linear_model_error_search_ODEsys_and_algSys

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1 Error search ODE system and algebra system Linear model

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This document contains an attempt of finding the error in the calculations of the symmetry generators for Hydon's model. The model at hand is the following two component ODE system:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u + v = \omega_1(t,u,v), \tag{1}$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = u + v = \omega_2(t,u,v). \tag{2}$$

To this model, the aim is to find the most general form of the *infinitesimal generator of the Lie group* denoted by X which is defined as follows:

$$X = \xi(t,u,v)\partial_t + \eta_1(t,u,v)\partial_u + \eta_2(t,u,v)\partial_v.$$

To find this generator, a set of *linear ansätze* is used for the three tangents as follows:

$$\xi(t,u,v) = c_{00}(t) + c_{01}(t)u + c_{02}(t)v, \tag{3}$$
$$\eta_1(t,u,v) = c_{10}(t) + c_{11}(t)u + c_{12}(t)v, \tag{4}$$
$$\eta_2(t,u,v) = c_{20}(t) + c_{21}(t)u + c_{22}(t)v. \tag{5}$$
$$\tag{6}$$

The aim is to find the nine arbitrary functions $c_{ij}(t)$ for the two indices $i,j \in \{0,1,2\}$. The equations required in order to find these constants are given by the two *linearised symmetry conditions* given by

$$X^{(1)}(y_k' - \omega_k(t,u,v)) = 0, \quad \text{for} \quad k \in \{1,2\}.$$

Here, $X^{(1)}$ corresponds to the prolonged generator given by

$$X^{(1)} = X + \eta_1^{(1)}\partial_{u'} + \eta_2^{(1)}\partial_{v'}$$

where the prolonged tangents are given by the *prolongation formula*:

$$\eta_k^{(1)} = D_t\eta_k - y'D_t\xi, \quad \text{for} \quad k \in \{1,2\}$$

What is nice about Hydon's model is that it has a known generator, namely the *scaling generator* given by

$$X = t\partial_t + u\partial_u + v\partial_v.$$

Thus, we now when the algorithm performs correctly in this case as the above generator should be returned as an output.

Moreover, plugging in these ansätze into the linearised symmetry conditions will result in a linear system of equations which can be formulated on matrix form as follows:

$$A\frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t} = B\mathbf{c}(t)$$

where the vector $\mathbf{c}(t) \in \mathcal{C}(\mathbb{R}^9)$ contains the nine arbitrary coefficients in the tangential ansätze. Typically, the number of equations are much larger than the number of unknowns meaning that if $A, B \in \mathcal{C}\left(\mathbb{R}^{n\times m}\right)$ then $n \gg m$ (in this case $m = 9$). After row reducing this system and simplifying it is (in the best of worlds) possible to write the system on the following form:

$$\frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t} = B\mathbf{c}(t), \tag{7}$$

$$B_{\mathrm{algebraic}}\mathbf{c}(t) = \mathbf{0}. \tag{8}$$

The first ODE system is a quadratic ODE system which can be solved using the Jordan decomposition. That is if

$$B = P^{-1}JP$$

then the solution to the ODE system is given by

$$\mathbf{c}(t) = P^{-1}e^{Jt}P\mathbf{c}_0$$

for some initial condition $\mathbf{c}_0$ composed of arbitrary integration constants. Then the solution of the system of ODEs is plugged in to the algebraic equations given by the second matrix equation above. This will result in certain algebraic equations that can simplify the results even further.

Now, the problem is that certain generators are obtained that do not solve the linearised symmetry conditions. This implies that the implementation of the algorithm is wrong, as the methodology of ansätze can never yield non-solutions. Therefore, the Hydon example will be used to see if the error is introduced in the solution of the ODE system or if it is when certain simplifications are made when the algebraic equations are solved.

What will be done in the subsequent cells is that all matrices will be printed out and then we will try to track down the error.

