Quantum I Assignment #5

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Q-1 (a)

$$
\begin{aligned}
\dot{[}\tilde{\mathbf{X}}, \tilde{\mathbf{H}}] |\phi_n\rangle &= \tilde{\mathbf{X}} \left(\frac{1}{2m} \tilde{\mathbf{P}}^2 + V(\tilde{\mathbf{X}}) \right) |\phi_n\rangle - \left(\frac{1}{2m} \tilde{\mathbf{P}}^2 + V(\tilde{\mathbf{X}}) \right) \tilde{\mathbf{X}} |\phi_n\rangle \\
&= \tilde{\mathbf{X}} \frac{1}{2m} \tilde{\mathbf{P}}^2 |\phi_n\rangle - \frac{1}{2m} \tilde{\mathbf{P}}^2 \tilde{\mathbf{X}} |\phi_n\rangle \\
&= [\tilde{\mathbf{X}}, \frac{1}{2m} \tilde{\mathbf{P}}^2] |\phi_n\rangle \\
&= \frac{1}{2m} \{ [\tilde{\mathbf{X}}, \tilde{\mathbf{P}}] \tilde{\mathbf{P}}, \tilde{\mathbf{P}} [\tilde{\mathbf{P}}, \tilde{\mathbf{X}}] \} |\phi_n\rangle \\
&= \frac{1}{2m} \{ i\hbar \tilde{\mathbf{P}}, \tilde{\mathbf{P}} i\hbar \} |\phi_n\rangle \\
\langle\phi_n| \tilde{\mathbf{P}} |\phi'_n\rangle &= \alpha \langle\phi_n| \tilde{\mathbf{X}} |\phi_n\rangle
\end{aligned}
$$

(b) Since $\alpha = E_n - E_{n'}$

$$
\begin{aligned}
\alpha^2 \langle\phi_n| \tilde{\mathbf{X}}^2 \sum_n{}' |\phi'_n\rangle \langle\phi'_n| |\phi_n\rangle &= \sum_n{}' \alpha^2 \langle\phi_n| \tilde{\mathbf{X}}^2 |\phi'_n\rangle \langle\phi'_n|\phi_n\rangle \\
&= \sum_n{}' \alpha^2 | \langle\phi'_n|\tilde{\mathbf{X}}|\phi_n\rangle |^2 \\
&= \sum_n{}' (E_n - E_{n'})^2 | \langle\phi'_n|\tilde{\mathbf{X}}|\phi_n\rangle |^2
\end{aligned}
$$

$$
\begin{aligned}
\langle\phi_n| \tilde{\mathbf{P}}^2 \sum_n{}' |\phi'_n\rangle \langle\phi'_n| |\phi_n\rangle &= \langle\phi_n| \tilde{\mathbf{P}}^2 \sum_n{}' |\phi'_n\rangle \langle\phi'_n|\phi_n\rangle \\
&= \langle\phi'_n|\phi_n\rangle \langle\phi_n| \tilde{\mathbf{P}}^2 \sum_n{}' |\phi'_n\rangle \\
&= \frac{\hbar^2}{m^2} \langle\phi_n| \tilde{\mathbf{P}}^2 \sum_n{}' |\phi'_n\rangle \\
&= \frac{\hbar^2}{m^2} \langle\phi_n| \tilde{\mathbf{P}}^2 |\phi'_n\rangle \\
\sum_n{}' (E_n - E_{n'})^2 | \langle\phi'_n|\tilde{\mathbf{X}}|\phi_n\rangle |^2 &= \frac{\hbar^2}{m^2} \langle\phi_n| \tilde{\mathbf{P}}^2 |\phi'_n\rangle
\end{aligned}
$$

1.33 Starting with the translation operator applied to the expectation value for $\mathbf{x}$:

$$\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle$$

By equation 1.207 we know:

$$\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x} = d\mathbf{x}'$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x}\right] &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathbf{x} + d\mathbf{x}' - \mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}'
\end{aligned}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{x}|\alpha\rangle + d\mathbf{x}'$.
Using the same process for $\mathbf{p}$: By equation 1.227 we know:

$$\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p} = 0$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p}\right] &= 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{p} &= 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathbf{p} &= 0 \\
\mathbf{p} + d\mathbf{p}' - \mathbf{p} &= 0
\end{aligned}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{p}|\alpha\rangle$.

