Math Methods Assignment #3

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1. (a)

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{l}\cos\theta)^2 + \dot{l}^2\sin^2\theta) \tag{1}$$

(b) Since M is confined to the x axis only m is dependent on gravity:

$$L = T - V \tag{2}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{l}\cos\theta)^2 + \dot{l}^2\sin^2\theta) + mgl\sin\theta \tag{3}$$

Expanding the terms:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2\cos^2\theta + \dot{l}^2\sin^2\theta) + mgl\sin\theta \tag{4}$$

Simplifying with trig identities:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2) + mgl\sin\theta \tag{5}$$

(c) Solving for $\ddot{x}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(M+m)\dot{x} + m\dot{l}\cos\theta\right] = (M+m)\ddot{x} + m\ddot{l}\cos\theta \tag{7}$$

$$\frac{\partial L}{\partial x} = 0 \tag{8}$$

$$(M+m)\ddot{x} + m\ddot{l}\cos\theta = 0 \tag{9}$$

$$\ddot{x} = -\mu\ddot{l}\cos\theta \tag{10}$$

Solving for $\ddot{l}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) - \frac{\partial L}{\partial l} = 0 \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = \frac{d}{dt}\left[m(\dot{x}\cos\theta + \dot{l})\right] = m(\ddot{x}\cos\theta + \ddot{l}) \tag{12}$$

$$\frac{\partial L}{\partial l} = mg\sin\theta \tag{13}$$

$$m(\ddot{x}\cos\theta + \ddot{l}) - mg\sin\theta = 0 \tag{14}$$

$$\ddot{l} = g\sin\theta - \ddot{x}\cos\theta \tag{15}$$

De-coupling the equations, starting with $\ddot{x}$:

$$\ddot{x} = -\mu\left(g\sin\theta - \ddot{x}\cos\theta\right)\cos\theta \tag{16}$$
$$\ddot{x} = -\mu g\cos\theta\sin\theta + \mu\ddot{x}\cos^2\theta \tag{17}$$
$$\ddot{x}\left(1-\mu\cos^2\theta\right) = -\mu g\cos\theta\sin\theta \tag{18}$$
$$\ddot{x} = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta} \tag{19}$$

De-coupling $\ddot{l}$:

$$\ddot{l} = g\sin\theta + \mu\ddot{l}\cos^2\theta \tag{20}$$
$$\ddot{l}\left(1-\mu\cos^2\theta\right) = g\sin\theta + \tag{21}$$
$$\ddot{l} = \frac{g\sin\theta}{1-\mu\cos^2\theta} \tag{22}$$

(d) No, $\ddot{l}$ will never be negative because μ will always be less than 1 and that means that equation (22) will always be positive.

(e) Integrating $\ddot{x}$:

$$\int -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}dt = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t + \dot{x}_0 \tag{23}$$
$$\int\left[-\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t_1 + c_1\right]dt = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 + \dot{x}_0 t + x_0 \tag{24}$$

Since both the wedge and the block start at rest:

$$x(t) = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 \tag{25}$$

Solving for when $\Delta x = \frac{h}{\tan\theta}$:

$$x(t) = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 = \frac{h}{\tan\theta} \tag{26}$$
$$t = -\sqrt{\frac{h}{\tan\theta}\frac{1-\mu\cos^2\theta}{\mu g\cos\theta\sin\theta}} \tag{27}$$

Repeating the same process for Δl results in:

$$l(t) = \frac{g\sin\theta}{1-\mu\cos^2\theta}t^2 = \frac{h}{\sin\theta} \tag{28}$$
$$t = -\sqrt{\frac{h}{\sin\theta}\frac{1-\mu\cos^2\theta}{g\sin\theta}} \tag{29}$$

(f) If $M\to\infty$ then $\mu\to 0$. This means that $\Delta x = 0$ because $t = -\sqrt{\frac{h}{\tan\theta}\frac{1-\mu\cos^2\theta}{\mu g\cos\theta\sin\theta}} \to \infty$, which makes sense since the wedge is infinitely heavy and wont move. Δl on the other hand goes to $t = -\sqrt{\frac{h}{\sin\theta}\frac{1-\mu\cos^2\theta}{g\sin\theta}} \to -\sqrt{\frac{h}{g\sin^2\theta}}$

(g) It doesn't.

