

Modelos Probabilistas Aplicados

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Tarea 12

1. Problemas a resolver

En el presente trabajo se realizaron las soluciones de diversos problemas del libro de Grinstead [1] sobre las funciones generadoras de momentos de una variable aleatoria (v.a), así como el valor esperado y varianza. Para el cálculo de estos valores se utilizaron las ecuaciones 1, 2, 3, donde X es una v.a con distribución de probabilidad $f(x)$:

Función generadora de momentos si X es continua,

$$g(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x) dx. \quad (1)$$

Valor esperado si X es continua,

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx. \quad (2)$$

Varianza de X ,

$$\sigma^2 = V(X) = E(X^2) - \mu^2 \quad (3)$$

1.1. Problema 1, página 392

Let Z_1, Z_2, \dots, Z_N describe a branching process in which each parent has j offspring with probability p_j . Find the probability d that the process eventually dies out if:

a) $p_0 = 1/2, p_1 = 1/4, p_2 = 1/4$

b) $p_0 = 1/3, p_1 = 1/3, p_2 = 1/3$

c) $p_0 = 1/3, p_1 = 0, p_2 = 2/3$

d) $p_j = 1/2^{j+1}$

e) $p_j = (1/3)(2/3)^j$

De acuerdo con el teorema 10.2, se tiene que si $m \leq 1$, entonces $d = 1$ y el proceso acaba con probabilidad 1; si $m > 1$, entonces $d < 1$ y el proceso acaba con probabilidad d . Para el cálculo del valor de m se utiliza las siguientes expresiones:

$$\begin{aligned} m &= p_1 + 2p_2 = 1 - p_0 - p_2 + 2p_2 = 1 - p_0 + p_2 \\ h(z) &= p_0 + p_1z + p_2z^2 + \dots \\ m &= h'(1). \end{aligned}$$

Inciso a)

$$\begin{aligned} m &= \frac{1}{4} + 2 \left(\frac{1}{4} \right) \\ m &= \frac{3}{4}. \end{aligned}$$

Como $m \leq 1$ y $p_0 > p_2$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

Inciso b)

$$\begin{aligned} m &= \frac{1}{3} + 2 \left(\frac{1}{3} \right) \\ m &= 1. \end{aligned}$$

Como $m \leq 1$ y $p_0 = p_2$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

Inciso c)

$$\begin{aligned} m &= 0 + 2 \left(\frac{2}{3} \right) \\ m &= \frac{4}{3}. \end{aligned}$$

Como $m > 1$ y $p_0 < p_2$, entonces $d < 1$ y el proceso acaba con una probabilidad de d . Para el cálculo del valor de d , se utiliza la ecuación 4:

$$\begin{aligned} d &= \frac{p_0}{p_2} \\ d &= \frac{\frac{1}{3}}{\frac{2}{3}} \\ d &= \frac{1}{2} \end{aligned} \tag{4}$$

Inciso d)

$$\begin{aligned}h(z) &= \frac{1}{2^{0+1}} + \frac{1}{2^{1+1}}z + \frac{1}{2^{2+1}}z^2 + \dots \\&= \frac{1}{2^1} + \frac{1}{2^2}z + \frac{1}{2^3}z^2 + \dots \\&= \frac{1}{2} \left(1 + \frac{1}{2^1}z + \frac{1}{2^2}z^2 + \dots \right) \\&= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{2}z} \right) \\&= \frac{1}{2 - z} \\h'(z) &= -\frac{\frac{d}{dz}(2 - z)}{(2 - z)^2} \\&= -\frac{0 - 1}{(2 - z)^2} \\&= \frac{1}{(2 - z)^2} \\m = h'(1) &= \frac{1}{(2 - 1)^2} \\&= 1\end{aligned}$$

Como $m \leq 1$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

Inciso e)

$$\begin{aligned}h(z) &= \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^0 + \left(\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^1\right) z + \left(\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2\right) z^2 + \dots \\&= \left(\frac{1}{3}\right) + \left(\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^1\right) z + \left(\left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2\right) z^2 + \dots \\&= \frac{1}{3} \left(1 + \left(\frac{2}{3}\right)^1 z + \left(\frac{2}{3}\right)^2 z^2 + \dots \right) \\&= \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}z} \right) \\&= \frac{1}{3 - 2z} \\h'(z) &= -\frac{\frac{d}{dz}(3 - 2z)}{(3 - 2z)^2} \\&= -\frac{\frac{d}{dz}(3 - 2z)}{(3 - 2z)^2} \\&= -\frac{0 - 2}{(3 - 2z)^2} \\&= \frac{2}{(3 - 2z)^2} \\m = h'(1) &= \frac{2}{(3 - 2)^2} \\&= 2\end{aligned}$$

