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5.3 6a) $f(0)=1$, $f(n)=-f(n-1)$ for $n \geq 1$

$f(1) = -f(0) = -1$

$f(2) = -f(1) = 1$

$f(3) = -f(2) = -1$

$$f(n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Proof by strong induction

If $k+1$ is even, then $k-1$ is odd

$f(k+1) = -f(k) = -(-f(k-1)) = f(k-1) = 1$

If $k+1$ is odd, then $k-1$ is even

$f(k+1) = -f(k) = -(-f(k-1)) = f(k-1) = -1$

Therefore $f(k+1)$ is correctly defined

$$f(n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

d) $f(0)=0$, $f(1)=1$, $f(n)=2f(n-1)$ for $n \geq 1$

$f(1) = 2f(0)$

$= 0$

$f(1) = 1 \neq f(1) = 0$. F is not a recursive.

14. $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ where n is positive

let $P(n)$: $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$, $n=1$

$f_2 f_0 - f_1^2 = (f_0+f_1)f_0 - f_1^2 = (0+1)(0) - 1^2$

$= (-1)^1$

$P(1)$ is true.

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Inductive step.

P(k+1).

$f_{(k+1)+1} f_{(k+1)-1} - f^2_{(k+1)} = f_{k+2} f_k - f^2_{k+1}$

$= (f_k + f_{k+1}) f_k - f^2_{k+1}$ $\qquad f_n = f_{n-1} + f_{n-2}$

$= f^2_k + f_{k+1} f_k - f^2_{k+1}$

$= f^2_k - f_{k+1} (f_{k+1} - f_k)$

$= f^2_k - f_{k+1} f_{k-1}$ $\qquad f_{n-2} = f_n - f_{n-1}$

$= -(f_{k+1} f_{k-1} - f^2_k)$

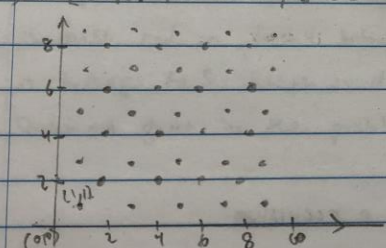
$= -(-1)^k$

$= (-1)(-1)^k$

$= (-1)^{k+1}$

P(k+1) is true

29a) $S = \{(a,b) \mid a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a+b \text{ is even}\}$



$a=1, b=1 \rightarrow$ True

$(a+1, b+1) \in S$

$(a+2, b) \in S$

$(a, b+2) \in S$

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5.4 29

```
Procedure term(n: positive integer)
if n=0 then
    return 1
else if n=1 then
    return 2
else return term(n-1) • term(n-2)
```

6.1 41. If n is even, $2^{n/2}$; if n is odd, $2^{(n+1/2)}$

56. [8 characters]

First character $2^{nd} - 8^{th}$

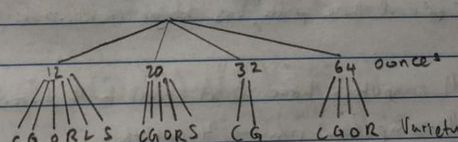
Upper/lower/_   63

26 upper
26 lower
10 digits
1 _
63

Total $= 53 + [(53)(63) + (53)(63)^2 + (53)(63)^3 + (53)(63)^4 + (53)(63)^5 + (53)(63)^6 + (53)(63)^7]$

68. a)

(C) Cola,	(G) Ginger ale,	(O) orange,	(R) root bear,	(L) lemonade,	(S) cream soda
12	12	12	12	12	12
20	20	20	20		20
32	32				
64	64	64	64		



Variety. Total=17.

b) Sum rule $= 6+5+2+4 = 17$.

Date

-2 16. Pigeonhole.

$j,k \in \{1,2,3,4,5\}$ $j \neq k$ such that (x_j, y_j) and (x_k, y_k) have same combination of parity.

$$\text{Midpoint} = \left(\frac{x_j + x_k}{2}, \frac{y_j + y_k}{2}\right)$$

If (odd, odd)

$x_j = 2a+1$ $x_k = 2c+1$

$y_j = 2b+1$ $y_k = 2d+1$

$$\text{Midpoint} = \left(\frac{2a+1+2c+1}{2}, \frac{2b+1+2d+1}{2}\right)$$

$$= (a+c+1, b+d+1)$$

If (odd, even)

$x_j = 2a+1$ $x_k = 2c+1$

$y_j = 2b$ $y_k = 2d$

$$M = \left(\frac{2a+1+2c+1}{2}, \frac{2b+2d}{2}\right)$$

$$= (a+c+1, b+d)$$

If (even, odd)

$x_j = 2a$ $x_k = 2c$

$y_j = 2b+1$ $y_k = 2d+1$

$$M = \left(\frac{2a+2c}{2}, \frac{2b+1+2d+1}{2}\right)$$

$$= (a+c, b+d+1)$$

If (even, even)

$x_j = 2a$ $x_k = 2c$

$y_j = 2b$ $y_k = 2d$

$$M = \left(\frac{2a+2c}{2}, \frac{2b+2d}{2}\right)$$

$$= (a+c, b+d)$$

The midpoint of (x_j, y_j) and (x_k, y_k) has integer coordinates therefore there is at least one pair of these points that has integer coordinates.

