We're studying the Lotka-Volterra model:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (1 - v(t))\, u(t),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = a\,(u(t) - 1)\, v(t).$$

We want to find an infinitesimal generator of the Lie group of the following kind:

$$X = \xi(t)\partial t + \eta_1(t, u, v)\partial_u + \eta_2\partial_v.$$

The infinitesimals or the tangents in this generator solves the following equations:

$$a\,(u-1)\,v\frac{\partial \eta_1}{\partial v} - (1-v)\,\eta_1 + (1-v)\,u\frac{\partial \eta_1}{\partial u} - (1-v)\,u\frac{d\xi}{dt} + \eta_2 u + \frac{\partial \eta_1}{\partial t} = 0,$$
$$-a\,(u-1)\,\eta_2 + a\,(u-1)\,v\frac{\partial \eta_2}{\partial v} - a\,(u-1)\,v\frac{d\xi}{dt} - a\eta_1 v + (1-v)\,u\frac{\partial \eta_2}{\partial u} + \frac{\partial \eta_2}{\partial t} = 0,$$

We use the following ansatze for the infinitesimals or the tangents:

$$\eta_1 = P_1 \exp\left(P_0\right),\ \eta_2 = P_2 \exp\left(P_0\right).$$

We use ansatze of the type

$$\eta_i = P_i \exp\left(P_0\right)$$

for some index $i \in \{1, \dots, n\}$ where n is the number of states. Here, our two polynomials P_1 and P_2 are multivariate polynomials of a given degree. In this case, the degree of our polynomials are: 2. Moreover, we have the following states or dependent variables:

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \tag{1}$$

and one independent variable t. Now, given all variables (independent and dependent) as well as the degree of the polynomials, we obtain the following monomials:

$$M = \begin{bmatrix} 1 \\ v(t) \\ v^2(t) \\ u(t) \\ u(t)v(t) \\ u^2(t) \\ t \\ tv(t) \\ tu(t) \\ t^2 \end{bmatrix}. \tag{2}$$

Furthermore, we have the following unknown coefficients:

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \\ c_{16} \\ c_{17} \\ c_{18} \\ c_{19} \\ c_{20} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \\ c_{25} \\ c_{26} \\ c_{27} \\ c_{28} \\ c_{29} \end{bmatrix}. \tag{3}$$

Now, given all of these unknowns and coefficient, this is the ansatze we get:

$\eta_1 = \xi(t),$

$\eta_2 = \left(c_{10} + c_{11}v(t) + c_{12}v^2(t) + c_{13}u(t) + c_{14}u(t)v(t) + c_{15}u^2(t) + c_{16}t + c_{17}tv(t) + c_{18}tu(t) + c_{19}t^2\right) e^{\frac{ac_{10}\left(c_{10}\left(c_0+c_2v^2(t)+c_3u(t)+c_4u(t)v(\right.\right.}{}}$

$\eta_3 = \left(c_{20} + c_{21}v(t) + c_{22}v^2(t) + c_{23}u(t) + c_{24}u(t)v(t) + c_{25}u^2(t) + c_{26}t + c_{27}tv(t) + c_{28}tu(t) + c_{29}t^2\right) e^{\frac{ac_{10}\left(c_{10}\left(c_0+c_2v^2(t)+c_3u(t)+c_4u(t)v(\right.\right.}{}}$

After plugging in these enormous ansatze, we organise these equations in terms of the various monomials (or linearly independent basis functions). Equations without coefficients in them:

$$\begin{aligned}
e^{-c_0}e^{-c_1v(t)}e^{-c_2v^2(t)}e^{-c_3u(t)}e^{-c_5u^2(t)}e^{-c_6t}e^{-c_9t^2}e^{-c_4u(t)v(t)}e^{-c_7tv(t)}e^{-c_8tu(t)} &: u(t)v(t)\frac{d}{dt}\xi(t) - u(t)\frac{d}{dt}\xi(t) &&= 0, \\
e^{-c_0}e^{-c_1v(t)}e^{-c_2v^2(t)}e^{-c_3u(t)}e^{-c_5u^2(t)}e^{-c_6t}e^{-c_9t^2}e^{-c_4u(t)v(t)}e^{-c_7tv(t)}e^{-c_8tu(t)} &: -au(t)v(t)\frac{d}{dt}\xi(t) + av(t)\frac{d}{dt}\xi(t) &&= 0,
\end{aligned}$$

