Sample Solutions on HW12 (*33 exercises in total*)

Sec.10.2 5, 24, 25, 42(b,f,h), 53, 60

5 By Theorem 2 in this section, the number of vertices of odd degree must be even. Hence there cannot be a graph with 15 vertices of odd degree 5. (We assume that the problem was meant to imply that the graph contained only these 15 vertices.)

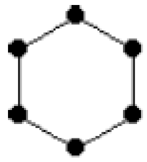
24 This is a complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

25 We can show that this graph is not bipartite by looking at a triangle, in this case the triangle formed by vertices b, d, and e. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.
(An alternative way to look at this is given by Theorem 4 in this section. Because of the existence of a triangle, it is impossible to color the three vertices of the triangle in two colors so that any two adjacent vertices are colored differently.)

42(a) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

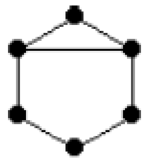
42(b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

42(c) A 6-cycle is such a graph.

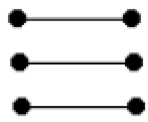


42(d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

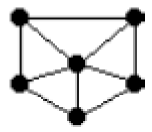
42(e) A 6-cycle with one of its diagonals added is such a graph.



42(f) A graph consisting of three edges with no common vertices is such a graph.



42(g) The 5-wheel is such a graph.



42(h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.

53(a) The complete graph K_n is regular for all values of $n \geqslant 1$, since the degree of each vertex is n-1.

53(b) The degree of each vertex of C_n is 2 for all n which C_n is defined, namely $n \geqslant 3$, so C_n is regular for all these values of n.

53(c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the rim is 3. Therefore W_n is regular if and only if n = 3. Of course W_3 is the same as K_4.

53(d) The cube Q_n is regular for all values of $n \geqslant 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

60 The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n$-1)/2 edges, we want to solve $n(n$-1)/2 = 28. Thus n = 8.

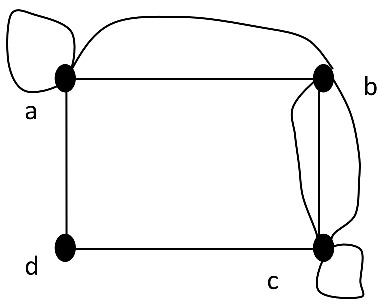
Sec. 10.3 8, 15, 17, 34-37

8 $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ with respect to the ordering of vertices a, b, c, d, e.

15 In this graph there are loops, which are represented by entries on the diagonal.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$ with respect to the ordering of vertices *a, b, c, d*.

17 Because of the numbers larger than 1, we need multiple edges in this graph.



34 These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_3$.

35 These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$, $f(u_5) = v_4$.

36 These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

37 These graphs are isomorphic, since each is the 7-cycle (this is just like Ex. 35).

Sec. 10.4 27(e), 28, 29, 62

27(e) There are two approaches here.
One is to use matrix multiplication on the adjacency matrix of this directed graph (by Theorem 2), which is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering

of vertices a, b, c, d, e. Thus we just need to compute A^6, and look at the $(1,5)^{th}$ entry to determine the number of paths from a to e of length 6, which is 5.

The other method is to examine the number of possibilities for a path of length 6. Since the only way to get to e is from b, we are asking for the number of paths of length 5 from a to b. We can go around the square (a,b,e,d,a,b), or else we can jog over to either b or d and back twice – there being 4 ways to choose where to do the jogging. Therefore there are 5 paths in all.

28 We show this by induction on n. For $n=1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geqslant 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degree is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

29 The definition given here makes it clear that u and v are related if and only if they are in the same component – in other words $f(u) = f(v)$ where $f(x)$ is the component in which x lies. Therefore this is an equivalence relation.

62 The adjacency matrix of G_1 is as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute A^2 and A^3, obtaining

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 & 1 & 2 & 1 \\ 5 & 5 & 4 & 6 & 1 & 6 & 1 \\ 2 & 2 & 6 & 2 & 3 & 5 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 6 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

Already every off-diagonal entry in A^3 is nonzero, so we know that there is a path of length 3 between every pair of distinct vertices in this graph. Therefore the graph G_1 is connected.

On the other hand, the adjacency matrix of G_2 is as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We compute A^2 through A^5, obtaining the following matrices:

$$A^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & 5 \\ 0 & 0 & 0 & 5 & 6 & 5 \\ 0 & 0 & 0 & 5 & 5 & 6 \end{bmatrix} \qquad A^5 = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 11 & 11 \\ 0 & 0 & 0 & 11 & 10 & 11 \\ 0 & 0 & 0 & 11 & 11 & 10 \end{bmatrix}$$

If we compute the sum $A + A^2 + A^3 + A^4 + A^5$ we obtain

$$\begin{bmatrix} 6 & 7 & 7 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 21 & 21 \\ 0 & 0 & 0 & 21 & 20 & 21 \\ 0 & 0 & 0 & 21 & 21 & 20 \end{bmatrix}.$$

There is a 0 in the (1,4) position, telling us that there is no path of length at most 5 from vertex a to vertex d. Since the graph only has six vertices, this tells us that there is no path at all from a to d. Thus the fact that there was a 0 as an off-diagonal entry in the sum told us that the graph was not connected.

Sec. 10.5 4, 6, 31, 34, 38, 41

4 The graph has no Euler circuit, since the degree of vertex *c* (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is *f, a, b, c, d, e, f, b, d, a, e, c*.

6 This graph has no Euler circuit, since the degree of vertex *b* (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is *b, c, d, e, f, d, g, i, d, a, h, i, a, b, i, c*.

31 It is clear that *a*, *b*, *c*, *d*, *e*, *a* is a Hamilton circuit.

34 This graph has no Hamilton circuit. If it did, then certainly the circuit would have to contain edges {*d*,*a*} and {*a*,*b*}, since these are the only edges incident to vertex *a*. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These eight edges already complete a circuit, and this circuit omits the nine vertices on the inside. Therefore there is no Hamilton circuit.

38 This graph has the Hamilton path *a*, *b*, *c*, *d*, *e*.

41 There are eight vertices of degree 2 in this graph. Only two of them can be the end vertices of a Hamilton path, so for each of the other six their two incident edges must be present in the path. Now if either all four of the “outside” vertices of degree 2 (*a*, *c*, *g*, and *e*) or all four of the “inside” vertices of degree 2 (*i*, *k*, *l*, and *n*) are not end vertices, then a circuit will be completed that does not include all the vertices – either the outside square or the middle square. Therefore if there is to be a Hamilton path then exactly one of the inside corner vertices must be an end vertex, and each of the other inside corner vertices must have its two incident edges in the path. Without loss of generality we can assume that vertex i is an end, and that the path begins i, o, n, m, l, q, k, j. At this point, either the path must visit vertex p, in which case it gets stuck, or else it must visit b, in which case it will never be able to reach p. Either case gives a contradiction, so there is no Hamilton path.