Lab 6: A Family of Competitive-Species Equations

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$$\frac{dx}{dt} = kx - 3x^2 - 4xy$$
$$\frac{dy}{dt} = 42y - 3y^2 - 2xy$$

1. Equilibrium points:

$$(0, 0)$$

$$(0, 14)$$

$$(\frac{k}{3}, 0)$$

$$(3k - 168, -2k + 126)$$

2. Bifurcation Values:

$$k = 0, 56, 63$$

3. Classifying Equilibrium Points:

$$Jacobian = \begin{bmatrix} k - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix}$$
$$(0, 0) = \begin{bmatrix} k & 0 \\ 0 & 42 \end{bmatrix}$$
$$(0, 14) = \begin{bmatrix} k - 56 & 0 \\ -28 & -42 \end{bmatrix}$$
$$(\frac{k}{3}, 0) = \begin{bmatrix} -k & -\frac{4}{3}k \\ 0 & 42 - \frac{2}{3}k \end{bmatrix}$$
$$(3k - 168, -2k + 126) = \begin{bmatrix} 504 - 9k & 672 - 12k \\ 4k - 252 & 6k - 378 \end{bmatrix}$$

Equilibrium Points	k=0	0<k<56	k=56	56<k<63	k=63	63<k<∞
(0,0)	Source*	Source	Source	Source	Source	Source
(0,14)	Sink	Sink	Sink*	Saddle	Saddle	Saddle
$(\frac{k}{3},0)$	Source*	Saddle	Saddle	Saddle	Sink*	Sink
$(3k-168,-2k+126)$	Saddle	Saddle	Sink*	Sink	Sink*	Saddle

*a * indicates a line of source or sink points*

4. Graphs of x(t) and y(t):

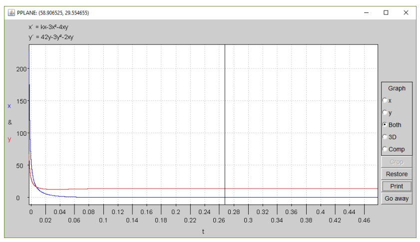

Figure 1: k=0

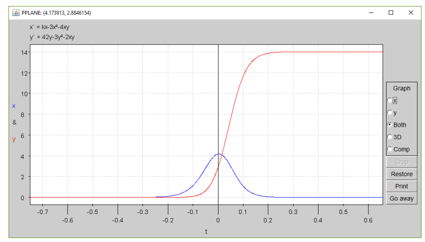

Figure 2: k=20

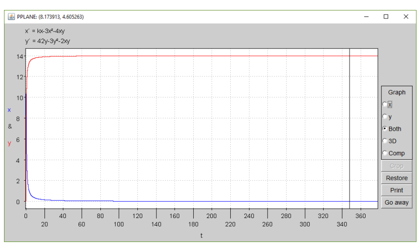

Figure 3: k=56

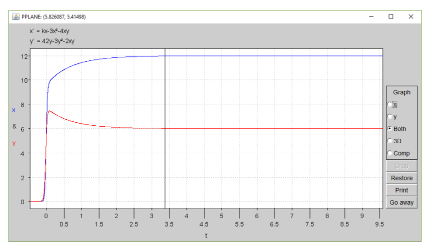

Figure 4: k=60

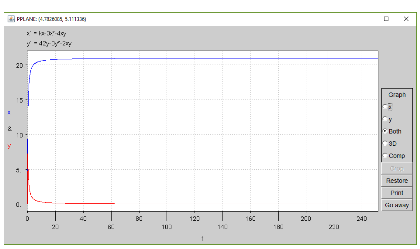

Figure 5: k=63

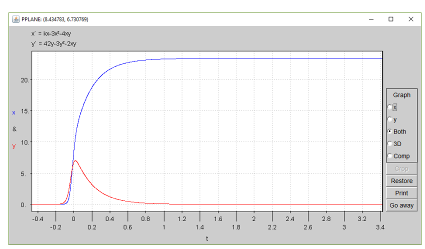

Figure 6: k=70

5. Examining the x-population:
The x population only survives if k>56. This is because, in the first quadrant the only equilibrium points where x does not equal 0 are $(\frac{k}{3},0)$ and

