Classical Mechanics Assignment #2

Johannes Byle

September 9, 2021

1. (a) Solving the first derivative to find the location of extrema:

$$V(x) = \frac{x^2}{2} - \frac{x_0\sqrt{1+gx^2}}{\sqrt{1+gx_0^2}}$$

$$\frac{\delta V(x)}{\delta x} = x\left[1 - \frac{gx_0}{\sqrt{gx^2+1}\sqrt{gx_0^2+1}}\right]$$

Solving for where $\frac{\delta V(x)}{\delta x} = 0$:

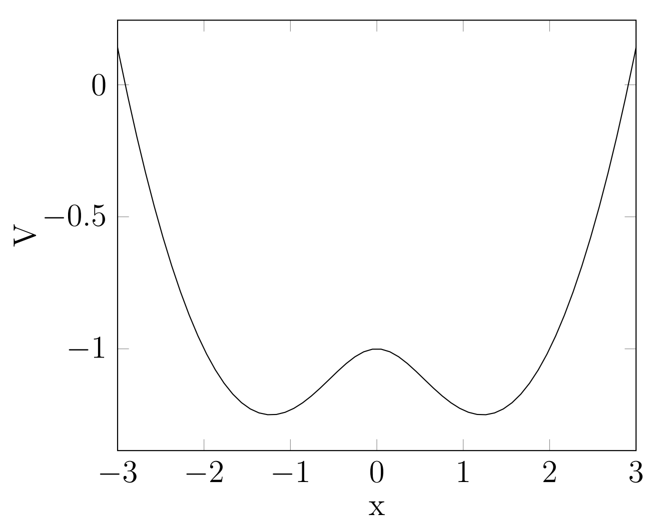
$$x = 0,\ \pm\frac{\sqrt{g^2x_0^2 - gx_0^2 - 1}}{\sqrt{g^2x_0^2+g}}$$

Taking the second derivative to evaluate each extremum:

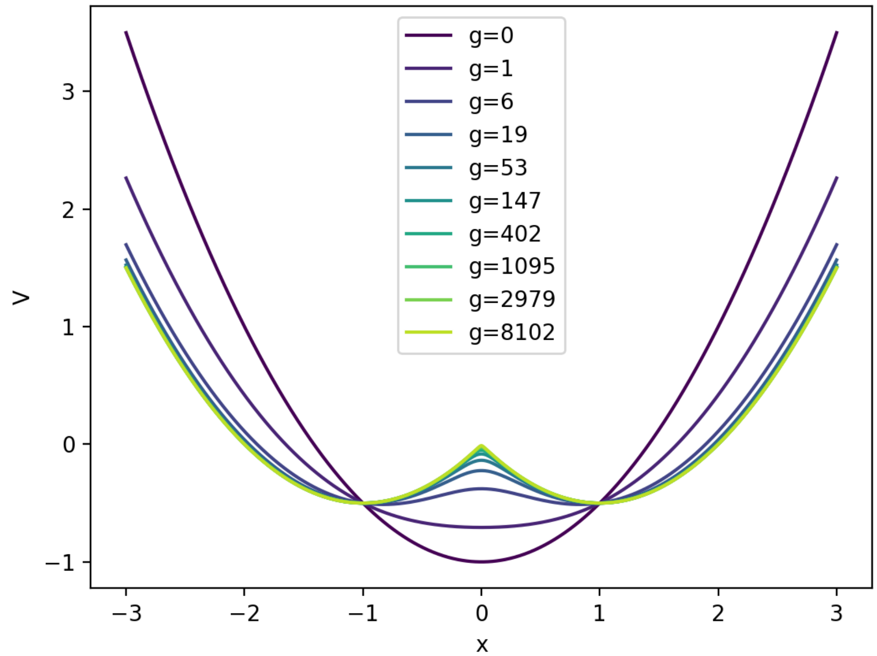
$$\frac{\delta V(x)}{\delta x^2} = \frac{x_0\left(\frac{g^2x^2}{(gx^2+1)^{3/2}} - \frac{g}{\sqrt{gx^2+1}}\right)}{\sqrt{gx_0^2+1}} + 1$$

$$\frac{\delta V(0)}{\delta x^2} = 1$$

This shows us that $x = 0$ is a local minimum. Not wanting to solve the algebra, simply by looking at the plot of $V(x)$ below we can see that as long as the other extrema exist they will be local minima:



These minimum will only be real (i.e. exist) if $(g^2x_0^2 - gx_0^2) \geq 1$



(b)

(c) The behaviour of the particle will depend on the exact value of g and x_0. If $g <= \frac{x_0+\sqrt{x_0^2+4}}{2x_0}$, or if $x_0 \gg 0$, then the particle will oscillate around $x = 0$. Otherwise, the particle will oscillate around the other extrema: $\pm\frac{\sqrt{g^2x_0^2-gx_0^2-1}}{\sqrt{g^2x_0^2+g}}$.