Como $m > 1$, entonces $d < 1$ y el proceso acaba con una probabilidad de d . Para el cálculo del valor de d , se utiliza la ecuación 5:

$$\begin{aligned} z &= h(z) \\ &= \frac{1}{3-2z} \\ z(3-2z) &= 1 \\ 2z^2 - 3z + 1 &= 0 \end{aligned} \tag{5}$$

Por lo tanto, resolviendo la ecuación cuadrática queda que $z_1 = 1$ y $z_2 = 1/2 = d$.

1.2. Problema 3, página 401

In the chain letter problem (see Example 10.14) find your expected profit if:

a) $p_0 = 1/2, p_1 = 0, p_2 = 1/2$

b) $p_0 = 1/2, p_1 = 0, p_2 = 1/2$

Show that if $p_0 > 1/2$, you cannot expect to make a profit.

Solución

En este problema se tiene que el número esperado de cartas que se pueden vender es $m = p_1 + 2p_2$ y el valor esperado de la ganancia es $E_{(\text{ganancia})} = 50(m + m^2) - 100$, entonces se tiene que:

Inciso a)

$$m = 0 + 2 \left(\frac{1}{2} \right)$$

$$m = 1.$$

$$E_{(\text{ganancia})} = 50(1 + 1^2) - 100$$

$$E_{(\text{ganancia})} = 0.$$

Inciso b)

$$m = \frac{1}{2} + 2 \left(\frac{1}{3} \right)$$

$$m = \frac{7}{6}.$$

$$E_{(\text{ganancia})} = 50 \left(\frac{7}{6} + \left(\frac{7}{6} \right)^{12} \right) - 100$$

$$E_{(\text{ganancia})} \approx 276.26.$$

Demostración

Se considera que $p_0 + p_1 + p_2 = 1$, entonces se contempla un $p_0 = 0.55$, $p_1 = 0.25$ y $p_2 = 0.2$, y se obtiene lo siguiente:

$$m = 0.25 + 2(0.2)$$

$$m = 0.65$$

$$E_{(\text{ganancia})} = 50(0.65 + 0.65^{12}) - 100$$

$$E_{(\text{ganancia})} \approx -67.22.$$

1.3. Problema 1, página 401

Let X be a continuous random variable with values in $[0, 2]$ and density f_X . Find the moment generating function $g(t)$ for X if:

a) $f_X(x) = 1/2$

$$\begin{aligned} g(t) &= \int_0^2 e^{xt} \frac{1}{2} dx \longrightarrow (\text{ecuación 1}) \\ &= \frac{1}{2} \int_0^2 e^{xt} dx \\ &= \frac{1}{2} \left[\frac{e^{xt}}{t} \right]_0^2 \\ &= \frac{1}{2} \left[\frac{e^{2t}}{t} - \frac{e^{0t}}{t} \right] \\ &= \frac{1}{2} \left[\frac{e^{2t} - 1}{t} \right] \\ &= \frac{e^{2t} - 1}{2t}. \end{aligned} \tag{6}$$

b) $f_X(x) = (1/2)x$

$$\begin{aligned}
g(t) &= \int_0^2 e^{xt} \frac{1}{2} x \, dx \longrightarrow (\text{ecuación 1}) \\
&= \frac{1}{2} \int_0^2 e^{xt} x \, dx \implies UV - \int_0^2 V \, dU, \quad U = x, \quad dU = dx, \quad V = \frac{e^{xt}}{t} \\
&= \frac{1}{2} \left[x \frac{e^{xt}}{t} - \int_0^2 \frac{e^{xt}}{t} \, dx \right] \\
&= \frac{1}{2} \left[x \frac{e^{xt}}{t} - \frac{1}{t} \int_0^2 e^{xt} \, dx \right] \\
&= \frac{1}{2} \left[x \frac{e^{xt}}{t} - \frac{e^{xt}}{t^2} \Big|_0^2 \right] \tag{7} \\
&= \frac{1}{2} \left[2 \frac{e^{2t}}{t} - \frac{e^{2t}}{t^2} - \left(0 \frac{e^{0t}}{t} - \frac{e^{0t}}{t^2} \right) \right] \\
&= \frac{1}{2} \left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2} \right] \\
&= \frac{1}{2} \left[\frac{2te^{2t} - e^{2t} + 1}{t^2} \right] \\
&= \frac{e^{2t}(2t - 1) + 1}{2t^2}. \tag{8}
\end{aligned}$$