$(3k-168, -2k+126)$. And if k≤56 then $(\frac{k}{3}, 0)$ is a saddle and x does not approach that point.
6. Examining the y-population:
The y population will only survive if k<63. This is because the only equilibrium points where y does not equal 0 are $(0, 14)$ and $(3k-168, -2k+126)$. And populations that start in the first quadrant will only approach $(0, 14)$ if k≤56.
7. Mutual Coexistence:
The only equilibrium point where both x and y are greater than 0 is $(3k-168, -2k+126)$. In this equation both x and y are only greater than 0 where 56<k<63.
Summary: The fastest way to analyze this system of differential equations is to examine the linearization of equilibrium points. We can ignore $(0, 0)$ for every value of k this point is a source, and no solutions will approach it. To examine the x population we can see that the only points we must examine are $(\frac{k}{3}, 0)$ and $(3k-168, -2k+126)$ since these are the only solutions where x has the possibility of not being equal to 0. For the first equilibrium point we can know that the value k=0 will not be important since at k=0 this equilibrium point is a source, this solutions will never approach this value. For the other values of k however we can only be certain of values of k greater than or equal to 63, since those are sinks. For values of k between 63 and 0 we need to consult the graph of x and y since there $(\frac{k}{3}, 0)$ is a saddle.

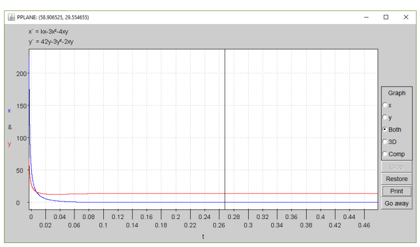

Figure 7: k=0

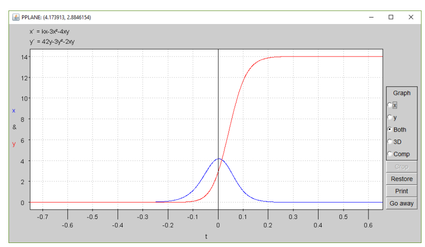

Figure 8: k=20

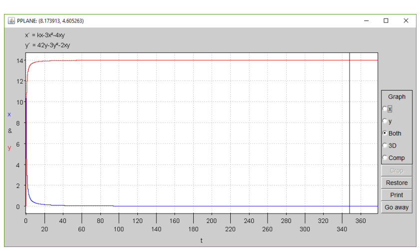

Figure 9: k=56

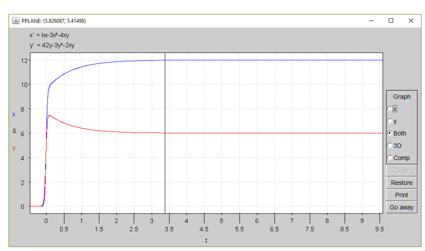

Figure 10: k=60

From these figures we can see that values that begin in the first quadrant will

never approach $(\frac{k}{3}, 0)$ when k<63.
Examining the y population can be done in the same way. The only points we must examine are $(0, 14)$ and $(3k - 168, -2k + 126)$ since these are the only solutions where y has the possibility of not being equal to 0. Looking at the table in part 3 we can see that all values of k are viable, since all values are either sinks or saddles. Thus we must look at the graphs of the values of k where $(0, 14)$ is a saddle to see whether solutions that begin in the first quadrant ever approach that equilibrium point. In both of these examples the y solution approaches 0, thus values that begin in the first quadrant will never approach $(0, 14)$ when k>63.

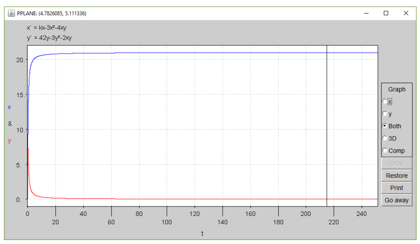

Figure 11: k=63

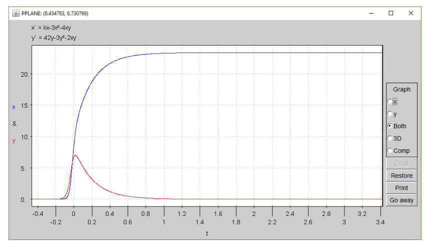

Figure 12: k=70

Since $(3k - 168, -2k + 126)$ is the only equilibrium point which has the possibility of having both x and y being greater than 0 examining this equilibrium point will tell us for which values of k both x and y can coexist. Looking at the table the only times where this equilibrium point is a true sink is for values of k where k is greater than 56 and less than 63. This makes sense, because above 63 the $-2k + 126$ is less than 0 and for values less than 56 $3k - 168$ is less than 0. Thus, simply by looking at the table we can see that k must be be greater than 56 and less than 63 for both species to be able to survive.