Fibre-preserving symmetries of time-invariant models in mathematical biology acting on the phase plane

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1 Introduction

There are numerous examples of biological systems that give rise to oscillatory behaviour. Some of these are the dynamics of a population consisting of predators and prey, chemical reactions such as the Belusov-Zhabotinskii reaction or the so called Brusselator reaction proposed by Prigogene and Lefever. In these cases, the oscillations have been modelled by mathematical models consisting of a two state system of first order ODEs describing the change in either populations sizes or the concentrations of chemical species over time. Mathematically, these oscillations have been described in terms of linear stability analysis, where the models are firstly linearised around their steady states and then the stability of these local linear systems is determined. However, this analytical tool can merely answer questions about the long term behaviour of the system and it does not reveal what properties that are conserved during these oscillations.

A well-known mathematical technique for deriving conserved properties in theoretical physics is that of symmetry methods. Symmetries are transformations which preserve the defining property of the objects they act on, and in the context of ODEs a symmetry is an operator mapping a solution curve to another solution curve. Moreover, symmetries have been used with huge success in theoretical physics to describe physical entities in terms of conservation laws. Here, we aim to apply these techniques on oscillatory dynamical systems modelled by a two state system of first order ODEs. In particular, we will focus on three models, namely the Lotka-Volterra (LV) model, the Belusov-Zhabotinskii (BZ) model and the so called Brusselator. The (dimensionless) LV model is given by

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\tau} &= u(1-v), \\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \alpha v(u-1). \end{aligned} \tag{1}$$

and it describes the "predator-prey" dynamics of the evolution of prey $u(\tau)$ and predators $v(\tau)$ at dimensionless time τ. Moreover, the BZ model is given by

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \frac{1}{\varepsilon}v - \frac{1}{\varepsilon}\left(\frac{1}{3}u^3 - u\right), \\
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= -u.
\end{aligned} \tag{2}
$$

describing the formation and degradation of the two chemical species $u(\tau)$ and $v(\tau)$ at time τ. Similarly, another oscillatory chemical system is the so called Brusselator given by

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= 1 - (b-1)u + au^2 v, \\
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= bu - au^2 v.
\end{aligned} \tag{3}
$$

where $u(\tau)$ and $v(\tau)$ are two chemical species. All these three systems can give rise to oscillatory behaviour, and our aim of this work is to be able to characterise oscillations in two state dynamical systems of first order ODEs in terms of their symmetries. Inspired by these three systems, we will restrict our focus to *autonomous* systems, specifically *time-invariant* systems, where the right hand sides are given by polynomials in the states u and v. For these type of systems, it is well-known that oscillations correspond to closed trajectories in the (u, v)-phase plane and therefore we will furthermore restrict our analysis to symmetries that act exclusively on the (u, v)-phase plane meaning that these symmetries are independent of time. Subsequently, we will derive the general equation defining fibre-preserving symmetries restricted to the (u, v)-phase plane, after that we will present the symmetries of each of these models and lastly we will interpret their meaning by deriving the differential invariants associated with these symmetries.

2 Mathematical theory of symmetries of differential equations

Here, we will present a condensed version of the mathematical theory of symmetries of differential equations. For the interested reader, there are many excellent introductory texts [1, 2, 3, 4, 5, 6] and here we will focus on general two component of first order ODEs

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \omega_1(t,u,v),$$
$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = \omega_2(t,u,v), \tag{4}$$

describing the time evolution of either the concentration profile of two proteins $u(\tau)$ and $v(\tau)$ or the evolution of two populations. Given this system, we subsequently will present the notions of Lie-transformations, their prolongations and symmetries of differential equations.

