Quantum I Assignment #7

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1. (a)

$$\tilde{\mathbf{H}}\left|\alpha\right\rangle = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \left|u_2\right\rangle\\ \left|u_3\right\rangle\end{bmatrix}$$

The values that can be found are $\left(\frac{1}{\sqrt{2}},\ 1,\ 1\right)$. The probabilities are $\left(\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4}\right)$. The mean value is:

$$\langle\tilde{\mathbf{H}}\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\frac{1}{3} = \frac{1}{2}$$

$$\Delta\tilde{\mathbf{H}} = \sqrt{\langle\tilde{\mathbf{H}}^2\rangle + \langle\tilde{\mathbf{H}}\rangle^2}$$

$$\langle\tilde{\mathbf{H}}^2\rangle = \left\langle\alpha\right|\tilde{\mathbf{H}}^2\left|\alpha\right\rangle = \begin{bmatrix}\left\langle u_1\right|\frac{1}{\sqrt{2}} & \left\langle u_2\right|\frac{1}{2} & \left\langle u_3\right|\frac{1}{2}\left|u_3\right\rangle\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = 2.5$$

$$\langle\tilde{\mathbf{H}}\rangle = \left\langle\alpha\right|\tilde{\mathbf{H}}\left|\alpha\right\rangle = \begin{bmatrix}\left\langle u_1\right|\frac{1}{\sqrt{2}} & \left\langle u_2\right|\frac{1}{2} & \left\langle u_3\right|\frac{1}{2}\left|u_3\right\rangle\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = 1.5$$

$$\Delta\tilde{\mathbf{H}} = \sqrt{1.5^2 + 2.5}$$

(b)

$$\tilde{\mathbf{A}}\left|\alpha\right\rangle = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix}$$

The values that can be found are $\left(\frac{1}{\sqrt{2}},\ 1/2,\ 1/2\right)$. The probabilities are $\left(\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4}\right)$. The state vector is: $\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle & \frac{1}{2}\left|u_2\right\rangle & \frac{1}{2}\left|u_3\right\rangle\end{bmatrix}$.

(c) The state vector is $\mathcal{U}(t)\left|\alpha\right\rangle = e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}}\left|u_1\right\rangle + e^{\frac{-it}{\hbar}}\left|u_2\right\rangle + e^{\frac{-it}{\hbar}}\left|u_3\right\rangle$

(d)

$$\langle\tilde{\mathbf{A}}\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}}\\ e^{\frac{-it}{\hbar}}\\ e^{\frac{-it}{\hbar}}\end{bmatrix}\frac{1}{3} = e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}}\frac{1}{\sqrt{2}} + e^{\frac{-it}{\hbar}}\frac{1}{2} + e^{\frac{-it}{\hbar}}\frac{1}{2}$$

$$\langle\tilde{\mathbf{B}}\rangle = \begin{bmatrix}\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}}\\ e^{\frac{-it}{\hbar}}\\ e^{\frac{-it}{\hbar}}\end{bmatrix}\frac{1}{3} = e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}}\frac{1}{2} + e^{\frac{-it}{\hbar}}\frac{1}{\sqrt{2}} + e^{\frac{-it}{\hbar}}\frac{1}{2}$$

(e)

$$\tilde{\mathbf{A}}\left|\alpha\right\rangle = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = \begin{bmatrix}e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{2}\left|u_2\right\rangle\\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{2}\left|u_3\right\rangle\end{bmatrix}$$

$$\tilde{\mathbf{B}}\left|\alpha\right\rangle = \begin{bmatrix} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\left|u_2\right\rangle\\ \frac{1}{\sqrt{2}}\left|u_1\right\rangle\\ \frac{1}{2}\left|u_3\right\rangle\end{bmatrix} = \begin{bmatrix}e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{2}\left|u_1\right\rangle\\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{\sqrt{2}}\left|u_2\right\rangle\\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}}\frac{1}{2}\left|u_3\right\rangle\end{bmatrix}$$

2.1 Starting with the expression $\frac{d\tilde{\mathbf{A}}}{d=}\frac{1}{i\hbar}\left[\tilde{\mathbf{A}},\tilde{\mathbf{H}}\right]$:

$$\frac{d\tilde{\mathbf{S}}_z}{dt} = \frac{1}{ih}\left[\tilde{\mathbf{S}}_z,\omega\tilde{\mathbf{S}}_z\right] = \frac{1}{i\hbar}0 = 0$$
$$\frac{d\tilde{\mathbf{S}}_x}{dt} = \frac{1}{ih}\left[\tilde{\mathbf{S}}_x,\omega\tilde{\mathbf{S}}_z\right] = \frac{\omega}{i\hbar}i\hbar S_y = -\omega S_y$$
$$\frac{d\tilde{\mathbf{S}}_y}{dt} = \frac{1}{ih}\left[\tilde{\mathbf{S}}_y,\omega\tilde{\mathbf{S}}_z\right] = \frac{\omega}{i\hbar}i\hbar S_x = \omega S_x$$

