New take after rejection from the Bulletin of Mathematical Biology

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1 Introduction

Okay, so we got rejected from the bulletin of Mathematical Biology. Overall, the two reviewers thought that the researchs aim and research questions were brilliant and they thought that our manuscript was very well-written. However, all the content was trivial and there was nothing new there. So they gave us three months to re-submit a new manuscript. The initial plan was to go through a bunch of famous models in mathematical biology and then present their symmetries as well as the differential invariants and so on. The thing that stopped us from doing this was the fact that our symbolic solver could not find the symmetries of these models.

So my suggestion here is that we try to find the symmetries of these models by hand essentially. Again, we study the following type of system of first order ODEs

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \omega_i(t, y_1, \ldots, y_k), \quad i = 1, \ldots, k, \tag{1}$$

where the infinitesimal generator of the Lie group is given by

$$X = \xi\partial_t + \eta_1\partial y_1 + \ldots + \eta_k\partial y_k \tag{2}$$

and the prolonged generator is given by

$$X^{(1)} = X + \eta_1^{(1)}\partial y_1 + \ldots + \eta_k^{(1)}\partial y_k. \tag{3}$$

Now, given this prolonged generator, the *linearised symmetry conditions* are defined as follows:

$$X^{(1)}\left(\frac{\mathrm{d}y_i}{\mathrm{d}t} - \omega_i(t, y_1, \ldots, y_k)\right) = 0, \quad i = 1, \ldots, k. \tag{4}$$

In particular, using the *total derivative*

$$D_t = \partial_t + y_1'\partial y_1 + \ldots + y_k'\partial y_k \tag{5}$$

these symmetry conditions can be written as follows

$$D_t\eta_i - \omega_i D_t\xi = X\left(\omega_i(t, y_1, \ldots, y_k)\right), \quad i = 1, \ldots, k\,. \tag{6}$$

So to increase the impact, we probably need to solve equation (6) for a bunch of biologically relevant models. This document is the start of that journey, and below I will list the models that I figured that we can focus on.

The Lotka-Volterra model:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= u(1-v), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \alpha v(u-1). \end{aligned} \tag{7}$$

The BZ model

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{\varepsilon}v - \frac{1}{\varepsilon}\left(\frac{1}{3}u^3 - u\right), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -u. \end{aligned} \tag{8}$$

The Lorenz equations:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= a(v-u), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -uw + bu - v, \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= uv - cw. \end{aligned} \tag{9}$$

The Brusselator:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= 1 - (b-1)u + au^2v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= bu - au^2v. \end{aligned} \tag{10}$$

The SIR model:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= -rSI, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= rSI - aI. \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= aI. \end{aligned} \tag{11}$$

The MM system:

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= -k_1 es + k_{-1}c,\\
\frac{\mathrm{d}e}{\mathrm{d}t} &= -k_1 es + (k_{-1}+k_2)c,\\
\frac{\mathrm{d}c}{\mathrm{d}t} &= k_1 es - (k_{-1}+k_2)c,\\
\frac{\mathrm{d}p}{\mathrm{d}t} &= k_2 c.
\end{aligned}
\tag{12}
$$

The Goodwin model (with n=1):

$$
\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}t} &= -b_1 R + \frac{K}{1+\beta T^n} = \omega_1(R,L,T)\\
\frac{\mathrm{d}L}{\mathrm{d}t} &= g_1 R - b_2 L = \omega_2(R,L,T)\\
\frac{\mathrm{d}T}{\mathrm{d}t} &= g_2 L - b_3 T = \omega_3(R,L,T)
\end{aligned}
\tag{13}
$$

So let's go through these models one by one and see if we can find any symmetries. Let's start with the Lotka-Volterra model!

