Numerical Solution of the Poisson problem in 2D with Finite Element Methods

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1 Introduction

The finite element method is a widely used numerical method for solving physical and mathematical problems. The main goal of the project is to solve the Poisson equation in two dimensions, as well as to build up a solid code base that can be adjusted later to solve more complex problems.

2 Theory

Gaussian quadrature

Using finite element methods to solve equations numerically usually involves evaluating definite integrals. Often, these need to calculated using numerical integration schemes. One method for approximating a definite integral is the Gaussian quadrature. In one dimension this takes the form

$$\int_a^b g(z)\,dz \approx \frac{b-a}{2}\sum_{q=1}^{N_q} \rho_q \cdot g(\frac{b-a}{2}z_q + \frac{b+a}{2}),$$

where N_q is the number of integration points, z_q are the Gaussian quadrature points and ρ_q are the associated Gaussian weights. In higher dimensions this can be extended to

$$\int_\Omega g(\boldsymbol{z})\,d\boldsymbol{z} \approx |\Omega| \sum_{q=1}^{N_q} \rho_q g(\boldsymbol{z}_q),$$

where $|\Omega|$ is the volume of the reference element Ω, and $\boldsymbol{z}_q$ are vector quadrature points. We use barycentric coordinates to map between the integral we want to solve and the reference element[1].

A sligtly modified version of the 1D quadrature can be used to evaluate line integrals in two-dimensions on the straight line between two points $\boldsymbol{a} = (a_1, a_2)$

and $\boldsymbol{b} = (b_1, b_2)$:

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} g(x,y)\,ds \approx \frac{\|\boldsymbol{b}-\boldsymbol{a}\|}{2}\sum_{q=1}^{N_q}\rho_q \cdot g(\frac{b_1-a_1}{2}z_q+\frac{b_1+a_1}{2}, \frac{b_2-a_2}{2}z_q+\frac{b_2+a_2}{2}),$$
$$\|\boldsymbol{b}-\boldsymbol{a}\| = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}.$$

Poisson in 2 dimensions

Consider the two-dimensional Poisson problem with both Neumann and homogeneous Dirichlet boundary conditions. The problem is given by:

$$\begin{aligned} \Delta u(x,y) &= -f(x,y) \quad \text{on} \quad \Omega \\ u(x,y)|_{\partial\Omega_D} &= 0 \\ \frac{\partial u(x,y)}{\partial n}|_{\partial\Omega_N} &= g(x,y), \end{aligned} \tag{1}$$

where $\Omega = \{(x,y) : x^2+y^2 \leq 1\}$ is the unit disc and $\partial\Omega_D, \partial\Omega_N$ are the parts of the boundary where respectively Dirichlet and Neumann conditions are given. f and g are given by:

$$f(r,\theta) = -8\pi\cos(2\pi r^2) + 16\pi^2 r^2 \sin(2\pi r^2), \tag{2}$$
$$g(x,y) = 4\pi r\cos(2\pi r^2). \tag{3}$$

We will consider both the case with only homogeneous Dirichlet conditions on the whole boundary and a case with both homogeneous Dirichlet and Neumann conditions on different parts of the boundary. In both cases,

$$u(x,y) = \sin(2\pi(x^2+y^2)) = \sin(2\pi r^2) \tag{4}$$

solves the problem. This is easily verified:

$$\begin{aligned} \Delta u(x,y) = \Delta u(r) &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \\ &= -16\pi^2 r^2 \sin(2\pi r^2) + 4\pi\cos(2\pi r^2) + \frac{1}{r}(4\pi r\cos(2\pi r)) \\ &= -(16\pi^2 r^2 \sin(2\pi r^2) - 8\pi\cos(2\pi r^2)) \\ &= -f(x,y), \\ u(x,y)|_{\partial\Omega} &= u(r)|_{r=1} = \sin(2\pi) = 0, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} &= \frac{\partial u}{\partial r}|_{r=1} \\ &= 4\pi r\cos(2\pi r^2) \\ &= g(r,\theta). \end{aligned}$$

