Classical Mechanics Assignment #1

Johannes Byle

September 1, 2021

1. (a) Starting with the forces, simplifying, and substituting variables:

$$
\begin{aligned}
F_{\text{total}} &= F_{\text{spring}} + F_{\text{damping}} + F_{\text{piston}} \\
m\ddot{x} &= -kx - m\nu\dot{x} + kX(t) \\
\ddot{x} + \frac{k}{m}x + \nu\dot{x} &= \frac{k}{m}(t) \\
\ddot{x} + \nu\dot{x} + \omega_0^2 x &= F_0(t)
\end{aligned}
$$

   (b) Complementary solution:

$$x(t) = e^{-\beta t}\left[A_1 e^{-i\sqrt{\omega_0^2-\beta^2 t}} + A_2 e^{i\sqrt{\omega_0^2-\beta^2 t}}\right]$$

   Starting the particular solution by finding the derivatives of $X(t)$:

$$
\begin{aligned}
X(t) &= X_0 e^{\alpha t}\cos(\omega t + \delta) \\
\dot{X}(t) &= e^{\alpha t} X_0 \left(\alpha\cos(\omega t + \delta) - \omega\sin(\omega t + \delta)\right) \\
\ddot{X}(t) &= X_0 e^{\alpha t}\left(\left(\alpha^2 - \omega^2\right)\cos(\omega t + \delta) - 2\alpha\omega\sin(\omega t + \delta)\right)
\end{aligned}
$$

   Substituting back into equation:

$$
\begin{aligned}
X_0 e^{\alpha t}\left(\left(\alpha^2 - \omega^2\right)\cos(t\omega) - 2\alpha\omega\sin(t\omega)\right) + \\
\nu e^{\alpha t} X_0 \left(\alpha\cos(\omega t) - \omega\sin(\omega t)\right) + \\
\omega_0^2 X_0 e^{\alpha t}\cos(\omega t) = \omega_0^2 e^{\alpha t}\cos(\omega t)
\end{aligned}
$$

   Solving for cos terms:

$$
\begin{aligned}
X_0 e^{\alpha t}\left(\alpha^2 - \omega^2\right) + \nu e^{\alpha t} X_0 \alpha + \omega_0^2 X_0 e^{\alpha t} &= \omega_0^2 e^{\alpha t} \\
X_0\left[e^{\alpha t}\left(\alpha^2 - \omega^2\right) + \nu e^{\alpha t}\alpha + \omega_0^2 e^{\alpha t}\right] &= \omega_0^2 e^{\alpha t} \\
X_0 &= \frac{\omega_0^2 e^{\alpha t}}{e^{\alpha t}\left(\alpha^2 - \omega^2\right) + \nu e^{\alpha t}\alpha + \omega_0^2 e^{\alpha t}} \\
X_0 &= \frac{\omega_0^2}{\alpha^2 + \alpha\nu - \omega^2 + \omega_0^2}
\end{aligned}
$$

   Particular solution:

$$e^{-\beta t}\left[A_1 e^{-i\sqrt{\omega_0^2-\beta^2 t}} + A_2 e^{i\sqrt{\omega_0^2-\beta^2 t}}\right] = \frac{\omega_0^2}{\alpha^2 + \alpha\nu - \omega^2 + \omega_0^2} e^{\alpha t}\cos(\omega t)$$

(c) At the steady state only the forcing function matters, and the amplitude of the steady state function will be maximum when:

$$\omega = \sqrt{\alpha^2 + \alpha\nu + \omega_0^2} = \omega_R$$

This does depend on the sign of α, as if α is negative as $t \to \infty$ it will tend toward 0.

2. (a) Assuming the "ripples" can be modeled by a sinusoidal function, the motion of the car can be described by the driven damped oscillator discussed in class. In this case the equation for the resonance frequency:

$$\omega_r = \sqrt{\omega_0^2 + 2\beta^2}$$
$$\frac{v}{x_{\text{spacing}}} = \sqrt{\omega_0^2 + 2\beta^2}$$
$$v = x_{\text{spacing}}\sqrt{\omega_0^2 + 2\beta^2}$$

Since $\omega_0 = \sqrt{\frac{k}{m}}$ and $\beta = \frac{b}{2m}$:

$$\omega_r = \sqrt{\omega_0^2 + 2\beta^2}$$
$$\frac{v}{x_{\text{spacing}}} = \sqrt{\omega_0^2 + 2\beta^2}$$
$$v = x_{\text{spacing}}\sqrt{\frac{k}{m} + \frac{b^2}{4m^2}}$$