2 Defining the tangents

The tangents are:

$$\begin{aligned}\xi &= u\,\mathrm{c}_{01}(t) + v\,\mathrm{c}_{02}(t) + \mathrm{c}_{00}(t)\\ \eta_1 &= u\,\mathrm{c}_{11}(t) + v\,\mathrm{c}_{12}(t) + \mathrm{c}_{10}(t)\\ \eta_2 &= u\,\mathrm{c}_{21}(t) + v\,\mathrm{c}_{22}(t) + \mathrm{c}_{20}(t)\end{aligned}$$

The unknown coefficients:

$$\mathbf{c} = \begin{bmatrix}\mathrm{c}_{00}\\ \mathrm{c}_{02}\\ \mathrm{c}_{01}\\ \mathrm{c}_{10}\\ \mathrm{c}_{12}\\ \mathrm{c}_{11}\\ \mathrm{c}_{20}\\ \mathrm{c}_{22}\\ \mathrm{c}_{21}\end{bmatrix} \tag{9}$$

3 The linearised symmetry conditions

Linearised symmetry condition 1:

$$\begin{aligned}0 = &\left(-\frac{d}{dt}\,\mathrm{c}_{10}(t)\right) + \left(u\frac{d}{dt}\,\mathrm{c}_{00}(t)\right) + (u\,\mathrm{c}_{21}(t)) + \left(v\frac{d}{dt}\,\mathrm{c}_{00}(t)\right) + (v\,\mathrm{c}_{22}(t)) + \left(u^2\frac{d}{dt}\,\mathrm{c}_{01}(t)\right)\\ &+ \left(u^2\,\mathrm{c}_{01}(t)\right) + \left(u^2\,\mathrm{c}_{02}(t)\right) + \left(v^2\frac{d}{dt}\,\mathrm{c}_{02}(t)\right) + \left(v^2\,\mathrm{c}_{01}(t)\right) + \left(v^2\,\mathrm{c}_{02}(t)\right)\\ &+ \left(-u\frac{d}{dt}\,\mathrm{c}_{11}(t)\right) + (-u\,\mathrm{c}_{12}(t)) + \left(-v\frac{d}{dt}\,\mathrm{c}_{12}(t)\right) + (-v\,\mathrm{c}_{11}(t)) + \left(uv\frac{d}{dt}\,\mathrm{c}_{01}(t)\right)\\ &+ \left(uv\frac{d}{dt}\,\mathrm{c}_{02}(t)\right) + (2uv\,\mathrm{c}_{01}(t)) + (2uv\,\mathrm{c}_{02}(t)) + (\mathrm{c}_{10}(t)) + (\mathrm{c}_{20}(t))\end{aligned}$$

Linearised symmetry condition 2:

$$
\begin{aligned}
0 = &\left(-\frac{d}{dt}\,\mathrm{c}_{20}\left(t\right)\right) + \left(u\frac{d}{dt}\,\mathrm{c}_{00}\left(t\right)\right) + \left(u\,\mathrm{c}_{11}\left(t\right)\right) + \left(v\frac{d}{dt}\,\mathrm{c}_{00}\left(t\right)\right) + \left(v\,\mathrm{c}_{12}\left(t\right)\right) + \left(u^2\frac{d}{dt}\,\mathrm{c}_{01}\left(t\right)\right)\\
&+ \left(u^2\,\mathrm{c}_{01}\left(t\right)\right) + \left(u^2\,\mathrm{c}_{02}\left(t\right)\right) + \left(v^2\frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\right) + \left(v^2\,\mathrm{c}_{01}\left(t\right)\right) + \left(v^2\,\mathrm{c}_{02}\left(t\right)\right)\\
&+ \left(-u\frac{d}{dt}\,\mathrm{c}_{21}\left(t\right)\right) + \left(-u\,\mathrm{c}_{22}\left(t\right)\right) + \left(-v\frac{d}{dt}\,\mathrm{c}_{22}\left(t\right)\right) + \left(-v\,\mathrm{c}_{21}\left(t\right)\right) + \left(uv\frac{d}{dt}\,\mathrm{c}_{01}\left(t\right)\right)\\
&+ \left(uv\frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\right) + \left(2uv\,\mathrm{c}_{01}\left(t\right)\right) + \left(2uv\,\mathrm{c}_{02}\left(t\right)\right) + \left(\mathrm{c}_{10}\left(t\right)\right) + \left(\mathrm{c}_{20}\left(t\right)\right)
\end{aligned}
$$

