Notes on determining equations

Johannes Borgqvist

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1 Introduction

We analyse the following second order travelling wave ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{u(z)^{\ell}}\frac{\mathrm{d}u}{\mathrm{d}z}\right)+c\frac{\mathrm{d}u}{\mathrm{d}z}+f(u(z))=0.$$

with $f(u)=c_0+c_1u+c_2u^2+c_3u^3$ and ℓ is an arbitrary index. This is a generalisation of the model in [1] that describes *cell migration in an epithelial tissue*, and this particular model is retrieved by the choices $\ell=2$ and $f(u)=u(1-u)$.

By denoting derivatives by $\mathrm{d}u/\mathrm{d}z=u'$ this ODE can be re-written as follows:

$$u''-\frac{\ell(u')^2}{u}+cu'u^{\ell}+u^{\ell}f(u)=0. \tag{1}$$

Now, we are interested in an infinitesimal generator of the Lie group

$$X=\xi(z,u)\partial_z+\eta(z,u)\partial_u$$

which has a second prolongation given by

$$X^{(2)}=\xi(z,u)\partial_z+\eta(z,u)\partial_u+\eta^{(1)}(z,u,u')\partial_{u'}+\eta^{(2)}(z,u,u',u'')\partial_{u''}.$$

Here, the two prolonged infinitesimals $\eta^{(1)}$ and $\eta^{(2)}$ are given by [2]

$$\eta^{(1)}(z,u,u')=\eta_z+(\eta_u-\xi_z)u'-\xi_u\left(u'\right)^2, \tag{2}$$

$$\begin{aligned}\eta^{(2)}(z,u,u',u'')=&\ \eta_{zz}+(2\eta_{zu}-\xi_{zz})u'+\left(\eta_{uu}-2\xi_{zu}\right)\left(u'\right)^2-\xi_{uu}\left(u'\right)^3\\&+\left\{\eta_u-2\xi_z-3\xi_uu'\right\}u''.\end{aligned} \tag{3}$$

The linearised symmetry condition for our ODE of interest is given by

$$\begin{aligned}&\eta^{(2)}+\left(cu^{\ell}-\frac{2\ell u'}{u}\right)\eta^{(1)}+\left(\frac{\ell(u')^2}{u^2}+c\ell u^{\ell-1}u'+u^{\ell-1}\left(\ell f(u)+u\frac{\mathrm{d}f}{\mathrm{d}u}\right)\right)\eta=0\\&\qquad\text{whenever}\quad u''-\frac{\ell(u')^2}{u}+cu'u^{\ell}+u^{\ell}f(u)=0.\end{aligned} \tag{4}$$

Now, we will try to solve the linearised symmetry condition.

2 Four determining equations

By calculating the prolonged infinitesimals $\eta^{(1)}$ and $\eta^{(2)}$, plugging these into this linearised symmetry condition, and then organising the resulting equation in terms of powers of u' results in the following four so called *determining equations*

$$\begin{aligned}
(u')^3: &\quad \xi_{uu}+\frac{\ell}{u}\xi_u &&=0, && (5)\\
(u')^2: &\quad 2c\xi_u u^{\ell}+\eta_{uu}-2\xi_{zu}-\frac{\ell\eta_u}{u}+\frac{\ell\eta}{u^2} &&=0, && (6)\\
u': &\quad cu^{\ell}\xi_z+3u^{\ell}\xi_u f(u)+2\eta_{zu}-\xi_{zz}-\frac{2\ell}{u}\eta_z+c\ell u^{\ell-1}\eta &&=0, && (7)\\
1: &\quad u^{\ell-1}\left(\ell f(u)+u\frac{\mathrm{d}f}{\mathrm{d}u}\right)\eta+cu^{\ell}\eta_z+\eta_{zz}+u^{\ell}f(u)(2\xi_z-\eta_u) &&=0. && (8)
\end{aligned}$$

Now, we will treat these four equations systematically, and solve them one by one. We will use our friend *SymPy* to do this.

