Intro to Differential Equations: A Subtle Subject

Big Question: Given a differential equation, does it have any solutions?
(Some have many solutions. Others don't have any at all.)
Which have solutions and which don't? This is a very difficult question.

Q: If our differential equation has solutions, can we find all of the solutions *explicitly?* (i.e.: Can we find actual formulas for the solutions?)
Usually no. Only in rare cases can we actually find *explicit* solutions.

Best Case Scenario: Would be finding **all explicit solutions**.
But to reiterate: This is usually not possible!

Next Best Thing: If we are lucky, we might be able to:
(1) Determine **qualitative behavior** of solutions.
(2) Prove an **abstract existence-uniqueness theorem**.
(3) Construct **approximations to solutions** by a numerical method.

Note: Even if a differential equation has solutions, different solution functions might have different domains! The domain need not be all of $\mathbb{R}$. For instance:

Example: The differential equation $y' = y^2$ has infinitely many solutions.
- ◦ One solution has its domain as all of $\mathbb{R}$.
- ◦ All other solutions have domains like $(-\infty, a) \cup (a, \infty)$ for some $a \in \mathbb{R}$.

Types of Differential Equations

There are four ways to distinguish differential equations:
- ◦ ODE vs PDE
- ◦ Single Equations vs Systems of Equations
- ◦ 1st-order vs 2nd-order vs 3rd-order etc.
- ◦ Linear vs Nonlinear

This Class: Only ODEs.* Mostly linear ones (but some nonlinear).
- ◦ 1st-Order ODEs. (Week 1)
- ◦ 1st-Order systems of ODEs. (Weeks 3-6)
- ◦ 2nd-Order ODEs. (Weeks 6-9)

(* At the end of the course, we'll examine a PDE (Laplace Equation).)

1st-Order ODEs

General Form: A first-order ODE looks like:

$$\frac{dy}{dt} = F(t, y).$$

There is **no general method** for finding **explicit solutions** to all first-order ODEs. (Life is unfair.)

However, there are a few things we can say about 1st-Order ODEs.

(0) Special Cases: Some special kinds of first-order ODEs can be solved explicitly (more or less). A couple of these are discussed below.

(1) Geometric Insight: Direction fields.

Direction fields are a geometric technique for gaining insight into the behavior of solutions to first-order ODEs.

(2) Abstract Existence-Uniqueness Theorem:

Q: Do all first-order ODE's have solutions? This is subtle. A first answer in this direction is given by an *abstract existence-uniqueness theorem.*

We will discuss this much later in the course.

(3) Approximations to Solutions: There are numerical methods (e.g.: Euler's Method) for approximating solutions.

We will discuss this much later in the course.

Special Kinds of 1st-Order ODEs

(A) Linear. These have the form

$$\frac{dy}{dt} = p(t)y + g(t).$$

They are called **homogeneous** if $g(t) \equiv 0$. Else, they are **non-homogeneous**.

To solve: Method of Integrating Factors.

(B) Separable. These have the form

$$\frac{dy}{dt} = f(t)g(y) \qquad \text{or} \qquad \frac{dy}{dt} = \frac{f(t)}{g(y)}.$$

To solve: Method of Separation of Variables.

Method of Integrating Factors

Goal: Explicitly solve first-order linear ODE. That is:

$$\frac{dy}{dt} + p(t)y = g(t).$$

Method:

0. Start with $y' + p(t)y = g(t)$.
1. Multiply both sides by the *integrating factor*,

$$\mu(t) = \exp\left(\int p(t)\,dt\right).$$

Result is:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t).$$

2. Write the result as

$$(\mu(t)y)' = \mu(t)g(t).$$

3. Integrate and solve for $y(t)$.

Method of Separation of Variables

Goal: Explicitly solve first-order separable ODE. That is:

$$\frac{dy}{dt} = f(t)g(y).$$

Method:

0. Start with $\frac{dy}{dt} = f(t)g(y)$.
1. Move all functions of y to one side, and all functions of t to the other:

$$\frac{dy}{g(y)} = f(t)\,dt.$$

2. Integrate and solve for y.

Note: Separable equations can also be written in the form $\frac{dy}{dt} = \frac{f(t)}{g(y)}$.

