Math Methods Assignment #6

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1. (a) Starting with Ampere's Law:

$$\begin{aligned}
\nabla \times \mathcal{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} \\
\nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c}\nabla \cdot \boldsymbol{J} + \frac{1}{c}\frac{\partial(\nabla \cdot \mathcal{E})}{\partial t} = 0 \quad \text{Using the identity: } \nabla\,(\nabla \cdot \times A) = 0 \\
\nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c}\nabla \cdot \boldsymbol{J} + \frac{4\pi}{c}\frac{\partial \rho}{\partial t} = 0 \quad \text{Using: } \nabla \cdot \mathcal{E} = 4\pi\rho \\
\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} &= 0
\end{aligned}$$

(b) Starting with the curl of the electric field:

$$\begin{aligned}
\nabla \times \mathcal{E} &= -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t} \\
\nabla \times \mathcal{E} &= -\frac{1}{c}\frac{\partial(\nabla \times \mathcal{A})}{\partial t} \\
\nabla \times \left(\mathcal{E} + \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t}\right) &= 0 \\
\mathcal{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t} \quad \text{Rewriting in terms of a scalar potential}
\end{aligned}$$

(c)

$$F = \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{1}{2}(B_1 + B_1) & \frac{1}{2}(B_2 + B_2) & \frac{1}{2}(B_3 + B_3) \end{bmatrix}$$

(d) Starting with $\mathcal{B} = \nabla \times A$:

$$\begin{aligned}
F_{ij} &= \epsilon_{ijk}(\nabla \times A)_k \\
F_{ij} &= \epsilon_{ijk}\epsilon_{jlm}\partial_l A_m \quad \text{Rewriting the curl using Levi-Civita} \\
F_{ij} &= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_l A_m \\
F_{ij} &= \partial_i A_j - \partial_j A_i
\end{aligned}$$

(e) Proving the first part:

$$F_{4j} = \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}$$
$$-(\frac{\partial A_4}{\partial x_j} - \frac{\partial A_j}{\partial x_4}) = \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}$$

Assuming this 4th term is time, $\mathcal{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t}$ and $\partial_t A_j = 0$ show that $F_{4j} = iE_j$.

(f) Starting with the definition given:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$
$$\partial_i(\partial_j A_j - \partial_k A_j) + \partial_j(\partial_k A_i - \partial_i A_k) + \partial_k(\partial_i A_j - \partial_j A_i) = 0$$