3 The pure diffusion case, $l=0$

Here, well summarise the symmetry calculations for the RD travelling wave equation given by

$$u''(z)+cu'(z)+f(u(z))=0 \tag{9}$$

where f is chosen to a general cubic function $f(u)=c_0+c_1u+c_2u^2+c_3u^3$. In this case, our determining equations become:

$$\begin{aligned}
(u')^3: &\quad \xi_{uu} &&=0, && (10)\\
(u')^2: &\quad 2c\xi_u+\eta_{uu}-2\xi_{zu} &&=0, && (11)\\
u': &\quad c\xi_z+3\xi_u f(u)+2\eta_{zu}-\xi_{zz} &&=0, && (12)\\
1: &\quad \frac{\mathrm{d}f}{\mathrm{d}u}\eta+c\eta_z+\eta_{zz}+f(u)(2\xi_z-\eta_u) &&=0. && (13)
\end{aligned}$$

So, let's tackle these systematically one by one. From (10), it follows that

$$\xi(z,u)=A(z)u+B(z). \tag{14}$$

By plugging this into (11), we get

$$\eta(z,u)=2(A'(z)-cA(z))u+C(z). \tag{15}$$

Now, plugging in (14) and (15) into (12) yields

$$c(A'(z)-B'(z))+3A(z)(c_0+c_1u+c_2u^2+c_3u^3)+4(A''(z)-cA'(z))-(A''(z)u+B''(z))=0. \tag{16}$$

Now, let's do everything we can to make sure that $A(z)\neq 0$. We see that the above equation decomposes into four sub equations since it amounts to finding the roots of a polynomial depending on the monomials $\{1,u,u^2,u^3\}$. The corresponding equations, we get are

$$\begin{aligned}
1: &-B''(z)-cB'(z)+4A''(z)-3cA'(z)+3c_0A(z) &&=0,\\
u: &3c_1A(z)-A''(z) &&=0,\\
u^2: &3c_2A(z) &&=0,\\
u^3: &3c_3A(z) &&=0.
\end{aligned}$$

Here, we get two clear cases. In the first case where $c_2 \neq 0$ or $c_3 \neq 0$ then $A(z) = 0$. If you follow all calculations through under these assumptions you get that the only generator we have left is the translation generator given by $X = \partial_z$.

Now, let's assume that $c_2 = c_3 = 0$ which gives us a linear reaction term $f(u) = c_0 + c_1 u$. In this case, the second equation above gives us that

$$A(z) = K_1 \exp\left(\sqrt{3c_1}z\right) + K_2 \exp\left(-\sqrt{3c_1}z\right)$$

for two arbitrary constants K_1 and K_2. Now, inserting this solution into the first equation above, we get that the first equation becomes

$$B''(z)+cB'(z) = \left(12\,c_1 - 3c\sqrt{3c_1} + 3c_0\right) K_1 \exp\left(\sqrt{3c_1}z\right)+\left(12\,c_1 + 3c\sqrt{3c_1} + 3c_0\right) K_2 \exp\left(-\sqrt{3c_1}z\right).$$

The solution to this equation is quite big, but it is given by the following equation

$$\begin{aligned} B(z) = & \frac{\left(12\,c_1 - 3c\sqrt{3c_1} + 3c_0\right)}{\sqrt{3c_1}(c+\sqrt{3c_1})} K_1 \exp\left(\sqrt{3c_1}z\right) + \frac{\left(12\,c_1 + 3c\sqrt{3c_1} + 3c_0\right)}{\sqrt{3c_1}(c-\sqrt{3c_1})} K_1 \exp\left(-\sqrt{3c_1}z\right) \\ & + K_3 \frac{\exp\left(-cz\right)}{c} + K_4. \end{aligned} \tag{17}$$