2.1 Lie groups of infinitesimal transformations

Here, a one parameter Lie-symmetry meaning a one parameter $\mathcal{C}^\infty$-diffeomorphism is a transformation

$$(\hat{\tau}(t,u(\tau),v(\tau);\epsilon),\hat{u}(t,u(\tau),v(\tau);\epsilon),\hat{v}(t,u(\tau),v(\tau);\epsilon)) = (\phi_1(\tau,u,v;\epsilon),\phi_2(\tau,u,v;\epsilon),\phi_3(\tau,u,v;\epsilon)) \tag{5}$$

parameterised by the parameter ϵ where the transformation is defined by the infinitely differentiable functions $\phi_i \in \mathcal{C}^\infty(\mathbb{R}^4), \quad i=1,2,3$. Now, we will parametrise such symmetries in terms of ϵ in the following way

$$\Gamma_\epsilon : (\tau,u(\tau),v(\tau)) \mapsto (\hat{\tau}(\epsilon),\hat{u}(\epsilon),\hat{v}(\epsilon)) \tag{6}$$

and the characterising feature of a symmetry of a differential equation is that if $(\tau,u(\tau),v(\tau))$ is a solution to the system of ODEs in Equation (4) then so is $(\hat{\tau}(\epsilon),\hat{u}(\epsilon),\hat{v}(\epsilon))$. The set of such one parameter Lie-transformations G together with a multiplication operation $\times$, forms a *one parameter Lie group of transformations* denoted by $(G,\times)$. Such a Lie-group has three defining properties:

1. *Multiplication*: For two transformation parameters $\epsilon,\delta \in \mathbb{R}$, multiplication of symmetries (meaning that we first transform with δ and then with ϵ) is defined by: $\Gamma_\epsilon \times \Gamma_\delta = \Gamma_{\epsilon+\delta}$,

2. *Identity element*: The trivial symmetry $\Gamma_0 = \Gamma_{\epsilon=0}$ acts trivially on curves: $\Gamma_0(\tau,u(\tau),v(\tau)) = (\tau,u(\tau),v(\tau))$,

3. *Inverse element*: The inverse symmetry Γ_ϵ^{-1} is defined by $\Gamma_\epsilon^{-1} = \Gamma_{-\epsilon}$.

Moreover, by the continuity of these transformations we can Taylor expand Γ_ϵ around $\epsilon \approx 0$ which gives us

$$\hat{\tau} = \tau + \xi(\tau, u, v)\epsilon + \mathcal{O}(\epsilon^2), \tag{7}$$

$$\hat{u} = u + \eta_1(\tau, u, v)\epsilon + \mathcal{O}(\epsilon^2), \tag{8}$$

$$\hat{v} = v + \eta_2(\tau, u, v)\epsilon + \mathcal{O}(\epsilon^2), \tag{9}$$

and here the tangents ξ, η_1 and η_2 are referred to as the *infinitesimals* which, in turn, are defined as follows:

$$\xi(\tau, u, v) = \left.\frac{\partial \phi_1(\tau, u, v; \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \tag{10}$$

$$\eta_1(\tau, u, v) = \left.\frac{\partial \phi_2(\tau, u, v; \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \tag{11}$$

$$\eta_2(\tau, u, v) = \left.\frac{\partial \phi_3(\tau, u, v; \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \tag{12}$$

where the functions $\phi_1, \phi_2, \phi_3 \in \mathcal{C}^\infty(\mathbb{R}^4)$ define the symmetry according to Equation (5). Now, one of the most powerful results from the theory of Lie-symmetries is that the global behaviour of the symmetry in Equation (6) can be retrieved from the local behaviour in terms of the infinitesimals in Equations (10) to (12). More precisely, the vector field defined by

$$X = \xi(\tau, u, v)\partial_t + \eta_1(\tau, u, v)\partial_u + \eta_2(\tau, u, v)\partial_v \tag{13}$$

is referred to as the *infinitesimal generator of the Lie group*. Using this vector field, *Lie's first fundamental theorem* [1] says that the symmetry in Equation (6), is in fact given by

$$\Gamma_\epsilon : (\tau, u, v) \mapsto \left(e^{\epsilon X}\tau, e^{\epsilon X}u, e^{\epsilon X}v\right) \tag{14}$$

where the exponential map is defined as follows

$$e^{\epsilon X} = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} X^j. \tag{15}$$

Accordingly, it is enough to know the local behaviour represented by the infinitesimal generator of the Lie group X in Equation (13) in order to also know the global behaviour of the symmetry according to Equation (14). Thus, to find the symmetries we must find the infinitesimals which is possible owing to the fact that the defining property of symmetries can be expressed in terms of their local action. Before, we are able to present the notion of a symmetry of a differential equation, we must introduce the idea of so called extended transformations or prolongations.