Weak formulation

The weak form of the problem is found by multiplying the Poisson problem (1) with a test function $v \in X$ and integrating over the domain

$$\iint_\Omega \Delta u \cdot v \, dA = \iint_\Omega -f \cdot v \, dA,$$

where the test space X is defined as

$$X = \{v \in H^1(\Omega) : v|_{\Omega_D} = 0\}.$$

Green's first identity gives that

$$\iint_\Omega \Delta u \cdot v \, dA = \int_{\partial\Omega} v \partial_n u \, ds - \iint_\Omega \nabla u \cdot \nabla v \, dA,$$

where n is the normal vector. Combining the two expressions gives

$$\iint_\Omega \nabla u \cdot \nabla v \, dA = \iint_\Omega f \cdot v \, dA + \int_{\partial\Omega} v \partial_n u \, ds. \tag{5}$$

where the left hand side is the bilinear functinal $a(u, v)$,

$$a(u, v) = \iint_\Omega \nabla u \cdot \nabla v \, dA.$$

The expression for $a(u, v)$ does not depend on whether we have Dirichlet or Neumann boundary conditions. With only homogeneous Dirichlet boundary conditions, $\partial\Omega_D = \partial\Omega$, the right hand side of (5) can be written as the linear functional l(v)

$$l(v) = \iint_\Omega f \cdot v \, dA. \tag{6}$$

In the case with Neumann boundary conditions on part of the boundary and homogeneous Dirichlet on the rest,

$$l(v) = \iint_\Omega f \cdot v \, dA + \int_{\partial\Omega_N} v \cdot g(x, y) \, ds \tag{7}$$

In both cases the weak forumlation of the problem is "find $u \in X$ such that

$$a(u, v) = l(v) \quad \forall \quad v \in X\text{"} \tag{8}$$

Galerkin projection

Now, define a triangulation by discretizing Ω into M triangles K_k, such that $\Omega = \cup_{k=1}^{M} K_k$. Each node in a trianglulation corresponds to one basis function. Let the space X_h be defined as

$$X_h = \{v \in X : v|_{K_k} \in \mathbb{P}_1(K_k), 1 \leq k \leq M\},$$

with basis functions $\{\phi\}_{i=1}^n$ satisfying $X_h = \text{span}\{\phi\}_{i=1}^n$ and $\phi_j(x_i) = \delta_{ij}$. Let u_h be the projeciton of the solution u onto the space X_h. Then u_h can be written as a weighted sum of the basis functions $u_h = \sum_{i=1}^n u_h^i \phi(x,y)$, and $v \in X_h$ is written as $v = \sum_{i=1}^n v^i \phi(x,y)$. Define $\boldsymbol{A} = [A_{ij}] = [a(\phi_i, \phi_j)], \boldsymbol{u} = [u_h^i], \boldsymbol{v} = [v^i]$ and $\boldsymbol{f} = [l(\phi_i)]$. Then we get

$$l(v) = \sum_{i=1^n} v^i \cdot l(\phi_i)$$

and

$$a(u_h, v) = \sum_{i=1}^n \sum_{j=1}^n u_h^i v^j a(\phi_i, \phi_j).$$

Hence the problem "Find $u_f \in X_h$ such that $a(u_h, v) = l(v) \forall v \in X_h$ becomes

$$\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{u} = \boldsymbol{v}^T \boldsymbol{f} \quad \forall \quad \boldsymbol{v} \in \mathbb{R}^n.$$

Since this must be true for any $\boldsymbol{v}$, the problem is equivalent to finding the $\boldsymbol{u}$ satisfying

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \tag{9}$$

The entries in the vector $\boldsymbol{u}$ are the coefficients in the weighted sum of the basis functions u_h, but since $\phi_j(x_i) = \delta_{ij}$, $\boldsymbol{u}$ gives the values of u_h in each node.

3 Numerical experiments

Gaussian quadrature was implemented in both 1D and 2D with up to 4 integration points. This was tested on the integrals

$$\int_1^2 e^x \, dx \tag{10}$$

$$\iint_\Omega \log(x+y) \, dx \, dy \tag{11}$$

where Ω is the triangle defined by the corner points (1,0), (3,1) and (3,2). Table 1 shows the difference between the exact and the apporoximated solution. Notice that the error becomes smaller with more integration points. Gaussian quadrature for line integrals in $\mathbb{R}^2$ as described in the theory section was also implemented.

