the i^{th} element in the list. If this is larger than the answer, reset the answer to be this value.