2.2 Prolongations: extended transformations

Our aim is to mathematically define a symmetry of a differential equation as a Lie-transformation that maps a solution curve of the differential equation to another solution curve. More precisely, given the Lie transformation

$$\Gamma_\epsilon : (\tau, u(\tau), v(\tau)) \mapsto (\hat{t}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon))$$

we would say that Γ_ϵ is a symmetry of the system of differential equations in Equation (4) if it maps a solutions curve $(\tau, u(\tau), v(\tau))$ of this system of ODEs to another solution curve $(\hat{t}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon))$. Here, $\tau \in \mathcal{B} \sim \mathbb{R}$ is called the *independent variable* and it defines the so called *base space* $\mathcal{B}$ of the symmetry Γ_ϵ. Also, the states $(u(\tau), v(\tau)) \in F \sim \mathbb{R}^2$ are called the *dependent variables* and they define the so called *fibre* F of the symmetry Γ_ϵ. Given these spaces, the symmetry Γ_ϵ acts on the so called *total space* E defined as $E = B \times F \sim \mathbb{R}^3$, and thus we have that $\Gamma_\epsilon : E \mapsto E$. Now, a transformation acting on solutions to differential equations must account for the derivatives of the states, e.g. $u'(\tau)$ and $v'(\tau)$, and to this end we introduce the notion of extended transforamtions.

There is a natural extension of a Lie-symmetry Γ_ϵ referred to as the prolongation of the symmetry and it is defined by the derivatives of the states. More precisely, *the first prolongation of the Lie-symmetry* $\Gamma_\epsilon^{(1)}$ is defined as follows

$$\Gamma_\epsilon^{(1)} : (\tau, u(\tau), v(\tau), u'(\tau), v'(\tau)) \mapsto (\hat{t}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon), \hat{u}'(\epsilon), \hat{v}'(\epsilon)) \tag{16}$$

where the derivatives of the states are defined by $u'(\tau) = \omega_1(t, u, v)$ and $v'(\tau) = \omega_1(t, u, v)$ according to Equation (4). Here, it is not entirely clear how the derivatives $\hat{u}'(\epsilon)$ and $\hat{v}'(\epsilon)$ are defined, and to this end we need to introduce the notion of the *total derivative* D_τ. This operator is defined as follows

$$D_\tau = \partial_\tau + u'(\tau)\partial_u + v'(\tau)\partial_v \tag{17}$$

and given the total derivative the derivatives of the transformed coordinates are defined as follows

$$\begin{aligned} \hat{u}'(\epsilon) &= \frac{D_\tau \hat{u}(\tau, u, v; \epsilon)}{D_\tau \hat{\tau}(\tau, u, v; \epsilon)}, \\ \hat{v}'(\epsilon) &= \frac{D_\tau \hat{v}(\tau, u, v; \epsilon)}{D_\tau \hat{\tau}(\tau, u, v; \epsilon)}. \end{aligned} \tag{18}$$

Moreover, the derivatives $(u'(\tau), v'(\tau)) \in F' \sim \mathbb{R}^2$ define the *prolonged fibre* F', and *the first jet space* $\mathcal{J}^{(1)}$ is defined by $\mathcal{J}^{(1)} = E \times F'$. Given the jet space, the prolonged symmetry can be succintly written in the following way $\Gamma_\epsilon^{(1)} : \mathcal{J}^{(1)} \mapsto \mathcal{J}^{(1)}$. Also, the operator $\Gamma_\epsilon \mapsto \Gamma_\epsilon^{(1)}$ is well-defined and is referred to as the *lift* of the symmetry Γ_ϵ. Previously, we showed that the infinitesimal action of the symmetry Γ_ϵ is expressed by the infinitesimal generator of the Lie group X in Equation (13), and similarly there is an infinitesimal representation of the prolonged symmetry $\Gamma_\epsilon^{(1)}$.