	1	2	3	4
(10)	1.89e-1	1.05e-3	2.24e-06	2.54e-09
(11)	3.86e-2	—	7.57e-3	2.50e-3

Table 1: Difference between numerical and anlytical solution when computing the integrals (10) and (11) with Gaussian quadrature, with up to 4 integration points.

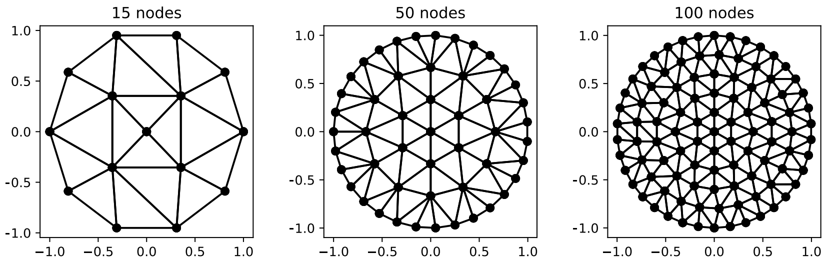
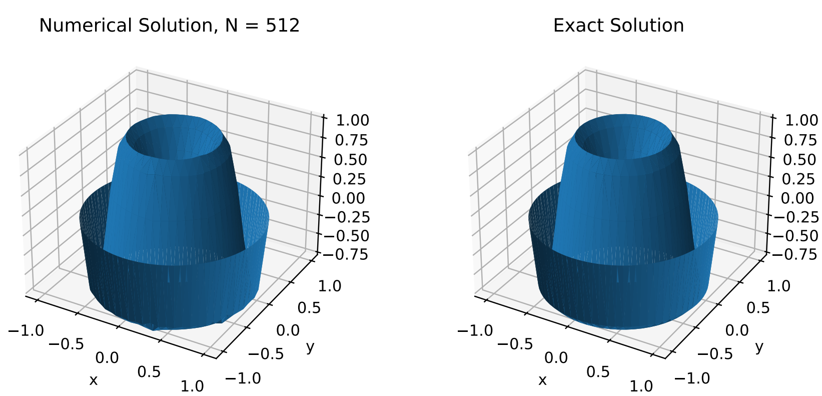


Figure 1: Meshes covering the unit disk with 15, 50 and 100 nodes.

The Poisson problem on the unit disk is solved using a triangulation like those depicted in Figure 1. The solution is approximated as described in the section "Galerkin projection" by solving (9). A generel method for creating A and F as described without considering boundary conditions is implemented. The gaussian quadrature already implemented is used to evaluate the definite integrals in $a(\phi_i, \phi_j), l(\phi_i)$. Here, we use $l(\phi_i)$ as in equation (6).

This procedure gives a matrix A with condition number $> 1/\eta$, where $\eta \ll 1$ is the machine limit for floating point types. I practice, this means that A i singular. This is as expected, since not considering boundary conditions actually corresponds to using $\partial_n u = 0$. With pure Neumann conditions the problem does not have a unique solution.

Homogeneous Dirichlet boundary conditions are implemented using the Big-Number approach[2] by setting $\boldsymbol{A}_{ii} = 1/\epsilon$ where $\epsilon << 1$ and $f_i = 0$ where i are the indices of edge nodes. This forces u_i to be ≈ 0 at these nodes.

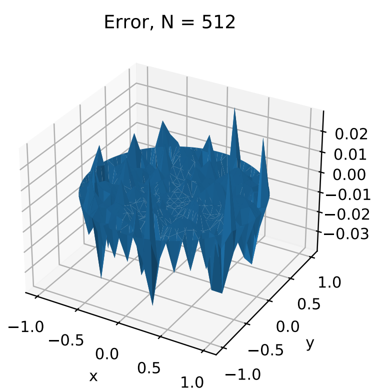


(a) Numerical solution with 512 nodes.