The text describes them in the form $f(t)\,dt + g(y)\,dy = 0$, which is also the same thing.

Eigenvalues and Eigenvectors: Motivation

The simplest matrices are the **diagonal matrices**. These are of the form

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

Geometrically, D acts as follows:

◦ Vectors along the x_1-axis are stretched/contracted/reflected by d_1.

⋮

◦ Vectors along the x_n-axis are stretched/contracted/reflected by d_n.

Note: Stretching along the x_i-axis happens if $|d_i| > 1$.
Contracting along the x_i-axis happens if $|d_i| < 1$.
Reflection happens if d_i is negative.

Hopes and Dreams:

◦ Maybe every matrix is really a diagonal matrix after a "change-of-basis."

◦ Maybe every matrix A is *similar* to a diagonal matrix, meaning that $A = TDT^{-1}$ for a diagonal matrix D.

◦ Maybe every matrix geometrically acts like a diagonal matrix, stretching some line L_1 by a factor d_1, some other line L_2 by a factor d_2, and so on.

Reality Check: None of these hopes are true as stated. (Life is hard.)

But there is hope in the following two theorems.

Def: A matrix A is **diagonalizable** iff A is similar to a diagonal matrix. That is: $A = TDT^{-1}$ for some diagonal D and some invertible T.

Diagonalization Criterion: Let A be an $n \times n$ matrix. Then:

$$A \text{ is diagonalizable} \iff A \text{ has } n \text{ linearly independent eigenvectors.}$$

Spectral Theorem: If A is a symmetric matrix, then A is diagonalizable.

Eigenvalues and Eigenvectors: Definitions

Def: Let A be an $n \times n$ matrix.

An **eigenvector** of A is a non-zero vector $\mathbf{v} \in \mathbb{C}^n$, $\mathbf{v} \neq \mathbf{0}$ such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some scalar $\lambda \in \mathbb{C}$.

The scale factor $\lambda \in \mathbb{C}$ is the **eigenvalue** associated to the eigenvector $\mathbf{v}$.

Note: If $\mathbf{v}$ is an eigenvector for A, then so too is every scalar multiple $c\mathbf{v}$.

$\therefore$ Every eigenvector of A determines an entire line of eigenvectors of A.

Q: How to find eigenvectors and their eigenvalues?
A: Observe that:

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} \text{ for some } \mathbf{v} \neq \mathbf{0} &\iff (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\iff \det(A - \lambda I) = 0. \end{aligned}$$

Def: The **characteristic polynomial** of A is

$$p(\lambda) = \det(A - \lambda I).$$

So: Eigenvalues are the zeros of the characteristic polynomial!

By the Fundamental Theorem of Algebra, we can factor $p(\lambda)$:

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}.$$

The **algebraic multiplicity** of λ_j is the corresponding exponent m_j.

Def: Let A be an $n \times n$ matrix. The **eigenspace of** $\lambda \in \mathbb{C}$ is

$$E(\lambda) = \{\mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda\mathbf{v}\} = N(A - \lambda I).$$

The **geometric multiplicity** of λ is:

$$\begin{aligned} \text{GeoMult}(\lambda) &= \dim(E(\lambda)) \\ &= \#\text{ of linearly independent eigenvectors with eigenvalue } \lambda. \end{aligned}$$

Fact: For every eigenvalue λ, we have:

$$\text{GeoMult}(\lambda) \leq \text{AlgMult}(\lambda).$$

Eigenvalues and Eigenvectors: More Facts

To Find Eigenvectors of A:

(1) First find the eigenvalues. They will be the zeros of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

(2) For each eigenvalue λ that you found in part (1), solve the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

for vectors $\mathbf{v}$. These vectors are the eigenvectors associated to λ.

(In fact, these vectors are the elements of $E(\lambda) = N(A - \lambda I)$.)

Fact: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (each repeated according to its algebraic multiplicity).

(a) $\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n$.

(b) $\det(A) = \lambda_1 \cdots \lambda_n$.

