Math Methods Assignment #4

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1. It would move in a parabola, which can be seen from the fact that if we rotate our coordinate system such that the rockets total acceleration is pointing down this becomes a simple projectile motion problem, which we know is parabolic.

2. If we take a rhombus with corners $A,\ B,\ C,\ D$ we can represent the sides as $\vec{AB},\ \vec{BC},\ \vec{CD},\ \vec{DA}$. We know that $|\vec{AB}| = |\vec{BC}| = |\vec{CD}| = |\vec{DA}|$, and that $\vec{AC} = \vec{AB} + \vec{BC}$ and $\vec{BD} = \vec{BC} + \vec{CD}$. Since the sides of a rhombus are parallel we can for the sake of this problem consider $\vec{AC} = \vec{AB} + \vec{AD}$ and $\vec{BD} = \vec{AD} - \vec{AB}$. To show that the diagonals are orthogonal we need to show that the dot products are equal to zero: $\vec{AC} \cdot \vec{BD} = \left(\vec{AB} + \vec{AD}\right) \cdot \left(\vec{AD} - \vec{AB}\right) = \vec{AD}^2 - \vec{AB}^2 = 0$ since the sides all have equal length.

3. This is equivalent to $\sum_i \sum_j \sum_k \epsilon_{ijk}\epsilon_{ijk}$. The number of permutations of n numbers is $n!$, which means there are $3! = 6$ permutations in our case. Since whether the permutation is even or odd the term $\epsilon_{ijk}\epsilon_{ijk} = 1$ and otherwise 0 the sum is equal to 6.

4. Using the definition of the cross product $\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk}\hat{\boldsymbol{e}}_i a_j b_k$, and the relation $\epsilon_{ijk}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$:

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m && (1)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m && (2)\\
&= a_j b_i c_j - a_j b_j c_i && (3)\\
&= b(a \cdot c) - c(a \cdot b) && (4)
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= \epsilon_{ijk} a_i b_j \epsilon_{klm} c_l && (5)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j c_l && (6)\\
&= a_i b_j c_i - a_i b_j c_j && (7)\\
&= b(a \cdot c) - a(b \cdot c) && (8)
\end{aligned}$$

5. The following matrix converts between x and x':

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9}$$

Multiplying the gradient of the scalar function we can see that they are both equal and thus covariant:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\hat{e}_1 + \left(\frac{1}{\sqrt{2}} + 1\right)\hat{e}_2 = \frac{1}{\sqrt{2}}\left(\hat{e}_1 + \hat{e}_2\right) + \hat{e}_2 \tag{10}$$

6. For an orthogonal transformation A the following is true: $AA^T = A^T A = \mathbb{1}$. This means that $(A^{-1})AA^T = A^{-1}\mathbb{1} = A^T = A^{-1}$ which means that $A = (A^T)^{-1}$, which means that a vector that transforms according to $(A^T)^{-1}$ is no different from a vector that transforms according to A.