2. (a) Starting with the equation for the center of mass $y_{cm} = \frac{1}{M}\sum_i m_i y_i$ and using L_r for the length of the right side and L_l for the left side:

$$L_r = \frac{L+y}{2} \quad L_l = \frac{L-y}{2} \tag{30}$$

$$y_{cm} = \frac{1}{\rho L}\left[(\rho L_r)\left(\frac{L_r}{2}\right) + (\rho L_l)\left(\frac{L_l}{2} + y\right)\right] \tag{31}$$

$$y_{cm} = \frac{1}{L}\left[\left(\frac{L_r^2}{2}\right) + \left(\frac{L_l^2}{2} + yL_L\right)\right] \tag{32}$$

$$y_{cm} = \frac{1}{L}\left[\left(\frac{\left(\frac{L+y}{2}\right)^2}{2}\right) + \left(\frac{\left(\frac{L-y}{2}\right)^2}{2} + y\frac{L-y}{2}\right)\right] \tag{33}$$

$$y_{cm} = \frac{1}{2L}\left[\frac{L^2+2yL+y^2}{4} + \frac{L^2-2yL+y^2}{4} + yL - y^2\right] \tag{34}$$

$$y_{cm} = \frac{1}{2L}\left[\frac{L^2-y^2}{2} + yL\right] \tag{35}$$

$$y_{cm} = \frac{L^2+2yL-y^2}{4L} \tag{36}$$

(b)

$$L = \frac{1}{2}m\frac{L-y}{2L}\dot{y}^2 - mgy_{cm} \tag{37}$$

$$L = m\frac{L-y}{4L}\dot{y}^2 - mg\left[\frac{L^2+2yL-y^2}{4L}\right] \tag{38}$$

(c) Since we know the initial conditions we can solve this using conservation of energy:

$$m\frac{L-y}{4L}\dot{y}^2 + mgy_{cm} = mg\frac{L}{4} \tag{39}$$

$$m\frac{L-y}{4L}\dot{y}^2 + mg\left[\frac{L^2+2yL-y^2}{4L}\right] = mg\frac{L}{4} \tag{40}$$

$$\dot{y} = \pm\frac{\sqrt{gy^2-2gLy}}{\sqrt{L-y}} \tag{41}$$

(d) We can differentiate the answer to the previous section to find the acceleration:

$$\ddot{y} = \frac{\dot{y}\sqrt{gy^2-2gLy}}{2(L-y)^{3/2}} + \frac{2gy\dot{y} - 2gL\dot{y}}{2\sqrt{L-y}\sqrt{gy^2-2gLy(t)}} \tag{42}$$

(e) This answer is greater than g because the entire system is gaining potential energy, yet only a smaller and smaller section of rope is accelerating, thus for energy to be conserved that section of the rope must gain velocity faster than the acceleration due to gravity.

3. (a)

$$V = \frac{1}{2}k(s-a)^2 - mgs \tag{43}$$

(b)

$$T = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}M\dot{s}^2 = m\dot{s}^2 \tag{44}$$

$$L = T - V = m\dot{s}^2 - \frac{1}{2}k(s-a)^2 + mgs \tag{45}$$

$$L = m\dot{s}^2 - \frac{1}{2}k(s^2 - 2sa + a^2) + mgs \tag{46}$$

(c)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \tag{47}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left(2m\dot{s}\right) = 2m\ddot{s} \tag{48}$$

$$\frac{\partial L}{\partial s} = -\frac{1}{2}k\left(2s - 2a\right) + mg \tag{49}$$

$$2m\ddot{s} + k(s-a) - mg = 0 \tag{50}$$

$$s(t) = a + c_1 \sin\left(\sqrt{\frac{k}{2}}t\right) + c_2 \cos\left(\sqrt{\frac{k}{2}}t\right) + \frac{gm}{k} \tag{51}$$

(d) The equilibrium position where $\ddot{s} = 0$:

$$\ddot{s} = \frac{mg - k(s-a)}{2m} \tag{52}$$

(e) The frequency is clear from the equations of motion:

$$\omega = \frac{k}{2} \tag{53}$$

4. (a) Starting with the Lagrangian:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{54}$$

And the constraint g:

$$g = z - \alpha\left(x^2 + y^2\right) = 0 \tag{55}$$

We get the following equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \lambda\frac{\partial g}{\partial x} = 0 \tag{56}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) + \lambda\frac{\partial g}{\partial y} = 0 \tag{57}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) + \lambda\frac{\partial g}{\partial z} = 0 \tag{58}$$

Solving for x:

$$\frac{\partial L}{\partial x} = 0 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \quad \lambda\frac{\partial g}{\partial x} = 2\lambda\alpha x \tag{59}$$

$$m\ddot{x} = -2\lambda\alpha x \tag{60}$$

$$x(t) = c_1 \cos\left(\sqrt{\frac{2\alpha\lambda}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{2\alpha\lambda}{m}}t\right) \tag{61}$$

Solving for y:

$$\frac{\partial L}{\partial y} = 0 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} \quad \lambda\frac{\partial g}{\partial y} = 2\lambda\alpha y \tag{62}$$

$$m\ddot{y} = -2\lambda\alpha y \tag{63}$$

$$y(t) = c_1 \cos\left(\sqrt{\frac{2\alpha\lambda}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{2\alpha\lambda}{m}}t\right) \tag{64}$$

