Math Methods Assignment #4

Johannes Byle

September 24, 2021

1. (a) Using the following conversions between cartesian coordinates and the coordinates of our system:

$$x_1 = l_1 \sin\varphi_1 \quad y_1 = -l_1 \cos\varphi_1$$
$$x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 \quad y_2 = -l_1 \cos\varphi_1 - l_2 \cos\varphi_2$$

Taking the derivatives:

$$\dot{x}_1 = l_1 \dot{\varphi}_1 \cos\varphi_1 \quad \dot{y}_1 = l_1 \dot{\varphi}_1 \sin\varphi_1$$
$$\dot{x}_2 = l_1 \dot{\varphi}_1 \cos\varphi_1 + l_2 \dot{\varphi}_2 \cos\varphi_2 \quad \dot{y}_2 = l_1 \dot{\varphi}_1 \sin\varphi_1 + l_2 \dot{\varphi}_2 \sin\varphi_2$$

This gives us the following values for the kinetic and potential energy:

$$V = m_1 g y_1 + m_2 g y_2 = -m_1 g l_1 \cos\varphi_1 + m_2 g \left(-l_1 \cos\varphi_1 - l_2 \cos\varphi_2\right)$$
$$T = \frac{1}{2}\left[m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)\right]$$
$$T = \frac{1}{2}\left[m_1\left[(l_1\dot{\varphi}_1\cos\varphi_1)^2 + (l_1\dot{\varphi}_1\sin\varphi_1)^2\right] + \right.$$
$$\left. m_2\left[(l_1\dot{\varphi}_1\cos\varphi_1 + l_2\dot{\varphi}_2\cos\varphi_2)^2 + (l_1\dot{\varphi}_1\sin\varphi_1 + l_2\dot{\varphi}_2\sin\varphi_2)^2\right]\right]$$
$$T = \frac{1}{2}\left[m_1 l_1^2 \dot{\varphi}_1^2 + m_2\left(l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right)\right]$$

(b) In the case where $l_1 = l_2 = l$ and $m_1 = m_2 = m$:

$$V = mgl(2\cos\varphi_1 - \cos\varphi_2)$$
$$T = \frac{1}{2}l^2 m\left[2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right]$$
$$L = \frac{1}{2}l^2 m\left[2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right] - mgl(2\cos\varphi_1 - \cos\varphi_2)$$

Solving the Lagrangian for φ_1:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) - \frac{\partial L}{\partial \varphi_1} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = \frac{1}{2}l^2 m\left(2\phi_2'\left(\phi_2' - \phi_1'\right)\sin\left(\phi_1 - \phi_2\right) + 4\phi_1'' + 2\phi_2''\cos\left(\phi_1 - \phi_2\right)\right)$$
$$\frac{\partial L}{\partial \varphi_1} = lm\left(2g\sin\left(\phi_1\right) - l\phi_1'\phi_2'\sin\left(\phi_1 - \phi_2\right)\right)$$
$$lm\left(-2g\sin\left(\phi_1\right) + l{\phi_2'}^2\sin\left(\phi_1 - \phi_2\right) + 2l\phi_1'' + l\phi_2''\cos\left(\phi_1 - \phi_2\right)\right) = 0$$

Repeating the same process for φ_2:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) - \frac{\partial L}{\partial \varphi_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = l^2 m \left(\phi_1' \left(\phi_2' - \phi_1'\right) \sin\left(\phi_1 - \phi_2\right) + \phi_2'' + \phi_1'' \cos\left(\phi_1 - \phi_2\right)\right)$$

$$\frac{\partial L}{\partial \varphi_1} = lm \left(l \phi_1' \phi_2' \sin\left(\phi_1 - \phi_2\right) - g \sin\left(\phi_2\right)\right)$$

$$lm \left(g \sin\left(\phi_2\right) + l \left(\phi_1'^{\,2} \left(-\sin\left(\phi_1 - \phi_2\right)\right) + \phi_2'' + \phi_1'' \cos\left(\phi_1 - \phi_2\right)\right)\right) = 0$$

(c) Both expressions seems similar, they are coupled, non-linear differential equations.

