Quantum I Assignment #2

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Q.1 (a)

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{i}{2}\frac{-i}{2} + \left(\frac{1}{2}\right)^2 = 1$$
$$|\psi_0\rangle \text{ is normalized}$$
$$\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{i}{\sqrt{3}}\frac{-i}{\sqrt{3}} = \frac{2}{3}$$
$$|\psi_1\rangle \text{ is not normalized}$$

(b)

$$\tilde{\mathbf{P}}_0 = |\psi_0\rangle\langle\psi_0|$$

Representing this as a matrix:

$$\begin{bmatrix} \frac{1}{2}|u_1\rangle\langle u_1| & \frac{-i}{2\sqrt{2}}|u_1\rangle\langle u_2| & \frac{1}{2\sqrt{2}}|u_1\rangle\langle u_3| \\ \frac{i}{2\sqrt{2}}|u_2\rangle\langle u_1| & \frac{1}{4}|u_2\rangle\langle u_2| & \frac{i}{4}|u_2\rangle\langle u_3| \\ \frac{1}{2\sqrt{2}}|u_3\rangle\langle u_1| & \frac{-i}{4}|u_3\rangle\langle u_2| & \frac{1}{4}|u_3\rangle\langle u_3| \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix}$$

This is hermitian because it is equal to it's complex conjugate transpose.

$$\tilde{\mathbf{P}}_1 = |\psi_1\rangle\langle\psi_1|$$

Representing this as a matrix:

$$\begin{bmatrix} \frac{1}{3}|u_1\rangle\langle u_1| & \frac{-i}{3}|u_1\rangle\langle u_2| \\ \frac{i}{3}|u_2\rangle\langle u_1| & \frac{1}{3}|u_2\rangle\langle u_2| \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{3} & \frac{-i}{3} \\ \frac{i}{3} & \frac{1}{3} \end{bmatrix}$$

This is hermitian because it is equal to it's complex conjugate transpose.

1.9 (a)

$$\prod_{a_i}(A - a') = (A - a_1)(A - a_2)\cdots(A - a_i)$$

When this operator operates on some eigenket of A since it's eigenvalue will be in the series, one of the terms will go to zero, making the whole expression zero.

(b)

$$\prod_{a''\neq a'}(A-a') = \frac{(a'-a'')}{(a'-a'')}\frac{(a'''-a'')}{(a'-a'')}\cdots\frac{(a_n-a'')}{(a'-a'')}$$

(c)

$$S_z = \frac{\hbar}{2}\left[(\left|+\right\rangle\left\langle+\right|)-(\left|-\right\rangle\left\langle-\right|)\right]$$
$$\prod_{a_i} S_z\left|+\right\rangle = \left(\frac{\hbar}{2}-\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}-0\right) = 0$$

1.11

$$\tilde{\mathbf{S}} = (S_x,\ S_y,\ S_z)$$
$$\tilde{\mathbf{n}} = (\cos\alpha\sin\beta,\ \sin\alpha\sin\beta,\ \cos\beta)$$
$$\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}} = (S_x\cos\alpha\sin\beta,\ S_y\sin\alpha\sin\beta,\ S_z\cos\beta)$$
$$\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}} = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\cos\alpha\sin\beta + \frac{\hbar}{2i}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\sin\alpha\sin\beta + \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\cos\beta$$
$$\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}} = \frac{\hbar}{2}\left(\begin{bmatrix}0 & \cos\alpha\sin\beta\\ \cos\alpha\sin\beta & 0\end{bmatrix} + \begin{bmatrix}0 & i\sin\alpha\sin\beta\\ -i\sin\alpha\cos\beta & 0\end{bmatrix} + \begin{bmatrix}\cos\beta & 0\\ 0 & -\cos\beta\end{bmatrix}\right)$$
$$\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}} = \begin{bmatrix}\cos\beta & \cos\alpha\sin\beta + i\sin\alpha\sin\beta\\ \cos\alpha\sin\beta - i\sin\alpha\sin\beta & -\cos\beta\end{bmatrix}$$

Solving for $\left|+\right\rangle$ as $\begin{bmatrix}a\\ b\end{bmatrix}$:

$$\left|\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}};+\right\rangle = \begin{bmatrix}\cos\beta & \cos\alpha\sin\beta + i\sin\alpha\sin\beta\\ \cos\alpha\sin\beta - i\sin\alpha\sin\beta & -\cos\beta\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}a\\ b\end{bmatrix}$$
$$\begin{bmatrix}\cos\beta & e^{i\alpha}\sin\beta\\ e^{-i\alpha}\sin\beta & -\cos\beta\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}a\\ b\end{bmatrix}$$

As a system of equations:

$$a\cos\beta + be^{i\alpha}\sin\beta = \frac{\hbar}{2}a$$
$$ae^{-i\alpha}\sin\beta - b\cos\beta = \frac{\hbar}{2}b$$

Solving for a:

$$b = \frac{a\sin\beta e^{-i\alpha}}{\left(\cos\beta + \frac{\hbar}{2}\right)}$$
$$a\left(\cos\beta - \frac{\hbar}{2}\right) + \frac{a\sin\beta e^{-i\alpha}}{\left(\cos\beta + \frac{\hbar}{2}\right)}\left(\sin\beta e^{i\alpha}\right)$$

1.17 We can show that for any base ket $|\psi_n\rangle$, $AB\,|\psi_n\rangle = BA\,|\psi_n\rangle$:

$$AB\,|\psi_n\rangle = Ab_n\,|\psi_n\rangle = a_n b_n\,|\psi_n\rangle = BA\,|\psi_n\rangle$$

Since this is true for any $|\psi_n\rangle$ and we know that the simultaneous eigenkets form a complete orthonormal set of base kets $[A, B] = 0$.[1]

1.18 If $|\psi\rangle$ is a eigenket of both A and B then:

$$\begin{gathered} A\,|\psi\rangle = a\,|\psi\rangle \\ B\,|\psi\rangle = b\,|\psi\rangle \\ (AB + BA)\,|\psi\rangle = 2ab\,|\psi\rangle \end{gathered}$$

The only way this is true is if either a or b is 0.[2]

[1] https://en.wikipedia.org/wiki/Complete_set_of_commuting_observables

[2] http://peeterjoot.com/2015/09/28/can-anticommuting-operators-have-a-simulaneous-eigenket/