Modelos Probabilistas Aplicados

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Tarea 12

# 1. Poblemas a resolver

En el presente trabajo se realizaron las soluciones de diversos problemas del libro de Grinstead [1] sobre las funciones generadoras de momentos de una variable aleatoria (v.a), así como el valor esperado y varianza. Para el cálculo de estos valores se utilizaron las ecuaciones 1, 2, 3, donde $X$ es una v.a con distribución de probabilidad $f(x)$:

Función generadora de momentos sí $X$ es continua,

$$g(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)\, dx. \tag{1}$$

Valor esperado si $X$ es continua,

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx. \tag{2}$$

Varianza de $X$,

$$\sigma^2 = V(X) = E(X^2) - \mu^2 \tag{3}$$

## 1.1. Problema 1, página 392

*Let $Z_1$, $Z_2$, ..., $Z_N$ describe a branching process in which each parent has $j$ offspring with probability $p_j$. Find the probability $d$ that the process eventually dies out if:*

a) $p_0 = 1/2$, $p_1 = 1/4$, $p_2 = 1/4$

b) $p_0 = 1/3$, $p_1 = 1/3$, $p_2 = 1/3$

c) $p_0 = 1/3$, $p_1 = 0$, $p_2 = 2/3$

d) $p_j = 1/2^{j+1}$

e) $p_j = (1/3)(2/3)^j$

De acuerdo con el teorema 10.2, se tiene que sí $m \leq 1$, entonces $d = 1$ y el proceso acaba con probabilidad 1; sí $m > 1$, entonces $d < 1$ y el proceso acaba con probabilidad $d$. Para el cálculo del valor de $m$ se utiliza las siguientes expresiones:

$$\begin{aligned}
m &= p_1 + 2p_2 = 1 - p_0 - p_2 + 2p_2 = 1 - p_0 + p_2 \\
h(z) &= p_0 + p_1 z + p_2 z^2 + \dots \\
m &= h'(1).
\end{aligned}$$

**Inciso a)**

$$\begin{aligned}
m &= \frac{1}{4} + 2\left(\frac{1}{4}\right) \\
m &= \frac{3}{4}.
\end{aligned}$$

Como $m \leq 1$ y $p_0 > p_2$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

**Inciso b)**

$$\begin{aligned}
m &= \frac{1}{3} + 2\left(\frac{1}{3}\right) \\
m &= 1.
\end{aligned}$$

Como $m \leq 1$ y $p_0 = p_2$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

**Inciso c)**

$$\begin{aligned}
m &= 0 + 2\left(\frac{2}{3}\right) \\
m &= \frac{4}{3}.
\end{aligned}$$

Como $m > 1$ y $p_0 < p_2$, entonces $d < 1$ y el proceso acaba con una probabilidad de $d$. Para el cálculo del valor de $d$, se utliza la ecuación 4:

$$d = \frac{p_0}{p_2} \tag{4}$$

$$d = \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$d = \frac{1}{2}$$

**Inciso d)**

$$\begin{aligned}
h(z) &= \frac{1}{2^{0+1}} + \frac{1}{2^{1+1}}z + \frac{1}{2^{2+1}}z^2 + \dots \\
&= \frac{1}{2^1} + \frac{1}{2^2}z + \frac{1}{2^3}z^2 + \dots \\
&= \frac{1}{2}\left(1 + \frac{1}{2^1}z + \frac{1}{2^2}z^2 + \dots\right) \\
&= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}z}\right) \\
&= \frac{1}{2-z} \\
h'(z) &= -\frac{\frac{d}{dz}(2-z)}{(2-z)^2} \\
&= -\frac{0-1}{(2-z)^2} \\
&= \frac{1}{(2-z)^2} \\
m = h'(1) &= \frac{1}{(2-1)^2} \\
&= 1
\end{aligned}$$

Como $m \leq 1$, entonces $d = 1$ y el proceso acaba con una probabilidad de 1.

**Inciso e)**

$$\begin{aligned}
h(z) &= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 + \left(\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1\right)z + \left(\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2\right)z^2 + \dots \\
&= \left(\frac{1}{3}\right) + \left(\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^1\right)z + \left(\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2\right)z^2 + \dots \\
&= \frac{1}{3}\left(1 + \left(\frac{2}{3}\right)^1 z + \left(\frac{2}{3}\right)^2 z^2 + \dots\right) \\
&= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}z}\right) \\
&= \frac{1}{3-2z} \\
h'(z) &= -\frac{\frac{d}{dz}(2-z)}{(2-z)^2} \\
&= -\frac{\frac{d}{dz}(3-2z)}{(3-2z)^2} \\
&= -\frac{0-2}{(3-2z)^2} \\
&= \frac{2}{(3-2z)^2} \\
m = h'(1) &= \frac{2}{(3-2)^2} \\
&= 2
\end{aligned}$$

Como $m > 1$, entonces $d < 1$ y el proceso acaba con una probabilidad de $d$. Para el cálculo del valor de $d$, se utliza la ecuación 5:

$$\begin{aligned} z &= h(z) \\ &= \frac{1}{3-2z} \\ z(3-2z) &= 1 \\ 2z^2 - 3z + 1 &= 0 \end{aligned} \tag{5}$$

Por lo tanto, resolviendo la ecuación cuadrática queda que $z_1 = 1$ y $z_2 = 1/2 = d$.

## 1.2. Problema 3, página 401

*In the chain letter problem (see Example 10.14) find your expected profit if:*

a) $p_0 = 1/2$, $p_1 = 0$, $p_2 = 1/2$

b) $p_0 = 1/2$, $p_1 = 0$, $p_2 = 1/2$

Show that if $p_0 > 1/2$, you cannot expect to make a profit.

**Solución**

En este problema se tiene que el número esperado de cartas que se pueden vender es $m$=$p_1 + 2p_2$ y el valor esperado de la ganancia es $\mathrm{E}_{(\text{ganancia})}$= $50(m + m^12) - 100$, entonces se tiene que:

**Caso a)**

$$\begin{aligned} m &= 0 + 2\left(\frac{1}{2}\right) \\ m &= 1. \\ \mathrm{E}_{(\text{ganancia})} &= 50(1 + 1^{12}) - 100 \\ \mathrm{E}_{(\text{ganancia})} &= 0. \end{aligned}$$

**Caso b)**

$$
\begin{aligned}
m &= \frac{1}{2} + 2\left(\frac{1}{3}\right) \\
m &= \frac{7}{6}. \\
\mathrm{E}_{(\text{ganancia})} &= 50\left(\frac{7}{6} + \left(\frac{7}{6}\right)^{12}\right) - 100 \\
\mathrm{E}_{(\text{ganancia})} &\approx 276.26.
\end{aligned}
$$

**Demostración**

Se considera que $p_0 + p_1 + p_2 = 1$, entonces se contempla un $p_0 = 0.55$, $p_1 = 0.25$ y $p_2 = 0.2$, y se obtiene lo siguiente:

$$
\begin{aligned}
m &= 0.25 + 2(0.2) \\
m &= 0.65 \\
\mathrm{E}_{(\text{ganancia})} &= 50(0.65 + 0.65^{12}) - 100 \\
\mathrm{E}_{(\text{ganancia})} &\approx -67.22.
\end{aligned}
$$

## 1.3. Problema 1, página 401

*Let $X$ be a continuous random variable with values in $[0, 2]$ and density $f_X$. Find the moment generating function $g(t)$ for $X$ if:*

a) $f_X(x) = 1/2$

$$
\begin{aligned}
g(t) &= \int_0^2 e^{xt}\frac{1}{2}\,dx \longrightarrow \text{(ecuación 1)} \\
&= \frac{1}{2}\int_0^2 e^{xt}\,dx \\
&= \frac{1}{2}\left[\frac{e^{xt}}{t}\Big|_0^2\right] \\
&= \frac{1}{2}\left[\frac{e^{2t}}{t} - \frac{e^{0t}}{t}\right] \\
&= \frac{1}{2}\left[\frac{e^{2t}-1}{t}\right] \\
&= \frac{e^{2t}-1}{2t}.
\end{aligned} \tag{6}
$$

