

Math Methods Assignment #4

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1. It would move in a parabola, which can be seen from the fact that if we rotate our coordinate system such that the rockets total acceleration is pointing down this becomes a simple projectile motion problem, which we know is parabolic.
2. If we take a rhombus with corners A, B, C, D we can represent the sides as $\vec{AB}, \vec{BC}, \vec{CD}, \vec{DA}$. We know that $|\vec{AB}| = |\vec{BC}| = |\vec{CD}| = |\vec{DA}|$, and that $\vec{AC} = \vec{AB} + \vec{BC}$ and $\vec{BD} = \vec{BC} + \vec{CD}$. Since the sides of a rhombus are parallel we can for the sake of this problem consider $\vec{AC} = \vec{AB} + \vec{AD}$ and $\vec{BD} = \vec{AD} - \vec{AB}$. To show that the diagonals are orthogonal we need to show that the dot products are equal to zero: $\vec{AC} \cdot \vec{BD} = (\vec{AB} + \vec{AD}) \cdot (\vec{AD} - \vec{AB}) = \vec{AD}^2 - \vec{AB}^2 = 0$ since the sides all have equal length.
3. This is equivalent to $\sum_i \sum_j \sum_k \epsilon_{ijk} \epsilon_{ijk}$. The number of permutations of n numbers is $n!$, which means there are $3! = 6$ permutations in our case. Since whether the permutation is even or odd the term $\epsilon_{ijk} \epsilon_{ijk} = 1$ and otherwise 0 the sum is equal to 6.
4. Using the definition of the cross product $\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} \hat{e}_i a_j b_k$, and the relation $\epsilon_{ijk} \epsilon_{klm} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})$:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \quad (1)$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \quad (2)$$

$$= a_j b_i c_j - a_j b_j c_i \quad (3)$$

$$= b(a \cdot c) - c(a \cdot b) \quad (4)$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \epsilon_{ijk} a_i b_j \epsilon_{klm} c_l \quad (5)$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_i b_j c_l \quad (6)$$

$$= a_i b_j c_i - a_i b_j c_j \quad (7)$$

$$= b(a \cdot c) - a(b \cdot c) \quad (8)$$

5. The following matrix converts between x and x' :

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (9)$$

Multiplying the gradient of the scalar function we can see that they are both equal and thus covariant:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \hat{e}_1 + \left(\frac{1}{\sqrt{2}} + 1 \right) \hat{e}_2 = \frac{1}{\sqrt{2}} (\hat{e}_1 + \hat{e}_2) + \hat{e}_2 \quad (10)$$

6. For an orthogonal transformation A the following is true: $AA^T = A^T A = \mathbb{1}$