Fact: Let A be an $n \times n$ matrix (with real entries).

If $\lambda = a + ib$ is an eigenvalue of A, then $\overline{\lambda} = a - ib$ is an eigenvalue of A.

Fact: If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors associated to *distinct* eigenvalues, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Diagonalization Criterion: Let A be an $n \times n$ matrix. Then:

$$\begin{aligned} A \text{ is diagonalizable} &\iff A \text{ has } n \text{ linearly independent eigenvectors} \\ &\iff \text{Every eigenvalue } \lambda \text{ of } A \text{ has} \\ &\ \ \mathrm{GeoMult}(\lambda) = \mathrm{AlgMult}(\lambda). \end{aligned}$$

Review: Dot Products

Recall: The **dot product** (or **inner product**) of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \cdots + u_n v_n.$$

Also: $\|\mathbf{v}\|^2 = v_1^2 + \cdots + v_n^2 = \langle \mathbf{v}, \mathbf{v} \rangle$.

Def: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an **orthogonal set** iff every pair of (distinct) vectors is orthogonal: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$.

Fact: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of non-zero vectors, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Symmetric Matrices

Def: An $n \times n$ matrix is **symmetric** iff $A^T = A$.

Theorem: Let A be a symmetric matrix. Then:

(a) All eigenvalues of A are real.

(b) If $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A corresponding to *different* eigenvalues, then $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$.

(c) (Spectral Theorem) There exists a set of n linearly independent eigenvectors for A. Thus, A is diagonalizable.

N.B.: Parts (a), (b), (c) are stated under the assumption that A is symmetric. If A is not symmetric, then (a), (b), (c) may or may not be true!

Jordan Form

Diagonalization Criterion: Let A be an $n \times n$ matrix. Then:

$$\begin{aligned} A \text{ is diagonalizable} &\iff A \text{ has } n \text{ linearly independent eigenvectors} \\ &\iff \text{Every eigenvalue } \lambda \text{ of } A \text{ has} \\ &\qquad \operatorname{GeoMult}(\lambda) = \operatorname{AlgMult}(\lambda). \end{aligned}$$

Q: If A is not diagonalizable, is A still similar to a "simple" matrix?
A: Yes: If we can't diagonalize, the next best thing is Jordan form.

Def: A **Jordan block** is a matrix of the form

$$\begin{pmatrix} \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ or } \dots \text{ or } \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

A matrix is in **Jordan form** if it is of the form

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix},$$

where each J_i is a Jordan block (possibly of different sizes and λ's).

Example: The following two matrices are both in Jordan form:

$$J = \begin{pmatrix} 4 & & & & & \\ & 7 & 1 & & & \\ & & 7 & & & \\ & & & 4 & 1 & \\ & & & & 4 & 1 \\ & & & & & 4 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 4 & & & & & \\ & 7 & & & & \\ & & 7 & & & \\ & & & 4 & & \\ & & & & 4 & \\ & & & & & 4 \end{pmatrix}$$

Notice: Diagonal matrices are exactly the Jordan form matrices in which every Jordan block has size 1.

Theorem: Every $n \times n$ matrix A is similar to a matrix in Jordan form.
That is: Every $n \times n$ matrix A can be written as

$$A = TJT^{-1},$$

where J is in Jordan form. Diagonal entries of J are the eigenvalues of A.

The Matrix Exponential

Def: Let A be an $n \times n$ matrix. The **matrix exponential** is

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Fortunately, this series always converges (so e^{tA} does make sense).

Theorem: The solution to the initial-value problem

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \qquad \text{is exactly } \ \mathbf{x}(t) = e^{tA}\mathbf{x}_0.$$

Main Facts:

(a) If P is invertible, then $e^{PAP^{-1}} = Pe^A P^{-1}$.
(b) If $AB = BA$, then $e^{A+B} = e^A e^B$
(c) $e^{(t+s)A} = e^{tA}e^{sA}$ for all $t, s \in \mathbb{R}$.
(d) $e^{nA} = (e^A)^n$ for all $n \in \mathbb{Z}$
(e) $e^{-A} = (e^A)^{-1}$.
(f) $\det(e^A) = e^{\mathrm{tr}(A)}$.