Locally, we can describe the action of the first prolongation of the symmetry $\Gamma_\epsilon^{(1)}$ by the *first prolongation of the infinitesimal generator of the Lie group* $X^{(1)}$. This operator is defined as follows

$$X^{(1)} = X + \eta_1^{(1)}(\tau, u, v)\partial_{u'} + \eta_2^{(1)}(\tau, u, v)\partial_{v'} \tag{19}$$

and here the prolonged infinitesimals $\eta_1^{(1)}$ and $\eta_2^{(1)}$ respectively are given by the *prolongation formula*

$$\eta_i^{(1)}(\tau, u, v) = D_\tau \eta_i(\tau, u, v) - \omega_i(\tau, u, v) D_\tau \xi(\tau, u, v), \quad i = 1, 2. \tag{20}$$

Now, given the notion of prolongations, we are now able to formulate the conditions that define a symmetry of a differential equation.

2.3 Symmetries of differential equations

Consider a one parameter Lie transformation $\Gamma_\epsilon : (\tau, u(\tau), v(\tau)) \mapsto (\hat{\tau}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon))$. Then, this transformation is a symmetry of the system of differential equations in Equation (4) if it maps a solution curve $(\tau, u(\tau), v(\tau))$ to another solution curve $(\hat{\tau}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon))$. Using the notions of jet spaces and prolongations, it can be shown that a Lie-transformation Γ_ϵ is a symmetry of the system of differential equations in Equation (4) if and only if the following so called *symmetry conditions* hold

$$\begin{aligned} u'(\epsilon) &= \omega_1(\hat{\tau}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon)) \quad \text{whenever} \quad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \omega_1(\tau, u, v), \\ v'(\epsilon) &= \omega_2(\hat{\tau}(\epsilon), \hat{u}(\epsilon), \hat{v}(\epsilon)) \quad \text{whenever} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \omega_2(\tau, u, v), \end{aligned} \tag{21}$$

where the derivatives $u'(\epsilon)$ and $v'(\epsilon)$ are defined by Equation (18). In general, it is difficult to use these symmetry conditions and instead one can formulate the same condition in terms of the infinitesimal action of the prolonged symmetry.

More precsely, a Lie transformation Γ_ϵ is a symmetry of the system of differential equations in Equation (4) if and only if the *linearised symmetry conditions* given by

$$\begin{aligned} X^{(1)}\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \omega_1(\tau, u, v)\right) &= 0 \quad \text{whenever} \quad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \omega_1(\tau, u, v), \\ X^{(1)}\left(\frac{\mathrm{d}v}{\mathrm{d}t} - \omega_2(\tau, u, v)\right) &= 0 \quad \text{whenever} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \omega_2(\tau, u, v), \end{aligned} \tag{22}$$

are satisfied. By the linearity of the prolonged generator $X^{(1)}$ in Equation (19), these equations can in fact be written as follows [7]:

$$\begin{aligned} \eta_1^{(1)}(\tau, u, v) &= X(\omega_1(\tau, u, v)) \quad \text{whenever} \quad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \omega_1(\tau, u, v), \\ \eta_2^{(1)}(\tau, u, v) &= X(\omega_2(\tau, u, v)) \quad \text{whenever} \quad \frac{\mathrm{d}v}{\mathrm{d}\tau} = \omega_2(\tau, u, v), \end{aligned} \tag{23}$$

where the prolonged tangents in the left hand sides are given by the prolongation formula in Equation (20). In general, the symmetries of a given differential equation are found by solving the linearised symmetry conditions for the infinitesimals and then the symmetry is retrieved using the exponential map. Next, we will focus on a common class of ODEs in mathematical biology, namely that of autonomous models and specifically time-invariant models.

3 Fibre-preserving symmetries of time-invariant models acting on the phase-plane

A common class of models in Mathematical Biology is that of two state time-invariant models. For example, these models typically describe the evolution of two competing populations or the evolution of two reacting proteins. To this end, we will now consider the following autonomous (specifically time-invariant) two state system of first order ODEs

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \omega_1(u,v),\\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \omega_2(u,v).\end{aligned} \tag{24}$$

where the right hand sides $\omega_1(u,v)$ and $\omega_2(u,v)$ referred to as the *reaction terms* are independent of the time τ. For these types of systems, there are two well-known symmetries. The first one is the time translation symmetry

$$\begin{gathered}\Gamma_\epsilon : (\tau,u,v) \mapsto (\tau+\epsilon,u,v)\\ \text{generated by}\\ X = \partial_\tau\end{gathered} \tag{25}$$

and this symmetry is common to all autonomous models. The second symmetry is the *trivial symmetry* generated by the infinitesimal generator of the Lie group given by

$$X = \partial_\tau + \omega_1(u,v)\partial_u + \omega_2(u,v)\partial_v \tag{26}$$

and this vector field is parallel to the vector field of the original system of ODEs in Equation (24). Consequently, this symmetry maps points on a solution curve onto other points on the *same solution curve*. Note that the infinitesimals in the time direction ξ of these two generators are *independent of the states* u and v meaning that these generators do not mix the time and state dependence. Such symmetries are called *fibre preserving*, and next we are interested in such symmetries that are restricted to the (u,v)-phase plane.

Definition 3.1 (Fibre preserving symmetries of time-invariant models restricted to the phase plane)**.** Consider the autonomous system of ODEs in Equation (24). A *fibre-preserving symmetry* Γ_ϵ that is restricted to the (u,v)-phase plane is a symmetry that only acts on the fibre:

$$\Gamma_\epsilon : (\tau, u(\tau), v(\tau)) \mapsto (\tau, \hat{u}(u,v;\epsilon), \hat{v}(u,v;\epsilon)). \tag{27}$$

Moreover, its infinitesimal generator of the Lie group X lacks an infinitesimal in the τ-direction, i.e. $\xi \equiv 0$, and it is given by

$$X = \eta_1(u,v)\partial_u + \eta_2(u,v)\partial_v. \tag{28}$$

Next, we demonstrate that the calculations of these fibre preserving symmetries of time-invariant models restricted to the phase plane are straightforward. This is due to the fact that the two linearised symmetry conditions in Equation (22), are condensed into a single solvable PDE (Thm 3.1) in this case.

Theorem 3.1 (The linearised symmetry condition of fibre preserving symmetries of time-invariant models restricted to the phase plane.)**.** *Consider the time-invariant system of ODEs in Equation* (24)*. Further, let* Γ_ϵ *be a fibre preserving symmetry of this model that is restricted to the* (u, v)*-phase plane according to Definition 3.1 and let the corresponding infinitesimal generator of the Lie group* X *be given by Equation* (28)*. Then, the infinitesimals* $\eta_1(u, v)$ *and* $\eta_2(u, v)$ *defining these symmetries solve the single PDE given by*

$$\omega_1^2\frac{\partial\eta_2}{\partial u}+\omega_1\omega_2\left(\frac{\partial\eta_2}{\partial v}-\frac{\partial\eta_1}{\partial u}\right)-\omega_2^2\frac{\partial\eta_1}{\partial v}=\left(\frac{\partial\omega_2}{\partial u}\omega_1-\omega_2\frac{\partial\omega_1}{\partial u}\right)\eta_1+\left(\frac{\partial\omega_2}{\partial v}\omega_1-\omega_2\frac{\partial\omega_1}{\partial v}\right)\eta_2. \quad (29)$$

. . . .

Proof. The dynamics in the (u, v)-phase plane of Equation (24) is described by a single first order ODE:

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\Omega(u,v)=\frac{\omega_2(u,v)}{\omega_1(u,v)}. \quad (30)$$

Moreover, the total derivative for the phase plane is given by $D_u = \partial_u + (\mathrm{d}v/\mathrm{d}u)\partial_v$, and the linearised symmetry condition according to Equation (23) is given by

$$D_u\eta_2-\Omega D_u\eta_1=\frac{\partial\Omega}{\partial u}\eta_1+\frac{\partial\Omega}{\partial v}\eta_2 \quad \text{whenever} \quad \frac{\mathrm{d}v}{\mathrm{d}u}=\Omega(u,v). \quad (31)$$

