Sample Solutions on HW7 (*19 exercises in total*)

Sec. 6.5 10, 16, 26, 32, 46, 50, 54, 61

10(a) $C(6+12-1, 12) = C(17, 12) = 6188$

10(b) $C(6+36-1, 36) = C(41, 36) = 749,398$

10(c) If we first pick the two of each kind, then we have picked 2 • 6 = 12 croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part (a), namely $C(6+12-1, 12) = C(17, 12) = 6188$

10(d) We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1, 21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1, 24) = C(29, 24) = 118755$ ways to pick
24 croissants without restriction, there must be 118755 – 65780 = 52,975 ways to choose two dozen croissants with no more than two broccoli.

10(e) Eight croissants are specified, so this problem is the same as choosing 24 – 8 =
16 croissants without restriction, which can be done in $C(6+16-1, 16) = C(21, 16)$ =20,349 ways.

10(f) First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are 24 – 9 = 15 left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1, 15) = C(20, 15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1, 11) = C(16, 11)$ =4368 choices. Therefore the number of ways to make the selection without violating the restriction is 15504 – 4368 = 11,136

16(a) We require each $x_i \geqslant 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x_1' + x_2' + x_3' + x_4' + x_5 + x_6' = 17$ with each x_i a nonnegative integer. The number of solutions is therefore $C(6+17-1,17) = C(22, 17)$ =26,334.

16(b) The restriction use up 22 of the total, leaving a free total of 7. Therefore the answer is $C(6+7-1, 7) = C(12, 7)$ =792.

16(c) The number of solutions without restriction is $C(6+29-1, 29) = C(34, 29) =$ 278,526. The number of solution violating the restriction by having $x_1 \geqslant 6$ is $C(6+23-1, 23) = C(28, 23) =$ 98,280. Therefore the answer is 278256 – 98280 = 179,976.

16(d) The number of solutions with $x_2>=9$ (as required) but without the restriction on x_1 is $C(6+20-1, 20) = C(25, 20) = 53130$. The number of solutions violating the additional restriction by having $x_1 \geqslant 8$ is $C(6+12-1, 12) = C(17, 12) = 6188$. Therefore the answer is 53130 – 6188 =46,942.

26 We can model this problem by letting x_i be the i^{th} digit of the number for i = 1,2,3,4,5,6 and asking for the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$, where each x_i is between 0 and 8, inclusive, except that one of them equals 9. First, there are 6 ways to decide which of the digits is 9. Without loss of generality assume that $x_6 = 9$. Then the number of ways to choose the remaining digits is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$ (note that the restriction that each $x_i \leqslant 8$ was moot, since the sum was only 4). By Theorem 2 there are $C(5+4-1, 4) = C(8, 4) =70$ solutions. Therefore the answer is $6 \bullet 70 = 420$.

32 We can treat the 3 consecutive A's as one letter. Thus we have 6 letters, of which 2 are the same (the two R's), so by Theorem 3 the answer is 6! / 2! = 360.

46 We follow the hint. There are 5 bars (chosen books), and therefore there 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars, …, after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be places in 6 slots. The number of ways to do this is therefore $C(6+3-1, 3) = C(8,3) = 56$.

50 This is actually a problem about partitions of sets. Let us cal the set of 5 objects {a,b,c,d,e}. We want to partition this set into three pairwise disjoint subsets (some possibly empty). We count in a fairly ad hoc way. First, we could put all five objects into one subset (i.e., all five objects go into one box, with the other two boxes empty). Second, we could put four of the objects into one subset and one into another, such as {a,b,c,d} together with {e}. There are **5** ways to do this, since each of the five objects can be the singleton. Third, we could put three of the objects into one set (box) and two into another; there are $C(5,2)$ = **10** ways to do this, since there are that many ways to choose which objects are to be the doubleton. Similarly, there are **10** ways to distribute the elements so that three go into one set and one each into the other two sets (for example, {a,b,c}. {d}, and {e}). Finally, we could put two items into one set, two into another, and one into the third (for example, {a,b}, {c,d}, and {e}). Here we need to choose the singleton (5 ways), and then we need to choose one of the 3 ways to separate the remaining four elements into pairs; this gives a total of **15** partitions. In all we have **41** different partitions.

54 We are asked for the partitions of 5 into at most 3 parts, notice that we are not required to use all three boxes. We can easily list these partitions explicitly: 5 = 5, 5 = 4+1, 5 = 3+2, 5 = 3+1+1, and 5 = 2+2+1. Therefore the answer is 5.

61 Without the restriction on site X, we are simply asking for the number of ways to order the ten symbols V, V, W, W, X, X, Y, Y, Z, Z (the ordering will give us the visiting schedule). By Theorem 3 this can be done in $10! / (2!)^5$ = 113,400 ways. If the inspector visits site X on consecutive days, then in effect we are ordering nine symbols (including only one X)., where now the X means to visit site X twice in a row. There are $9! / (2!)^4$ = 22,680 ways to do this. Therefore the answer is 113,400 - 22,680 = 90,720.

Sec. 6.6 6(f), 7, 9

6(f) The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is a_7=1 and a_8=6. The least integer to the right of 1 that is greater than 1 is 6. Hence 6 is placed in the 7th position. The integer 1 is then placed in the 8th positions, giving the permutation 23587461

7 We begin with the permutation **1234**. Then we apply Algorithm 1 23 times in succession, giving us the other 23 permutations in lexicographic order: **1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312,** and **4321**. The last permutation is the one entirely in decreasing order.

9 We begin with the first 3-combination, namely **{1,2,3}**. Let us trace through Algorithm 3 to find the next. Note that n = 5 and r = 3; also a_1=1, a_2=2, and a_3=3. We set i equal to 3 and then decrease i until $a_i \neq$ 5-3+i. This inequality is already satisfied for i = 3, since $a_3 \neq$ 5. At this point we increment a_i by 1 (so that now a_3 = 4), and fill the remaining spaces with consecutive integers following a_i (in this case there are no more remaining spaces). Thus our second 3-combination is **{1,2,4}**. The next call to Algorithm 3 works the same way, producing the third 3-combination, namely **{1,2,5}**. To get the fourth 3-combination, we again call Algorithm 3. This time the i that we end up with is 2, since 5 = a_3 = 5-3+3. Therefore the second element in the list is incremented, namely goes from a 2 to a 3, and the third element is the next larger element after 3, namely 4. Thus this 3-combination is **{1,3,4}**. Another call to the algorithm gives us **{1,3,5}**, and another call gives us **{1,4,5}**. Now when we call the algorithm, we find i =1 at the end of the **while** loop, since in this case the last two elements are the two largest elements in the set. Thus a_1 is increased to 2, and the remainder of the list is filled with the next two consecutive integers, giving us **{2,3,4}**. Continuing in this manner, we get the rest of the 3-combinations: **{2,3,5}**, **{2,4,5}**, **{3,4,5}**.