Classical Mechanics Assignment #1

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1. (a) Starting with the forces, simplifying, and substituting variables:

$$\begin{gathered}
F_{\text{total}} = F_{\text{spring}} + F_{\text{damping}} + F_{\text{piston}} \\
m\ddot{x} = -kx - m\nu\dot{x} + kX(t) \\
\ddot{x} + \frac{k}{m}x + \nu\dot{x} = \frac{k}{m}(t) \\
\ddot{x} + \nu\dot{x} + \omega_0^2 x = F_0(t)
\end{gathered}$$

(b) Complementary solution:

$$x(t) = e^{-\beta t}\left[A_1 e^{-i\sqrt{\omega_0^2-\beta^2}t} + A_2 e^{i\sqrt{\omega_0^2-\beta^2}t}\right]$$

Particular solution:

$$\begin{gathered}
X(t) = X_0 e^{\alpha t}\cos(\omega t) \\
\dot{X}(t) = e^{\alpha t} X_0 \left(\alpha\cos(\omega t) - \omega\sin(\omega t)\right) \\
\ddot{X}(t) = X_0 e^{\alpha t}\left(\left(\alpha^2 - \omega_0^2\right)\cos(t\omega) - 2\alpha\omega\sin(t\omega)\right)
\end{gathered}$$

Particular solution continued: substituting back into equation:

$$\begin{gathered}
X_0 e^{\alpha t}\left(\left(\alpha^2 - \omega_0^2\right)\cos(t\omega) - 2\alpha\omega\sin(t\omega)\right) + \\
\nu e^{\alpha t} X_0 \left(\alpha\cos(\omega t) - \omega\sin(\omega t)\right) + \\
\omega_0^2 X_0 e^{\alpha t}\cos(\omega t) = \omega^2 e^{\alpha t}\cos(\omega t)
\end{gathered}$$

Particular solution continued: finding X_0:

$$\omega_0^2 X_0 e^{\alpha t}\cos(\omega t) = \omega^2 e^{\alpha t}\cos(\omega t)$$

2. (a) Assuming the "ripples" can be modeled by a sinusoidal function, the motion of the car can be described by the driven damped oscillator discussed in class. In this case the

equation for the resonance frequency:

$$\omega_r = \sqrt{\omega_0^2 + 2\beta^2}$$
$$\frac{v}{x_{\text{spacing}}} = \sqrt{\omega_0^2 + 2\beta^2}$$
$$v = x_{\text{spacing}}\sqrt{\omega_0^2 + 2\beta^2}$$

Since $\omega_0 = \sqrt{\frac{k}{m}}$ and $\beta = \frac{b}{2m}$:

$$\omega_r = \sqrt{\omega_0^2 + 2\beta^2}$$
$$\frac{v}{x_{\text{spacing}}} = \sqrt{\omega_0^2 + 2\beta^2}$$
$$v = x_{\text{spacing}}\sqrt{\frac{k}{m} + \frac{b^2}{4m^2}}$$

Plugging in some reasonable values we can check whether or not our equations make physical sense. I used the mass of my car $m = 1565$ kg, and damping $b = 2000$ N s/m and stiffness $k = 22,000$ N/m parameters from a research paper.[1]

$$v = 2\sqrt{\frac{22000}{1565} + \frac{2000^2}{4 \cdot 1565^2}} \approx 7.6 \text{ m/s} \approx 17 \text{ mph}$$

This is a reasonable number and the units make sense:

$$v = m\sqrt{\frac{N}{m \cdot kg} + \frac{N^2 \cdot s^2}{m^2 \cdot kg^2}}$$
$$v = m\sqrt{\frac{kg \cdot m}{m \cdot kg \cdot s^2} + \frac{kg^2 \cdot m^2 \cdot s^2}{m^2 \cdot kg^2 \cdot s^4}}$$
$$v = m\sqrt{\frac{1}{s^2} + \frac{1}{s^2}} = \frac{m}{s}$$

(b) So that the suspension doesn't fall apart I probably want a damping constant that only allows resonances at very high speeds, say above 100 mph:

$$\sqrt{4m^2\left(\frac{v^2}{x_{\text{spacing}}^2} - \frac{k}{m}\right)} = b$$
$$\sqrt{4 \cdot 1565^2\left(\frac{45}{2} - \frac{22000}{1565}\right)} \approx 9100 \text{ N s/m}$$

[1]https://journals.sagepub.com/doi/full/10.1177/1687814016648638 Analysis of suspension with variable stiffness and variable damping force for automotive applications

3. (a) Since $F = \frac{dU}{ds}$ the change in energy of the system is simply:

$$m\ddot{x} = F = \dot{E} = -\left(x^2 + \dot{x}^2 - 1\right)\dot{x} - x$$

(b) Simply plugging in the variables $x = r\cos\theta$ and $\dot{x} = r\sin\theta$:

$$\dot{E} = -\left((r\cos\theta)^2 + (r\sin\theta)^2 - 1\right) r\sin\theta - r\cos\theta$$

Solving for $\dot{r}$: