Quantum I Assignment #5

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Q-1

$$[\tilde{\mathbf{X}}, \tilde{\mathbf{H}}]$$

1.33 Starting with the translation operator applied to the expectation value for $\mathbf{x}$:

$$\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle$$

By equation 1.207 we know:

$$\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x} = d\mathbf{x}'$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{gathered}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x}\right] = \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{x} = \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathbf{x} = \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathbf{x} + d\mathbf{x}' - \mathbf{x} = \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}'
\end{gathered}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{x}|\alpha\rangle + d\mathbf{x}'$.
Using the same process for $\mathbf{p}$: By equation 1.227 we know:

$$\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p} = 0$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{gathered}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p}\right] = 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{p} = 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathbf{p} = 0 \\
\mathbf{p} + d\mathbf{p}' - \mathbf{p} = 0
\end{gathered}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{p}|\alpha\rangle$.

1.34 Satisfies unitary property because $\mathbf{W}$ is hermitian:

$$\begin{aligned}
\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}') &= (1 - i\mathbf{W}\cdot d\mathbf{p})(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= (1 - i\mathbf{W}\cdot d\mathbf{p}^\dagger)(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= 1 - i(\mathbf{W} - \mathbf{W}^\dagger) \\
&\simeq 1
\end{aligned}$$

Satisfies the associative property:

$$\begin{aligned}
\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}'') &= (1 + i\mathbf{W}\cdot d\mathbf{p}')\cdot(1 + i\mathbf{W}\cdot d\mathbf{p}'') \\
&\simeq 1 - i\mathbf{W}\cdot(d\mathbf{p}'d\mathbf{p}'') \\
&= \mathscr{B}(d\mathbf{p}' + d\mathbf{p}'')
\end{aligned}$$

Satisfies the inverse property trivially:

$$\begin{gathered}
\mathscr{B}(-d\mathbf{p}') = \mathscr{B}^{-1}(d\mathbf{p}') \\
1 + i\mathbf{W}\cdot d\mathbf{p} = -(-1 - i\mathbf{W}\cdot d\mathbf{p})
\end{gathered}$$

Since $d\mathbf{p}$ has units of $\frac{\text{kg m}}{\text{s}^2}$

1.35 (a)

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} dx' \,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})\,\langle x'|\alpha\rangle \\
&= \int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx' - \frac{x'^2}{2d^2}\right](-i\hbar\frac{\partial}{\partial x'})\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[ikx' - \frac{x'^2}{2d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\int_{-\infty}^{\infty} dx' \exp\left[-ikx' - \frac{x'^2}{2d^2}\right]\frac{\partial}{\partial x'}\exp\left[ikx' - \frac{x'^2}{2d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\int_{-\infty}^{\infty} dx' \left[ik - \frac{x'}{d^2}\right]\exp\left[-\frac{x'^2}{d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\left(\frac{-k\pi^{1/2}d}{i}\right) \\
&= \hbar k
\end{aligned}$$

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} dx' \,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})^2\,\langle x'|\alpha\rangle \\
&= \hbar^2\int_{-\infty}^{\infty} dx' \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx' - \frac{x'^2}{2d^2}\right]\frac{\partial^2}{\partial x'^2}\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx' - \frac{x'^2}{2d^2}\right] \\
&= \frac{\hbar^2}{2d^2} + \hbar^2k^2
\end{aligned}$$

(b)

$$\begin{aligned}
\langle p|\alpha\rangle &= \int_{-\infty}^{\infty} dx' \,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})\,\langle x'|\alpha\rangle \\
&= \int_{-\infty}^{\infty} dp\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right](-i\hbar\frac{\partial}{\partial x'})\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right] \\
&= \hbar k
\end{aligned}$$

$$\begin{aligned}
\langle p^2|\alpha\rangle &= \int_{-\infty}^{\infty} dx' \,\langle \alpha|x'\rangle \,(-i\hbar\frac{\partial}{\partial x'})^2 \,\langle x'|\alpha\rangle \\
&= \hbar^2 \int_{-\infty}^{\infty} dp \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right] \frac{\partial^2}{\partial x'^2} \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right. \\
&= \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}$$

1.36 (a) i. Using equation (1.265a) $\langle p'|\alpha\rangle = \int dx' \,\langle p'|x'\rangle \,\langle x'|\alpha\rangle$:

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx' \,\langle p'|x|x'\rangle \,\langle x'|\alpha\rangle \\
&= \int dx' x' \,\langle p'|x'\rangle \,\langle x'|\alpha\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx' x' \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle \quad \text{using (1.264)} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \frac{\partial}{\partial p'} i\hbar \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle \\
&= i\hbar\frac{\partial}{\partial p'} \int dx' \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle \\
\langle p'|x|\alpha\rangle &= i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle
\end{aligned}$$

ii. This holds true from the previous result:

$$\begin{aligned}
\langle \beta|x|\alpha\rangle &= \int dp' \,\langle \beta|p'\rangle \,\langle p'|x|\alpha\rangle = \int dp' x \,\langle \beta|p'\rangle \,\langle p'\alpha\rangle \\
&= \int dp' \,\langle \beta|p'\rangle \, i\hbar\frac{\partial}{\partial p'} \langle p'\alpha\rangle \quad \text{from the previous part} \\
\langle \beta|x|\alpha\rangle &= \int dp' \phi_\beta(p') i\hbar\frac{\partial}{\partial p'} \phi_\alpha(p')
\end{aligned}$$

(b)