Quantum I Assignment #5

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Q-1

$$[\tilde{\mathbf{X}}, \tilde{\mathbf{H}}]$$

1.33 Starting with the translation operator applied to the expectation value for $\mathbf{x}$:

$$\langle \alpha | \mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') |\alpha\rangle$$

By equation 1.207 we know:

$$\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x} = d\mathbf{x}'$$

Since the translation operator is unitary we can apply $\mathscr{J}^{\dagger}(d\mathbf{x}')$ to both sides:

$$\mathscr{J}^{\dagger}(d\mathbf{x}')\left[\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x}\right] = \mathscr{J}^{\dagger}(d\mathbf{x}')d\mathbf{x}'$$
$$\mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^{\dagger}(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{x} = \mathscr{J}^{\dagger}(d\mathbf{x}')d\mathbf{x}'$$
$$\mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathbf{x} = \mathscr{J}^{\dagger}(d\mathbf{x}')d\mathbf{x}'$$
$$\mathbf{x} + d\mathbf{x}' - \mathbf{x} = \mathscr{J}^{\dagger}(d\mathbf{x}')d\mathbf{x}'$$

This means that $\langle \alpha | \mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') |\alpha\rangle \rightarrow \langle \alpha | \mathbf{x} |\alpha\rangle + d\mathbf{x}'$.
Using the same process for $\mathbf{p}$: By equation 1.227 we know:

$$\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p} = 0$$

Since the translation operator is unitary we can apply $\mathscr{J}^{\dagger}(d\mathbf{x}')$ to both sides:

$$\mathscr{J}^{\dagger}(d\mathbf{x}')\left[\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p}\right] = 0$$
$$\mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^{\dagger}(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{p} = 0$$
$$\mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathbf{p} = 0$$
$$\mathbf{p} + d\mathbf{p}' - \mathbf{p} = 0$$

This means that $\langle \alpha | \mathscr{J}^{\dagger}(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') |\alpha\rangle \rightarrow \langle \alpha | \mathbf{p} |\alpha\rangle$.

1.34 Satisfies unitary property because **W** is hermitian:

$$\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}') = (1 - i\mathbf{W} \cdot d\mathbf{p})(1 + i\mathbf{W} \cdot d\mathbf{p})$$
$$= (1 - i\mathbf{W} \cdot d\mathbf{p}^\dagger)(1 + i\mathbf{W} \cdot d\mathbf{p})$$
$$= 1 - i(\mathbf{W} - \mathbf{W}^\dagger)$$
$$\simeq 1$$

Satisfies the associative property:

$$\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}'') = (1 + i\mathbf{W} \cdot d\mathbf{p}') \cdot (1 + i\mathbf{W} \cdot d\mathbf{p}'')$$
$$\simeq 1 - i\mathbf{W} \cdot (d\mathbf{p}'d\mathbf{p}'')$$
$$= \mathscr{B}(d\mathbf{p}' + d\mathbf{p}'')$$

Satisfies the inverse property trivially:

$$\mathscr{B}(-d\mathbf{p}') = \mathscr{B}^{-1}(d\mathbf{p}')$$
$$1 + i\mathbf{W} \cdot d\mathbf{p} = -(-1 - i\mathbf{W} \cdot d\mathbf{p})$$

Since $d\mathbf{p}$ has units of $\frac{\text{kg m}}{\text{s}^2}$

1.35 (a)

$$\langle p \rangle = \int_{-\infty}^{\infty} dx' \, \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'} \right) \langle x' | \alpha \rangle$$
$$= \int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4}\sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \left(-i\hbar \frac{\partial}{\partial x'} \right) \left[\frac{1}{\pi^{1/4}\sqrt{d}} \right] \exp \left[ikx' - \frac{x'^2}{2d^2} \right]$$
$$= \left[\frac{-i\hbar}{\pi^{1/2}d} \right] \int_{-\infty}^{\infty} dx' \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \frac{\partial}{\partial x'} \exp \left[ikx' - \frac{x'^2}{2d^2} \right]$$
$$= \left[\frac{-i\hbar}{\pi^{1/2}d} \right] \int_{-\infty}^{\infty} dx' \left[ik - \frac{x'}{d^2} \right] \exp \left[-\frac{x'^2}{d^2} \right]$$
$$= \left[\frac{-i\hbar}{\pi^{1/2}d} \right] \left(\frac{-k\pi^{1/2}d}{i} \right)$$
$$= \hbar k$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx' \, \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'} \right)^2 \langle x' | \alpha \rangle$$
$$= \hbar^2 \int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4}\sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \frac{\partial^2}{\partial x'^2} \left[\frac{1}{\pi^{1/4}\sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right]$$
$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

(b)

$$\langle p | \alpha \rangle = \int_{-\infty}^{\infty} dx' \, \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'} \right) \langle x' | \alpha \rangle$$
$$= \int_{-\infty}^{\infty} dp \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right] \left(-i\hbar \frac{\partial}{\partial x'} \right) \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right]$$
$$= \hbar k$$

$$\langle p^2|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, \langle\alpha|x'\rangle \left(-i\hbar\frac{\partial}{\partial x'}\right)^2 \langle x'|\alpha\rangle$$

$$= \hbar^2 \int_{-\infty}^{\infty} dp \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right] \frac{\partial^2}{\partial x'^2} \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right]$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

1.36 (a)   i. Using equation (1.265a) $\langle p'|\alpha\rangle = \int dx' \, \langle p'|x'\rangle \langle x'|\alpha\rangle$:

$$\langle p'|x|\alpha\rangle = \int dx' \, \langle p'|x|x'\rangle \langle x'|\alpha\rangle$$

$$= \int dx' x' \, \langle p'|x'\rangle \langle x'|\alpha\rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' x' \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle \quad \text{using (1.264)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \frac{\partial}{\partial p'} i\hbar \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle$$

$$= i\hbar\frac{\partial}{\partial p'} \int dx' \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

$$\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

ii. This holds true from the previous result:

$$\langle \beta|x|\alpha\rangle = \int dp' \, \langle\beta|p'\rangle \langle p'|x|\alpha\rangle = \int dp' x \, \langle\beta|p'\rangle \langle p'\alpha\rangle$$

$$= \int dp' \, \langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'} \langle p'\alpha\rangle \quad \text{from the previous part}$$

$$\langle \beta|x|\alpha\rangle = \int dp' \phi_\beta(p') i\hbar\frac{\partial}{\partial p'} \phi_\alpha(p')$$

(b)