c) $f_X(x) = 1 - (1/2)x$

$$\begin{aligned}
g(t) &= \int_0^2 e^{xt} \left[1 - \frac{1}{2}x \right] \, dx \longrightarrow (\text{ecuación 1}) \\
&= \int_0^2 e^{xt} - \frac{e^{xt}}{2} x \, dx \\
&= \int_0^2 e^{xt} \, dx - \int_0^2 \frac{e^{xt}}{2} x \, dx \\
&= \int_0^2 e^{xt} \, dx - \frac{1}{2} \int_0^2 e^{xt} x \, dx \\
&= \frac{e^{2t} - 1}{t} - \frac{e^{2t}(2t - 1) + 1}{2t^2} \longrightarrow (\text{ecuación 6 y ecuación 8}) \\
&= \frac{2e^{2t} - 2t - 2e^{2t} + e^{2t} - 1}{2t^2} \\
&= \frac{e^{2t} - 2t - 1}{2t^2}.
\end{aligned}$$

d) $f_X(x) = |1 - x|$

$$\begin{aligned}
g(t) &= \int_0^2 e^{xt} (|1 - x|) dx \longrightarrow (\text{ecuación 1}) \\
&= \int_0^1 e^{xt} (|1 - x|) dx + \int_1^2 e^{xt} (|x - 1|) dx \\
&= \int_0^1 e^{xt} - e^{xt} x dx + \int_1^2 e^{xt} x - e^{xt} dx \\
&= \left[\int_0^1 e^{xt} - \int_0^1 e^{xt} x dx \right] + \left[\int_1^2 e^{xt} x - \int_1^2 e^{xt} dx \right] \\
&= \left[\left[\frac{e^{xt}}{t} - \left(\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} \right) \right]_0^1 \right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t} \right]_1^2 \right] \longrightarrow (\text{ecuación 6 y ecuación 7}) \\
&= \left[\left[\frac{e^{xt}}{t} - \frac{xe^{xt}}{t} + \frac{e^{xt}}{t^2} \right]_0^1 \right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t} \right]_1^2 \right] \\
&= \left[\left[\frac{e^{xt} - xe^{xt}}{t} + \frac{e^{xt}}{t^2} \right]_0^1 \right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t} \right]_1^2 \right] \\
&= \left[\frac{e^{1t} - 1e^{1t}}{t} + \frac{e^{1t}}{t^2} - \left(\frac{e^{0t} - 0e^{0t}}{t} + \frac{e^{0t}}{t^2} \right) \right] + \left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} - \frac{e^{2t}}{t} - \left(\frac{1e^{1t}}{t} - \frac{e^{1t}}{t^2} - \frac{e^{1t}}{t} \right) \right] \\
&= \left[\frac{e^t}{t^2} - \frac{1}{t} - \frac{1}{t^2} \right] + \left[\frac{e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{e^t}{t^2} \right] \\
&= \left[\frac{e^t - t - 1}{t^2} \right] + \left[\frac{te^{2t} - e^{2t} + e^t}{t^2} \right] \\
&= \frac{te^{2t} - e^{2t} + 2e^t - t - 1}{t^2} \\
&= \frac{e^{2t}(t - 1) + 2e^t - t - 1}{t^2}.
\end{aligned}$$