The left hand side of the above equation can be written as:

$$D_u\eta_2-\Omega D_u\eta_1=\frac{\partial\eta_2}{\partial u}+\Omega\left(\frac{\partial\eta_2}{\partial v}-\frac{\partial\eta_1}{\partial u}\right)-\Omega^2\frac{\partial\eta_1}{\partial v} \quad \text{whenever} \quad \frac{\mathrm{d}v}{\mathrm{d}u}=\Omega(u,v).$$

The partial derivatives in the right hand side of Equation (31) are given by

$$\frac{\partial\Omega}{\partial u}=\frac{\frac{\partial\omega_2}{\partial u}\omega_1-\omega_2\frac{\partial\omega_1}{\partial u}}{\omega_1^2},$$
$$\frac{\partial\Omega}{\partial v}=\frac{\frac{\partial\omega_2}{\partial v}\omega_1-\omega_2\frac{\partial\omega_1}{\partial v}}{\omega_1^2}.$$

By plugging in these partial derivatives into the right hand side of Equation (31), equating the resulting expression with the left hand side and lastly multiplying the resulting equation with the factor ω_1^2 results in the PDE in Equation (29). $\square$

. . . .

Morover, a subgroup of time-invariant models as in Equation (24) that are common in mathematical biology are definied by polynomial reaction terms ω_1 and ω_2 respectively. In this case the linearised symmetry condition in Equation (29) decomposes into a system of coupled first order PDEs where the number of equations depends on the degree of the reaction terms.

Corollary 3.1.1 (An algorithm for solving the linearised symmetry condition in the case of polynomial reaction terms.)**.** Consider the time-invariant system of first order ODEs in Equation (24) where the reaction terms $\omega_1(u,v)$ and $\omega_2(u,v)$ are two polynomials of degree d_1 and d_2 respectively. Furthermore, assume that two polynomial ansätze are used for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$ where the degree of these ansätze are d_3 and d_4 respectively. Then, the linearised symmetry condition in Equation (29) decomposes into a system of $d \in \mathbb{N}_+$ non-linear algebraic equations for the unknown coefficients in the polynomial ansätze for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$ where the number of equations are bounded by $0 \leq d \leq D$ where the upper bound is given by

$$D = \binom{2+\delta}{\delta} = \frac{(2+\delta)!}{2!\delta!}, \quad \delta = \max\left(2\,\max\left(d_1, d_2\right), d_3, d_4\right). \tag{32}$$

. . . .

Proof. In the case when the reaction terms $\omega_1(u,v)$ and $\omega_2(u,v)$ are polynomials of degree d_1 and d_2 respectively and where two polynomial ansätze of degree d_3 and d_4 are used for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$, the linearised symmetry condition in Equation (29) corresponds to finding the roots of a multivariate polynomial in two varibles, namely u and v. The degree of this polynomial will either be determined by the term ω_1^2, the term ω_2^2 or the degrees d_3 and d_4. Therefore, the number of monomials composed of u and v in this polynomial is given by Equation (32). Since these monomials are linearly independent it follows that all their coefficients in Equation (29) equal zero. Lastly, all coefficients of the monomials composed of u and v are non-linear equations involving the unknown constants in the polynomial ansätze for $\eta_1(u,v)$ and $\eta_2(u,v)$, and hence the claim of the corollary follows. $\square$

. . . .

Remark. Note that the family of trivial generators are given by

$$X_0 = f(u,v)\left[\omega_1(u,v)\partial_u + \omega_2(u,v)\partial_v\right]$$

where f is an arbitrary function [1]. Thus, if we choose the degrees d_3 and d_4 to be larger than $\max(d_1, d_2)$, we are guaranteed to find trivial generators.

Subsequently, we will try to solve this linearised symmetry condition for the three models that were presented previously starting with the LV-model.

