

Math Methods Assignment #5

Johannes Byle

October 2, 2021

1. (a) From the definition of orthogonality given by Arfken (page 105), to show that they are orthogonal we must show that $g_{ij} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = 0$ for all $i \neq j$:

$$\frac{\partial \mathbf{r}}{\partial q_\eta} \cdot \frac{\partial \mathbf{r}}{\partial q_\phi} = \left(\sinh \eta \cos \hat{\phi} + \cosh \eta \sin \hat{\phi} \right) \cdot \left(-\cosh \eta \sin \hat{\phi} + \sinh \eta \cos \hat{\phi} \right) = 0$$

$$\frac{\partial \mathbf{r}}{\partial q_\eta} \cdot \frac{\partial \mathbf{r}}{\partial q_z} = \left(\sinh \eta \cos \hat{\phi} + \cosh \eta \sin \hat{\phi} \right) \cdot \left(\hat{k} \right) = 0$$

$$\frac{\partial \mathbf{r}}{\partial q_\phi} \cdot \frac{\partial \mathbf{r}}{\partial q_z} = \left(-\cosh \eta \sin \hat{\phi} + \sinh \eta \cos \hat{\phi} \right) \cdot \left(\hat{k} \right) = 0$$

- (b) The metric tensor is defined as $ds^2 = g_{ij} dq_i dq_j$. Since we know the coordinate system is orthogonal we only need to calculate the diagonal terms:

$$ds^2 = \begin{bmatrix} d\eta & d\phi & dz \end{bmatrix} \begin{bmatrix} \sinh^2 \eta \cos^2 \phi + \cosh^2 \eta \sin^2 \phi & 0 & 0 \\ 0 & \cosh^2 \eta \sin^2 \phi + \sinh^2 \eta \cos^2 \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\eta \\ d\phi \\ dz \end{bmatrix}$$

- (c) Defining $g_{ii} = h_i^2$ we can use the relation from Arfken: $\nabla f = \hat{q}_i \frac{1}{h_i} \frac{\partial f}{\partial q_i}$:

$$\nabla f = \frac{\hat{\eta}}{\sinh \eta \cos \phi + \cosh \eta \sin \phi} \frac{\partial f}{\partial \eta} + \frac{\hat{\phi}}{\cosh \eta \sin \phi + \sinh \eta \cos \phi} \frac{\partial f}{\partial \phi} + \hat{z} \frac{\partial f}{\partial z}$$

- (d) Using equation (2.21) from Arfken:

$$\nabla \cdot \mathcal{E} = \frac{\frac{\partial}{\partial \eta} \mathcal{E}_\eta + \frac{\partial}{\partial \phi} \mathcal{E}_\phi + \frac{\partial}{\partial z} \mathcal{E}_z (\cosh \eta \sin \phi + \sinh \eta \cos \phi)}{(\cosh \eta \sin \phi + \sinh \eta \cos \phi)}$$

- (e) Using equation (2.22) from Arfken:

$$\nabla^2 \cdot \mathcal{E} = \frac{\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial}{\partial z} [(\cosh \eta \sin \phi + \sinh \eta \cos \phi)^2 \frac{\partial f}{\partial z}]}{(\cosh \eta \sin \phi + \sinh \eta \cos \phi)^2}$$