e) $f_X(x) = (3/8)x^2$

$$\begin{aligned}
g(t) &= \int_0^2 e^{xt} \frac{3}{8} x^2 dx \longrightarrow (\text{ecuación 1}) \\
&= \int_0^2 \frac{3x^2 e^{xt}}{8} dx \\
&= \frac{3}{8} \int_0^2 x^2 e^{xt} dx \implies UV - \int_0^2 V dU, \quad U = x^2, \quad dU = 2x dx, \quad V = \frac{e^{xt}}{t} \\
&= \frac{3}{8} \left[x^2 \frac{e^{xt}}{t} - \int_0^2 \frac{e^{xt}}{t} 2x dx \right] \\
&= \left[x^2 \frac{e^{xt}}{t} - \frac{2}{t} \int_0^2 e^{xt} x dx \right] \\
&= \frac{3}{8} \left[\left[x^2 \frac{e^{xt}}{t} - \frac{2}{t} \left(x \frac{e^{xt}}{t} - \frac{e^{xt}}{t^2} \right) \right] \right]_0^2 \\
&= \frac{3}{8} \left[\left[x^2 \frac{e^{xt}}{t} - \frac{2}{t} \left(x \frac{e^{xt}}{t} - \frac{e^{xt}}{t^2} \right) \right] \right]_0^2 \longrightarrow (\text{ecuación 7}) \\
&= \frac{3}{8} \left[\left[\frac{x^2 e^{xt}}{t} - \frac{2}{t} \left(\frac{tx e^{xt}}{t} - \frac{e^{xt}}{t^2} \right) \right] \right]_0^2 \\
&= \frac{3}{8} \left[\left[\frac{x^2 e^{xt}}{t} - \frac{2tx e^{xt} + 2e^{xt}}{t^3} \right] \right]_0^2 \\
&= \frac{3}{8} \left[\left[\frac{t^2 x^2 e^{xt} - 2tx e^{xt} + 2e^{xt}}{t^3} \right] \right]_0^2 \\
&= \frac{3}{8} \left[\frac{t^2(2)^2 e^{2t} - 2t(2)e^{2t} + 2e^{2t}}{t^3} - \left(\frac{t^2(0)^2 e^{0t} - 2t(0)e^{0t} + 2e^{0t}}{t^3} \right) \right] \\
&= \frac{3}{8} \left[\frac{4t^2 e^{2t} - 4te^{2t} + 2e^{2t} - 2}{t^3} \right] \\
&= \frac{12t^2 e^{2t} - 12te^{2t} + 6e^{2t} - 6}{8t^3} \\
&= \frac{6t^2 e^{2t} - 6te^{2t} + 3e^{2t} - 3}{4t^3} \\
&= \frac{3}{4} \left[\frac{e^{2t}(2t^2 - 2t + 1) - 1}{t^3} \right].
\end{aligned}$$

1.4. Problema 6, página 402

Let X be a continuous random variable whose characteristic function $k = X(\tau)$ is:

$$k_X(\tau) = e^{-|\tau|} \quad -\infty < \tau < \infty. \quad (9)$$

Show directly that the density f_X of X is

$$f_X(x) = \frac{1}{\pi(1+x^2)}. \quad (10)$$

Solución

Se tiene que la función de densidad $f_X(x)$ es:

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\tau} k_X(\tau) d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\tau} e^{-|\tau|} d\tau \longrightarrow (\text{ecuación 9}) \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{-ix\tau} e^{-(-\tau)} d\tau + \int_0^{\infty} e^{-ix\tau} e^{-\tau} d\tau \right] \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{-ix\tau+\tau} d\tau + \int_0^{\infty} e^{-ix\tau-\tau} d\tau \right] \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{\tau(1-ix)} d\tau + \int_0^{\infty} e^{-\tau(ix+1)} d\tau \right] \\ &= \frac{1}{2\pi} \left[\left[\frac{e^{\tau(1-ix)}}{(1-ix)} \right] \Big|_{-\infty}^0 + \left[-\frac{e^{-\tau(ix+1)}}{(ix+1)} \right] \Big|_0^{\infty} \right] \\ &= \frac{1}{2\pi} \left[\left[\frac{e^{0(1-ix)}}{(1-ix)} - \frac{e^{-\infty(1-ix)}}{(1-ix)} \right] - \left[\frac{e^{-\infty(ix+1)}}{(ix+1)} - \frac{e^{-0(ix+1)}}{(ix+1)} \right] \right] \\ &= \frac{1}{2\pi} \left[\frac{1}{(1-ix)} + \frac{1}{(ix+1)} \right] \\ &= \frac{1}{2\pi} \left[\frac{ix+1+1-ix}{(ix+1-(ix)^2-ix)} \right] \\ &= \frac{1}{2\pi} \left[\frac{2}{(1-(ix)^2)} \right] \\ &= \frac{1}{\pi} \left[\frac{1}{(1-(-x^2))} \right] \longrightarrow (i = \sqrt{-1}, (\text{teorema 10.4})) \\ &= \frac{1}{\pi(1+x^2)} \longrightarrow (\text{ecuación 10}). \end{aligned}$$

