Date

3.2

8(c) $f(x) = \left(\frac{x^4+x^2+1}{x^4+1}\right) = 1 + \frac{x^2}{x^4+1}$

The largest power of x of the quotient is the smallest n for which $f(x)$ is $O(x^n)$

$n=0$

$|f(x)| = \left|1 + \frac{x^2}{x^4+1}\right|$

$\leq |1| + \left|\frac{x^2}{x^4+1}\right|$

$= 1 + \frac{x^2}{x^4+1}$

$\leq 1+1$

≤ 2

$= 2 \cdot |1|$

$= 2|x^0|$

Thus we need $C \geq 2$.

let $C=2$

By definition of the Big-O notation,

$f(x) = O(x^0) = O(1)$ with $k=0$ and $C=2$.

26(a) $(n^3+n^2\log n)(\log n+1) + (17\log n+19)(n^3+2)$

$= n^3\log n + n^2(\log n)^2 + n^3 + n^2\log n + 17n^3\log n + 19n^3 + 34\log n + 38$

$= 18n^3\log n + n^2(\log n)^2 + 20n^3 + n^2\log n + 34\log n + 38$

Lets assume, $g(n) = n^3\log n$

$k = 10^{20}$ thus $n > 10^{20}$

$|f(n)| = |18n^3\log n + n^2(\log n)^2 + 20n^3 + n^2\log n + 34\log n + 38|$

$= 18n^3\log n + n^2(\log n)^2 + 20n^3 + n^2\log n + 34\log n + 38$

Using $\log n \leq n$ and $n > 10^{20}$ and $\log n > \log 10^{20} = 20$

$\leq 18n^3\log n + n^3\log n + n^3\log n + n^3\log n + n\log n + n$

$\leq 18n^3\log n + n^3\log n + n^3\log n + n^3\log n + n^3\log n + n^3\log n$

$= 23n^3\log n$

$= 23|n^3\log n|$

C must be at least 23.

By definition of the Big-O notation, $f(n) = (n^3+n^2\log n)(\log n+1) + (17\log n+19)(n^3+2)$ is $O(n^3\log n)$ with $k=10^{20}$ and $C=23$.

54. Show that $x^5y^3 + x^4y^4 + x^3y^5$ is $\Omega(x^3y^3)$

Let $k_1 = k_2 = 1$. For $x \geq 1$ and $y \geq 1$,

$|f(x,y)| = |x^5y^3 + x^4y^4 + x^3y^5|$

$= x^5y^3 + x^4y^4 + x^3y^5$

Since $x \geq 1$ and $y \geq 1$, $x^5 \geq x^3$, $x^4 \geq x^3$, $y^4 \geq y^3$ and $y^5 \geq y^4$

$\geq x^3y^3 + x^3y^3 + x^3y^3$

$= 3x^3y^3$

$= 3|x^3y^3|$

C then needs to be at most 3. Let $C=3$.

$f(x,y) = x^5y^3 + x^4y^4 + x^3y^5$ is $\Omega(x^3y^3)$ with constants $C=3$, $k_1 = k_2 = 1$

56. $f(x,y) = \lceil x, y \rceil$

let $k_1 = k_2 = 1$. For $x > k_1$ and $y > k_2$ then.

$|f(x,y)| = \lceil xy \rceil$

$= \lceil xy \rceil$

Property ceiling function: $\lceil a \rceil \geq a$

$\geq xy$

$= |xy|$

C needs to be at most 1, let $C=1$

$f(x,y) = \lceil xy \rceil$ is $\Omega(xy)$ with constants $C=1$, $k_1 = k_2 = 1$

33 7. Linear search - Linear search would take at most 4 iterations.

Binary search will divide the set until there is one element

$32 \to 16 \to 8 \to 4 \to 2 \to 1$, taking 5 iterations.

Date

10. a) On every iteration of the while-loop a 1 of S is changed to 0 and the count is increased by 1. Thus the count will keep track of the number of ones in the bit string

b) The number of bitwise and operations is equal to the number of 1 bits in the string S.

5.1 46 $\frac{n(n-1)(n-2)}{6}$, let $n=3$

$$\frac{3(3-1)(3-2)}{6} = 1$$

P(3) has only 1 element. P(3) is true.

Induction step: let P(k) be true

P(k+1)

$$\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} = k(k-1)\left(\frac{k-2}{6} + \frac{1}{2}\right)$$

$$= k(k-1)\left(\frac{k-2}{6} + \frac{3}{6}\right)$$

$$= k(k-1)\left(\frac{k-2+3}{6}\right)$$

$$= k(k-1)\left(\frac{k+1}{6}\right)$$

$$= \frac{(k+1)(k)(k-1)}{6}$$

$$= \frac{(k+1)((k+1)-1)((k+1)-2)}{6}.$$

Thus P(k+1) is true.

A set with n elements has $\frac{n(n-1)(n-2)}{6}$ subsets containing exactly 3 elements.

47. Procedure towers($x_1, \ldots x_d$: nonnegative integers with $d \geq 2$)

```
Sort the list x1,...,xd in increasing order
S := {x1 + 1}
i := x1 + 1
for j := 2 to d
    if xj > last {
        S := S ∪ {xj + 1}
        last := xj + 1
    }
Return S
```

48. Let $P(n)$ be "The algorithm is the previous exercise provides the fewest towers possible when there are n buildings."

$n=2$, $P(2)$ is true

Induction step: Let $P(k)$ be true.

The algorithm provides the fewest towers possible when there are k buildings.

5.2 8) Every amount divisible by \$5 above \$140 is available. Therefore, $P(n)$ is the proposition that the above amounts and every amount is divisible by \$5 from \$140 upwards. Therefore the possible amounts are in (\$) : 25, 40, 50, 65, 75, 80, 90, 100, 105, 115, 120, 125, 130, every multiple of 5 from 140 upwards.

25	10	
1	0	25
0	1	40
2	0	50
1	1	65
3	0	75
0	2	80
2	1	90
4	0	100
1	2	105
3	1	115
0	3	120
5	0	125
2	2	130
4	1	140
1	3	145
6	0	150
3	2	155
0	4	160

Date

18 $n=3$

A convex polygon P with 3 vertices V_1, V_2, V_3 is a triangle. Let the triangle be v_1.

This implies that $P(3)$ is true.

Inductive step. Let $P(3), P(4) \cdots P(k)$ be true

Then $n-2$ triangles can be numbered $1, 2, \cdots, n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, \cdots, n-2$.

39. Consider the set S of positive integers that cannot be described using no more than fifteen english words. Suppose S is non-empty. By the well-ordering principle, there must exist a least element of this set. If n is that least element then n can be described as "the smallest positive integer that cannot be described using no more than fifteen english words". However this description itself contains only fifteen english words, hence the statement is false and n doesn't belong to S. But according to our assumption, n belongs to S which is a contradiction. Hence S is empty and the statement given in the problem is true.