

Quantum I Assignment #5

Johannes Byle

October 4, 2021

1.33 Starting with the translation operator applied to the expectation value for \mathbf{x} :

$$\langle \alpha | \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{x} \mathcal{J}(d\mathbf{x}') | \alpha \rangle$$

By equation 1.207 we know:

$$\mathbf{x} \mathcal{J}(d\mathbf{x}') - \mathcal{J}(d\mathbf{x}') \mathbf{x} = d\mathbf{x}'$$

Since the translation operator is unitary we can apply $\mathcal{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned} \mathcal{J}^\dagger(d\mathbf{x}') [\mathbf{x} \mathcal{J}(d\mathbf{x}') - \mathcal{J}(d\mathbf{x}') \mathbf{x}] &= \mathcal{J}^\dagger(d\mathbf{x}') d\mathbf{x}' \\ \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{x} \mathcal{J}(d\mathbf{x}') - \mathcal{J}^\dagger(d\mathbf{x}') \mathcal{J}(d\mathbf{x}') \mathbf{x} &= \mathcal{J}^\dagger(d\mathbf{x}') d\mathbf{x}' \\ \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{x} \mathcal{J}(d\mathbf{x}') - \mathbf{x} &= \mathcal{J}^\dagger(d\mathbf{x}') d\mathbf{x}' \\ \mathbf{x} + d\mathbf{x}' - \mathbf{x} &= \mathcal{J}^\dagger(d\mathbf{x}') d\mathbf{x}' \end{aligned}$$

This means that $\langle \alpha | \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{x} \mathcal{J}(d\mathbf{x}') | \alpha \rangle \rightarrow \langle \alpha | \mathbf{x} | \alpha \rangle + d\mathbf{x}'$.

Using the same process for \mathbf{p} : By equation 1.227 we know:

$$\mathbf{p} \mathcal{J}(d\mathbf{x}') - \mathcal{J}(d\mathbf{x}') \mathbf{p} = 0$$

Since the translation operator is unitary we can apply $\mathcal{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned} \mathcal{J}^\dagger(d\mathbf{x}') [\mathbf{p} \mathcal{J}(d\mathbf{x}') - \mathcal{J}(d\mathbf{x}') \mathbf{p}] &= 0 \\ \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{p} \mathcal{J}(d\mathbf{x}') - \mathcal{J}^\dagger(d\mathbf{x}') \mathcal{J}(d\mathbf{x}') \mathbf{p} &= 0 \\ \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{p} \mathcal{J}(d\mathbf{x}') - \mathbf{p} &= 0 \\ \mathbf{p} + d\mathbf{p}' - \mathbf{p} &= 0 \end{aligned}$$

This means that $\langle \alpha | \mathcal{J}^\dagger(d\mathbf{x}') \mathbf{p} \mathcal{J}(d\mathbf{x}') | \alpha \rangle \rightarrow \langle \alpha | \mathbf{p} | \alpha \rangle$.

1.34 Satisfies unitary property because \mathbf{W} is hermitian:

$$\begin{aligned} \mathcal{B}^\dagger(d\mathbf{p}') \mathcal{B}(d\mathbf{p}') &= (1 - i\mathbf{W} \cdot d\mathbf{p})(1 + i\mathbf{W} \cdot d\mathbf{p}) \\ &= (1 - i\mathbf{W} \cdot d\mathbf{p}^\dagger)(1 + i\mathbf{W} \cdot d\mathbf{p}) \\ &= 1 - i(\mathbf{W} - \mathbf{W}^\dagger) \\ &\simeq 1 \end{aligned}$$

Satisfies the associative property:

$$\begin{aligned}\mathcal{B}^\dagger(d\mathbf{p}')\mathcal{B}(d\mathbf{p}'') &= (1 + i\mathbf{W} \cdot d\mathbf{p}') \cdot (1 + i\mathbf{W} \cdot d\mathbf{p}'') \\ &\simeq 1 - i\mathbf{W} \cdot (d\mathbf{p}'d\mathbf{p}'') \\ &= \mathcal{B}(d\mathbf{p}' + d\mathbf{p}'')\end{aligned}$$

Satisfies the inverse property trivially:

$$\begin{aligned}\mathcal{B}(-d\mathbf{p}') &= \mathcal{B}^{-1}(d\mathbf{p}') \\ 1 + i\mathbf{W} \cdot d\mathbf{p} &= -(-1 - i\mathbf{W} \cdot d\mathbf{p})\end{aligned}$$

Since $d\mathbf{p}$ has units of $\frac{\text{kg m}}{\text{s}^2}$

1.35 (a)

$$\begin{aligned}\langle p \rangle &= \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle (-i\hbar \frac{\partial}{\partial x'}) \langle x' | \alpha \rangle \\ &= \int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4} \sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] (-i\hbar \frac{\partial}{\partial x'}) \left[\frac{1}{\pi^{1/4} \sqrt{d}} \right] \exp \left[ikx' - \frac{x'^2}{2d^2} \right] \\ &= \left[\frac{-i\hbar}{\pi^{1/2} d} \right] \int_{-\infty}^{\infty} dx' \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \frac{\partial}{\partial x'} \exp \left[ikx' - \frac{x'^2}{2d^2} \right] \\ &= \left[\frac{-i\hbar}{\pi^{1/2} d} \right] \int_{-\infty}^{\infty} dx' \left[ik - \frac{x'}{d^2} \right] \exp \left[-\frac{x'^2}{d^2} \right] \\ &= \left[\frac{-i\hbar}{\pi^{1/2} d} \right] \left(\frac{-k\pi^{1/2} d}{i} \right) \\ &= \hbar k\end{aligned}$$

$$\begin{aligned}\langle p^2 \rangle &= \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle (-i\hbar \frac{\partial}{\partial x'})^2 \langle x' | \alpha \rangle \\ &= \hbar^2 \int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4} \sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \frac{\partial^2}{\partial x'^2} \left[\frac{1}{\pi^{1/4} \sqrt{d}} \right] \exp \left[-ikx' - \frac{x'^2}{2d^2} \right] \\ &= \frac{\hbar^2}{2d^2} + \hbar^2 k^2\end{aligned}$$

(b)

$$\begin{aligned}\langle p | \alpha \rangle &= \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle (-i\hbar \frac{\partial}{\partial x'}) \langle x' | \alpha \rangle \\ &= \int_{-\infty}^{\infty} dp \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right] (-i\hbar \frac{\partial}{\partial x'}) \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right] \\ &= \hbar k\end{aligned}$$

$$\begin{aligned}\langle p^2 | \alpha \rangle &= \int_{-\infty}^{\infty} dx' \langle \alpha | x' \rangle (-i\hbar \frac{\partial}{\partial x'})^2 \langle x' | \alpha \rangle \\ &= \hbar^2 \int_{-\infty}^{\infty} dp \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right] \frac{\partial^2}{\partial x'^2} \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp \left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2} \right] \\ &= \frac{\hbar^2}{2d^2} + \hbar^2 k^2\end{aligned}$$