1.5. Problema 10, página 403

Let X_1, X_2, \dots, X_n , be an independent trials process with density

$$f(x) = \frac{1}{2} e^{-|x|} \quad -\infty < x < \infty.$$

a) Find the mean and variance of $f(x)$

$$\begin{aligned}\mu = E(X) &= \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|} dx \longrightarrow (\text{ecuación 2}) \\ &= \frac{1}{2} \left[\int_{-\infty}^0 x e^{-(-x)} dx + \int_0^{\infty} x e^{-(x)} dx \right] \\ &= \frac{1}{2} \left[\int_{-\infty}^0 x e^{(x)} dx + \int_0^{\infty} x e^{-(x)} dx \right] \\ &= \frac{1}{2} [-1 + 1] \\ &= 0.\end{aligned}$$

Ahora, para calcular la varianza, se necesita el valor de $E(X^2)$, entonces:

$$\begin{aligned}\mu = E(X) &= \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx \longrightarrow (\text{ecuación 2}) \\ &= \frac{1}{2} \left[\int_{-\infty}^0 x^2 e^{-(-x)} dx + \int_0^{\infty} x^2 e^{-(x)} dx \right] \\ &= \frac{1}{2} \left[\int_{-\infty}^0 x^2 e^{(x)} dx + \int_0^{\infty} x^2 e^{-(x)} dx \right] \\ &= \frac{1}{2} [2 + 2] \\ &= 2.\end{aligned}$$

Por lo tanto, para calcular la varianza se utiliza la ecuación 3:

$$\begin{aligned}V(X) &= 2 - (0)^2 \\ &= 2 - 0 \\ &= 2.\end{aligned}$$

b) Find the moment generating function for X_1 , S_n , A_n , and S_n^*

$$\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} e^{xt} \left[\frac{e^{-|x|}}{2} \right] dx \longrightarrow (\text{ecuación 1}) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} e^{xt} e^{-|x|} dx \\
&= \frac{1}{2} \left[\int_{-\infty}^0 e^{xt} e^{-(-x)} dx + \int_0^{\infty} e^{xt} e^{-(x)} dx \right] \\
&= \frac{1}{2} \left[\int_{-\infty}^0 e^{xt+x} dx + \int_0^{\infty} e^{xt-x} dx \right] \\
&= \frac{1}{2} \left[\int_{-\infty}^0 e^{x(t+1)} dx + \int_0^{\infty} e^{-x(-t+1)} dx \right] \\
&= \frac{1}{2} \left[\left[\frac{e^{x(t+1)}}{(t+1)} \right]_{-\infty}^0 + \left[-\frac{e^{-x(1-t)}}{(1-t)} \right]_0^{\infty} \right] \\
&= \frac{1}{2} \left[\left[\frac{e^{0(t+1)}}{(t+1)} - \frac{e^{-\infty(t+1)}}{t+1} \right] - \left[\frac{e^{-\infty(1-t)}}{(1-t)} - \frac{e^{-0(1-t)}}{(1-t)} \right] \right] \\
&= \frac{1}{2} \left[\frac{1}{(t+1)} + \frac{1}{(1-t)} \right] \\
&= \frac{1}{2} \left[\frac{1-t+t+1}{(t-t^2+1-t)} \right] \\
&= \frac{1}{2} \left[\frac{2}{(1-t^2)} \right] \\
&= \frac{1}{(1-t^2)}.
\end{aligned}$$

$$\begin{aligned}
S_n &= (g(t))^n \\
&= \left(\frac{1}{(1-t^2)} \right)^n \\
&= \frac{1}{(1-t^2)^n}.
\end{aligned}$$

$$\begin{aligned}
S_n^* &= \left(g\left(\frac{t}{\sqrt{n}}\right) \right)^n \\
&= \left(\frac{1}{\left(1 - \left(\frac{t}{\sqrt{n}}\right)^2\right)} \right)^n \\
&= \frac{1}{\left(1 - \left(\frac{t}{\sqrt{n}}\right)^2\right)^n}.
\end{aligned}$$

c) *What can you say about the moment generating function of S_n^* as $n \rightarrow \infty$?*

$$\lim_{n \rightarrow \infty} \frac{1}{\left(1 - \left(\frac{t}{\sqrt{n}}\right)^2\right)^n}$$

Por lo tanto, la función generadora de $S_n^* \rightarrow 1$.

Referencias

- [1] Grinstead, Charles M., Snell, J. Laurie. *Introduction to Probability*. American Mathematical Society, 2006.