b) $f_X(x) = (1/2)x$

$$
\begin{aligned}
g(t) &= \int_0^2 e^{xt}\frac{1}{2}x\,dx \longrightarrow \text{(ecuación 1)} \\
&= \frac{1}{2}\int_0^2 e^{xt}x\,dx \Longrightarrow UV - \int_0^2 V\,dU,\;\; U = x,\;\; dU = dx,\;\; V = \frac{e^{xt}}{t} \\
&= \frac{1}{2}\left[x\frac{e^{xt}}{t} - \int_0^2 \frac{e^{xt}}{t}\,dx\right] \\
&= \frac{1}{2}\left[x\frac{e^{xt}}{t} - \frac{1}{t}\int_0^2 e^{xt}\,dx\right] \\
&= \frac{1}{2}\left[x\,\frac{e^{xt}}{t} - \frac{e^{xt}}{t^2}\bigg|_0^2\right] \qquad (7) \\
&= \frac{1}{2}\left[2\frac{e^{2t}}{t} - \frac{e^{2t}}{t^2} - \left(0\frac{e^{0t}}{t} - \frac{e^{0t}}{t^2}\right)\right] \\
&= \frac{1}{2}\left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2}\right] \\
&= \frac{1}{2}\left[\frac{2te^{2t} - e^{2t} + 1}{t^2}\right] \\
&= \frac{e^{2t}(2t-1)+1}{2t^2}. \qquad (8)
\end{aligned}
$$

c) $f_X(x) = 1 - (1/2)x$

$$
\begin{aligned}
g(t) &= \int_0^2 e^{xt}\left[1 - \frac{1}{2}x\right]\,dx \longrightarrow \text{(ecuación 1)} \\
&= \int_0^2 e^{xt} - \frac{e^{xt}}{2}x\,dx \\
&= \int_0^2 e^{xt}\,dx - \int_0^2 \frac{e^{xt}}{2}x\,dx \\
&= \int_0^2 e^{xt}\,dx - \frac{1}{2}\int_0^2 e^{xt}x\,dx \\
&= \frac{e^{2t}-1}{t} - \frac{e^{2t}(2t-1)+1}{2t^2} \longrightarrow \text{(ecuación 6 y ecuación 8)} \\
&= \frac{2e^{2t} - 2t - 2e^{2t} + e^{2t} - 1}{2t^2} \\
&= \frac{e^{2t} - 2t - 1}{2t^2}.
\end{aligned}
$$

d) $f_X(x) = |1-x|$

$$
\begin{aligned}
g(t) &= \int_0^2 e^{xt}(|1-x|)\,dx \longrightarrow \text{(ecuación 1)}\\
&= \int_0^1 e^{xt}(|1-x|)\,dx + \int_1^2 e^{xt}(|x-1|)\,dx\,dx\\
&= \int_0^1 e^{xt} - e^{xt}x\,dx + \int_1^2 e^{xt}x - e^{xt}\,dx\\
&= \left[\int_0^1 e^{xt} - \int_0^1 e^{xt}x\,dx\right] + \left[\int_1^2 e^{xt}x - \int_1^2 e^{xt}\,dx\right]\\
&= \left[\left[\frac{e^{xt}}{t} - \left(\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2}\right)\right]\Bigg|_0^1\right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t}\right]\Bigg|_1^2\right] \longrightarrow \text{(ecuación 6 y ecuación 7)}\\
&= \left[\left[\frac{e^{xt}}{t} - \frac{xe^{xt}}{t} + \frac{e^{xt}}{t^2}\right]\Bigg|_0^1\right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t}\right]\Bigg|_1^2\right]\\
&= \left[\left[\frac{e^{xt} - xe^{xt}}{t} + \frac{e^{xt}}{t^2}\right]\Bigg|_0^1\right] + \left[\left[\frac{xe^{xt}}{t} - \frac{e^{xt}}{t^2} - \frac{e^{xt}}{t}\right]\Bigg|_1^2\right]\\
&= \left[\frac{e^{1t} - 1e^{1t}}{t} + \frac{e^{1t}}{t^2} - \left(\frac{e^{0t} - 0e^{0t}}{t} + \frac{e^{0t}}{t^2}\right)\right] + \left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} - \frac{e^{2t}}{t} - \left(\frac{1e^{1t}}{t} - \frac{e^{1t}}{t^2} - \frac{e^{1t}}{t}\right)\right]\\
&= \left[\frac{e^t}{t^2} - \frac{1}{t} - \frac{1}{t^2}\right] + \left[\frac{e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{e^t}{t^2}\right]\\
&= \left[\frac{e^t - t - 1}{t^2}\right] + \left[\frac{te^{2t} - e^{2t} + e^t}{t^2}\right]\\
&= \frac{te^{2t} - e^{2t} + 2e^t - t - 1}{t^2}\\
&= \frac{e^{2t}(t-1) + 2e^t - t - 1}{t^2}.
\end{aligned}
$$