Note: The most important are Facts (a) and (b).
Notice that Fact (b) implies Facts (c), (d), (e) as special cases.

Example 1: Suppose D is a diagonal matrix.

$$\text{If } D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \text{ then } e^D = \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix}.$$

Example 2: Suppose B satisfies $B^N = 0$. Then:

$$e^B = I + B + \frac{1}{2!}B^2 + \cdots + \frac{1}{(N-1)!}B^{N-1}.$$

Diagonalizable Case: Suppose A is a diagonalizable matrix. Then we can write $A = TDT^{-1}$ with T invertible and D diagonal. Therefore:

$$e^A = e^{TDT^{-1}} = Te^D T^{-1}.$$

Advanced: Jordan Form

Recall: Let A be an $n \times n$ matrix.
The **eigenspace** of $\lambda \in \mathbb{C}$ is

$$\begin{aligned} E(\lambda) &= \{\mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda\mathbf{v}\} \\ &= N(A - \lambda I). \end{aligned}$$

A vector $\mathbf{v} \in E(\lambda)$ is an **eigenvector** of **eigenvalue** λ.

Def: The **generalized eigenspace of rank** k of $\lambda \in \mathbb{C}$ is

$$E_k(\lambda) = N((A - \lambda I)^k).$$

A vector $\mathbf{v} \in E_k(\lambda)$ is a **generalized eigenvector**.

Fact: For a given $\lambda \in \mathbb{C}$, the generalized eigenspaces are nested:

$$E(\lambda) = E_1(\lambda) \subset E_2(\lambda) \subset \cdots$$

Fact: Let A be an $n \times n$ matrix. Then the Jordan form of A has:
- ∘ $\dim(E_1(\lambda))$ Jordan blocks of eigenvalue λ size ≥ 1
- ∘ $\dim(E_2(\lambda)) - \dim(E_1(\lambda))$ Jordan blocks of eigenvalue λ of size ≥ 2
- ∘ $\dim(E_3(\lambda)) - \dim(E_2(\lambda))$ Jordan blocks of eigenvalue λ of size ≥ 3
- ∘ etc.

Advanced: The Matrix Exponential

Let A be any $n \times n$ matrix. Write A in Jordan form as $A = TJT^{-1}$, where J has Jordan blocks $J_1, \dots, J_k$. Then:

$$e^A = e^{TJT^{-1}} = Te^J T^{-1} = T \begin{pmatrix} e^{J_1} & & & \\ & e^{J_2} & & \\ & & \ddots & \\ & & & e^{J_k} \end{pmatrix} T^{-1}$$

Q: How do we calculate $e^{J_1}, \dots, e^{J_k}$?

Let J_i be a Jordan block of eigenvalue λ_i of size N_i. We can write

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_i \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{pmatrix}}_{\lambda_i I} + \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix}}_{B_i}.$$

Notice that the matrix B_i, which is of size $N_i \times N_i$, is such that $B_i^{N_i} = 0$. Therefore:

$$e^{J_i} = e^{\lambda_i I + B_i} = e^{\lambda_i} e^{B_i} = e^{\lambda_i} \left[I + B_i + \frac{1}{2!} B_i^2 + \cdots + \frac{1}{(N_i - 1)!} B_i^{N_i - 1} \right].$$

1st-Order Linear Systems: Intro

Def: A **first-order linear ODE system** is one of the form

$$\frac{d\mathbf{x}}{dt} = P(t)\mathbf{x} + \mathbf{g}(t). \tag{$\dagger$}$$

Here, $P(t)$ is a matrix of functions.

In detail, this is

$$\begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

For now, this is too difficult. So, we simplify things with two assumptions:

(1) Constant Coefficients: This is the special case of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}. \tag{$\ddagger$}$$

That is, $P(t) \equiv A$ is a constant matrix and $\mathbf{g}(t) = \mathbf{b}$ is a constant function.

These are sometimes called **autonomous**.

