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Cell Migration

Symmetry

v. 33

$$\frac{\partial}{\partial z}\left(\frac{1}{u^l}\cdot\frac{\partial u}{\partial z}\right) + c\frac{\partial u}{\partial z} + f(u) = 0 \qquad (*)$$

Cell migration model: with $l = 0, 1, 2, 3$.

~~The~~ diffusive term becomes:

$$\frac{\partial}{\partial z}\left(u^{-l}\cdot u_z\right) = -l\,u^{-(l+1)}(u_z)^2 + u^{-l}u_{zz}$$

and then we multiply by u^l which gives us the new travelling wave

$$u_{zz} - \frac{l\cdot(u_z)^2}{u} + c\,u_z\cdot u^l + u^l f(u) = 0. \qquad (**)$$

So, now we want to derive our linearised symmetry condition. Let the infinitesimal generator of the Lie group

$$X = \xi(z,u)\partial_z + \eta(z,u)\partial_u. \qquad (1)$$

Now, we know at least one generator is the translation generator ~~[illegible]~~ $X = \partial_z$.

Our second prolongation is given by

$$X^{(2)} = \xi(z,u)\partial_z + \eta(z,u)\partial_u + \eta^{(1)}(z,u,u')\partial_{u'} + \eta^{(2)}(z,u,u',u'')\partial_{u''} \qquad (2)$$

where our two prolonged infinitesimal $\eta^{(1)}$ and $\eta^{(2)}$ are given by:

$$\eta^{(1)} = \eta_z + (\eta_u - \xi_z)(u') - \xi_u (u')^2, \qquad (3)$$

$$\eta^{(2)} = \eta_{zz} + (2\eta_{zu} - \xi_{zz})(u') + (\eta_{uu} - 2\xi_{zu})(u')^2 - \xi_{uu}(u')^3 + \{\eta_u - 2\xi_z - 3\xi_u (u')\}u''. \qquad (4)$$

the linearised symmetry condition

$$X^{(2)}\left(u_{zz} - \frac{l(u_z)^2}{u} + c u_z u^l + u^l f(u)\right) = 0 \text{ whenever } u_{zz} - \frac{l(u_z)^2}{u} + c u_z u^l + u^l f(u) = 0$$

(5).

Now, the linearity of $X^{(2)}$ gives

$$\eta^{(2)} - l X^{(2)}\left(\frac{(u_z)^2}{u}\right) + c X^{(2)}\left(u_z u^l\right) + X^{(2)}\left(u^l f(u)\right) = 0.$$

So, let's evaluate each of these terms.

$$X^{(2)}(u_{zz}) = \eta^{(2)},$$

$$X^{(2)}\left((u_z)^2 u^{-1}\right) = -u^{-2} u_z \eta + u^{-1} \eta^{(1)} \cdot 2(u_z) = \eta^{(1)} \frac{2u_z}{u} - \frac{u_z}{u^2}\eta,$$

$$X^{(2)}(u_z u^l) = l u^{l-1} \eta u_z + u^l \eta^{(1)} = u^l \eta^{(1)} + l u^{l-1} u_z \eta,$$

$$X^{(2)}(u^l f(u)) = \left(l u^{l-1} f(u) + u^l \frac{df}{du}\right)\eta = u^{l-1}\left(l f(u) + u \frac{df}{du}\right)\eta.$$

Now, we can assemble our linearised symmetry condition:

$$\eta^{(2)} - l\left(\eta^{(1)}\frac{2u_z}{u} - \frac{(u_z)^2}{u^2}\eta\right) + c\left(u^l \eta^{(1)} + l u^{l-1} u_z \eta\right) + u^{l-1}\left(l f(u) + u\frac{df}{du}\right)\eta$$

$$= \eta^{(2)} + \left(c u^l - \frac{2l u_z}{u}\right)\eta^{(1)} + \left(\frac{l(u_z)^2}{u^2} + c l u^{l-1} + u^{l-1}\left(l f(u) + u \frac{df}{du}\right)\right)\eta = 0$$

So our linearised symmetry condition:

$$\eta^{(2)} + \left(cu^{l} - \frac{2l u_z}{u}\right)\eta^{(1)} + \left(\frac{l(u_z)^2}{u^2} + clu^{l-1}u_z + u^{l-1}\left(lf(u) + u\frac{df}{du}\right)\right)\eta = 0 \qquad (6)$$

whenever

$$u_{zz} + \frac{l(u_z)^2}{u} + cu_z u^l + u^l f(u) = 0,$$

So, this seems to be our linearised symmetry condition. Let's see how we can write out our prolonged infinitesimals: Let's start with $\eta^{(2)}$, and, in particular, the last term of the prolonged infinitesimal:

$$\{\eta_u - 2\xi_z - 3\xi_u(u')\}u'' = (\eta_u - 2\xi_z)u'' - 3\xi_u(u')u''$$

$$= (\eta_u - 2\xi_z)\left(\frac{l(u')^2}{u} - cu'u^l - u^l f(u)\right) - 3\xi_u(u')\left(\frac{l(u')^2}{u} - cu'u^l - u^l f(u)\right)$$