2 Lotka-Volterra

We remind ourselves that we want to study the Lotka-Volterra model:

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= u(1-v),\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \alpha v(u-1),
\end{aligned}
\tag{14}
$$

and we are looking for a generator of the following kind:

$$X = \xi(t,u,v)\partial_t + \eta_1(t,u,v)\partial_u + \eta_2(t,u,v)\partial_v. \tag{15}$$

Now, derive the linearised symmetry conditions in equation (6) by plugging in our model in equation (14). Given that we have autnomous reaction terms, our linearised symmetry conditions can be written as follows:

$$
\begin{aligned}
&\frac{\partial\eta_1}{\partial t} + u(1-v)\frac{\partial\eta_1}{\partial u} + \alpha v(u-1)\frac{\partial\eta_1}{\partial v}\\
&-[u(1-v)]\left(\frac{\partial\xi}{\partial t} + u(1-v)\frac{\partial\xi}{\partial u} + \alpha v(u-1)\frac{\partial\xi}{\partial v}\right)\\
&= \eta_1(1-v) - \eta_2 u,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\frac{\partial\eta_2}{\partial t} + u(1-v)\frac{\partial\eta_2}{\partial u} + \alpha v(u-1)\frac{\partial\eta_2}{\partial v}\\
&-[\alpha v(u-1)]\left(\frac{\partial\xi}{\partial t} + u(1-v)\frac{\partial\xi}{\partial u} + \alpha v(u-1)\frac{\partial\xi}{\partial v}\right)\\
&= \alpha\eta_1 v + \alpha\eta_2(u-1).
\end{aligned} \tag{17}
$$

Now, let's expand this as much as possible and then try to derive the determining equations. Equation (16) is expanded as follows

$$
\begin{aligned}
\alpha u^2 v^2\frac{\partial\xi}{\partial v} - \alpha u^2 v\frac{\partial\xi}{\partial v} - \alpha u v^2\frac{\partial\xi}{\partial v} + \alpha uv\frac{\partial\eta_1}{\partial v}&\\
+\alpha uv\frac{\partial\xi}{\partial v} - \alpha v\frac{\partial\eta_1}{\partial v} + \eta_1 v - \eta_1 + \eta_2 u&\\
-u^2v^2\frac{\partial\xi}{\partial u} + 2u^2 v\frac{\partial\xi}{\partial u} - u^2\frac{\partial\xi}{\partial u} - uv\frac{\partial\eta_1}{\partial u}&\\
+uv\frac{\partial\xi}{\partial t} + u\frac{\partial\eta_1}{\partial u} - u\frac{\partial\xi}{\partial t} + \frac{\partial\eta_1}{\partial t}& = 0
\end{aligned} \tag{18}
$$

and here we see that we essentially have a polynomial where the different monomials are powers of the states u and v. Now, since all these monomials are *linearly independent* it follows that all coefficients in front of these monomials must be zero. This is what gives us our *determining equations*, and more precisely here are our determining equations:

$$
\begin{aligned}
1 :& - \eta_1 + \frac{\partial \eta_1}{\partial t} = 0, & (19)\\
v :& - \alpha\frac{\partial \eta_1}{\partial v} + \eta_1 = 0, & (20)\\
u :& \eta_2 + \frac{\partial \eta_1}{\partial u} - \frac{\partial \xi}{\partial t} = 0, & (21)\\
uv :& \alpha\frac{\partial \eta_1}{\partial v} + \alpha\frac{\partial \xi}{\partial v} - \frac{\partial \eta_1}{\partial u} + \frac{\partial \xi}{\partial t} = 0, & (22)\\
uv^2 :& - \alpha\frac{\partial \xi}{\partial v} = 0, & (23)\\
u^2 :& - \frac{\partial \xi}{\partial u} = 0, & (24)\\
u^2v :& - \alpha\frac{\partial \xi}{\partial v} + 2\frac{\partial \xi}{\partial u} = 0, & (25)\\
u^2v^2 :& \alpha\frac{\partial \xi}{\partial v} - \frac{\partial \xi}{\partial u} = 0. & (26)
\end{aligned}
$$

