

Classical Mechanics Assignment #3

Johannes Byle

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1. i These constraints can be found by defining the equations for the position of each of the elements and then solving for their velocities. Defining the center of the first wheel as (x_1, y_1) and the second as (x_2, y_2) we get:¹

$$\begin{aligned}x_1 &= x - \frac{l}{2} \cos \theta & y_1 &= y - \frac{l}{2} \sin \theta \\x_2 &= x + \frac{l}{2} \cos \theta & y_2 &= y + \frac{l}{2} \sin \theta\end{aligned}$$

Taking the derivatives:

$$\begin{aligned}\dot{x}_1 &= \dot{x} + \frac{l}{2} \dot{\theta} \sin \theta & \dot{y}_1 &= \dot{y} - \frac{l}{2} \dot{\theta} \cos \theta \\ \dot{x}_2 &= \dot{x} - \frac{l}{2} \dot{\theta} \sin \theta & \dot{y}_2 &= \dot{y} + \frac{l}{2} \dot{\theta} \cos \theta\end{aligned}$$

We also know the velocities of the center of each wheel is constrained by the velocity of the wheel:

$$\begin{aligned}\dot{x}_1 &= R\dot{\phi} \sin \theta & \dot{y}_1 &= -R\dot{\phi} \cos \theta \\ \dot{x}_2 &= R\dot{\phi}' \sin \theta & \dot{y}_2 &= -R\dot{\phi}' \cos \theta\end{aligned}$$

Pretending dt is just part of a fraction to piss off mathematicians:

$$\begin{aligned}dx_1 &= dx + \frac{l}{2} d\theta \sin \theta & dy_1 &= dy - \frac{l}{2} d\theta \cos \theta \\ dx_2 &= dx - \frac{l}{2} d\theta \sin \theta & dy_2 &= dy + \frac{l}{2} d\theta \cos \theta \\ dx_1 &= R d\phi \sin \theta & dy_1 &= -R d\phi \cos \theta \\ dx_2 &= R d\phi' \sin \theta & dy_2 &= -R d\phi' \cos \theta\end{aligned}$$

¹<https://physics.stackexchange.com/questions/105864/is-there-a-systematic-way-to-derive-constraint-equations>

Substituting the different equations to solve for dx and dy :

$$\begin{aligned}
dx + \frac{l}{2}d\theta \sin \theta &= Rd\phi \sin \theta \\
dx &= \sin \theta \left(Rd\phi - \frac{l}{2}d\theta \right) \\
dy - \frac{l}{2}d\theta \cos \theta &= -Rd\phi \cos \theta \\
dy &= \cos \theta \left(\frac{l}{2}d\theta - Rd\phi \right)
\end{aligned}$$

The first equation is easy to see from this by substituting for dx and dy :

$$\begin{aligned}
\sin \theta dy &= -\cos \theta dx \\
\sin \theta \left[\cos \theta \left(\frac{l}{2}d\theta - Rd\phi \right) \right] &= \cos \theta \left[-\sin \theta \left(Rd\phi - \frac{l}{2}d\theta \right) \right] \\
\sin \theta \left[\cos \theta \left(\frac{l}{2}d\theta - Rd\phi \right) \right] &= \cos \theta \left[\sin \theta \left(\frac{l}{2}d\theta - Rd\phi \right) \right]
\end{aligned}$$

We can derive the second equation using the equations for dx and dy using the midpoint:

$$\begin{aligned}
dx &= \frac{dx_2 + dx_1}{2} & dy &= \frac{dy_2 + dy_1}{2} \\
dx &= \frac{R \sin \theta (d\phi + d\phi')}{2} & dy &= -\frac{R \cos \theta (d\phi + d\phi')}{2} \\
\sin \theta dx - \cos \theta dy &= R(d\phi + d\phi') \\
\sin \theta \frac{R \sin \theta (d\phi + d\phi')}{2} + \cos \theta \frac{R \cos \theta (d\phi + d\phi')}{2} &= R(d\phi + d\phi') \\
\frac{R(d\phi + d\phi')}{2} + \frac{R(d\phi + d\phi')}{2} &= R(d\phi + d\phi') \\
R(d\phi + d\phi') &= R(d\phi + d\phi')
\end{aligned}$$

ii This constraint states that the derivatives must be equal to 0:

$$\begin{aligned}
\frac{d}{dt} \left(\theta + \frac{R}{l}(\phi - \phi') \right) &= d\theta + \frac{R}{l}(d\phi - d\phi') = 0 \\
d\theta &= -\frac{R}{l}(d\phi - d\phi')
\end{aligned}$$