Solving for z:

$$\frac{\partial L}{\partial z} = -mg \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z} \quad \lambda\frac{\partial g}{\partial y} = 1 \tag{65}$$

$$m\ddot{z} = 1 - mg \tag{66}$$

$$z(t) = c_2 t + c_1 - \left(g + \frac{1}{m}\right)t^2 \tag{67}$$

(b) In this case $\alpha = \frac{z_0}{x^2+y^2}$ so we can solve for λ using the system of equations:

$$m\ddot{y} = -2\lambda y \frac{z_0}{x^2+y^2} \tag{68}$$

$$m\ddot{x} = -2\lambda x \frac{z_0}{x^2+y^2} \tag{69}$$

$$\lambda \to -\frac{(x^2+y^2)(m\ddot{x} - m\ddot{y})}{2z_0(x-y)} \tag{70}$$

(c) i. Removing the need for the x coordinate using the constraint:

$$x = \sqrt{\frac{z}{\alpha}} \quad \dot{x} = \frac{\dot{z}}{2\sqrt{\alpha z}} \tag{71}$$

$$\frac{1}{2}m(\dot{z}^2 + \dot{x}^2) + mgz = mgz_1 \tag{72}$$

$$\frac{1}{2}m(\dot{z}^2 + \frac{\dot{z}^2}{4\alpha z}) + mgz = mgz_1 \tag{73}$$

$$\dot{z} = \pm\frac{2\sqrt{g(z_1 - z)}}{\sqrt{\frac{1}{2\alpha z} + 2}} \tag{74}$$

ii.

5. There isn't since there isn't any conserved quantity under infinitesimally small changes.

6. Solving this problem in dimension, starting with the following equations:

$$r_i = r_i + \epsilon v_0 t \quad \dot{r}_i = \dot{r}_i + \epsilon v_0 \tag{75}$$

$$L = \frac{1}{2}\left(m_1\dot{r}_1^2 + m_2\dot{r}_2^2\right) - V(r_1 - r_2) \tag{76}$$

$$\tag{77}$$

Combining both equations:

$$L = \frac{1}{2}\left(m_1\left(\dot{r}_1^2 + 2\dot{r}_1\epsilon v_0 + \epsilon^2 v_0^2\right) + m_2\left(\dot{r}_2^2 + 2\dot{r}_2\epsilon v_0 + \epsilon^2 v_0^2\right)\right) - V(r_1 + \epsilon v_0 t - r_2 - \epsilon v_0 t) \tag{78}$$

$$L = \frac{1}{2}\left(m_1\left(\dot{r}_1^2 + 2\dot{r}_1\epsilon v_0 + \epsilon^2 v_0^2\right) + m_2\left(\dot{r}_2^2 + 2\dot{r}_2\epsilon v_0 + \epsilon^2 v_0^2\right)\right) - V(r_1 - r_2) \tag{79}$$

$$\tag{80}$$

To show that the Lagrangian is unchanged we have to consider its path:

$$S[\bar{r}] = \int \frac{1}{2}\left(m_1\left(\dot{r}_1^2 + 2\dot{r}_1\epsilon v_0 + \epsilon^2 v_0^2\right) + m_2\left(\dot{r}_2^2 + 2\dot{r}_2\epsilon v_0 + \epsilon^2 v_0^2\right)\right) - V(r_1 - r_2)dt \tag{81}$$

We know we can ignore the ϵ^2 terms because we only care about changes up to order ϵ.

$$S[\bar{r}] = \int \frac{1}{2}\left(m_1\left(\dot{r}_1^2 + 2\dot{r}_1\epsilon v_0\right) + m_2\left(\dot{r}_2^2 + 2\dot{r}_2\epsilon v_0\right)\right) - V(r_1 - r_2)dt \tag{82}$$

Separating the $\dot{r}$ terms:

$$S[\bar{r}] = \int \frac{1}{2}\left(m_1\dot{r}_1^2 + m_2\dot{r}_2^2\right) - V(r_1 - r_2)dt + \int \frac{1}{2}\left(m_1\dot{r}_1\epsilon v_0 + m_2\dot{r}_2\epsilon v_0\right)dt \tag{83}$$

If the second term is zero then the expression is equal to the original Lagrangian. It can shown that this second term is zero by integrating it by parts, which works because the expression is true up to a total derivative.

$$\int \frac{1}{2}\left(m_1\dot{r}_1\epsilon v_0 + m_2\dot{r}_2\epsilon v_0\right)dt = \left(m_1\dot{r}_1\epsilon v_0 + m_2\dot{r}_2\epsilon v_0\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \epsilon\left(m_1\ddot{r}_1 v_0 + m_2\ddot{r}_2 v_0\right) \tag{84}$$

The first term is zero because ϵ has to be zero at the boundaries, and the second term has to be zero because ϵ is an arbitrary function, thus for the integral to be zero the term inside the parentheses must be zero.

7. Since $(m_1\ddot{r}_1 v_0 + m_2\ddot{r}_2 v_0)$ must be zero, the quantity that is quantity that is conserved is:

$$\left(m_1\ddot{r}_1 v_0 + m_2\ddot{r}_2 v_0\right) = \frac{d}{dt}\left(m_1\dot{r}_1 v_0 + m_2\dot{r}_2 v_0\right) \tag{85}$$

8. This quantity is the momentum, which my intro physics classes have claimed is useful.