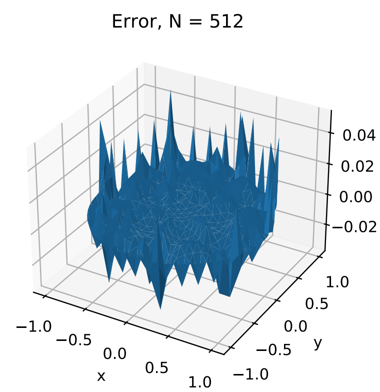
(b) Exact solution.

Figure 2: Solution plots for the Poisson problem in 2D on the unit disk with homogeneous Dirichlet boundary conditions.

We test the implementation by solving the problem (1) with f as in (2) on the unit disc. The numerical solution is shown next to the exact solution (4) in figure 2 and the error is shown in figure 3(a). The numerical solution is close to the analytical solution. Increasing N improved the result. The error is largest where the function has its minimum.



(a) Homogeneous Dirichlet on $\partial\Omega = \{(x, y) : x^2 + y^2 = 1\}$



(b) Neumann on $\partial\Omega_N = \{(x, y) : x^2 + y^2 = 1, y > 0\}$, Dirichlet on $\partial\Omega_D = \{(x, y) : x^2 + y^2 = 1, y \leq 0\}$

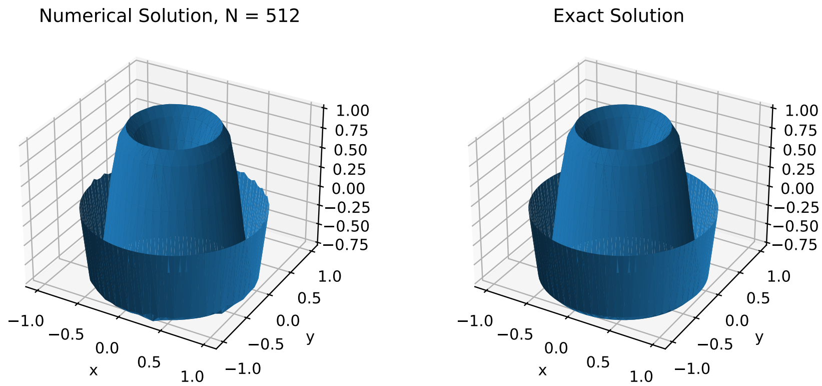
Figure 3: Error given by $u_{num} - u_{ex}$ for the solution of the Poisson problem in 2D on the unit disk with 512 nodes.

When using Neumann conditions on part of the boundary, $l(\phi_i)$ should be as in (7). For nodes not on $\partial\Omega_N$, this makes no difference, as the integral $\int_{\partial\Omega_N} \phi_i \cdot g(x, y)\, ds$ is zero for their corresponding basis function. However, for the nodes at the Neumann boundary, the integral is not zero. This is fixed by setting $F_i = F_i + \int \phi_i \cdot g\, ds$ where i is the index of egde nodes with Neumann conditions. Only the line integrals over the edges connected to given node needs to be calcualted as $\phi_i = 0$ otherwise. This integral is found using Gaussian quadrature. Note that we assume that $\partial_n u = g$ on the boundary of the numerical domain given by the triangulation. To test the implementation the problem (1) is solved on the unit disk with f and g as in (2) and (3) respectively with $\partial\Omega_N = \{(x, y) : x^2 + y^2 = 1, y > 0\}$ and $\partial\Omega_D = \{(x, y) : x^2 + y^2 = 1, y \leq 0\}$.

The numerical solution is shown next to the exact solution (4) in figure 4 and the error is shown in 3(b). The numerical solution in the interior is similar to the solution using Dirichlet conditions on the whole boundary. At the boundary with Neumann conditions, the error is larger than with only Dirichlet.

References

[1] Project description: tma4220-project1.pdf



(a) Numerical solution with 512 nodes.

(b) Exact solution.

Figure 4: Solution plots for the Poisson problem in 2D on the unit disk with Neumann boundary conditions on $\partial\Omega_N = \{(x, y) : x^2 + y^2 = 1, y > 0\}$ and Dirichlet boundary conditions on $\partial\Omega_D = \{(x, y) : x^2 + y^2 = 1, y \leq 0\}$.

[2] *Discretization of the Poisson Problem in R1: Theory and Implementation*