Notes on determining equations

Johannes Borgqvist

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1 Introduction

We analyse the following second order travelling wave ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{u(z)^{\ell}}\frac{\mathrm{d}u}{\mathrm{d}z}\right)+c\frac{\mathrm{d}u}{\mathrm{d}z}+f(u(z))=0.$$

with $f(u)=c_0+c_1u+c_2u^2+c_3u^3$ and ℓ is an arbitrary index. This is a generalisation of the model in [1] that describes *cell migration in an epithelial tissue*, and this particular model is retrieved by the choices $\ell=2$ and $f(u)=u(1-u)$.

By denoting derivatives by $\mathrm{d}u/\mathrm{d}z=u'$ this ODE can be re-written as follows:

$$u''-\frac{\ell(u')^2}{u}+cu'u^{\ell}+u^{\ell}f(u)=0. \tag{1}$$

Now, we are interested in an infinitesimal generator of the Lie group

$$X=\xi(z,u)\partial_z+\eta(z,u)\partial_u$$

which has a second prolongation given by

$$X^{(2)}=\xi(z,u)\partial_z+\eta(z,u)\partial_u+\eta^{(1)}(z,u,u')\partial_{u'}+\eta^{(2)}(z,u,u',u'')\partial_{u''}.$$

Here, the two prolonged infinitesimals $\eta^{(1)}$ and $\eta^{(2)}$ are given by [2]

$$\begin{aligned}\eta^{(1)}(z,u,u') &= \eta_z + (\eta_u - \xi_z)u' - \xi_u\,(u')^2\,, &(2)\\ \eta^{(2)}(z,u,u',u'') &= \eta_{zz} + (2\eta_{zu} - \xi_{zz})u' + (\eta_{uu} - 2\xi_{zu})\,(u')^2 - \xi_{uu}\,(u')^3 \\ &\quad + \{\eta_u - 2\xi_z - 3\xi_u u'\}\,u''. &(3)\end{aligned}$$

The linearised symmetry condition for our ODE of interest is given by

$$\begin{aligned}&\eta^{(2)} + \left(cu^{\ell} - \frac{2\ell u'}{u}\right)\eta^{(1)} + \left(\frac{\ell(u')^2}{u^2} + c\ell u^{\ell-1}u' + u^{\ell-1}\left(\ell f(u) + u\frac{\mathrm{d}f}{\mathrm{d}u}\right)\right)\eta = 0 \\ &\qquad\qquad \text{whenever}\quad u'' - \frac{\ell(u')^2}{u} + cu'u^{\ell} + u^{\ell}f(u) = 0.\end{aligned} \tag{4}$$

Now, we will try to solve the linearised symmetry condition.

2 Four determining equations

By calculating the prolonged infinitesimals $\eta^{(1)}$ and $\eta^{(2)}$, plugging these into this linearised symmetry condition, and then organising the resulting equation in terms of powers of u' results in the following four so called *determining equations*

$$\begin{aligned}(u')^3:&\quad \xi_{uu} + \frac{\ell}{u}\xi_u &&= 0, &(5)\\ (u')^2:&\quad 2c\xi_u u^{\ell} + \eta_{uu} - 2\xi_{zu} - \frac{(\eta_u - 2\xi_z)\ell}{u} + \frac{\ell\eta}{u^2} &&= 0, &(6)\\ u':&\quad cu^{\ell}\xi_z + 3u^{\ell}\xi_u f(u) + 2\eta_{zu} - \xi_{zz} - \frac{2\ell}{u}\eta_z + c\ell u^{\ell-1}\eta &&= 0, &(7)\\ 1:&\quad u^{\ell-1}\left(\ell f(u) + u\frac{\mathrm{d}f}{\mathrm{d}u}\right)\eta + cu^{\ell}\eta_z + \eta_{zz} + u^{\ell}f(u)(2\xi_z - \eta_u) &&= 0. &(8)\end{aligned}$$

Now, we will treat these four equations systematically, and solve them one by one. We will use our friend *SymPy* to do this.