Repeating to get the second derivative:

$$\frac{d^2\tilde{\mathbf{S}}_z}{dt^2} = \frac{d}{dt}\frac{1}{i\hbar}0 = 0$$
$$\frac{d^2\tilde{\mathbf{S}}_x}{dt^2} = \frac{d}{dt}\frac{\omega}{i\hbar}i\hbar S_y = -\omega^2 S_y$$
$$\frac{d^2\tilde{\mathbf{S}}_y}{dt^2} = \frac{d}{dt}\frac{\omega}{i\hbar}i\hbar S_x = -\omega^2 S_x$$

These two equations are basic differential equations, so it is clear that $\tilde{\mathbf{S}}_x = \alpha e^{-i\omega t}$ and $\tilde{\mathbf{S}}_y = \beta e^{-i\omega t}$

2.3 (a) Starting with the definition of $\tilde{\mathbf{S}} \cdot \vec{\tilde{\mathbf{n}}}$:

$$\tilde{\mathbf{S}} \cdot \vec{\tilde{\mathbf{n}}} = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle$$
$$\tilde{\mathbf{S}}_x = \frac{\hbar}{2}\left(|+\rangle + |-\rangle\right)$$

Applying the time evolution operator:

$$\mathcal{U}(t)\,|\tilde{\mathbf{S}}_n;+\rangle = \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right)|+\rangle + \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)|-\rangle$$
$$\mathcal{U}(t)\,|\tilde{\mathbf{S}}_n;+\rangle = \exp\left(\frac{-i\omega t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right)|+\rangle + \exp\left(\frac{-i\omega t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)|-\rangle$$
$$P(t) = |\,\langle\tilde{\mathbf{S}}_x;+|\tilde{\mathbf{S}}_n\rangle\,|^2 = \frac{1}{2}\left[\exp\left(\frac{-i\omega t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right) + \exp\left(\frac{-i\omega t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)\right]^2$$
$$P(t) = \frac{1}{2}\left(1 + 2\sin\beta\cos\omega t\right)$$

(b) The probability of being in $|\tilde{\mathbf{S}}_n;-\rangle = 1 - \frac{1}{2}\left(1 + 2\sin\beta\cos\omega t\right)$

$$\langle\tilde{\mathbf{S}}_x\rangle = \frac{\hbar}{2}\left(\frac{1}{2}\left(1 + 2\sin\beta\cos\omega t\right)\right) - \frac{\hbar}{2}\left(1 - \frac{1}{2}\left(1 + 2\sin\beta\cos\omega t\right)\right)$$
$$\langle\tilde{\mathbf{S}}_x\rangle = \frac{\hbar}{4}\sin\beta\cos\omega t$$

(c)

$$\beta \to 0 \quad P(t) \to \frac{1}{2} \quad \langle\tilde{\mathbf{S}}_x\rangle \to 0$$
$$\beta \to \pi/2 \quad P(t) \to \frac{1}{2}\left(1 + 2\cos\omega t\right) \quad \langle\tilde{\mathbf{S}}_x\rangle \to \frac{\hbar}{4}\cos\omega t$$

2.4 Starting with the definitions (equations 2.63a-b):

$$|v_e\rangle = \cos\theta\,|v_1\rangle - \sin\theta\,|v_2\rangle$$
$$|v_\mu\rangle = \sin\theta\,|v_1\rangle + \cos\theta\,|v_2\rangle$$

Applying the time evolution operator:

$$\mathcal{U}(t)\,|v_e\rangle = \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right)\cos\theta + \exp\left(\frac{i\tilde{\mathbf{H}}t}{\hbar}\right)\sin\theta$$
$$\mathcal{U}(t)\,|v_e\rangle = \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\cos\theta + \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\sin\theta$$
$$P(t) = |\,\langle v_e|v_e\rangle\,|^2 = \left(\exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\cos\theta + \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\sin\theta\right)^2$$
$$P(v_e \to v_e) = 1 - \sin^2 2\theta\,\sin^2\left(\Delta m^2c^4\frac{L}{4E\hbar c}\right)$$

2.9 (a)

$$\int_{-a}^{a} A^2(x'-a)^2(x'+a)^{2e^{-ikx'}}(x'-a)^2(x'+a)^{2e^{ikx'}}\,dx' = \frac{256a^9A^2}{315} = 1$$

$$A = \pm\frac{3\sqrt{35}}{16a^{9/2}}$$

(b)

$$\langle x\rangle = \int_{-a}^{a} A^2(x'-a)^2(x'+a)^{2e^{-ikx'}}x(x'-a)^2(x'+a)^{2e^{ikx'}}\,dx' = 0$$

$$\langle p\rangle = -i\hbar\int_{-a}^{a} A^2(x'-a)^2(x'+a)^{2e^{-ikx'}}\frac{\partial}{\partial x}(x'-a)^2(x'+a)^{2e^{ikx'}}\,dx' = -\hbar k$$

$$\langle x^2\rangle = \int_{-a}^{a} A^2(x'-a)^2(x'+a)^{2e^{-ikx'}}x^2(x'-a)^2(x'+a)^{2e^{ikx'}}\,dx' = \frac{a^2}{11}$$

$$\langle p^2\rangle = \int_{-a}^{a} A^2(x'-a)^2(x'+a)^{2e^{-ikx'}}\frac{\partial^2}{\partial x^2}(x'-a)^2(x'+a)^{2e^{ikx'}}\,dx' = \hbar\frac{3}{a^2}+k^2$$

(c)