Similarly, the second linearised symmetry condition in equation (17) is expanded as follows:

$$
\begin{aligned}
&-\alpha^2u^2v^2\frac{\partial \xi}{\partial v} + 2\alpha^2uv^2\frac{\partial \xi}{\partial v} - \alpha^2v^2\frac{\partial \xi}{\partial v} - \alpha\eta_1 v - \alpha\eta_2 u\\
&+\alpha\eta_2 + \alpha u^2v^2\frac{\partial \xi}{\partial u} - \alpha u^2v\frac{\partial \xi}{\partial u} - \alpha uv^2\frac{\partial \xi}{\partial u} + \alpha uv\frac{\partial \eta_2}{\partial v}\\
&\qquad +\alpha uv\frac{\partial \xi}{\partial u} - \alpha uv\frac{\partial \xi}{\partial t} - \alpha v\frac{\partial \eta_2}{\partial v} + \alpha v\frac{\partial \xi}{\partial t}\\
&\qquad\qquad -uv\frac{\partial \eta_2}{\partial u} + u\frac{\partial \eta_2}{\partial u} + \frac{\partial \eta_2}{\partial t} = 0
\end{aligned}
\tag{27}
$$

and again equation can be viewed as finding the roots of a polynomial which has monomials determined by the states u and v. Since these monomials are linearly independent we can derive the determining equation stemming from this linearised symmetry condition:

$$
\begin{aligned}
1 :& \alpha\eta_2 + \frac{\partial \eta_2}{\partial t} = 0, && (28)\\
v :& -\alpha\eta_1 - \alpha\frac{\partial \eta_2}{\partial v} + \alpha\frac{\partial \xi}{\partial t} = 0, && (29)\\
v^2 :& -\alpha^2\frac{\partial \xi}{\partial v} = 0, && (30)\\
u :& -\alpha\eta_2 + \frac{\partial \eta_2}{\partial u} = 0, && (31)\\
uv :& \alpha\frac{\partial \eta_2}{\partial v} + \alpha\frac{\partial \xi}{\partial u} - \alpha\frac{\partial \xi}{\partial t} - \frac{\partial \eta_2}{\partial u} = 0, && (32)\\
uv^2 :& 2\alpha^2\frac{\partial \xi}{\partial v} - \alpha\frac{\partial \xi}{\partial u} = 0, && (33)\\
u^2v :& -\alpha\frac{\partial \xi}{\partial u} = 0, && (34)\\
u^2v^2 :& -\alpha^2\frac{\partial \xi}{\partial v} + \alpha\frac{\partial \xi}{\partial u} = 0. && (35)
\end{aligned}
$$

3 BZ

We remind ourselves that we want to study the BZ model

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{\varepsilon}v - \frac{1}{\varepsilon}\left(\frac{1}{3}u^3 - u\right),\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -u.
\end{aligned}
\tag{36}
$$

and we are looking for a generator of the following kind:

$$
X = \xi(t,u,v)\partial_t + \eta_1(t,u,v)\partial_u + \eta_2(t,u,v)\partial_v. \tag{37}
$$

The linearised symmetry condition 1 yields the following equations:

$$
\begin{aligned}
-u\frac{d}{dv}\eta_1 + \frac{\partial\eta_1}{\partial t} + \frac{\eta_1 u^2}{\varepsilon} - \frac{\eta_1}{\varepsilon} - \frac{\eta_2}{\varepsilon} - \frac{u^4\frac{d}{dv}\xi}{3\varepsilon} \\
-\frac{u^3\frac{d}{du}\eta_1}{3\varepsilon} + \frac{u^3\frac{\partial\xi}{\partial t}}{3\varepsilon} + \frac{u^2\frac{d}{dv}\xi}{\varepsilon} + \frac{uv\frac{d}{dv}\xi}{\varepsilon} \\
+\frac{u\frac{d}{du}\eta_1}{\varepsilon} - \frac{u\frac{\partial\xi}{\partial t}}{\varepsilon} + \frac{v\frac{d}{du}\eta_1}{\varepsilon} - \frac{v\frac{\partial\xi}{\partial t}}{\varepsilon} \\
-\frac{u^6\frac{d}{du}\xi}{9\varepsilon^2} + \frac{2u^4\frac{d}{du}\xi}{3\varepsilon^2} + \frac{2u^3v\frac{d}{du}\xi}{3\varepsilon^2} - \frac{u^2\frac{d}{du}\xi}{\varepsilon^2} \\
-\frac{2uv\frac{d}{du}\xi}{\varepsilon^2} - \frac{v^2\frac{d}{du}\xi}{\varepsilon^2} = 0
\end{aligned} \tag{38}
$$