3 The first determining equation

Ok, so we are interested in the following PDE

$$\xi_{uu} + \frac{\ell}{u}\xi_u = 0.$$

This one we solve by hand which gives us the following equation

$$\xi(z,u) = A(z)\left(\frac{1}{1-\ell}\right)u^{1-\ell} - B(z) \tag{9}$$

where $A, B \in \mathcal{C}^{\infty}(\mathbb{R})$.

4 The second determining equation

Ok, so we are interested in solving the following PDE

$$2c\xi_u u^\ell + \eta_{uu} - 2\xi_{zu} - \frac{(\eta_u - 2\xi_z)\ell}{u} + \frac{\ell\eta}{u^2} = 0$$

for the unknown tangent $\eta(z, u)$. We plugged this PDE into Wolphram Alpha, and out came the following suggested solution

$$\eta(z,u) = 2cA(z)\left(\frac{1}{\ell-2}\right)u^2 + A'(z)\left(\frac{2\ell-1}{(\ell-1)^2}\right)u^{2-\ell} + 2B'(z)\left(\frac{\ell}{\ell-1}\right)u\ln(u) + C(z)u^\ell + D(z)u \tag{10}$$

where $C, D \in \mathcal{C}^\infty(\mathbb{R})$ are two new arbitrary functions.

5 The third determining equation

Ok, thus far we have landed at the following infinitesimals:

$$\xi(z,u) = A(z)\left(\frac{1}{1-\ell}\right)u^{1-\ell} - B(z), \tag{11}$$

$$\eta(z,u) = 2cA(z)\left(\frac{1}{\ell-2}\right)u^2 + A'(z)\left(\frac{2\ell-1}{(\ell-1)^2}\right)u^{2-\ell} + 2B'(z)\left(\frac{\ell}{\ell-1}\right)u\ln(u) + C(z)u^\ell + D(z)u. \tag{12}$$

Next, we want to use following PDE

$$cu^\ell\xi_z + 3u^\ell\xi_u f(u) + 2\eta_{zu} - \xi_{zz} - \frac{2\ell}{u}\eta_z + c\ell u^{\ell-1}\eta = 0$$

in order to get equations that we can solve for the four unknown functions $A, B, C, D \in \mathcal{C}^\infty(\mathbb{R})$. So what we can do here is to plug in our unknown functions, and the we see that the above equation entails finding the roots of a polynomial in u. Hence, the equation decomposes into a set of sub equations which we can solve individually.

Here, we are going to assume a general cubic reaction term

$$f(u) = c_0 + c_1u + c_2u^2 + c_3u^3. \tag{13}$$

where $c_0, c_1, c_2, c_3 \in \mathbb{R}$ are four arbitrary constants.

By plugging in the tangents, into the third linearised symmetry condition, we get the following equation:

$$
\begin{aligned}
2A(z)c^2l^4u^4u^l - 6A(z)c^2l^3u^4u^l \\
+6A(z)c^2l^2u^4u^l - 2A(z)c^2lu^4u^l \\
-4A(z)cl^4u^4 + 20A(z)cl^3u^4 - 36A(z)cl^2u^4 \\
+28A(z)clu^4 - 8A(z)cu^4 + 3A(z)c_0l^4u^3 \\
-15A(z)c_0l^3u^3 + 27A(z)c_0l^2u^3 - 21A(z)c_0lu^3 + 6A(z)c_0u^3 \\
+3A(z)c_1l^4u^4 - 15A(z)c_1l^3u^4 + 27A(z)c_1l^2u^4 \\
-21A(z)c_1lu^4 + 6A(z)c_1u^4 + 3A(z)c_2l^4u^5 \\
-15A(z)c_2l^3u^5 + 27A(z)c_2l^2u^5 - 21A(z)c_2lu^5 + 6A(z)c_2u^5 + 3A(z)c_3l^4u^6 \\
-15A(z)c_3l^3u^6 + 27A(z)c_3l^2u^6 - 21A(z)c_3lu^6 + 6A(z)c_3u^6 \\
-7A''(z)l^3u^4u^{-l} + 24A''(z)l^2u^4u^{-l} - 23A''(z)lu^4u^{-l} + 6A''(z)u^4u^{-l} \\
+A'(z)cl^3u^4 - A'(z)cl^2u^4 - 3A'(z)clu^4 + 2A'(z)cu^4 - 4B''(z)l^5u^3\ln(u) \\
+20B''(z)l^4u^3\ln(u) + 3B''(z)l^4u^3 \\
-36B''(z)l^3u^3\ln(u) - 11B''(z)l^3u^3 + 28B''(z)l^2u^3\ln(u) \\
+11B''(z)l^2u^3 - 8B''(z)lu^3\ln(u) - B''(z)lu^3 - 2B''(z)u^3 \\
+2B'(z)cl^5u^3u^l\ln(u) \\
-8B'(z)cl^4u^3u^l\ln(u) + B'(z)cl^4u^3u^l \\
+10B'(z)cl^3u^3u^l\ln(u) - 5B'(z)cl^3u^3u^l \\
-4B'(z)cl^2u^3u^l\ln(u) + 9B'(z)cl^2u^3u^l \\
-7B'(z)clu^3u^l + 2B'(z)cu^3u^l + C(z)cl^5u^2u^{2l} - 5C(z)cl^4u^2u^{2l} \\
+9C(z)cl^3u^2u^{2l} - 7C(z)cl^2u^2u^{2l} + 2C(z)clu^2u^{2l} + D(z)cl^5u^3u^l \\
-5D(z)cl^4u^3u^l + 9D(z)cl^3u^3u^l \\
-7D(z)cl^2u^3u^l + 2D(z)clu^3u^l - 2D'(z)l^5u^3 \\
+12D'(z)l^4u^3 - 28D'(z)l^3u^3 \\
+32D'(z)l^2u^3 - 18D'(z)lu^3 + 4D'(z)u^3 = 0.
\end{aligned}
\tag{14}
$$

We see that in Equation (14), we have a lot of terms involving the function $\ln(u)$. So, we can actually further split this equation into two smaller equations based on the terms involving $\ln(u)$ and the terms not involving $\ln(u)$. Here, is the equation where the terms involve $\ln(u)$:

$$
\begin{aligned}
-4B''(z)l^5u^3 + 20B''(z)l^4u^3 - 36B''(z)l^3u^3 \\
+28B''(z)l^2u^3 - 8B''(z)lu^3 + 2B'(z)cl^5u^3u^l \\
-8B'(z)cl^4u^3u^l + 10B'(z)cl^3u^3u^l - 4B'(z)cl^2u^3u^l = 0.
\end{aligned}
\tag{15}
$$

Here, is the equation where the terms do not involve $\ln(u)$:

$$
\begin{aligned}
&2A(z)c^2l^4u^4u^l - 6A(z)c^2l^3u^4u^l + 6A(z)c^2l^2u^4u^l \\
&-2A(z)c^2lu^4u^l - 4A(z)cl^4u^4 + 20A(z)cl^3u^4 \\
&-36A(z)cl^2u^4 + 28A(z)clu^4 - 8A(z)cu^4 \\
&+3A(z)c_0l^4u^3 - 15A(z)c_0l^3u^3 + 27A(z)c_0l^2u^3 \\
&-21A(z)c_0lu^3 + 6A(z)c_0u^3 + 3A(z)c_1l^4u^4 \\
&-15A(z)c_1l^3u^4 + 27A(z)c_1l^2u^4 - 21A(z)c_1lu^4 \\
&+6A(z)c_1u^4 + 3A(z)c_2l^4u^5 - 15A(z)c_2l^3u^5 \\
&+27A(z)c_2l^2u^5 - 21A(z)c_2lu^5 + 6A(z)c_2u^5 + \\
&3A(z)c_3l^4u^6 - 15A(z)c_3l^3u^6 + 27A(z)c_3l^2u^6 \\
&-21A(z)c_3lu^6 + 6A(z)c_3u^6 - 7A''(z)l^3u^4u^{-l} \\
&+24A''(z)l^2u^4u^{-l} - 23A''(z)lu^4u^{-l} + 6A''(z)u^4u^{-l} \\
&+A'(z)cl^3u^4 - A'(z)cl^2u^4 - 3A'(z)clu^4 \\
&+2A'(z)cu^4 + 3B''(z)l^4u^3 - 11B''(z)l^3u^3 \\
&+11B''(z)l^2u^3 - B''(z)lu^3 - 2B''(z)u^3 \\
&+B'(z)cl^4u^3u^l - 5B'(z)cl^3u^3u^l + 9B'(z)cl^2u^3u^l \\
&-7B'(z)clu^3u^l + 2B'(z)cu^3u^l + C(z)cl^5u^2u^{2l} \\
&-5C(z)cl^4u^2u^{2l} + 9C(z)cl^3u^2u^{2l} - 7C(z)cl^2u^2u^{2l} \\
&+2C(z)clu^2u^{2l} + D(z)cl^5u^3u^l - 5D(z)cl^4u^3u^l \\
&+9D(z)cl^3u^3u^l - 7D(z)cl^2u^3u^l + 2D(z)clu^3u^l \\
&-2D'(z)l^5u^3 + 12D'(z)l^4u^3 - 28D'(z)l^3u^3 \\
&+32D'(z)l^2u^3 - 18D'(z)lu^3 + 4D'(z)u^3 = 0.
\end{aligned}
\tag{16}
$$

Next, we will plug in the values $l = 0, 1, 2$ and then further split these equations in terms of the monomials.

5.1 $l = 0$

$$
\begin{aligned}
u^3 &: 6A(z)c_0 - 2B''(z) + 2B'(z)c + 4D'(z) &&= 0, \\
u^4 &: -8A(z)c + 6A(z)c_1 + 6A''(z) + 2A'(z)c &&= 0, \\
u^5 &: 6A(z)c_2 &&= 0, \\
u^6 &: 6A(z)c_3 &&= 0.
\end{aligned}
$$

From these determining equations, we see that

$$A(z) = 0 \forall z \in \mathbb{R}$$

and that

$$D(z) = \frac{1}{2}\left(B'(z) + B(z)\right) + K \quad K \in \mathbb{R}.$$

5.2 $l = 1$

In the case when $l = 1$, all sub equations were set to 0, so we gain no extra information from the third determining equation.

5.3 $l = 2$

In the case when $l = 2$, all sub equations were set to 0, so we gain no extra information from the third determining equation.

6 The fourth determining equation

In summary, we have two different infinitesimal depending on whether $l = 0$ or if $l > 0$. In the former case, the infinitesimals are given by

$$\xi(z,u) = -B(z), \tag{17}$$

$$\eta(z,u) = -2B'(z)u\ln(u) + C(z) + \left(\frac{1}{2}\left(B'(z) + B(z)\right) + K\right)u, \quad K \in \mathbb{R}. \tag{18}$$

In the latter case, when $l > 0$ the infinitesimals are still given by

$$\xi(z,u) = A(z)\left(\frac{1}{1-\ell}\right)u^{1-\ell} - B(z), \tag{19}$$

$$\eta(z,u) = 2cA(z)\left(\frac{1}{\ell-2}\right)u^2 + A'(z)\left(\frac{2\ell-1}{(\ell-1)^2}\right)u^{2-\ell} + 2B'(z)\left(\frac{\ell}{\ell-1}\right)u\ln(u) + C(z)u^\ell + D(z)u. \tag{20}$$