4 The determining equations

Determining equations from linearised symmetry condition 1:

$$
\begin{aligned}
u^0v^0 &: 0 = \mathrm{c}_{10}\left(t\right) + \mathrm{c}_{20}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{10}\left(t\right)\\
u^0v^1 &: 0 = -\,\mathrm{c}_{11}\left(t\right) + \mathrm{c}_{22}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{00}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{12}\left(t\right)\\
u^0v^2 &: 0 = \mathrm{c}_{01}\left(t\right) + \mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\\
u^1v^0 &: 0 = -\,\mathrm{c}_{12}\left(t\right) + \mathrm{c}_{21}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{00}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{11}\left(t\right)\\
u^1v^1 &: 0 = 2\,\mathrm{c}_{01}\left(t\right) + 2\,\mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{01}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\\
u^2v^0 &: 0 = \mathrm{c}_{01}\left(t\right) + \mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{01}\left(t\right)
\end{aligned}
$$

Determining equations from linearised symmetry condition 2:

$$
\begin{aligned}
u^0v^0 &: 0 = \mathrm{c}_{10}\left(t\right) + \mathrm{c}_{20}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{20}\left(t\right)\\
u^0v^1 &: 0 = \mathrm{c}_{12}\left(t\right) - \mathrm{c}_{21}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{00}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{22}\left(t\right)\\
u^0v^2 &: 0 = \mathrm{c}_{01}\left(t\right) + \mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\\
u^1v^0 &: 0 = \mathrm{c}_{11}\left(t\right) - \mathrm{c}_{22}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{00}\left(t\right) - \frac{d}{dt}\,\mathrm{c}_{21}\left(t\right)\\
u^1v^1 &: 0 = 2\,\mathrm{c}_{01}\left(t\right) + 2\,\mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{01}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{02}\left(t\right)\\
u^2v^0 &: 0 = \mathrm{c}_{01}\left(t\right) + \mathrm{c}_{02}\left(t\right) + \frac{d}{dt}\,\mathrm{c}_{01}\left(t\right)
\end{aligned}
$$

5 Solving the determining equations

5.1 Step 1 of 6: the initial matrices

Dimension of matrices: 12X9
Matrix A

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Step 2 of 6: the reduced based on col(M^T) where M=[-A|B]
Dimension of matrices: 8X9
Matrix A

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{12}$$

Matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix} \tag{13}$$

Step 3 of 6: Splitting up to A, B and B_algebraic
Dimension of matrices A and B: 8X9
Dimension of matrices B_algebraic: 0X9
Matrix A

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{14}$$

Matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix} \tag{15}$$

Matrix B_algebraic

$$B_{\text{algebraic}} = [\,] \tag{16}$$

Coefficient matrix c:

$$\mathbf{c} = \begin{bmatrix} c_{00} \\ c_{02} \\ c_{01} \\ c_{10} \\ c_{12} \\ c_{11} \\ c_{20} \\ c_{22} \\ c_{21} \end{bmatrix} \tag{17}$$

Dimensions of A: 8X9

Dimensions of B_algebraic: 0X9

Non-Pivot columns [8] The non-homogeneous source term:

$$\left[\begin{matrix}- \frac{d}{d x_{0}} \operatorname{c_{21}}{\left(x_{0} \right)}\\0\\0\\0\\- \frac{d}{d x_{0}} \operatorname{c_{21}}{\left(x_{0} \right)}\\- \operatorname{c_{21}}{\left(x_{0} \right)} - \frac{d}{d x_{0}} \operatorname{c_{21}}{\left(x_{0} \right)}\\0\\\operatorname{c_{21}}{\left(x_{0} \right)} - \frac{d}{d x_{0}} \operatorname{c_{21}}{\left(x_{0} \right)}\end{matrix}\right]$$

Dimensions: (8, 1) The algebraic source term:

$$\left]\right[$$

Dimensions: (0, 1) ## Step 4 of 6: Removing potential extra pivot columns Dimension of matrices A and B: 8X8 Dimension of matrices B_algebraic: 0X8 Matrix A

$$A = \left[\begin{matrix}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\end{matrix}\right] \tag{18}$$

Matrix B

$$B = \left[\begin{matrix}0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & -1 & 0 & 0 & -1 & 0\\0 & 0 & 0 & 0 & 0 & 2 & 0 & -2\\0 & 0 & 0 & 0 & 1 & 1 & 0 & -1\\0 & 0 & 0 & -1 & 0 & 0 & -1 & 0\\0 & 0 & 0 & 0 & -1 & 1 & 0 & -1\end{matrix}\right] \tag{19}$$