Ok, so what we have left is to determine is the function $C(z)$ in (15) which we will do using the determining equation in (13). We have that (13) can be written as follows

$$\begin{aligned} c_1\left[2(A'(z) - cA(z))u + C(z)\right] + c\left[2(A''(z) - cA'(z))u + C'(z)\right] \\ + \left[2(A'''(z) - cA''(z))u + C''(z)\right] \\ + (c_0 + c_1 u)\left(2A'(z)u + 2B'(z) - 2(A'(z) - cA(z))\right) = 0. \end{aligned} \tag{18}$$

Now, if we study this equation in terms of the monomials $\{1, u, u^2\}$, we see that the coefficient in front of u^2 is $2c_1 A'(z)$. Again, we get two cases here. Either, we assume that $c_1 = 0$ which corresponds to a very boring case of $f(u) = c_0$. So instead we will now impose the condition that $c_1 \neq 0$. This yields that $A'(z) = 0$ which, in turn, yields that $A''(z) = A'''(z) = 0$. In addition, since $A'(z) = \sqrt{3c_1}K_1 \exp\left(\sqrt{3c_1}z\right) - \sqrt{3c_1}K_2 \exp\left(-\sqrt{3c_1}z\right) = 0$, we must have that $K_1 = K_2 = 0$. So at this point we have that

$$A(z) = 0, \quad B(z) = K_3 \frac{\exp\left(-cz\right)}{c} + K_4$$

which means that

$$\xi(z,u) = B(z), \quad \eta(z,u) = C(z).$$

If we plug this into (18) we get

$$c_1 C(z) + cC'(z) + C''(z) + (c_0 + c_1 u)\,2B'(z) = 0.$$

which we can split up in terms of the monomials u

$$\begin{aligned} 1 :& c_1 C(z) + cC'(z) + C''(z) + 2c_0 B'(z) &= 0, \\ u :& 2c_1 B'(z) &= 0. \end{aligned}$$

From this we get that $B'(z) = 0$ which means that $K_3 = 0$ which means that $B(z) = K_4$. Moreover, if we assume that $C(z) = \exp\left(\sigma z\right)$ for some σ, the characteristic polynomial corresponding to the top equation is given by

$$c_1 + c\sigma + \sigma^2 = 0$$

and the solutions to this equation is given by

$$\sigma = \frac{1}{2}\left(-c \pm \sqrt{c^2 - 4c_1}\right).$$

So all in all we seem to have three generators: $X_1 = \partial_z$, $X_2 = \exp\left(\frac{1}{2}\left(-c + \sqrt{c^2 - 4c_1}\right) z\right)\partial_u$ and $X_3 = \exp\left(-\frac{1}{2}\left(c + \sqrt{c^2 - 4c_1}\right) z\right)\partial_u$. Let's summarise all of this in a theorem.

Theorem 3.1 (**Symmetries of the RD travelling wave equation**). *Consider the travelling wave equation in* (9) *with a cubic reaction term:*

$$f(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3. \tag{19}$$

Then, the infinitesimal generators of the Lie group are given by:

1. ***The quadratic*** $(c_2 \neq 0)$ ***and cubic case*** $(c_3 \neq 0)$*:*

$$X_1 = \partial_z. \tag{20}$$

2. ***The linear case*** $(c_0 \neq 0$***,*** $c_1 \neq 0$ ***and*** $c_2 = c_3 = 0)$*:*
In addition to the translation generator X_1 *in* (20) *we have two generators given by*

$$X_2 = \exp\left(\frac{1}{2}\left(-c + \sqrt{c^2 - 4c_1}\right) z\right)\partial_u, \tag{21}$$

$$X_3 = \exp\left(-\frac{1}{2}\left(c + \sqrt{c^2 - 4c_1}\right) z\right)\partial_u, \tag{22}$$

where c *is the wave speed in* $z = x - ct$*.*

In particular, if we look at the growth term $f(u) = u$ where $c_1 = 1$ which corresponds to exponential growth, the travelling wave solution to (9) is in this case given by

$$u(z) = K_1 \exp\left(\frac{1}{2}\left(-c + \sqrt{c^2 - 4}\right) z\right) + K_2 \exp\left(-\frac{1}{2}\left(c + \sqrt{c^2 - 4}\right) z\right), \quad K_1, K_2 \in \mathbb{R}. \tag{23}$$

Now, we would like to find symmetries for a second order growth term $f(u) = u(1-u)$ corresponding to logistic growth and even a third order growth term $f(u) = u(K-u)(1-u)$ corresponding to a growth term with an Allee effect.

