A symmetry-based structural SI analysis of the SEI model

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1 Automated notes generated by the SymPy-based script

ODE for S:

$$\frac{d}{dt}S(t) = -\beta I(t)S(t) + c - \mu_S S(t) \tag{1}$$

ODE for E:

$$\frac{d}{dt}E(t) = \beta\left(1-\upsilon\right)I(t)S(t) - \delta E(t) - \mu_E E(t) \tag{2}$$

ODE for I:

$$\frac{d}{dt}I(t) = \beta \upsilon I(t)S(t) + \delta E(t) - \mu_I I(t) \tag{3}$$

2 Symmetry-based local SI analysis of the SEI model

We consider the following SEI model of epidemiological transmission of tuberculosis:

$$\dot{S} = -\beta I S + c - \mu_S S\,, \tag{4}$$

$$\dot{E} = \beta\left(1-\upsilon\right)IS - \delta E - \mu_E E\,, \tag{5}$$

$$\dot{I} = \beta \upsilon I S + \delta E - \mu_I I\,. \tag{6}$$

We also observe the following two outputs

$$y_E = k_E E\,, \tag{7}$$

$$y_I = k_I I\,, \tag{8}$$

and their interpretation is that we observe proportions k_E and k_I of the exposed and infected populations, respectively. In total, we have nine parameters collected in the vector $\boldsymbol{\theta} \in \mathbb{R}^9$

which are given by

$$\boldsymbol{\theta} = \begin{pmatrix} c \\ \beta \\ \mu_S \\ \mu_E \\ \mu_I \\ \delta \\ \upsilon \\ k_E \\ k_I \end{pmatrix} . \tag{9}$$

We are looking for a family of infinitesimal generators of the Lie group given by

$$\begin{aligned} X = {}& \xi(t,S,E,I,\boldsymbol{\theta})\partial_t + \eta_S(t,S,E,I,\boldsymbol{\theta})\partial_t + \eta_E(t,S,E,I,\boldsymbol{\theta})\partial_t + \eta_I(t,S,E,I,\boldsymbol{\theta})\partial_t \\ & + \chi_c(\boldsymbol{\theta})\partial_c + \chi_\beta(\boldsymbol{\theta})\partial_\beta + \chi_{\mu_S}(\boldsymbol{\theta})\partial_{\mu_S} + \chi_{\mu_E}(\boldsymbol{\theta})\partial_{\mu_E} + \chi_{\mu_I}(\boldsymbol{\theta})\partial_{\mu_I} + \chi_\delta(\boldsymbol{\theta})\partial_{\mu_\delta} \\ & + \chi_\upsilon(\boldsymbol{\theta})\partial_{\mu_\upsilon} + \chi_{k_E}(\boldsymbol{\theta})\partial_{k_E} + \chi_{k_I}(\boldsymbol{\theta})\partial_{k_I} . \end{aligned} \tag{10}$$

The three first prolongations are given by

$$\eta_S^{(1)}(t,S,E,I,\dot{S},\boldsymbol{\theta}) = D_t\eta_S(t,S,E,I,\boldsymbol{\theta}) - \dot{S}D_t\xi(t,S,E,I,\boldsymbol{\theta}) , \tag{11}$$

$$\eta_E^{(1)}(t,S,E,I,\dot{E},\boldsymbol{\theta}) = D_t\eta_E(t,S,E,I,\boldsymbol{\theta}) - \dot{E}D_t\xi(t,S,E,I,\boldsymbol{\theta}) , \tag{12}$$

$$\eta_I^{(1)}(t,S,E,I,\dot{S},\boldsymbol{\theta}) = D_t\eta_I(t,S,E,I,\boldsymbol{\theta}) - \dot{I}D_t\xi(t,S,E,I,\boldsymbol{\theta}) , \tag{13}$$

where the total derivative is defined by: $D_t = \partial_t + \dot{S}\partial_S + \dot{E}\partial_E + \dot{I}\partial_I$. These prolongations define the first prolongation of the infinitesimal generator $X^{(1)}$ according to

$$X^{(1)} = X + \eta_S^{(1)}\partial_{\dot{S}} + \eta_E^{(1)}\partial_{\dot{E}} + \eta_I^{(1)}\partial_{\dot{I}} . \tag{14}$$

Before, we define the linearised symmetry conditions, we make two critical simplifications. First, the model of interest is autonomous which implies that the time infinitesimal is a constant, i.e. $\xi(t,S,E,I) = K$ for some $K \in \mathbb{R}$ and thus $D_t\xi = 0$.

Second, the fact that the observed outputs are differential invariants of our generator yields equations for the infinitesimals η_E and η_I, respectively. Starting with the infinitesimal for E, we have that

$$X(y_E) = 0 \Longrightarrow \quad k_E\eta_E + \chi_{K_E}E = 0 ,$$

which gives us the following equation for η_E

$$\eta_E = -\left(\frac{\chi_{k_E}}{k_E}\right)E . \tag{15}$$

Analogously, the equation for η_I is given by

$$\eta_I = -\left(\frac{\chi_{k_I}}{k_I}\right)I . \tag{16}$$

Thus the corresponding prolongations simplify to

$$\eta_E^{(1)} = -\left(\frac{\chi_{k_E}}{k_E}\right)\dot{E} . \tag{17}$$

$$\eta_I^{(1)} = -\left(\frac{\chi_{k_I}}{k_I}\right)\dot{I} . \tag{18}$$