Plugging in some reasonable values we can check whether or not our equations make physical sense. I used the mass of my car $m = 1565$ kg, and damping $b = 2000$ N s/m and stiffness $k = 22,000$ N/m parameters from a research paper.[1]

$$v = 2\sqrt{\frac{22000}{1565} + \frac{2000^2}{4 \cdot 1565^2}} \approx 7.6 \text{ m/s} \approx 17 \text{ mph}$$

This is a reasonable number and the units make sense:

$$v = m\sqrt{\frac{N}{m \cdot kg} + \frac{N^2 \cdot s^2}{m^2 \cdot kg^2}}$$
$$v = m\sqrt{\frac{kg \cdot m}{m \cdot kg \cdot s^2} + \frac{kg^2 \cdot m^2 \cdot s^2}{m^2 \cdot kg^2 \cdot s^4}}$$
$$v = m\sqrt{\frac{1}{s^2} + \frac{1}{s^2}} = \frac{m}{s}$$

[1]https://journals.sagepub.com/doi/full/10.1177/1687814016648638 Analysis of suspension with variable stiffness and variable damping force for automotive applications

(b) So that the suspension doesn't fall apart I probably want a damping constant that only allows resonances at very high speeds, say above 100 mph:

$$\sqrt{4m^2\left(\frac{v^2}{x_{\text{spacing}}^2}-\frac{k}{m}\right)} = b$$

$$\sqrt{4\cdot 1565^2\left(\frac{45}{2}-\frac{22000}{1565}\right)} \approx 9100 \text{ N s/m}$$

3. (a) Since $F = \frac{dU}{ds}$ the change in energy of the system is simply:

$$m\ddot{x} = F = \dot{E} = -\left(x^2+\dot{x}^2-1\right)\dot{x} - x$$

(b) By the definitions that $x = r\cos\theta$ and $\dot{x} = r\sin\theta$:

$$r^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$r^2 = x^2 + \dot{x}^2$$

Taking a time derivative and solving for $\ddot{x}$:

$$\frac{d}{dt}(r^2) = \frac{d}{dt}\left(x^2+\dot{x}^2\right)$$
$$2r\dot{r} = 2x\dot{x} + 2\dot{x}\ddot{x}$$
$$\ddot{x} = \frac{r\dot{r}}{\dot{x}} - x$$

Substituting back in for x and $\dot{x}$

$$\ddot{x} = \frac{r\dot{r}}{r\sin\theta} - r\cos\theta$$
$$\ddot{x} = \frac{\dot{r}}{\sin\theta} - r\cos\theta$$

Plugging it back into the original equation of motion:

$$\frac{\dot{r}}{\sin\theta} - r\cos\theta = -\left(x^2+\dot{x}^2-1\right)\dot{x} - x$$
$$\frac{\dot{r}}{\sin\theta} - r\cos\theta = -\left((r\cos\theta)^2+(r\sin\theta)^2-1\right)r\sin\theta - r\cos\theta$$
$$\frac{\dot{r}}{\sin\theta} - r\cos\theta = -\left(r^2-1\right)r\sin\theta - r\cos\theta$$
$$\dot{r} = -\left(r^2-1\right)r\sin^2\theta$$
$$\dot{r} = r\left(1-r^2\right)\sin^2\theta$$

Repeating the same process to find $\dot{\theta}$:
By the definitions that $x = r\cos\theta$ and $\dot{x} = r\sin\theta$:

$$\frac{\dot{x}}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Taking a time derivative and solving for $\dot{\theta}$:

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{\dot{x}}{x}$$
$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2}$$
$$\dot{\theta} = \cos^2\theta\left(\frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2}\right)$$

Substituting back in the equation of motion from part (a) for $\ddot{x}$:

$$\dot{\theta} = \cos^2\theta\left(\frac{-\left(x^2+\dot{x}^2-1\right)\dot{x} - x}{x} - \frac{\dot{x}^2}{x^2}\right)$$

Substituting back in for x and $\dot{x}$ and simplifying:

$$\dot{\theta} = \cos^2\theta\left(\frac{-\left((r\cos\theta)^2+(r\sin\theta)^2-1\right)r\sin\theta - r\cos\theta}{r\cos\theta} - \frac{(r\sin\theta)^2}{(r\cos\theta)^2}\right)$$
$$\dot{\theta} = \cos^2\theta\left(\frac{-\left(r^2-1\right)\sin\theta - \cos\theta}{\cos\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right)$$
$$\dot{\theta} = \cos^2\theta\left(\frac{-\left(r^2-1\right)\sin\theta}{\cos\theta} - \left[1+\frac{\sin^2\theta}{\cos^2\theta}\right]\right)$$
$$\dot{\theta} = \cos^2\theta\left(\frac{-\left(r^2-1\right)\sin\theta}{\cos\theta} - \sec^2\theta\right)$$
$$\dot{\theta} = -\left(r^2-1\right)\sin\theta\cos\theta - 1$$

(c)