Equations with coefficients in them:

$$
\begin{aligned}
1 :& c_{10}c_6 + c_{16} &= 0,\\
v(t) :& -ac_1c_{10} - ac_{11} + c_{10}c_7 + c_{10} + c_{11}c_6 - c_{11} + c_{17} &= 0,\\
v^2(t) :& -ac_1c_{11} - 2ac_{10}c_2 - 2ac_{12} + c_{11}c_7 + c_{11} + c_{12}c_6 - c_{12} &= 0,\\
u(t) :& c_{10}c_3 + c_{10}c_8 + c_{13}c_6 + c_{18} + c_{20} &= 0,\\
u(t)v(t) :& ac_1c_{10} - ac_1c_{13} - ac_{10}c_4 + ac_{11} - ac_{14} - c_{10}c_3 + c_{10}c_4 + c_{11}c_3 + c_{11}c_8 + c_{13}c_7 + c_{14}c_6 + c_{21} &= 0,\\
u^2(t) :& 2c_{10}c_5 + c_{13}c_3 + c_{13}c_8 + c_{15}c_6 + c_{15} + c_{23} &= 0,\\
t :& 2c_{10}c_9 + c_{16}c_6 - c_{16} + 2c_{19} &= 0,\\
tv(t) :& -ac_1c_{16} - ac_{10}c_7 - ac_{17} + 2c_{11}c_9 + c_{16}c_7 + c_{16} + c_{17}c_6 - c_{17} &= 0,\\
tu(t) :& c_{10}c_8 + 2c_{13}c_9 + c_{16}c_3 + c_{16}c_8 + c_{18}c_6 + c_{26} &= 0,\\
t^2 :& 2c_{16}c_9 + c_{19}c_6 - c_{19} &= 0,\\
1 :& ac_{20} + c_{20}c_6 + c_{26} &= 0,\\
v(t) :& -ac_1c_{20} - ac_{10} + c_{20}c_7 + c_{21}c_6 + c_{27} &= 0,\\
v^2(t) :& -ac_1c_{21} - ac_{11} - 2ac_2c_{20} - ac_{22} + c_{21}c_7 + c_{22}c_6 &= 0,\\
u(t) :& -ac_{20} + ac_{23} + c_{20}c_3 + c_{20}c_8 + c_{23}c_6 + c_{23} + c_{28} &= 0,\\
u(t)v(t) :& ac_1c_{20} - ac_1c_{23} - ac_{13} - ac_{20}c_4 - c_{20}c_3 + c_{20}c_4 + c_{21}c_3 + c_{21}c_8 + c_{23}c_7 - c_{23} + c_{24}c_6 + c_{24} &= 0,\\
u^2(t) :& -ac_{23} + ac_{25} + 2c_{20}c_5 + c_{23}c_3 + c_{23}c_8 + c_{25}c_6 + 2c_{25} &= 0,\\
t :& ac_{26} + 2c_{20}c_9 + c_{26}c_6 + 2c_{29} &= 0,\\
tv(t) :& -ac_1c_{26} - ac_{16} - ac_{20}c_7 + 2c_{21}c_9 + c_{26}c_7 + c_{27}c_6 &= 0,\\
tu(t) :& -ac_{26} + ac_{28} + c_{20}c_8 + 2c_{23}c_9 + c_{26}c_3 + c_{26}c_8 + c_{28}c_6 + c_{28} &= 0,\\
t^2 :& ac_{29} + 2c_{26}c_9 + c_{29}c_6 &= 0,
\end{aligned}
$$

The coefficients we calculated were:

$$\mathbf{c} = \begin{bmatrix} c_0 \\ \frac{-ac_{10}c_{11}+c_{10}^2(c_7+1)-c_{10}(c_{11}-c_{17})-c_{11}c_{16}}{ac_{10}^2} \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ -\frac{c_{16}}{c_{10}} \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \\ c_{16} \\ c_{17} \\ c_{18} \\ c_{19} \\ c_{20} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \\ c_{25} \\ c_{26} \\ c_{27} \\ c_{28} \\ c_{29} \end{bmatrix}. \tag{4}$$

The tangents we calculated were:

$$\xi = \xi(t),$$

$$\eta_1 = c_{10}e^{c_0}e^{\frac{v(t)}{a}}e^{c_2v^2(t)}e^{c_3u(t)}e^{c_5u^2(t)}e^{c_9t^2}e^{\frac{c_7v(t)}{a}}e^{-\frac{c_{11}v(t)}{c_{10}}}e^{-\frac{c_{16}t}{c_{10}}}e^{c_4u(t)v(t)}e^{c_7tv(t)}e^{c_8tu(t)}e^{-\frac{c_{11}v(t)}{ac_{10}}}e^{\frac{c_{17}v(t)}{ac_{10}}}e^{-\frac{c_{11}c_{16}v(t)}{ac_{10}^2}} + c_{11}v(t)e^{c_0}e^{\frac{v(t)}{a}}e^{c_2}$$

$$\eta_2 = c_{20}e^{c_0}e^{\frac{v(t)}{a}}e^{c_2v^2(t)}e^{c_3u(t)}e^{c_5u^2(t)}e^{c_9t^2}e^{\frac{c_7v(t)}{a}}e^{-\frac{c_{11}v(t)}{c_{10}}}e^{-\frac{c_{16}t}{c_{10}}}e^{c_4u(t)v(t)}e^{c_7tv(t)}e^{c_8tu(t)}e^{-\frac{c_{11}v(t)}{ac_{10}}}e^{\frac{c_{17}v(t)}{ac_{10}}}e^{-\frac{c_{11}c_{16}v(t)}{ac_{10}^2}} + c_{21}v(t)e^{c_0}e^{\frac{v(t)}{a}}e^{c_2}$$