38. 8 computers. 4 printers

4 computers have direct access to 4 printers $= 4 \times 4 = 16$ cables

4 computers rem. $\rightarrow$ 4 cables

Total = 20 cables.

12. Let x_i denote the number of matches played after the i^{th} hour for $1 \leq i \leq 75$ where $1 \leq x_i \leq 125$. , $1+n \leq x_i + n \leq 125 + n$.

150 positive integers $x_1 \dots x_{75}, x_{1+n} \dots x_{75+n}$ $\therefore$ between 1 and $(125+n)$.

Pigeonhole principle.

if no numbers are repeated

$n \leq 24$. a) True 2 b) True 23 c) True 25 d) False 30.

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6.3. 20 length=10.

a) $\binom{10}{3} = 120$ 3-zeros.

b) $\binom{10}{4}+\binom{10}{3}+\binom{10}{2}+\binom{10}{1}+\binom{10}{0} = 386$ more 0s than 1s.

c) $\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} = 176$ at least 7 1s.

d) $\binom{10}{3}+\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} = 968$ at least 3 1s.

42. $P = \frac{n!}{(n-r)!} \times 2r$ if $r > 2$ or $P = \frac{n!}{(n-r)!} \times r$ is $r < 3$.

44. 4 horses.

No tie $= P(4,4) = 24$

Two horses tie $= \binom{4}{2} = 6 \times P(3,3) = 6$
$= 36$

Two ties $= \binom{4}{2} = 6$

3 horses tie $= \binom{4}{3} = 4 \times P(2,2) = 2$
$= 8$

All horses tie $= 1$

Total $= 24 + 36 + 6 + 2 + 8 + 1 = 75 \cancel{9}$

6.4 14. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $\binom{n}{k} - \binom{n}{k-1} = \binom{n}{k-1}\left(\frac{n-k+1}{k} - 1\right)$
$= \binom{n}{k-1}\left(\frac{n-2k+1}{k}\right) > 0$

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22. $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ $\quad r \leq n,\ k \leq r$

a) $n \to n-r \ +r \to r-k+k$ remove r then k

$= \binom{n}{r}\binom{r}{k}$

$n \to k + n-k \to n-r + r-k$ remove k then r

$= \binom{n}{k}\binom{n-k}{r-k}$

Therefore $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

b) $\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{(r-k)!(n-r)!}$

$= \frac{n!}{k!(r-k)!(n-r)!}$

$= \frac{n!}{r!(n-r)!} \times \frac{r!}{(r-k)!k!}$

$= \binom{n}{k}\binom{r}{k}$

26. $\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}\frac{1}{2} - \binom{2n}{n}$ $\quad$ RHS $= \binom{2n+2}{n+1}\frac{1}{2} - \binom{2n}{n} = \binom{2n}{n+1}$

$\binom{2n}{n+1} = \sum_{k=0}^{n+1}\binom{n}{k}\binom{n}{n+1-k} = \sum_{k=1}^{n}\binom{n}{k}\binom{n}{n+1-k}$

$= \sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}$

where $k=0$ and $k=n+1$

$\binom{n}{n+1-k} = \binom{n}{n-(n+1-k)} = \binom{n}{n-n-1+k}$

$= \binom{n}{k-1}$

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30. $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

I can either choose the chairsperson first or last. If I choose first then there can be n choices of chairperson. Then I'll choose the rest of n-1 population from the two groups of professors, which is 2n-1. Thus, the choice can be written as $n\binom{2n-1}{n-1}$.

If I choose last, then I'll choose k members from mathematics prof. first, then n-k members from computer science prof. $k \geq 1$ for I need the chairsperson to be from math profs. Thus, the equation can be written as

$$\sum_{k=1}^{n} k\binom{n}{k}\binom{n}{n-k} = \sum_{k=1}^{n} k\binom{n}{k}^2 \quad \text{since } \binom{n}{n-k} = \binom{n}{k}$$

Therefore $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$