Math Methods Assignment #2

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1.

$$f(x,y) = 2x^2 + \frac{1}{2}y^2 - xy$$
$$g(x,y) = 4x^2 + y^2 - 4 = 0$$

Using a Lagrange multiplier:

$$h = f + \lambda g = 2x^2 + \frac{1}{2}y^2 - xy + \lambda\left(4x^2 + y^2 - 4\right)$$
$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0$$
$$\frac{\partial}{\partial y}\left(2x^2 + \frac{1}{2}y^2 - xy + \lambda\left(4x^2 + y^2 - 4\right)\right) - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0$$
$$-x + y + 2y\lambda - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(2x^2 + \frac{1}{2}y^2 - xy + \lambda\left(4x^2 + y^2 - 4\right)\right)\right) = 0$$
$$-x + y + 2y\lambda = 0$$

Plugging back into original equations:

$$y = \frac{x}{1+2\lambda}$$
$$4x^2 + y^2 - 4 = 4x^2 + \left(\frac{x}{1+2\lambda}\right)^2 - 4 = 0$$
$$x = \pm\frac{2}{\sqrt{\frac{1}{(2\lambda+1)^2} + 4}}$$
$$\left(\pm\frac{2}{(2\lambda+1)\sqrt{\frac{1}{(2\lambda+1)^2} + 4}}, \pm\frac{2}{\sqrt{\frac{1}{(2\lambda+1)^2} + 4}}\right)$$

2. The quantity that needs to be minimized is the length $ds = \sqrt{dx^2 + dy^2 + dz^2}$. Re-writing

this in terms cylindrical coordinates:

$$
\begin{aligned}
dx &= \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta \\
dy &= \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta \\
dx^2 + dy^2 &= \dot{p}^2 + p^2\dot{\theta}^2 \\
dz &= \frac{\dot{\rho}}{\rho} \\
ds &= \sqrt{\dot{p}^2 + p^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}}
\end{aligned}
$$

Plugging this into the Lagrangian:

$$
f = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}}
$$

$$
\dot{\rho}\frac{\partial f}{\partial \dot{\rho}} - f = 0
$$

$$
\dot{\rho}\frac{\partial}{\partial \dot{\rho}}\left(\sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}}\right) - \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}} = 0
$$

$$
\dot{\rho}\left(2\dot{\rho} + \frac{2\dot{\rho}}{\rho^2}\right) - \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}} = 0
$$

$$
2\dot{\rho}^2 + \frac{2\dot{\rho}^2}{\rho^2} - \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \frac{\dot{\rho}^2}{\rho^2}} = 0
$$

3. The constraints of the problem can be written described using the following two equations:

$$
I = \int_{x_a}^{x_b} y\,dx
$$

$$
L = \int_{x_a}^{x_b} \sqrt{1 + y'^2}\,dx
$$

Using the Lagrange multiplier:

$$
\begin{gathered}
h = y - \lambda\sqrt{1 + y'^2} \\
\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0 \\
\frac{\partial}{\partial y}\left(y - \lambda\sqrt{1 + y'^2}\right) - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0 \\
1 - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(y - \lambda\sqrt{1 + y'^2}\right)\right) = 0 \\
1 - \frac{d}{dx}\left(-\frac{\lambda y'}{\sqrt{(y')^2 + 1}}\right) = 0 \\
\frac{d}{dx}\left(-\frac{\lambda y'}{\sqrt{(y')^2 + 1}}\right) = 1 \\
\frac{\lambda y'}{\sqrt{(y')^2 + 1}} = x + x_0 \\
\frac{\lambda y'^2}{y^2 + 1} = (x + x_0)^2 \\
y' = \frac{x + x_0}{\sqrt{\lambda^2 - (x + x_0)^2}} \\
\int y' dx = \int \frac{x + x_0}{\sqrt{\lambda^2 - (x + x_0)^2}} dx = \sqrt{-(x + x_a)^2 + \lambda^2} + x_2 \\
y + y_0 = \sqrt{-(x + x_0)^2 + \lambda^2}
\end{gathered}
$$

This is the equation for a semicircle, and it intuitively makes sense that if L is too big then to make a shape that most closely resembles a circle would require the function that has two values of y for a given value of x.

4. (a) Starting from the functions from Byron and Fuller:

$$
\begin{gathered}
I = \int_{x_a}^{x_b} \sqrt{\frac{1 + y'^2}{y}} dx \\
L = \int_{x_a}^{x_b} \sqrt{1 + y'^2} dx
\end{gathered}
$$

Substituting these functions into the Euler-Lagrange equation with Lagrange multipliers:

$$h = \sqrt{\frac{1+y'^2}{y}} - \lambda\sqrt{1+y'^2}$$

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0$$

$$\frac{\partial h}{\partial y} = \frac{\sqrt{\frac{1+y'^2}{y}}}{2y}$$

$$\frac{\partial h}{\partial y'} = \frac{\lambda y'}{\sqrt{(y')^2+1}} + \frac{y'}{y\sqrt{\frac{(y')^2+1}{y}}}$$

$$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{\left(\frac{y'^2+1}{y}\right)^{3/2}\left(2y(y''-y'^2\left(y'^2+1\right)\right)}{2\left(y'^2+1\right)^3} - \frac{\lambda y''}{\left(y'^2+1\right)^{3/2}}$$

$$\frac{\sqrt{\frac{1+y'^2}{y}}}{2y} - \frac{\left(\frac{y'^2+1}{y}\right)^{3/2}\left(2y(y''-y'^2\left(y'^2+1\right)\right)}{2\left(y'^2+1\right)^3} - \frac{\lambda y''}{\left(y'^2+1\right)^{3/2}} = 0$$