e) $f_X(x) = (3/8)x^2$

$$
\begin{aligned}
g(t) &= \int_0^2 e^{xt}\frac{3}{8}x^2\,dx \longrightarrow (\text{ecuación } 1)\\
&= \int_0^2 \frac{3x^2e^{xt}}{8}\,dx\\
&= \frac{3}{8}\int_0^2 x^2e^{xt}\,dx \Longrightarrow UV - \int_0^2 V\,dU,\quad U = x^2,\quad dU = 2x\,dx,\quad V = \frac{e^{xt}}{t}\\
&= \frac{3}{8}\left[x^2\frac{e^{xt}}{t} - \int_0^2 \frac{e^{xt}}{t}2x\,dx\right]\\
&= \left[x^2\frac{e^{xt}}{t} - \frac{2}{t}\int_0^2 e^{xt}x\,dx\right]\\
&= \frac{3}{8}\left[\left[x^2\frac{e^{xt}}{t} - \frac{2}{t}\left(x\,\frac{e^{xt}}{t} - \frac{e^{xt}}{t^2}\right)\right]\Bigg|_0^2\right]\\
&= \frac{3}{8}\left[\left[x^2\frac{e^{xt}}{t} - \frac{2}{t}\left(x\,\frac{e^{xt}}{t} - \frac{e^{xt}}{t^2}\right)\right]\Bigg|_0^2\right] \longrightarrow (\text{ecuación } 7)\\
&= \frac{3}{8}\left[\left[\frac{x^2e^{xt}}{t} - \frac{2}{t}\left(\frac{txe^{xt} - e^{xt}}{t^2}\right)\right]\Bigg|_0^2\right]\\
&= \frac{3}{8}\left[\left[\frac{x^2e^{xt}}{t} - \frac{2txe^{xt} + 2e^{xt}}{t^3}\right]\Bigg|_0^2\right]\\
&= \frac{3}{8}\left[\left[\frac{t^2x^2e^{xt} - 2txe^{xt} + 2e^{xt}}{t^3}\right]\Bigg|_0^2\right]\\
&= \frac{3}{8}\left[\frac{t^2(2)^2e^{2t} - 2t(2)e^{2t} + 2e^{2t}}{t^3} - \left(\frac{t^2(0)^2e^{0t} - 2t(0)e^{0t} + 2e^{0t}}{t^3}\right)\right]\\
&= \frac{3}{8}\left[\frac{4t^2e^{2t} - 4te^{2t} + 2e^{2t} - 2}{t^3}\right]\\
&= \frac{12t^2e^{2t} - 12te^{2t} + 6e^{2t} - 6}{8t^3}\\
&= \frac{6t^2e^{2t} - 6te^{2t} + 3e^{2t} - 3}{4t^3}.\\
&= \frac{3}{4}\left[\frac{e^{2t}(2t^2 - 2t + 1) - 1}{t^3}\right].
\end{aligned}
$$

## 1.4. Problema 6, página 402

*Let $X$ be a continuous random variable whose characteristic function $k - X(\tau)$ is:*

$$k_X(\tau) = e^{-|\tau|} \quad -\infty < \tau < \infty. \tag{9}$$

Show directly that the density $f_X$ of $X$ is

$$f_X(x) = \frac{1}{\pi(1 + x^2)}. \tag{10}$$

## Solución

Se tiene que la función de densidad $f_X(x)$ es:

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\tau}k_X(\tau)\,d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\tau}e^{-|\tau|}\,d\tau \longrightarrow (\text{ecuación } 9) \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-ix\tau}e^{-(-\tau)}\,d\tau + \int_{0}^{\infty} e^{-ix\tau}e^{-\tau}\,d\tau\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-ix\tau+\tau}\,d\tau + \int_{0}^{\infty} e^{-ix\tau-\tau}\,d\tau\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{\tau(1-ix)}\,d\tau + \int_{0}^{\infty} e^{-\tau(ix+1)}\,d\tau\right] \\
&= \frac{1}{2\pi}\left[\left[\frac{e^{\tau(1-ix)}}{(1-ix)}\right]\Bigg|_{-\infty}^{0} + \left[-\frac{e^{-\tau(ix+1)}}{(ix+1)}\right]\Bigg|_{0}^{\infty}\right] \\
&= \frac{1}{2\pi}\left[\left[\frac{e^{0(1-ix)}}{(1-ix)} - \frac{e^{-\infty(1-ix)}}{(1-ix)}\right] - \left[\frac{e^{-\infty(ix+1)}}{(ix+1)} - \frac{e^{-0(ix+1)}}{(ix+1)}\right]\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{(1-ix)} + \frac{1}{(ix+1)}\right] \\
&= \frac{1}{2\pi}\left[\frac{ix+1+1-ix}{(ix+1-(ix)^2-ix)}\right] \\
&= \frac{1}{2\pi}\left[\frac{2}{(1-(ix)^2)}\right] \\
&= \frac{1}{\pi}\left[\frac{1}{(1-(-x^2)}\right] \longrightarrow (i=\sqrt{-1}, (\text{teorema } 10.4)) \\
&= \frac{1}{\pi(1+x^2)} \longrightarrow (\text{ecuación } 10).
\end{aligned}
$$