We also know that the general formula for $d\theta$ of a vector is:

$$d\theta = \frac{xdy - ydx}{x^2 + y^2}$$

But since this is the relative vector, and the distance is constant:

$$d\theta = \frac{l \cos \theta dy - l \sin \theta dx}{l}$$

$$d\theta = \cos \theta dy - \sin \theta dx$$

Plugging in the various variables:

$$d\theta = -\cos \theta \cos \theta \left(\frac{l}{2} d\theta + R d\phi \right) - \sin \theta \sin \theta \left(R d\phi + \frac{l}{2} d\theta \right)$$

$$d\theta = -\cos^2 \theta \left(\frac{l}{2} d\theta + R d\phi \right) - \sin^2 \theta \left(R d\phi + \frac{l}{2} d\theta \right)$$

$$d\theta = -\frac{R}{l} (d\phi - d\phi')$$

2. (a) Using d'Alembert's principle

$$((T - mg) - m\ddot{y}_m) \delta y_m = 0$$

$$((T - Mg) - (-M\ddot{y}_m)) \delta y_M = 0$$

(b) Since we know:

$$\delta y_m = -\delta y_M$$

Solving that equation

$$-m\ddot{y}_m + (T - mg) = (T - Mg) - (-M\ddot{y}_m)$$

$$-m\ddot{y}_m - mg = -Mg - (-M\ddot{y}_m)$$

$$-m\ddot{y}_m - M\ddot{y}_m = -Mg + mg$$

$$\ddot{y}_m = \frac{M - m}{M + m} g$$

(c) Using the same steps above:

$$((T - mg \sin \beta) - (-m\ddot{L}_m)) \delta y_m = 0$$

$$((T - Mg \sin \alpha) - M\ddot{L}_m) \delta y_M = 0$$

Solving that equation:

$$-m\ddot{L}_m + (T - (-mg \sin \beta)) = (T - Mg \sin \alpha) - M\ddot{L}_m$$

$$-m\ddot{L}_m + mg \sin \beta = -Mg \sin \alpha - M\ddot{L}_m$$

$$-m\ddot{L}_m - M\ddot{L}_m = Mg \sin \alpha - mg \sin \beta$$

$$\ddot{L}_m = \frac{m \sin \alpha - M \sin \beta}{M + m} g$$

- (d) Using d'Alembert's principle, and specifying the distance between the first pulley and m_1 as y_1 , the distance between the first pulley and the second pulley as y_p , the distance between the second pulley and m_2 as y_2 , and the distance between the second pulley and m_3 as y_3 :

$$\begin{aligned}((T_1 - m_1g) + m_1\ddot{y}_1)\delta y_1 &= 0 \\ ((T_2 - m_2g) + m_2\ddot{y}_2)\delta y_2 &= 0 \\ ((T_2 - m_3g) - m_3\ddot{y}_3)\delta y_3 &= 0\end{aligned}$$

We also know the following constraints:

$$\begin{aligned}y_1 + y_p &= l \\ y_2 + y_3 &= L \\ \delta y_1 &= -\delta y_p \\ \delta y_p &= \frac{\delta y_2 + \delta y_3}{2}\end{aligned}$$

- (e) Re-arranging the equations from part (d):

$$\begin{aligned}\ddot{y}_1 &= -\frac{T_1 - m_1}{m_1} \\ \ddot{y}_2 &= -\frac{T_2 - m_2}{m_2} \\ \ddot{y}_3 &= \frac{T_2 - m_3}{m_3}\end{aligned}$$

- (f) Starting with d'Alembert's principle and doing algebra:

$$\begin{aligned}(m_1(g - \ddot{y}_1))\delta y_1 + (m_2(g - \ddot{y}_2))\delta y_2 + (m_3(g - \ddot{y}_3))\delta y_3 &= 0 \\ -(m_1(g - \ddot{y}_1))\frac{\delta y_2 + \delta y_3}{2} + (m_2(g - \ddot{y}_2))\delta y_2 + (m_3(g - \ddot{y}_3))\delta y_3 &= 0\end{aligned}$$

We can split up the δy_3 and δy_2 terms because they both independently have to be 0:

$$\begin{aligned}\frac{-(m_1(g - \ddot{y}_1))}{2} + (m_2(g - \ddot{y}_2)) &= 0 \\ \frac{-(m_1(g - \ddot{y}_1))}{2} + (m_3(g - \ddot{y}_3)) &= 0\end{aligned}$$

Since $\delta y_p = \frac{\delta y_2 + \delta y_3}{2}$:

$$\begin{aligned}\frac{-(m_1(g + \frac{\ddot{y}_2 + \ddot{y}_3}{2}))}{2} + (m_2(g - \ddot{y}_2)) &= 0 \\ \frac{-(m_1(g + \frac{\ddot{y}_2 + \ddot{y}_3}{2}))}{2} + (m_3(g - \ddot{y}_3)) &= 0\end{aligned}$$

Using Mathematica:

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In[3]= Solve[ToExpression["\\frac{-(m_1 (g+\\frac{y_2+y_3}{2}))}{2}+(m_2 (g-y_2))=\\frac{-(m_1 (g+\\frac{y_2+y_3}{2}))}{2}+(m_3 (g-y_3))",TeXForm]]
Out[3]= {{m3[g-y3] -> m2[g-y2]}}
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$$m_3 (g - y_3) = m_2 (g - y_2)$$