(b) Starting from the functions from Byron and Fuller:

$$I = \int_{x_a}^{x_b} \sqrt{1+y'^2}dx$$

$$L = \int_{x_a}^{x_b} \sqrt{\frac{1+y'^2}{y}}dx$$

Substituting these functions into the Euler-Lagrange equation with Lagrange multipliers:

$$h = \sqrt{1+y'^2} - \lambda\sqrt{\frac{1+y'^2}{y}}$$

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0$$

$$\frac{\partial h}{\partial y} = \frac{\lambda\sqrt{\frac{y'(x)^2+1}{y(x)}}}{2y(x)}$$

$$\frac{\partial h}{\partial y'} = y'(x)\left(-\frac{\lambda}{y(x)\sqrt{\frac{y'(x)^2+1}{y(x)}}} - \frac{1}{\sqrt{1-y'(x)^2}}\right)$$

$$\frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = \frac{\lambda\left(\frac{y'(x)^2+1}{y(x)}\right)^{3/2}\left(-2y(x)y''(x)+y'(x)^4+y'(x)^2\right)}{2\left(y'(x)^2+1\right)^3} + \frac{y''(x)}{\left(1-y'(x)^2\right)^{3/2}}$$

$$\frac{1}{2}1(x)\left(\frac{\lambda\left(\frac{y'(x)^2+1}{y(x)}\right)^{3/2}\left(2y(x)y''(x)-y'(x)^2\left(y'(x)^2+1\right)\right)}{\left(y'(x)^2+1\right)^3} + \frac{2y''(x)}{\left(1-y'(x)^2\right)^{3/2}}\right) = 0$$

(c) Starting from the functions from Byron and Fuller, and using adding constraint equations:

$$I = 2\pi \int_{\rho_a}^{\rho_b} \rho\sqrt{1+\rho'^2}dz$$
$$V = \pi \int_{\rho_a}^{\rho_b} \rho^2\left(1+\rho'^2\right)dz$$
$$M = \pi\alpha \int_{\rho_a}^{\rho_b} \rho^2\left(1+\rho'^2\right)dz$$

5. (a)

$$h = f^2 - \lambda_1 g_1 - \lambda_2 g_2$$
$$\frac{\partial h}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial h}{\partial y_1'}\right) = 0$$
$$2ff' - \lambda_1 g_1' - \lambda_2 g_2' - \frac{d}{dx}\left(2ff'\right) = 0$$

This is not the same curve as if it were simply a function of f

(b)

$$h = f^2 - \lambda_1 g_1^2 - \lambda_2 g_2^2$$
$$\frac{\partial h}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial h}{\partial y_1'}\right) = 0$$
$$f' - \lambda_1 2 g_1 g_1' - \lambda_2 2 g_2 g_2' - \frac{d}{dx}\left(f'\right) = 0$$

This also does not give you the same curve.

6. (a)

$$f = \alpha(x,y)\sqrt{1+z_x^2+z_y^2} - \lambda z$$
$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

(b) In the case where $\alpha(x,y) = \alpha_0$

$$\frac{\partial f}{\partial z} = -\lambda$$
$$\frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} = \frac{\partial}{\partial x}\frac{z_x}{\sqrt{1+z_x^2+z_y^2}} = \frac{(z_y^2+1)z_x'}{\left(1+z_x^2+z_y^2\right)^{3/2}}$$
$$\frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = \frac{\partial}{\partial y}\frac{z_y}{\sqrt{1+z_x^2+z_y^2}} = \frac{(z_x^2+1)z_y'}{\left(1+z_x^2+z_y^2\right)^{3/2}}$$
$$-\lambda - \frac{(z_y^2+1)z_x'}{\left(1+z_x^2+z_y^2\right)^{3/2}} - \frac{(z_x^2+1)z_y'}{\left(1+z_x^2+z_y^2\right)^{3/2}} = 0$$

If $z(x,y) = \sqrt{R^2 - x^2 - y^2}$:

$$
\begin{aligned}
z_x &= -\frac{x}{\sqrt{1-x^2-y^2}} \\
z_x' &= \frac{y^2-1}{(1-x^2-y^2)^{3/2}} \\
z_y &= -\frac{y}{\sqrt{1-x^2-y^2}} \\
z_y' &= \frac{x^2-1}{(1-x^2-y^2)^{3/2}} \\
-\lambda - \frac{(z_y^2+1)z_x' - (z_x^2+1)z_y'}{\left(1+z_x^2+z_y^2\right)^{3/2}} &= 0
\end{aligned}
$$

Plugging in variables and simplifying using Mathematica we get a circle:

$$\lambda + \frac{2(x-y)(x+y)\sqrt{R^2-x^2-y^2}}{R^2}$$

The relationship between λ and R is:

$$\lambda = \frac{2(x-y)(x+y)\sqrt{R^2-x^2-y^2}}{R^2}$$

7. Using the Euler-Lagrange equation, and treating ∇ as a simple derivative:

$$
\begin{aligned}
f &= \frac{1}{8\pi}\phi'^2 - \rho\phi \\
\frac{\partial f}{\partial \phi} - \frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) &= 0 \\
\frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right) &= \frac{1}{4\pi}\phi'' \\
\frac{\partial f}{\partial \phi} &= -\rho \\
-\rho - \frac{1}{4\pi}\phi'' &= 0 \\
\phi'' = -4\pi\rho \rightarrow \vec{\nabla}^2\phi(\vec{r}) &= -4\pi\rho(\vec{r})
\end{aligned}
$$

Taking a derivative of the above result gives:

8. `FactorInteger[113517805*113517805+1]`→`{{2,1},{373,1},{1312889,1},{13157129,1}}`