Dimensions of B: 8X8 Coefficients:

$$\left[\begin{matrix}C_{1}\\C_{2}\\C_{3}\\C_{4}\\C_{5}\\C_{6}\\C_{7}\\C_{8}\end{matrix}\right] \tag{20}$$

Number of unknowns: 8 ## Step 5 of 6: Solving the ODE system Dimension of the matrix B: 8X8 Dimension of the matrix B_algebraic: 0X8 ODE system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathrm{c}_{00}\\\mathrm{c}_{02}\\\mathrm{c}_{01}\\\mathrm{c}_{10}\\\mathrm{c}_{12}\\\mathrm{c}_{11}\\\mathrm{c}_{20}\\\mathrm{c}_{22}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&1&0&-1\\0&1&1&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&-1&0&0&-1&0\\0&0&0&0&0&2&0&-2\\0&0&0&0&1&1&0&-1\\0&0&0&-1&0&0&-1&0\\0&0&0&0&-1&1&0&-1\end{bmatrix}\begin{bmatrix}\mathrm{c}_{00}\\\mathrm{c}_{02}\\\mathrm{c}_{01}\\\mathrm{c}_{10}\\\mathrm{c}_{12}\\\mathrm{c}_{11}\\\mathrm{c}_{20}\\\mathrm{c}_{22}\end{bmatrix}\tag{21}$$

Solve the ODE system: Initial conditions for $\mathbf{c}$ denoted by $\mathbf{c}_0$ in terms of arbitrary integration constants:

$$\mathbf{c}_0=\begin{bmatrix}\mathrm{c}_{00}\\\mathrm{c}_{02}\\\mathrm{c}_{01}\\\mathrm{c}_{10}\\\mathrm{c}_{12}\\\mathrm{c}_{11}\\\mathrm{c}_{20}\\\mathrm{c}_{22}\end{bmatrix}=\begin{bmatrix}C_1\\C_2\\C_3\\C_4\\C_5\\C_6\\C_7\\C_8\end{bmatrix}\tag{22}$$

Jordan form:

$$P=\begin{bmatrix}0&0&0&0&0&1&0&-1\\0&1&1&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&-1&0&0&-1&0\\0&0&0&0&0&2&0&-2\\0&0&0&0&1&1&0&-1\\0&0&0&-1&0&0&-1&0\\0&0&0&0&-1&1&0&-1\end{bmatrix}\tag{23}$$

$$J=\begin{bmatrix}-2&0&0&0&0&0&0&0\\0&-2&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2\end{bmatrix}\tag{24}$$

Exponential form:

$$\exp\left(J\cdot t\right)=\begin{bmatrix}e^{-2t}&0&0&0&0&0&0&0\\0&e^{-2t}&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&e^{2t}&0\\0&0&0&0&0&0&0&e^{2t}\end{bmatrix}\tag{25}$$

Solution to the ODE system:

$$P \exp\left(J \cdot t\right) P^{-1} \mathbf{c}_0 = \begin{bmatrix} C_1 + C_5 \left(\frac{e^{2t}}{4} - \frac{1}{2} + \frac{e^{-2t}}{4}\right) + C_6 \left(\frac{e^{2t}}{4} - \frac{e^{-2t}}{4}\right) + C_8 \left(-\frac{e^{2t}}{4} + \frac{e^{-2t}}{4}\right) \\ C_2 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) + C_3 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) \\ C_2 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) + C_3 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) \\ C_4 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_7 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(\frac{e^{2t}}{2} + \frac{e^{-2t}}{2}\right) + C_6 \left(\frac{e^{2t}}{2} - \frac{e^{-2t}}{2}\right) + C_8 \left(-\frac{e^{2t}}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) + C_6 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) + C_8 \left(\frac{1}{2} - \frac{e^{2t}}{2}\right) \\ C_4 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_7 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_6 \left(\frac{1}{2} - \frac{e^{-2t}}{2}\right) + C_8 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) \end{bmatrix} \tag{26}$$