$$= (u')^3\left(-\frac{3l}{u}\xi_u\right) + (u')^2\left(\frac{(\eta_u - 2\xi_z)l}{u} + 3c\xi_u u^l\right)$$

$$+ (u')\left(cu^l(2\xi_z - \eta_u) + 3u^l\xi_u f(u)\right)$$

$$+ (u')^0\left(u^l f(u)(2\xi_z - \eta_u)\right).$$

Ok, so that is pretty neat. It turned out, I had miscalculated one term previously which might change things dramatically. ~~Also~~ Also, we should state that $u' = u_{zz}$ and $u'' = u_{zz}$. Now, our second prolonged infinitesimal becomes:

$$\eta^{(2)} = \left(\eta_{zz} + u^l f(u)(2\xi_z - \eta_u)\right) + (u')\left(cu^l(2\xi_z - \eta_u) + 3u^l\xi_u f(u) + (2\eta_{zu} - \xi_{zz})\right)$$

$$+ (u')^2\left(\frac{(\eta_u - 2\xi_z)l}{u} + 3c\xi_u u^l + \eta_{uu} - 2\xi_{zu}\right)$$

$$+ (u')^3\left(-\xi_{uu} - \frac{3l}{u}\xi_u\right) =$$

$\dots$ Now, let's move on to the next term involving the first prolongation.

$$\left(cu^l + \frac{2l u'}{u}\right)\eta^{(1)} = \left(cu^l - \frac{2l u'}{u}\right)\left(\eta_z + (\eta_u - \xi_z)(u') - \xi_u (u')^2\right)$$

$$= (u')^3\left(+\frac{2l\,\xi_u}{u}\right) + (u')^2\left(-cu^l\xi_u - \frac{2l(\eta_u - \xi_z)}{u}\right)$$

$$+ (u')\left(cu^l(\eta_u - \xi_z) - \frac{2l}{u}\eta_z\right) + (u')^0\left(cu^l\eta_z\right).$$

Lastly, the final term is given by:

$$\left(\frac{l(u')^2}{u^2} + cl u^{l-1}(u') + u^{l-1}\left(l f(u) + u\frac{df}{du}\right)\right)\eta$$

$$= (u')^2\left(\frac{l\eta}{u^2}\right) + (u')\left(clu^{l-1}\eta\right) + \left(u^{l-1}\left(l f(u) + u\frac{df}{du}\right)\eta\right)(u')^0.$$

So, now we can assemble our ~~[illegible]~~ Sorry our determining equations I mean...

$\underline{(u')^3}$: $-\xi_{uu} - \frac{3l}{u}\xi_u + \frac{2l\,\xi_u}{u} = -\xi_{uu} - \frac{l}{u}\xi_u = 0,$

$\underline{(u')^2}$: $\frac{(\eta_u - 2\xi_z)l}{u} + 3c\xi_u u^l + \eta_{uu} - 2\xi_{zu} - cu^l\xi_u - \frac{2l(\eta_u - \xi_z)}{u} + \frac{l\eta}{u^2}$

$= 2c\xi_u u^l + \eta_{uu} - 2\xi_{zu} + \frac{(\eta_u - 2\xi_z)l}{u} + \frac{l\eta}{u^2} = 0$

$\underline{(u')^1}$: $cu^l(2\xi_z - \eta_u) + 3u^l\xi_u f(u) + (2\eta_{zu} - \xi_{zz}) + cu^l(\eta_u - \xi_z) - \frac{2l}{u}\eta_z$

$+ clu^{l-1}\eta = cu^l\xi_z + 3u^l\xi_u f(u) + 2\eta_{zu} - \xi_{zz} - \frac{2l}{u}\eta_z + clu^{l-1}\eta$

$= 0,$

$(u')^0$: $u^{l-1}\left(lf(u) + u\frac{df}{du}\right)\eta + cu^l\eta_z + \eta_{zz} + u^l f(u)(2\xi_z - \eta_u) = 0,$

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So after some tidying up, here comes our determining equations:

$$(u')^3:\quad \xi_{uu} + \frac{1}{u}\xi_u = 0, \qquad (7)$$

$$(u')^2:\quad 2c\,\xi_u u^l + \eta_{uu} - 2\xi_{zu} - \frac{(\eta_u - \lambda\xi_z)\,l}{u} + \frac{l\eta}{u^2} = 0, \qquad (8)$$

$$(u')^1:\quad c u^l \xi_z + 3u^l \xi_u f(u) + 2\eta_{zu} - \xi_{zz} - \frac{2l}{u}\eta_z + c\,l\,u^{l-1}\eta = 0, \qquad (9)$$

$$(u')^0:\quad u^{l-1}\left(l f(u) + u\frac{df}{du}\right)\eta + c u^l \eta_z + \eta_{zz} + u^l f(u)(\lambda\xi_z - \eta_u) = 0, \qquad (10)$$