Math Methods Assignment #1

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1. (a)

$$E = \frac{1}{2}mv^2$$
$$P_0 = \frac{E}{\Delta t} = \frac{\Delta m v^2}{2\Delta t}$$
$$v = \sqrt{\frac{2P_0\Delta t}{\Delta m}}$$
$$p = \Delta m\sqrt{\frac{2P_0\Delta t}{\Delta m}} = \sqrt{2P_0\Delta m\Delta t}$$

(b) Tsiolkovsky rocket equation from wikipedia:

$$\Delta v = v_e \ln\frac{m_0}{m_f}$$

Substituting in the answer from part (a):

$$v_f = \sqrt{2P_0\frac{\Delta t}{\Delta m}}\ln\left(\frac{m}{m-\Delta m}\right)$$

(c) From the same wikipedia article, substituting in the answer from part (a) and converting Δ to d:

$$mv' = -v_e m'$$
$$mv' + v_e m' = 0$$
$$v' = \sqrt{2P_0\frac{dt}{dm}}\frac{m'}{m}$$

Assuming $\frac{dt}{dm} = (m')^{-1}$:

$$v' = \sqrt{\frac{2P_0}{m'}}\frac{m'}{m}$$
$$v = \sqrt{2P_0}\int_0^{t_f}\sqrt{\frac{1}{m'}}\frac{m'}{m}dt$$

Using the Euler-Lagrange equation:

$$m'\frac{\delta f}{\delta m'} - f = c$$

$$m'\frac{\delta}{\delta m'}\left(\sqrt{\frac{1}{m'}\frac{m'}{m}}\right) - \sqrt{\frac{1}{m'}\frac{m'}{m}} = c$$

$$m'\frac{1}{2m}\sqrt{\frac{1}{m'}} - \sqrt{\frac{1}{m'}\frac{m'}{m}} = c$$

$$-\frac{m'}{2m}\sqrt{\frac{1}{m'}} = c$$

$$m' = 4c^2m^2$$

(d) Final velocity in part (c), using the equation for v derived in the first part of (c):

$$v = \sqrt{2P_0}\int_0^{t_f}\sqrt{\frac{1}{m'}\frac{m'}{m}}dt$$

$$v = \sqrt{2P_0}\int_0^{t_f}\sqrt{\frac{1}{4c^2m^2}\frac{4c^2m^2}{m}}dt$$

$$v = \sqrt{2P_0}\int_0^{t_f}2c^2m\sqrt{\frac{1}{c^2m^2}}dt$$

$$v = 2c^2m\sqrt{\frac{2P_0}{c^2m^2}}t_f$$

If c is positive:

$$v = 2ct_f\sqrt{2P_0}$$

This differs from the answer in part (b) because it does not depend on m' or m?

2. (a) Using the Euler-Lagrange equation:

$$y'\frac{\delta f}{\delta y'} - f = c$$

$$y'\frac{\delta}{\delta y'}\left[\left(\frac{\delta y}{\delta x}\right)^2 + \alpha y\frac{\delta y}{\delta x}\right] - \left[\left(\frac{\delta y}{\delta x}\right)^2 + \alpha y\frac{\delta y}{\delta x}\right] = c$$

$$y'\frac{\delta}{\delta y'}\left[(y')^2 + \alpha yy'\right] - \left[(y')^2 + \alpha yy'\right] = c$$

$$y'\cdot(2y' + \alpha y) - \left[(y')^2 + \alpha yy'\right] = c$$

$$y' = \sqrt{c}$$

Since y' is constant:

$$y = Ax + B$$

Since $y(0) = 0$ and $y(x_f) = y_f$:

$$B = 0$$

$$y_f = A\cdot x_f A = \frac{y_f}{x_f}$$

(b) It doesn't because $\alpha y \frac{\delta y}{\delta x}$ is a total derivative and thus is path independent.

3. Differentiating $I(\epsilon)$ with respect to ϵ:

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B} \left[\frac{\delta f}{\delta y}\frac{dy}{d\epsilon} + \frac{\delta f}{\delta y'}\frac{dy'}{d\epsilon} + \frac{\delta f}{\delta y''}\frac{dy''}{d\epsilon} \right]$$

Since y' endpoints are prescribed, the integration by parts trick on page 46 will work on both the $\frac{\delta f}{\delta y'}\frac{dy'}{d\epsilon}$ and $\frac{\delta f}{\delta y''}\frac{dy''}{d\epsilon}$ terms.

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B} \left[\frac{\delta f}{\delta y} - \frac{d}{dx}\left(\frac{\delta f}{\delta y'}\right) - \frac{d^2}{dx^2}\left(\frac{\delta f}{\delta y'}\right) \right] \frac{dy}{d\epsilon} dx$$

This requires that $y(x, \epsilon)$ and all its derivatives through third order are continuous functions of x and ϵ

4. (a) From the definition of ds:

$$ds = \sqrt{1 + y'^2} dx$$

Using the Euler-Lagrange equation:

$$f = \frac{\sqrt{1+y'^2}}{u}$$

$$y' \frac{\delta}{\delta y'}\left[\frac{\sqrt{1+y'^2}}{u}\right] - \frac{\sqrt{1+y'^2}}{u} = \text{const}$$

$$\frac{y'^2}{u\sqrt{y'^2+1}} - \frac{\sqrt{1+y'^2}}{u} = \frac{y'^2 - (1+y'^2)}{u\sqrt{y'^2+1}} = \text{const}$$

$$\frac{1}{u\sqrt{y'^2+1}} = \frac{1}{u\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} = \frac{1}{u\sqrt{\cot^2\phi + 1}} = \text{const}$$

$$\frac{1}{u\sqrt{\csc^2\phi}} = \frac{\sin\phi}{u} = \text{const}$$

(b) Since we assumed u above was a function of y we can simply take our result from part (a):

$$\frac{\sin\phi}{\alpha y} = \text{const}$$

$$\sin\phi = \alpha y$$

$$\sin(\arctan y') = \alpha y$$

Using WolframAlpha to solve this differential equation gives:

$$\alpha x + x_0 = \sqrt{1 - \alpha^2 y^2} - \tanh^{-1}\left(\sqrt{1-\alpha^2 y^2}\right)$$

Assuming the $\tanh^{-1}$ can be ignored as a phase or something similar, it's clear that $\alpha x + x_0 = \sqrt{1-\alpha^2 y^2}$ is the equation of a circle centered on the x axis.