(2) Homogeneous: This is the special case that $\mathbf{b} = \mathbf{0}$. That is, we study:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{$\star$}$$

Note: Studying ($\star$) will ultimately be the key to understanding ($\dagger$) and ($\ddagger$).

Remark: Geometry of Solutions

A solution to an ODE system consists of n functions $x_1(t), \ldots, x_n(t)$ of t. Geometrically, there are two ways to visualize solutions:

(a) The graph of $x_1(t)$ is a curve in the (t, x_1)-plane, the graph of $x_2(t)$ is a curve in the (t, x_2)-plane, etc.

So: A solution consists of n graphical curves (called **component plots**), each in a different plane.

(b) The function $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ is a parametric curve in the **phase space** (or **state space**) $\mathbb{R}^n$, which has coordinates $(x_1, \ldots, x_n)$.

So: A solution consists of one parametric curve in $\mathbb{R}^n$.

Remark: Equilibrium Solutions of $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$

Def: An **equilibrium solution** (or **critical point**) is a solution $\mathbf{x}(t)$ that is constant. That is, a solution for which $\frac{d\mathbf{x}}{dt} = 0$.

Observation: Consider the first-order autonomous linear system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}.$$

An equilibrium solution must satisfy $\mathbf{x}' = \mathbf{0}$, so

$$A\mathbf{x} = -\mathbf{b}.$$

There are three possibilities:

(i) $A\mathbf{x} = -\mathbf{b}$ has ∞-many solutions $\iff$ ∞-many equilibrium solutions.

(ii) $A\mathbf{x} = -\mathbf{b}$ has one solution $\iff$ Exactly one equilibrium solution.

(iii) $A\mathbf{x} = -\mathbf{b}$ has no solutions $\iff$ No equilibrium solutions.

Notice: In the special case of $\mathbf{x}' = A\mathbf{x}$ (i.e., if $\mathbf{b} = \mathbf{0}$), then case (iii) cannot occur. So, in this case, there only two possibilities:

(i) $\det A = 0 \iff A\mathbf{x} = \mathbf{0}$ has ∞-many solutions.

(ii) $\det A \neq 0 \iff A\mathbf{x} = \mathbf{0}$ has exactly one solution: the origin.

1st-Order Linear Systems: General Observations

Goal: Study 1st-order linear systems of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (\star)$$

for n dependent variables $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$.

1. Superposition Principle: Consider the 1st-order linear system $\mathbf{x}' = A\mathbf{x}$.

If $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are solutions on an interval I, then any linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution on I. (Here, $c_1, \ldots, c_k \in \mathbb{R}$ are any real numbers.)

Def: Consider the 1st-order linear system $\mathbf{x}' = A\mathbf{x}$.

A **fundamental set of solutions** on an interval I is a set of solutions $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ that is **linearly independent on** I.

That is: The only constants $c_1, \ldots, c_k$ with

$$c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0} \text{ for all } t \in I$$

are $c_1 = \cdots = c_k = 0$.

2. Theorem: Consider the 1st-order linear system $\mathbf{x}' = A\mathbf{x}$.

If $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ is a fundamental set of solutions on an interval I, then every solution of $\mathbf{x}' = A\mathbf{x}$ on I is of the form

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t).$$

We then call $c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ the **general solution** of the system.

Def: Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be n solutions of $\mathbf{x}' = A\mathbf{x}$.

The **Wronskian** of these n solutions is the function

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det\begin{pmatrix} | & & | \\ \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \\ | & & | \end{pmatrix}$$

3. Fact: Consider the 1st-order linear system $\mathbf{x}' = A\mathbf{x}$.

If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are n solutions with $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) \neq 0$ for all t, then $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)\}$ is a fundamental set of solutions.

1st-Order Linear Systems: Methods of Solution

Goal: Find solutions to 1st-order linear systems of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \qquad (\star)$$

There are essentially two methods.

1. Matrix Exponential Method: Every solution to $\mathbf{x}' = A\mathbf{x}$ is of the form $\mathbf{x}(t) = e^{At}\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

In other words: Every solution is

$$\mathbf{x}(t) = e^{At}\mathbf{c} = \begin{pmatrix} | & & | \\ \mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t) \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t),$$

where $e^{At} = \begin{pmatrix} \mathbf{y}_1(t) & \cdots & \mathbf{y}_n(t) \end{pmatrix}$ and $c_1, \ldots, c_n \in \mathbb{R}$ are constants.

Note: This method is cool because it gives **<u>all</u>** solutions to the system. But it is annoying because calculating e^{At} can be a pain. Here's an alternative:

2. Eigenvalue Method: Consider $\mathbf{x}' = A\mathbf{x}$. If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Proof: Let $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. Notice that

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v} \quad \text{and}$$
$$A\mathbf{x} = A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v} = e^{\lambda t}\lambda\mathbf{v} = \lambda e^{\lambda t}\mathbf{v},$$

so $\mathbf{x}' = A\mathbf{x}$ holds. Yay. $\Diamond$

Concern: Does this second method give us **<u>all</u>** solutions? Not necessarily.

Special Case: Two Dependent Variables

Going forward: We will only consider the case of two dependent variables: $\mathbf{x}(t) = (x(t), y(t))$ and A will be a 2×2 matrix. There will be three cases to consider:

(1) Eigenvalues of A are real and distinct.
(2) Eigenvalues of A are real and repeated ($\lambda_1 = \lambda_2$).
(3) Eigenvalues of A are complex.

The Eigenvalue Method

Goal: Find a fundamental set of solutions to the 1st-order ODE system

$$\mathbf{x}' = A\mathbf{x},$$

in two dependent variables $\mathbf{x}(t) = (x(t), y(t))$.

The starting point will be the following:

Fact: If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Case 1: A has real eigenvalues & two independent eigenvectors

The Fact above gives a fundamental set: $\begin{cases} \mathbf{x}_1(t) = e^{\lambda_1 t}\,\mathbf{v}_1, \text{ where } A\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{x}_2(t) = e^{\lambda_2 t}\,\mathbf{v}_2, \text{ where } A\mathbf{v}_2 = \lambda_2\mathbf{v}_2. \end{cases}$

Case 2: A has a real eigenvalue & one independent eigenvector

Let λ be the real (repeated) eigenvalue of A, with $A\mathbf{v} = \lambda\mathbf{v}$. Then the Fact above gives only one solution:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}.$$

A second solution (independent from the first) is

$$\mathbf{x}_2(t) = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w},$$

where $\mathbf{w}$ is any solution of $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

Case 3: A has complex eigenvalues

Let $\lambda = \mu + i\nu$ and $\overline{\lambda} = \mu - i\nu$ be the complex eigenvalues of A, with eigenvectors $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ and $\overline{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$, respectively. The Fact above gives two solutions:

$$\mathbf{z}_1(t) = e^{(\mu+i\nu)t}\,\mathbf{v} = e^{\mu t}e^{i\nu t}(\mathbf{a} + i\mathbf{b}) = e^{\mu t}(\cos(\nu t) + i\sin(\nu t))(\mathbf{a} + i\mathbf{b})$$
$$\mathbf{z}_2(t) = e^{(\mu-i\nu)t}\,\overline{\mathbf{v}} = e^{\mu t}e^{-i\nu t}(\mathbf{a} - i\mathbf{b}) = e^{\mu t}(\cos(\nu t) - i\sin(\nu t))(\mathbf{a} - i\mathbf{b})$$

The real and imaginary parts of $\mathbf{z}_1(t)$ (or of $\mathbf{z}_2(t)$) are a fundamental set:

$$\mathbf{x}_1(t) = e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t))$$
$$\mathbf{x}_2(t) = e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t)).$$

For Fun: What even is Stability?

Setup: Consider a system $\mathbf{x}' = A\mathbf{x}$. Notice that the origin $\mathbf{x}(t) = \mathbf{0}$ is an equilibrium solution.

Def: The origin is **stable** if for every choice of disk $\{x^2 + y^2 < \epsilon\}$, there is a smaller "threshold disk" $\{x^2 + y^2 < \delta\}$ such that:

Every solution that starts inside the threshold disk exists for all $t > 0$ and stays inside the chosen disk forever.

