Math Methods Assignment #7

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1. It satisfies associativity:

$$
\begin{gathered}
e_k = e^{\frac{2ik\pi}{n}} \\
(e_a \cdot e_b)e_c = e_a(e_b \cdot e_c) \\
(e^{\frac{2ia\pi}{n}} \cdot e^{\frac{2ib\pi}{n}})e^{\frac{2ic\pi}{n}} = e^{\frac{2ia\pi}{n}}(e^{\frac{2ib\pi}{n}} \cdot e^{\frac{2ic\pi}{n}}) \\
e^{\frac{2ia\pi}{n}+\frac{2ib\pi}{n}+\frac{2ic\pi}{n}} = e^{\frac{2ia\pi}{n}+\frac{2ib\pi}{n}+\frac{2ic\pi}{n}}
\end{gathered}
$$

There is an identity element: e^0
Each element has an inverse: $e^{\frac{2i(n-k)\pi}{n}}$
The group is abelian:

$$
\begin{gathered}
e_a \cdot e_b = e_b \cdot e_a \\
e^{\frac{2ia\pi}{n}} \cdot e^{\frac{2ib\pi}{n}} = e^{\frac{2ib\pi}{n}} \cdot e^{\frac{2ia\pi}{n}} \\
e^{\frac{2i\pi(a+b)}{n}} = e^{\frac{2i\pi(a+b)}{n}}
\end{gathered}
$$

2. (a) $AB - BA = -(BA - AB)$

   (b)

$$
\begin{aligned}
A[B,C] - [A,C]B &= A(BC - CB) - (AC - CA)B \\
&= ABC - ACB - ACB - CAB \\
[AB,C] &= ABC - CAB
\end{aligned}
$$

   (c)

$$
\begin{aligned}
A\{B,C\} - \{A,C\}B &= A(BC + CB) - (AC + CA)B \\
&= ABC + ACB - ACB - CAB \\
[AB,C] &= ABC - CAB
\end{aligned}
$$

   (d)

$$
\begin{aligned}
[A,[B,C]] + [B,[C,A]] + [C,[A,B]] &= 0 \\
[A,(BC - CB)] + [B,(CA - AC)] + [C,(AB - BA)] &= 0 \\
A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C &= 0 \\
ABC - ACB - BCA + CBA + BCA - BAC - CAB + CAB + ACB - CBA - ABC + BAC &= 0 \\
0 &= 0
\end{aligned}
$$

3. Clear[Global`*]
   $\mathrm{a}_1 = \{\{0,1\},\{1,0\}\};$
   $a_2 = \{\{0,-I\},\{I,0\}\};$
   $a_3 = \{\{1,0\},\{0,-1\}\};$
   MatrixForm [Table [MatrixForm [$a_i.a_j + a_j.a_i$] , $\{j,1,3\},\{i,1,3\}$]]
   MatrixForm [Table [$a_i.a_j + a_j.a_i ==$ 2KroneckerDelta[j,i]IdentityMatrix[2], $\{j,1,3\},\{i,1,3\}$]]
   MatrixForm [Table [MatrixForm [$a_i.a_j - a_j.a_i$] , $\{j,1,3\},\{i,1,3\}$]]
   MatrixForm [Table [$a_i.a_j - a_j.a_i$==2ITotal [Table [LeviCivitaTensor[3][[i,j,k]]a_k, $\{k,1,3\}$]] , $\{j,1,3\},\{i,1,3\}$]]

$$\begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\ \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

4. Clear[Global`*]
   a = $\{\{\mathrm{Cos}[\theta],\mathrm{Sin}[\theta]\},\{-\mathrm{Sin}[\theta],\mathrm{Cos}[\theta]\}\};$
   values = Eigenvalues[a];
   vectors = Eigenvectors[a];
   p = Normalize/@Transpose[vectors];
   MatrixForm[values]
   MatrixForm[vectors]

MatrixForm[p]
MatrixForm[FullSimplify[Inverse[p].a.p]]

$$\begin{pmatrix} \text{Cos}[\theta] - i\text{Sin}[\theta] \\ \text{Cos}[\theta] + i\text{Sin}[\theta] \end{pmatrix}$$

$$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

5. Because $AA^{-1} = \mathbb{1}$:

$$\begin{aligned}
\mathbb{1} &= (C^{-1} - C^{-1}D(C+D)^{-1})(C+D) \\
&= C^{-1}(C+D) - C^{-1}D(C+D)^{-1}(C+D) \\
&= C^{-1}(C+D) - C^{-1}D\mathbb{1} \\
&= C^{-1}C + C^{-1}D - C^{-1}D \\
&= C^{-1}C \\
&= \mathbb{1}
\end{aligned}$$

6. Starting from the definition of the derivative of a polynomial as $\frac{\partial^i}{\partial t^i} f^{(n)}(x) = \frac{n!}{(n-i)!} x^{n-i}$:[1]

$$Tf(t) = \sum_{in} \frac{(x\frac{\partial}{\partial t})^n}{n!} f^{(i)}(t) = \sum_{ni} \frac{x^n}{n!} \frac{\partial^n}{\partial t^n} f^{(i)}(t)$$
$$Tf(t) = \sum_{ni} \frac{x^n}{n!} \frac{n!}{(n-i)!} t^{n-i}$$
$$Tf(t) = \sum_{ni} \binom{n}{x} t^{n-i} x^n$$

Since this last terms is simply the binomial expansion we have:

$$Tf(t) = \sum_i (t+x)^i = f(t+x)$$

[1] https://en.wikipedia.org/wiki/Binomial_coefficient#Binomial_coefficients_as_polynomials