Step 6 of 6: Solving the algebraic system Number of algebraic equations: 0

Matrix B_algebraic

$$B_{\text{algebraic}} =\!| \; [\tag{27}$$

Algebraic equations:

$$] \; [\begin{bmatrix} c_{00} \\ c_{02} \\ c_{01} \\ c_{10} \\ c_{12} \\ c_{11} \\ c_{20} \\ c_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

Algebraic equations after substitution of the solution to the ODE system:

$$] \; [\begin{bmatrix} C_1 + C_5 \left(\frac{e^{2t}}{4} - \frac{1}{2} + \frac{e^{-2t}}{4}\right) + C_6 \left(\frac{e^{2t}}{4} - \frac{e^{-2t}}{4}\right) + C_8 \left(-\frac{e^{2t}}{4} + \frac{e^{-2t}}{4}\right) \\ C_2 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) + C_3 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) \\ C_2 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) + C_3 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) \\ C_4 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_7 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(\frac{e^{2t}}{2} + \frac{e^{-2t}}{2}\right) + C_6 \left(\frac{e^{2t}}{2} - \frac{e^{-2t}}{2}\right) + C_8 \left(-\frac{e^{2t}}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(\frac{e^{2t}}{2} - \frac{1}{2}\right) + C_6 \left(\frac{e^{2t}}{2} + \frac{1}{2}\right) + C_8 \left(\frac{1}{2} - \frac{e^{2t}}{2}\right) \\ C_4 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_7 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) \\ C_5 \left(-\frac{1}{2} + \frac{e^{-2t}}{2}\right) + C_6 \left(\frac{1}{2} - \frac{e^{-2t}}{2}\right) + C_8 \left(\frac{1}{2} + \frac{e^{-2t}}{2}\right) \end{bmatrix} =\!| =\!|\!| \; [\tag{29}$$

Solution *after* algebraic substitution: Could not make this printing work...

6 The very final step

The very final step: substituting the solution into the tangents and print the results: Arbitrary integration constants in the final solution:

$$\begin{bmatrix} C_6 \\ C_2 \\ C_7 \\ C_3 \\ C_4 \\ C_1 \\ C_5 \\ C_8 \end{bmatrix}$$

Number of generators which are divided based on the number of constants: 8

Number of component tangents before removing: 9 Generator 1 out of 9:

$$\begin{aligned} \xi &= \frac{e^{2t}}{4} - \frac{e^{-2t}}{4} \\ \eta_1 &= \frac{ue^{2t}}{2} + \frac{u}{2} + \frac{ve^{2t}}{2} - \frac{ve^{-2t}}{2} \\ \eta_2 &= \frac{v}{2} - \frac{ve^{-2t}}{2} \end{aligned}$$

Checking the 2 linearised symmetry conditions of generator X_1: Lin syms

$$\left[-x_1e^{4x_0} + x_1 - x_2e^{4x_0} - x_2,\ x_1e^{4x_0} + x_1 + x_2e^{4x_0} - x_2\right]$$

Generator 2 out of 9:

$$\begin{aligned} \xi &= \frac{ue^{2t}}{2} - \frac{u}{2} + \frac{ve^{2t}}{2} + \frac{v}{2} \\ \eta_1 &= 0 \\ \eta_2 &= 0 \end{aligned}$$

Checking the 2 linearised symmetry conditions of generator X_2: Lin syms

$$\left[2x_1^2e^{2x_0} + 4x_1x_2e^{2x_0} + 2x_2^2e^{2x_0},\ 2x_1^2e^{2x_0} + 4x_1x_2e^{2x_0} + 2x_2^2e^{2x_0}\right]$$

Generator 3 out of 9:

$$\xi = 0$$
$$\eta_1 = -\frac{1}{2} + \frac{e^{-2t}}{2}$$
$$\eta_2 = \frac{1}{2} + \frac{e^{-2t}}{2}$$

Checking the 2 linearised symmetry conditions of generator X_3: Lin syms

$$[2,\ 2]$$

Generator 4 out of 9:

$$\xi = \frac{ue^{2t}}{2} + \frac{u}{2} + \frac{ve^{2t}}{2} - \frac{v}{2}$$
$$\eta_1 = 0$$
$$\eta_2 = 0$$

Checking the 2 linearised symmetry conditions of generator X_4: Lin syms

$$\left[2x_1^2e^{2x_0} + 4x_1x_2e^{2x_0} + 2x_2^2e^{2x_0},\ 2x_1^2e^{2x_0} + 4x_1x_2e^{2x_0} + 2x_2^2e^{2x_0}\right]$$