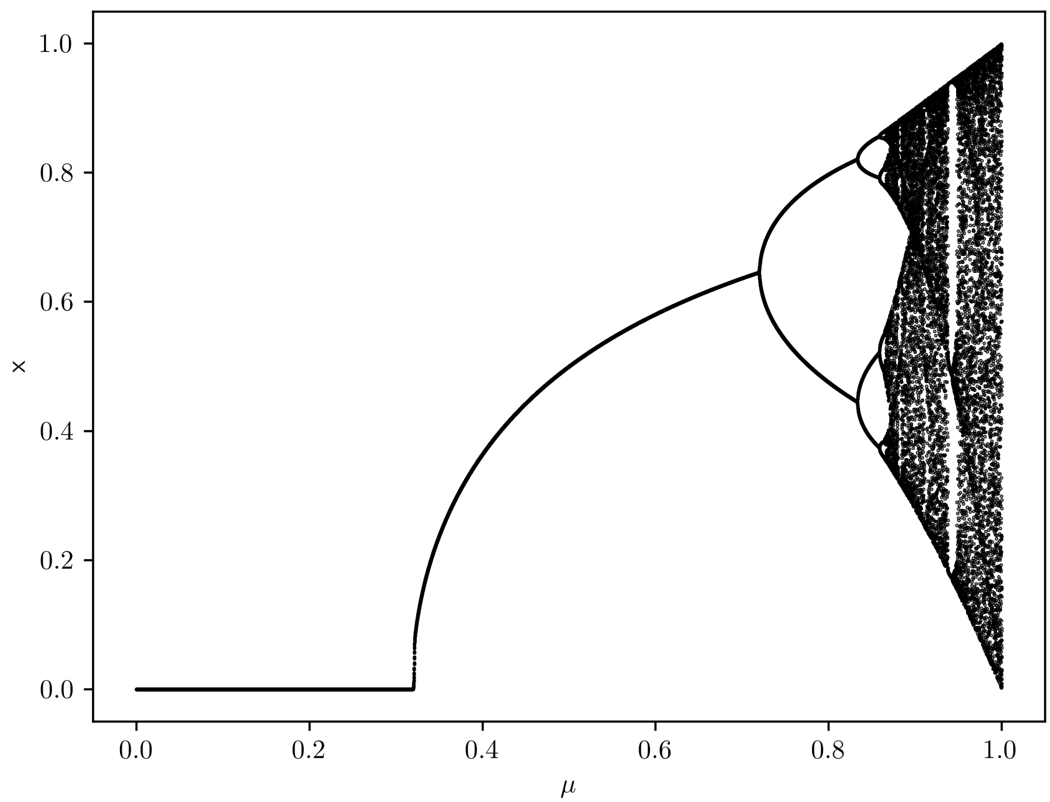
2.
```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib

matplotlib.rcParams['text.usetex'] = True
from math import pi, sin
from tqdm import tqdm


def f(x_, mu_=0.5):
    return mu_ * sin(pi * x_)


def f2(x_, a=0.5):
    if x_ < 0.5:
        return 2 * a * x_
    else:
        return 2 * a * (1 - x_)
```

```
num_points = 1000
mu_array = []
x_array = []
for mu in tqdm(np.linspace(0, 1, num_points * 10)):
    x0 = 10 ** (-5)
    x = [x0]
    for _ in range(num_points):
        x0 = f(x0, mu=mu)
        x.append(x0)
    final_slice = int(num_points / 100)
    mu_array += [mu] * final_slice
    x_array += x[-final_slice:]
plt.scatter(mu_array, x_array, c="black", s=0.1)
plt.ylabel("x")
plt.xlabel(r"$\mu$")
plt.savefig("bifurcation.png", dpi=500)
plt.show()
```



3. Fixed points are where $x_{n+1} = x_n = \mu \sin(\pi x_n)$. The first trivial solution is where $x_n = 0$ as $\mu \sin(\pi \cdot 0) = 0$. The second solution is more complicated; taking a derivative in terms of x_n

and solving for μ:

$$\frac{\delta}{\delta x_n}\left(\mu \sin\left(\pi x_n\right) - x_n\right) = \pi\mu\cos(\pi x) - 1$$

$$\mu = \frac{\sec(\pi x_n)}{pi}$$

$$\mu_0 = \frac{1}{\pi}$$