and we see that this equation amounts to finding the roots of a polynomial of the states u and v. Since these monomials are linearly independent we obtain the following determining equations:

$$
1 : \frac{\partial\eta_1}{\partial t} - \frac{\eta_1}{\varepsilon} - \frac{\eta_2}{\varepsilon} = 0, \tag{39}
$$

$$
v : \frac{\partial}{\partial u}\frac{\eta_1}{\varepsilon} - \frac{\frac{\partial\xi}{\partial t}}{\varepsilon} = 0, \tag{40}
$$

$$
v^2 : -\frac{\partial}{\partial u}\frac{\xi}{\varepsilon^2} = 0, \tag{41}
$$

$$
u : -\frac{d}{dv}\eta_1 + \frac{\partial}{\partial u}\frac{\eta_1}{\varepsilon} - \frac{\frac{\partial\xi}{\partial t}}{\varepsilon} = 0, \tag{42}
$$

$$
uv : \frac{\partial}{\partial v}\frac{\xi}{\varepsilon} - 2\frac{\partial}{\partial u}\frac{\xi}{\varepsilon^2} = 0, \tag{43}
$$

$$
u^2 : \frac{\eta_1}{\varepsilon} + \frac{\partial}{\partial v}\frac{\xi}{\varepsilon} - \frac{\partial}{\partial u}\frac{\xi}{\varepsilon^2} = 0, \tag{44}
$$

$$
u^3 : -\frac{\partial}{\partial u}\frac{\eta_1}{3\varepsilon} + \frac{\frac{\partial\xi}{\partial t}}{3\varepsilon} = 0, \tag{45}
$$

$$
u^3v : \frac{\partial}{\partial u}\frac{2\xi}{3\varepsilon^2} = 0, \tag{46}
$$

$$
u^4 : -\frac{\partial}{\partial v}\frac{\xi}{3\varepsilon} + \frac{\partial}{\partial u}\frac{2\xi}{3\varepsilon^2} = 0, \tag{47}
$$

$$
u^6 : -\frac{\partial}{\partial u}\frac{\xi}{9\varepsilon^2} = 0. \tag{48}
$$

$$
\tag{49}
$$

Similarly, the second linearised symmetry condition is given by

$$
\eta_1 - u^2\frac{\partial}{\partial v}\xi - u\frac{\partial}{\partial v}\eta_2 + u\frac{\partial\xi}{\partial t} + \frac{\partial\eta_2}{\partial t} - \frac{u^4\frac{\partial}{\partial u}\xi}{3\varepsilon} - \frac{u^3\frac{\partial}{\partial u}\eta_2}{3\varepsilon} + \frac{u^2\frac{\partial}{\partial u}\xi}{\varepsilon} + \frac{uv\frac{\partial}{\partial u}\xi}{\varepsilon} + \frac{u\frac{\partial}{\partial u}\eta_2}{\varepsilon} + \frac{v\frac{\partial}{\partial u}\eta_2}{\varepsilon} = 0 \tag{50}
$$

and the corresponding determining equations

$$
\begin{aligned}
1 :&\eta_1 + \frac{\partial \eta_2}{\partial t} = 0\\
v :&\frac{\partial}{\partial u}\frac{\eta_2}{\varepsilon} = 0\\
u :&-\frac{\partial}{\partial v}\eta_2 + \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial u}\frac{\eta_2}{\varepsilon} = 0\\
uv :&\frac{\partial}{\partial u}\frac{\xi}{\varepsilon} = 0\\
u^2 :&-\frac{\partial}{\partial v}\xi + \frac{\partial}{\partial u}\frac{\xi}{\varepsilon} = 0\\
u^3 :&-\frac{\partial}{\partial u}\frac{\eta_2}{3\varepsilon} = 0\\
u^4 :&-\frac{\partial}{\partial u}\frac{\xi}{3\varepsilon} = 0
\end{aligned}
$$