Def: The origin is **unstable** if it is not stable.

Def: The origin is **asymptotically stable** if:

(1) It is stable, and:

(2) There exists a "threshold disk" $\{x^2 + y^2 < \eta\}$ such that every solution $\mathbf{y}(t)$ starting inside the threshold disk has the property that $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$.

Classification of Critical Points of $\mathbf{x}' = A\mathbf{x}$

Q: Given a first-order ODE system

$$\mathbf{x}' = A\mathbf{x}.$$

How do solutions behave near the equilibrium solution?

Note: We only deal with systems of two equations. That is, $\mathbf{x}(t) = (x(t), y(t))$ and A is a 2×2 matrix.

We also suppose that $\det A \neq 0$.

A: Depends on the eigenvalues of A. There are 10 possibilities:

(1) Eigenvalues are real and distinct.

(1a) $\lambda_1, \lambda_2 > 0 \implies$ Unstable Node

(1b) $\lambda_1, \lambda_2 < 0 \implies$ Asymp Stable Node

(1c) $\lambda_1 < 0$ and $\lambda_2 > 0 \implies$ (Unstable) Saddle point

(2) Eigenvalues are real and repeated ($\lambda_1 = \lambda_2$).

(2a) $\lambda_1 > 0 \implies$ Unstable Proper/Improper Node

(2b) $\lambda_1 < 0 \implies$ Asymp Stable Proper/Improper Node

(3) Eigenvalues are complex: $\lambda_1, \lambda_2 = \mu \pm i\nu$.

(3a) $\mu > 0 \implies$ Unstable Spiral

(3b) $\mu < 0 \implies$ Asymp Stable Spiral

(3c) $\mu = 0 \implies$ (Stable) Center.

Corollary: Let $T = \text{tr}(A)$ and $D = \det(A)$ and $\Delta = T^2 - 4D$. Then the characteristic polynomial of A is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

Thus, the eigenvalues are: $\lambda = \frac{1}{2}(T \pm \sqrt{\Delta})$.

(1) $\Delta > 0 \implies$ Node or Saddle.

(2) $\Delta = 0 \implies$ Proper or Improper Node

(3) $\Delta < 0$ and $T \neq 0 \implies$ Spiral

(3′) $\Delta < 0$ and $T = 0 \implies$ Center.

Moreover:

(i) $D > 0$ and $T < 0 \implies$ Asymp stable

(ii) $D > 0$ and $T = 0 \implies$ Stable.

(iii) $D < 0$ or $T > 0 \implies$ Unstable.

Stability Diagram: Page 190.

Solving $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$

Goal: Want to study

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}.$$

∘ A invertible $\implies$ Exactly one equilibrium solution.
∘ A not invertible $\implies$ No equil solutions OR ∞-many equil solutions.

Suppose A is invertible. Equilibrium solutions those with $\mathbf{x}' = \mathbf{0}$, so $A\mathbf{x} + \mathbf{b} = \mathbf{0}$, so

$$\mathbf{x}_{\text{equil}} = -A^{-1}\mathbf{b}.$$

Fact: Suppose A is invertible.
If $\mathbf{y}(t)$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then

$$\mathbf{x}(t) = \mathbf{y}(t) - A^{-1}\mathbf{b}$$

is a solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$.

Q: What if A is not invertible?
A: Can use a method called “variation of parameters.” (next time)

Reminder on Jordan Form

Recall: Let A be an $n \times n$ matrix.
The **eigenspace** of $\lambda \in \mathbb{C}$ is

$$E(\lambda) = \{\mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda\mathbf{v}\} = N(A - \lambda I).$$

The **geometric multiplicity of** λ is

$$\begin{aligned} \text{GeoMult}(\lambda) &= \dim(E(\lambda)) \\ &= \#\text{ of linearly independent eigenvectors with eigenvalue } \lambda. \end{aligned}$$

Fact: Consider the Jordan form of A.
The number of Jordan blocks of eigenvalue λ is exactly $\text{GeoMult}(\lambda)$.