

Sample Solutions on HW3 (25 exercises in total)

Sec. 1.6 12, 14(d), 18, 24, 29, 34(a)

12. Exercise 11 states that $((p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow r) \rightarrow ((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r))$

Applying exercise 11, we just need to show that the conclusion r follows

from the five premises

$(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, $\neg s$, and q

Step	Reason
1. q	Premise
2. $q \rightarrow (u \wedge t)$	Premise
3. $u \wedge t$	Modus ponens, 1,2
4. u	Simplification, 3
5. t	Simplification, 3
6. $u \rightarrow p$	Premise
7. p	Modus ponens, 4,,6
8. $p \wedge t$	Conjunction, 5,7
9. $(p \wedge t) \rightarrow (r \vee s)$	Premise
10. $r \vee s$	Modus ponens, 8,9
11. $\neg s$	Premise
12. r	Disjunctive syllogism, 10,11

Q.E.D.

14(d) Let $C(x)$ be “ x is in this class,” let $F(x)$ be “ x has been to France,” and let $L(x)$ be “ x has visited the Louvre.” We are given premises $\exists x(C(x) \wedge F(x))$, $\forall x(F(x) \rightarrow L(x))$, and we want to conclude $\exists x(C(x) \wedge L(x))$.

In the following proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(C(x) \wedge F(x))$	Premise
2. $C(y) \wedge F(y)$	EI, 1
3. $F(y)$	Simplification, 2
4. $C(y)$	Simplification, 2
5. $\forall x(F(x) \rightarrow L(x))$	Premise
6. $F(y) \rightarrow L(y)$	UI, 5
7. $L(y)$	Modus ponens, 3, 6
8. $C(y) \wedge L(y)$	Conjunction, 4, 7
9. $\exists x(C(x) \wedge L(x))$	EG, 8
Q.E.D.	

18. We know that some s exists that makes $S(s, \text{Max})$ true, but we cannot conclude that Max is one such s . Therefore this first step is invalid.

24. Steps 3 and 5 are incorrect; simplification applies to conjunction, not disjunction.

29.

Step	Reason
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	EI, 1
3. $\forall x (P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	UI, 3
5. $Q(c)$	Disjunctive syllogism, 4,2
6. $\forall x (\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	UI, 6
8. $S(c)$	Disjunctive syllogism, 5,7
9. $\forall x (R(x) \rightarrow \neg S(x))$	Premise
10. $R(c) \rightarrow \neg S(c)$	UI, 9
11. $\neg R(c)$	Modus tollens, 8,10
12. $\exists x \neg R(x)$	EG, 11
Q.E.D.	

34(a) Let d be “logic is difficult,” s be “many students like logic,” and e for “mathematics is easy.”. Then the assumptions are $d \vee \neg s$ and $e \rightarrow \neg d$.

Note that the first of these is equivalent to

$s \rightarrow d$. This exercise asks whether we can conclude $s \rightarrow \neg e$

Step	Reason
1. $d \vee \neg s$	Premise
2. $s \rightarrow d$	Implication Rule, 1
3. $e \rightarrow \neg d$	Premise
4. $d \rightarrow \neg e$	Contrapositive, 3
5. $s \rightarrow \neg e$	Hypothetical syllogism, 2,4
Q.E.D.	

Sec. 1.7 7, 8, 34

7. Assume n is odd. By the definition of an odd integer, it follows that

$$n=2k+1, \text{ where } k \text{ is some integer.}$$

$$\text{Then } (k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$$

Therefore, every odd integer can be expressed as the difference of two squares.

8. Let $n = m^2$. If $m = 0$, then $n+2 = 2$, which is not a perfect square, so we can assume that $m \geq 1$. The smallest perfect square greater than n is

$(m+1)^2$, and we have

$(m+1)^2 = m+2m+1 = n+2m+1 > n+2 \cdot 1+1 > n+1$. Therefore $n+2$ cannot be a perfect square.

34. No. The line of reasoning shows that if $\sqrt{2x^2 - 1} = x$, then we must have $x = 1$ or $x = -1$. These are therefore the only possible solutions, but we have no guarantee that they are solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we must substitute these values back into the original equation to determine whether they do indeed satisfy it.

Sec. 1.8 22, 24

22. We follow the hint. The square of every real number is nonnegative, so $(x - 1/x)^2 \geq 0$. Multiplying this out and simplifying, we obtain $x^2 - 2 + 1/x^2 \geq 0$, so $x^2 + 1/x^2 \geq 2$, as desired.

24. Let $x = 1$ and $y = 10$. Then their arithmetic is 5.5 and their quadratic mean is $\sqrt{50.5} \approx 7.11$. Similarly, if $x = 5$ and $y = 8$, then the arithmetic mean is $(5+8)/2 = 6.5$ and the quadratic mean is $\sqrt{(5^2 + 8^2)/2} \approx 6.67$. So we conjecture that the quadratic mean is always greater than or equal to the arithmetic mean. Thus we want to prove that

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2}$$

for all positive real numbers x and y . Doing some algebra, we find that this inequality is equivalent to the true statement that $(x - y)^2 \geq 0$:

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2}$$

$$\equiv 2x^2 + 2y^2 \geq x^2 + 2xy + y^2$$

$$\equiv x^2 - 2xy + y^2 \geq 0$$

$$(x - y)^2 \geq 0$$

In fact, our argument also shows that equality holds if and only if $x = y$.

Sec. 2.1 11, 18, 22, 24, 32(a),(c)

11. (a) True (b) True (c) False (d) True (e) True (f) False

18. Since the empty set is a subset of every set, we just need to take a set B that contains ϕ as an element. Thus we can let $A = \phi$ and $B = \{\phi\}$ as the simplest example.

22. The union of all the sets in the power set of a set X must be exactly X . In other words, we can recover X from its power set, uniquely. Therefore the answer is Yes.

24.

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus ϕ is not the power set of any set.

b) This is the power set of $\{a\}$

c) This set has three elements. Since 3 is not a power of any integer, this set cannot be the power set of any set.

d) This is the power set of $\{a,b\}$.

32.

a) $\{(a,x,0), (a,x,1), (a,y,0), (a,y,1), (b,x,0), (b,x,1), (b,y,0), (b,y,1), (c,x,0), (c,x,1), (c,y,0), (c,y,1)\}$

c) $\{(0,a,x), (0,a,y), (0,b,x), (0,b,y), (0,c,x), (0,c,y), (1,a,x), (1,a,y), (1,b,x), (1,b,y), (1,c,x), (1,c,y)\}$