4 The travelling wave equation when $l = 2$

Now, we will look at the following travelling wave ODE:

$$u'' - \frac{2(u')^2}{u} + cu'u^2 + u^2 f(u) = 0. \tag{24}$$

The corresponding determining equations we want to solve are the following:

$$(u')^3: \quad \xi_{uu} + \frac{2}{u}\xi_u = 0, \tag{25}$$

$$(u')^2: \quad 2c\xi_u u^2 + \eta_{uu} - 2\xi_{zu} - \frac{2\eta_u}{u} + \frac{2\eta}{u^2} = 0, \tag{26}$$

$$u': \quad cu^2\xi_z + 3u^2\xi_u f(u) + 2\eta_{zu} - \xi_{zz} - \frac{4}{u}\eta_z + 2cu\eta = 0, \tag{27}$$

$$1: \quad \left(2uf(u) + u^2\frac{\mathrm{d}f}{\mathrm{d}u}\right)\eta + cu^2\eta_z + \eta_{zz} + u^2 f(u)(2\xi_z - \eta_u) = 0. \tag{28}$$

Starting by solving (25), we have that the solution is given by:

$$\xi(z,u) = -\frac{A(z)}{u} + B(z). \tag{29}$$

By plugging in this equation into (27), we obtain the following PDE

$$\eta_{uu} - 2\frac{\eta_u}{u} + 2\frac{\eta}{u^2} = 2\frac{A'(z)}{u^2} - 2cA(z). \tag{30}$$

The solution to this equation is given by

$$\eta(z,u) = C(z)u^2 + D(z)u - 2cA(z)u^2\ln(u) + A'(z). \tag{31}$$

Ok, so now we will plug in our two tangents into (27) and see what we get. The third determining equation is:

$$\begin{aligned} -4A(z)c^2u^3\ln(u) + 3A(z)c_0u + 3A(z)c_1u^2 + 3A(z)c_2u^3 + \\ 3A(z)c_3u^4 - 3A''(z) - 5A'(z)cu^2 + 2A'(z)cu - B''(z)u + \\ B'(z)cu^3 + 2C(z)cu^3 + 2D(z)cu^2 - 2D''(z)u = 0. \end{aligned} \tag{32}$$

Now, the coefficient in front of the basis function $u^3\ln(u)$ is given by $-4A(z)c^2$, and thus it follows that $A(z) = 0 \implies A'(z) = 0 \implies A''(z) = 0$. After plugging this into (32), we obtain

$$-B''(z)u + B'(z)cu^3 + 2C(z)cu^3 + 2D(z)cu^2 - 2D''(z)u = 0$$

which can be split up with respect to the monomials $\{u, u^2, u^3\}$. This yields the following equations

$$\begin{aligned} u :& -B''(z) - 2D''(z) &&= 0, \\ u^2 :& 2D(z)c &&= 0, \\ u :& cB'(z) + 2cC(z) &&= 0. \end{aligned}$$

The second equation yields that $D(z) = 0 \implies D'(z) = 0 \implies D''(z) = 0$. Inserting this into the first equation yields that $B''(z) = 0 \implies B'(z) = K_1 \implies B(z) = K_1z + K_2$. And inserting this into the last equation gives us that $C(z) = -B'(z)/2 = -K_1/2$. All in all, this gives us the following tangents

$$\xi(z,u) = K_1z + K_2, \tag{33}$$

$$\eta(z,u) = -\frac{K_1}{2}u^2. \tag{34}$$