This equation would be stationary if the masses equal each other:

$$m_2 = \frac{m_3 (g - y_3)}{(g - y_2)}$$

$$m_3 = \frac{m_2 (g - y_2)}{(g - y_3)}$$

3. (a) Using cylindrical coordinates the Lagrangian is:

$$L = \frac{1}{2}m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) + \frac{1}{2}M \dot{z}^2 - Mgz$$

We can impose the constraints:

$$\dot{\rho} = -\dot{z}$$

$$z = l - \rho$$

This gives us the following equation:

$$L = \frac{1}{2}m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) + \frac{1}{2}M \dot{\rho}^2 - Mg(l - \rho)$$

Solving for the equations of motion in terms of ρ :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\rho}} \right) - \frac{dL}{d\rho} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\rho}} \right) = (m + M)\ddot{\rho}$$

$$\frac{dL}{d\rho} = m\rho\dot{\phi}^2 + Mg$$

$$(m + M)\ddot{\rho} - m\rho\dot{\phi}^2 - Mg = 0$$

$$\rho(t) = \rho_0 e^{\sqrt{\frac{m}{m+M}}\dot{\phi}t} + \rho_1 e^{-\sqrt{\frac{m}{m+M}}\dot{\phi}t} - \frac{gM}{m\dot{\phi}^2}$$

Solving for the equations of motion in terms of ϕ :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{dL}{d\phi} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = m\rho^2\ddot{\phi}$$

$$\frac{dL}{d\phi} = 0$$

$$m\rho^2\ddot{\phi} = 0$$

$$\phi(t) = \phi_1 t + \phi_0$$

- (b) The significance of these equations is that the rotation of the block on the table does not change. The distance of the block from the hole does change however, and is even influenced by the rotation of the block. The energy, as well as the angular acceleration, are conserved quantities.
4. (a) The situation in the problem can be described by:

$$\ddot{x} = -x$$

Solving this equation gives:

$$\begin{aligned}x(t) &= c_1 \sin(t) + c_2 \cos(t) \\ \dot{x}(t) &= c_1 \cos(t) - c_2 \sin(t)\end{aligned}$$

This results in the following phase portrait:

- (b) Each phase portrait is a circle, so it's radius will never change, so if it begins in the area it will stay in the area.
- (c) Starting with the energy equation:

$$E = \frac{p^2}{2m} + mgq$$

Solving for q :

$$\begin{aligned}q &= \frac{1}{mg} \left[E - \frac{p^2}{2m} \right] \\ q_1 &= \frac{1}{mg} \left[E' - \frac{p^2}{2m} \right] \\ q_2 &= \frac{1}{mg} \left[E'' - \frac{p^2}{2m} \right]\end{aligned}$$

Integrating to find the area:

$$\begin{aligned}
& \int_{p_1}^{p_2} \int_{q(p_1)}^{q(p_2)} dq dp \\
& \int_{p_1}^{p_2} \int_{\frac{1}{mg} \left[E' - \frac{p^2}{2m} \right]}^{\frac{1}{mg} \left[E'' - \frac{p^2}{2m} \right]} dq dp \\
& \int_{p_1}^{p_2} \left[\frac{1}{mg} \left[E'' - \frac{p^2}{2m} \right] - \frac{1}{mg} \left[E' - \frac{p^2}{2m} \right] \right] dp \\
& \int_{p_1}^{p_2} \frac{1}{mg} \left[E' - \frac{p^2}{2m} \right] - \frac{1}{mg} \left[E'' - \frac{p^2}{2m} \right] dp \\
& \frac{1}{mg} (E' - E'') \int_{p_1}^{p_2} dp \\
& A = \frac{1}{mg} (E' - E'') (p_2 - p_1)
\end{aligned}$$

(d) Starting with the Lagrangian to solve the equations of motion:

$$\begin{aligned}
L &= \frac{1}{2} m \dot{q}^2 - mgq \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{dL}{dq} &= 0 \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) &= m\ddot{q} \\
\frac{dL}{dq} &= -mg \\
m\ddot{q} + mg &= 0 \\
q(t) &= c_2 t + c_1 - \frac{gt^2}{2}
\end{aligned}$$

Integrating $q(t)$ to get the momentum:

$$\dot{q}(t) = \int \left(c_2 t + c_1 - \frac{gt^2}{2} \right) dt = c_2 t + c_1 - \frac{gt^2}{2}$$

Plugging this back into the equation for A :

$$\begin{aligned}
A &= \frac{1}{mg} (E' - E'') (p_2 - p_1) \\
A &= \frac{1}{mg} (E' - E'') (c_1 - gt - c_2 + gt) \\
A &= \frac{1}{mg} (E' - E'') (c_1 - c_2)
\end{aligned}$$

Since we already know E' and E'' are constant, and the rest of the equation is constant A is constant.