## 1.5. Problema 10, página 403

*Let $X_1, X_2, \ldots, X_n$, be an independent trials process with density*

$$f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty.$$

a) *Find the mean and variance of $f(x)$*

$$
\begin{aligned}
\mu = E(X) &= \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|}\, dx \longrightarrow \text{(ecuación 2)} \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} x e^{-(-x)}\, dx + \int_{0}^{\infty} x e^{-(x)}\, dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} x e^{(x)}\, dx + \int_{0}^{\infty} x e^{-(x)}\, dx\right] \\
&= \frac{1}{2}\left[-1+1\right] \\
&= 0.
\end{aligned}
$$

Ahora, para calcular la varianza, se necesita el valor de $E(X^2)$, entonces:

$$
\begin{aligned}
\mu = E(X) &= \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|}\, dx \longrightarrow \text{(ecuación 2)} \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{-(-x)}\, dx + \int_{0}^{\infty} x^2 e^{-(x)}\, dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{(x)}\, dx + \int_{0}^{\infty} x^2 e^{-(x)}\, dx\right] \\
&= \frac{1}{2}\left[2+2\right] \\
&= 2.
\end{aligned}
$$

Por lo tanto, para calcular la varianza se utiliza la ecuación 3:

$$
\begin{aligned}
V(X) &= 2-(0)^2 \\
&= 2-0 \\
&= 2.
\end{aligned}
$$

b) *Find the moment generating function for $X_1$, $S_n$, $A_n$, and $S_n^*$*

$$
\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} e^{xt}\left[\frac{e^{-|x|}}{2}\right] dx \longrightarrow \text{(ecuación 1)} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{xt}e^{-|x|}\, dx \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{xt}e^{-(-x)}\, dx + \int_{0}^{\infty} e^{xt}e^{-(x)}\, dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{xt+x}\, dx + \int_{0}^{\infty} e^{xt-x}\, dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(t+1)} + \int_{0}^{\infty} e^{-x(-t+1)}\right] \\
&= \frac{1}{2}\left[\left[\frac{e^{x(t+1)}}{(t+1)}\right]\Bigg|_{-\infty}^{0} + \left[-\frac{e^{-x(1-t)}}{(1-t)}\right]\Bigg|_{0}^{\infty}\right] \\
&= \frac{1}{2}\left[\left[\frac{e^{0(t+1)}}{(t+1)} - \frac{e^{-\infty(t+1)}}{t+1}\right] - \left[\frac{e^{-\infty(1-t)}}{(1-t)} - \frac{e^{-0(1-t)}}{(1-t)}\right]\right] \\
&= \frac{1}{2}\left[\frac{1}{(t+1)} + \frac{1}{(1-t)}\right] \\
&= \frac{1}{2}\left[\frac{1-t+t+1}{(t-t^2+1-t)}\right] \\
&= \frac{1}{2}\left[\frac{2}{(1-t^2)}\right] \\
&= \frac{1}{(1-t^2)}.
\end{aligned}
$$

$$
\begin{aligned}
S_n &= (g(t))^n \\
&= \left(\frac{1}{(1-t^2)}\right)^n \\
&= \frac{1}{(1-t^2)^n}.
\end{aligned}
$$

$$
\begin{aligned}
S_n^* &= \left(g\left(\frac{t}{\sqrt{n}}\right)\right)^n \\
&= \left(\frac{1}{\left(1-\left(\frac{t}{\sqrt{n}}\right)^2\right)}\right)^n \\
&= \frac{1}{\left(1-\left(\frac{t}{\sqrt{n}}\right)^2\right)^n}.
\end{aligned}
$$

# Referencias

[1] Grinstead, Charles M., Snell, J. Laurie. *Introduction to Probability.* American Mathematical Society, 2006.