(d) In the case where $\dot{\varphi}_1 = 0$ (and by definition $\ddot{\varphi}_1 = 0$) the first equation is trivially zero, the second equation is:

$$lm \left(g \sin\left(\phi_2\right) + l \phi_2''\right) = 0$$

(e) Applying the small angle approximation, and starting with the following Lagrangian:

$$L = \frac{1}{2} l^2 m \left(2 \phi_1'^{\,2} + 2 \left(\phi_1 - \phi_2\right) \phi_2' \phi_1' + \phi_2'^{\,2}\right) - glm \left(2\phi_1 - \phi_2\right)$$

Results in the following equation of motion for φ_2:

$$lm \left(-2 \left(g + l \phi_1''\right) + l \phi_2'^{\,2} + l \left(\phi_2 - \phi_1\right) \phi_2''\right) = 0$$

And the following equation of motion for φ_2:

$$lm \left(l \left(\phi_1'^{\,2} + \left(\phi_1 - \phi_2\right) \phi_1'' + \phi_2''\right) - g\right) = 0$$

2. (a) Using conservation of energy we can find the velocity:

$$\frac{1}{2} m v^2 = mgy$$

$$v = \sqrt{2gy}$$

This gives the following time of descent:

$$t = \int \frac{ds}{v} = \frac{1}{2g} \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}$$

Converting to an integral of dy:

$$t = \frac{1}{2g} \int dy \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}}{\sqrt{y}} = \int_{y_1}^{y_2} dy \sqrt{\frac{x'^2 + 1}{y}}$$

(b) Taking the lagrangian:

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial \dot{x}}\right)$$
$$\frac{\partial F}{\partial x} = 0$$
$$\frac{d}{dy}\left(\frac{\partial F}{\partial \dot{x}}\right) = \frac{d}{dy}\left(\frac{1}{\sqrt{2g}}\frac{x'}{y}\sqrt{\frac{y}{x'^2+1}}\right)$$

Since $\frac{\partial F}{\partial x} = 0$ is zero we can simply integrate:

$$\frac{1}{\sqrt{2g}}\frac{x'}{y}\sqrt{\frac{y}{x'^2+1}} = c$$

Solving for x':

$$\frac{dx}{dy} = \frac{c\sqrt{2gy}}{\sqrt{1-2c^2gy}}$$

Integrating:

$$y_1 - y_2 = \int \frac{\sqrt{1-2c^2gy}}{c\sqrt{2gy}}dx$$

Im pretty sure this integral can be solved with the change of variables $x = \frac{c^2}{4g}(\theta - \sin\theta)$ and $y = \frac{c^2}{4g}(1-\cos\theta)$.

3. Since this is the Lagrangian, writing it in the form of the action I and differentiating $I(\epsilon)$ with respect to ϵ:

$$\frac{dI}{d\epsilon} = \int_{t_A}^{t_B}\left[\frac{\delta f}{\delta q}\frac{dq}{d\epsilon} + \frac{\delta f}{\delta q'}\frac{dq'}{d\epsilon} + \frac{\delta f}{\delta q''}\frac{dq''}{d\epsilon}\right]$$

Since q' endpoints are prescribed, the integration by parts trick used when deriving the Lagrangian will work on both the $\frac{\delta f}{\delta y'}\frac{dy'}{d\epsilon}$ and $\frac{\delta f}{\delta y''}\frac{dy''}{d\epsilon}$ terms.

$$\frac{dI}{d\epsilon} = \int_{t_A}^{t_B}\left[\frac{\delta f}{\delta q} - \frac{d}{dt}\left(\frac{\delta f}{\delta q'}\right) - \frac{d^2}{dt^2}\left(\frac{\delta f}{\delta q'}\right)\right]\frac{dq}{d\epsilon}dx$$

This requires that $y(x,\epsilon)$ and all its derivatives through third order are continuous functions of x and ϵ

4. (a) The particle is confined to the hoop and thus can only move around the hoop and thus is:

$$T = \frac{1}{2}m\left(R^2\sin^2\theta\omega^2 + R\dot{\theta}^2\right)$$
$$V = -mgy = -mgR\cos\theta$$
$$L = T - V = \frac{1}{2}m\left(R^2\sin^2\theta\omega^2 + R\dot{\theta}^2\right) - mgR\cos\theta$$

Solving the Lagrangian:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR\ddot{\theta}$$

$$mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta - mR\ddot{\theta} = 0$$

$$R\omega^2 \sin\theta\cos\theta + g\sin\theta - \ddot{\theta} = 0$$

(b) The bead is stationary when $\ddot{\theta} = 0$ which means:

$$R\omega^2 \sin\theta\cos\theta = g\sin\theta$$

This means that either $\sin\theta = 0$ or $\cos\theta = \frac{g}{R\omega^2}$.
Solving for ω:

$$\omega = \sqrt{\frac{\ddot{\theta} - g\sin\theta}{R\sin\theta\cos\theta}}$$