Generator 5 out of 9:

$$\xi = 0$$
$$\eta_1 = \frac{1}{2} + \frac{e^{-2t}}{2}$$
$$\eta_2 = -\frac{1}{2} + \frac{e^{-2t}}{2}$$

Checking the 2 linearised symmetry conditions of generator X_5: Lin syms

$$[2,\ 2]$$

Generator 6 out of 9:

$$\xi = 1$$
$$\eta_1 = 0$$
$$\eta_2 = 0$$

Checking the 2 linearised symmetry conditions of generator X_6: Lin syms

$$[0,\ 0]$$

Generator 7 out of 9:

$$\xi = \frac{e^{2t}}{4} - \frac{1}{2} + \frac{e^{-2t}}{4}$$
$$\eta_1 = \frac{ue^{2t}}{2} - \frac{u}{2} + \frac{ve^{2t}}{2} + \frac{ve^{-2t}}{2}$$
$$\eta_2 = -\frac{v}{2} + \frac{ve^{-2t}}{2}$$

Checking the 2 linearised symmetry conditions of generator X_7: Lin syms

$$\left[-x_1e^{4x_0} - x_1 - x_2e^{4x_0} + x_2,\ x_1e^{4x_0} - x_1 + x_2e^{4x_0} + x_2\right]$$

Generator 8 out of 9:

$$\xi = -\frac{e^{2t}}{4} + \frac{e^{-2t}}{4}$$
$$\eta_1 = -\frac{ue^{2t}}{2} + \frac{u}{2} - \frac{ve^{2t}}{2} + \frac{ve^{-2t}}{2}$$
$$\eta_2 = \frac{v}{2} + \frac{ve^{-2t}}{2}$$

Checking the 2 linearised symmetry conditions of generator X_8: Lin syms

$$\left[x_1e^{4x_0} - x_1 + x_2e^{4x_0} + x_2,\ -x_1e^{4x_0} - x_1 - x_2e^{4x_0} + x_2\right]$$

Generator 9 out of 9:

$$\xi = \frac{\left(-2\,\mathrm{c}_{21}(t)e^{2t} - 2e^{4t}\int \mathrm{c}_{21}(t)e^{-2t}\,dt - e^{4t}\int e^{-2t}\tfrac{d}{dt}\mathrm{c}_{21}(t)\,dt + 2\int \mathrm{c}_{21}(t)e^{2t}\,dt - \int e^{2t}\tfrac{d}{dt}\mathrm{c}_{21}(t)\,dt\right)e^{-2t}}{4}$$
$$\eta_1 = \frac{\left(-u\,\mathrm{c}_{21}(t)e^{2t} - 2ue^{4t}\int \mathrm{c}_{21}(t)e^{-2t}\,dt - ue^{4t}\int e^{-2t}\tfrac{d}{dt}\mathrm{c}_{21}(t)\,dt - 2ve^{4t}\int \mathrm{c}_{21}(t)e^{-2t}\,dt - ve^{4t}\int e^{-2t}\tfrac{d}{dt}\mathrm{c}_{21}(t)\,dt + \right.}{2}$$
$$\eta_2 = \frac{v\left(-\mathrm{c}_{21}(t)e^{2t} + 2\int \mathrm{c}_{21}(t)e^{2t}\,dt - \int e^{2t}\tfrac{d}{dt}\mathrm{c}_{21}(t)\,dt\right)e^{-2t}}{2}$$

Checking the 2 linearised symmetry conditions of generator X_9: Lin syms

$$\left[x_1\,\mathrm{c}_{21}(x_0)e^{2x_0} + 2x_1e^{4x_0}\int \mathrm{c}_{21}(x_0)e^{-2x_0}\,dx_0 + x_1e^{4x_0}\int e^{-2x_0}\frac{d}{dx_0}\mathrm{c}_{21}(x_0)\,dx_0 - 2x_1\int \mathrm{c}_{21}(x_0)e^{2x_0}\,dx_0 + x_1\int e^{2x_0}\right.$$

Number of component tangents after removing: 1 The final generators are given by:

$$X_1 = (1)\,\partial t.$$

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