4 The fibre-preserving symmetries of the LV-model restricted to the phase plane

The infinitesimal generators of the Lie group that where found with polynomial ansätze of degree $d_3 = d_4 = 5$ for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$ were the following:

$$
\begin{aligned}
X_1 &= -\frac{uv^2(u+1)(v-1)}{a}\partial_u + v^3(u-1)(u+1)\partial_v,\\
X_2 &= -\frac{uv(v-1)(u^2+u+1)}{a}\partial_u + v^2(u-1)\left(u^2+u+1\right)\partial_v,\\
X_3 &= -\frac{u(v-1)(u^2+u+1)}{a}\partial_u + v(u-1)\left(u^2+u+1\right)\partial_v,\\
X_4 &= -\frac{u(v-1)}{a}\partial_u + v(u-1)\partial_v,\\
X_5 &= -\frac{uv^2(v-1)}{a}\partial_u + v^3(u-1)\partial_v,\\
X_6 &= -\frac{uv^3(v-1)}{a}\partial_u + v^4(u-1)\partial_v,\\
X_7 &= -\frac{uv(v-1)}{a}\partial_u + v^2(u-1)\partial_v,\\
X_8 &= -\frac{u(u+1)(u^2+1)(v-1)}{a}\partial_u + v(u-1)(u+1)\left(u^2+1\right)\partial_v,\\
X_9 &= -\frac{u(u+1)(v-1)}{a}\partial_u + v(u-1)(u+1)\partial_v,\\
X_{10} &= -\frac{uv(u+1)(v-1)}{a}\partial_u + v^2(u-1)(u+1)\partial_v.
\end{aligned}
$$

Note that the reaction terms are:

$$
\begin{aligned}
\omega_1(u,v) &= u(1-v),\\
\omega_2(u,v) &= av(u-1),
\end{aligned}
$$

so all these generators are trivial.

5 The fibre-preserving symmetries of the BZ-model restricted to the phase plane

The infinitesimal generators of the Lie group that where found with polynomial ansätze of degree $d_3 = d_4 = 5$ for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$ were the following:

$$
\begin{aligned}
X_1 &= \frac{u\left(u^3-3u-3v\right)}{3e}\partial_u + u^2\partial_v,\\
X_2 &= \frac{v^2\left(u^3-3u-3v\right)}{3e}\partial_u + uv^2\partial_v,\\
X_3 &= \frac{uv\left(u^3-3u-3v\right)}{3e}\partial_u + u^2v\partial_v,\\
X_4 &= \frac{u^3-3u-3v}{3e}\partial_u + u\partial_v,\\
X_5 &= \frac{v\left(u^3-3u-3v\right)}{3e}\partial_u + uv\partial_v,\\
X_6 &= \frac{u^2\left(u^3-3u-3v\right)}{3e}\partial_u + u^3\partial_v.
\end{aligned}
$$

Note that the reaction terms are:

$$
\begin{aligned}
\omega_1(u,v) &= \frac{v}{e} - \frac{\frac{u^3}{3}-u}{e},\\
\omega_2(u,v) &= -u.
\end{aligned}
$$

so all these generators are trivial.

6 The fibre-preserving symmetries of the Brusselator restricted to the phase plane

The infinitesimal generators of the Lie group that where found with polynomial ansätze of degree $d_3 = d_4 = 5$ for the infinitesimals $\eta_1(u,v)$ and $\eta_2(u,v)$ were the following:

$$
\begin{aligned}
X_1 &= -\frac{u\left(au^2v - bu + u + 1\right)}{a}\partial_u + \frac{u^2\left(auv - b\right)}{a}\partial_v,\\
X_2 &= -\frac{v\left(au^2v - bu + u + 1\right)}{a}\partial_u + \frac{uv\left(auv - b\right)}{a}\partial_v,\\
X_3 &= -\frac{u^2\left(au^2v - bu + u + 1\right)}{a}\partial_u + \frac{u^3\left(auv - b\right)}{a}\partial_v,\\
X_4 &= -\frac{v^2\left(au^2v - bu + u + 1\right)}{a}\partial_u + \frac{uv^2\left(auv - b\right)}{a}\partial_v,\\
X_5 &= -\frac{uv\left(au^2v - bu + u + 1\right)}{a}\partial_u + \frac{u^2v\left(auv - b\right)}{a}\partial_v.
\end{aligned}
$$

Note that the reaction terms are:

$$
\begin{aligned}
\omega_1(u,v) &= au^2v - u\left(b-1\right) + 1,\\
\omega_2(u,v) &= -au^2v + bu.
\end{aligned}
$$

so all these generators are trivial.

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