Math Methods Assignment #5

Johannes Byle

October 5, 2021

1. (a) Using spherical coordinates, and setting our coordinates such that the mass can be described by only θ and r we get the following equations for the kinetic and potential energies and the constraint f:

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad U = mgr\cos\theta$$
$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$
$$f = r - R = 0$$

Solving the Lagrangian for r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0$$
$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - gm\cos\theta, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}, \quad \lambda\frac{\partial f}{\partial r} = 1$$
$$mr\dot{\theta}^2 - gm\cos\theta - m\ddot{r} + \lambda 1 = 0$$

Solving the Lagrangian for θ:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$
$$\frac{\partial L}{\partial \theta} = gmr\sin\theta, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$
$$gmr\sin\theta - 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$$

Since we know that $r = R$ and does not change, i.e. $\dot{r} = \ddot{r} = 0$, the equations simplify to:

$$mR\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$
$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

Solving for $\ddot{\theta}$ and integrating we can use substitution to solve for λ:

$$\ddot{\theta} = \frac{g\sin\theta}{R}$$
$$\int \ddot{\theta} d\dot{\theta} = \frac{g}{R}\int \sin\theta d\theta \quad \text{from Marion page 253}$$
$$\frac{\dot{\theta}^2}{2} = \frac{-g\cos\theta}{R} + \frac{g}{R}$$

$$
\begin{aligned}
\lambda &= mg\cos\theta - 2mR\left(\frac{-g\cos\theta}{R} + \frac{g}{R}\right) \\
&= mg\cos\theta + 2mg\cos\theta - 2mg \\
\lambda &= mg(3\cos\theta - 2)
\end{aligned}
$$

(b) We can find the height at which the ball will fall off the sphere using the fact that once the force of constraint goes to zero the ball will fall off the sphere, so solving for where $\lambda = 0$:

$$
\begin{aligned}
0 &= mg(3\cos\theta - 2) \\
0 &= 3\cos\theta - 2 \\
\theta_f &= \cos^{-1}\left(\frac{2}{3}\right)
\end{aligned}
$$

The height is now trivial to find from the angle: $h = R + R\sin\theta_f$

(c) Repeating the same process as above, just with an extra term and extra constraint, and a slightly modified original constraint:

$$
\begin{aligned}
T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) &\quad U = mgr\cos\theta \\
L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) &- mgr\cos\theta \\
f = r - (R + a) &= 0
\end{aligned}
$$