1.34 Satisfies unitary property because $\mathbf{W}$ is hermitian:

$$\begin{aligned}
\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}') &= (1 - i\mathbf{W}\cdot d\mathbf{p})(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= (1 - i\mathbf{W}\cdot d\mathbf{p}^\dagger)(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= 1 - i(\mathbf{W} - \mathbf{W}^\dagger) \\
&\simeq 1
\end{aligned}$$

Satisfies the associative property:

$$\begin{aligned}
\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}'') &= (1 + i\mathbf{W}\cdot d\mathbf{p}')\cdot(1 + i\mathbf{W}\cdot d\mathbf{p}'') \\
&\simeq 1 - i\mathbf{W}\cdot(d\mathbf{p}'d\mathbf{p}'') \\
&= \mathscr{B}(d\mathbf{p}' + d\mathbf{p}'')
\end{aligned}$$

Satisfies the inverse property trivially:

$$\begin{aligned}
\mathscr{B}(-d\mathbf{p}') &= \mathscr{B}^{-1}(d\mathbf{p}') \\
1 + i\mathbf{W}\cdot d\mathbf{p} &= -(-1 - i\mathbf{W}\cdot d\mathbf{p})
\end{aligned}$$

Since $d\mathbf{p}$ has units of $\frac{\text{kg m}}{\text{s}^2}$

1.35 (a)

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} dx'\,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})\,\langle x'|\alpha\rangle \\
&= \int_{-\infty}^{\infty} dx'\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx'-\frac{x'^2}{2d^2}\right](-i\hbar\frac{\partial}{\partial x'})\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[ikx'-\frac{x'^2}{2d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\int_{-\infty}^{\infty} dx'\exp\left[-ikx'-\frac{x'^2}{2d^2}\right]\frac{\partial}{\partial x'}\exp\left[ikx'-\frac{x'^2}{2d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\int_{-\infty}^{\infty} dx'\left[ik-\frac{x'}{d^2}\right]\exp\left[-\frac{x'^2}{d^2}\right] \\
&= \left[\frac{-i\hbar}{\pi^{1/2}d}\right]\left(\frac{-k\pi^{1/2}d}{i}\right) \\
&= \hbar k
\end{aligned}$$

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} dx'\,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})^2\,\langle x'|\alpha\rangle \\
&= \hbar^2\int_{-\infty}^{\infty} dx'\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx'-\frac{x'^2}{2d^2}\right]\frac{\partial^2}{\partial x'^2}\left[\frac{1}{\pi^{1/4}\sqrt{d}}\right]\exp\left[-ikx'-\frac{x'^2}{2d^2}\right] \\
&= \frac{\hbar^2}{2d^2}+\hbar^2k^2
\end{aligned}$$

(b)

$$\begin{aligned}
\langle p|\alpha\rangle &= \int_{-\infty}^{\infty} dx'\,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})\,\langle x'|\alpha\rangle \\
&= \int_{-\infty}^{\infty} dp\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right](-i\hbar\frac{\partial}{\partial x'})\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right] \\
&= \hbar k
\end{aligned}$$

$$\begin{aligned}
\langle p^2|\alpha\rangle &= \int_{-\infty}^{\infty} dx'\,\langle\alpha|x'\rangle\,(-i\hbar\frac{\partial}{\partial x'})^2\,\langle x'|\alpha\rangle \\
&= \hbar^2\int_{-\infty}^{\infty} dp\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right]\frac{\partial^2}{\partial x'^2}\sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left[\frac{-(p'-\hbar k)^2d^2}{2\hbar^2}\right. \\
&= \frac{\hbar^2}{2d^2}+\hbar^2k^2
\end{aligned}$$

1.36 (a) i. Using equation (1.265a) $\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle \langle x'|\alpha\rangle$:

$$\begin{aligned}
\langle p'|x|\alpha\rangle &= \int dx' \langle p'|x|x'\rangle \langle x'|\alpha\rangle \\
&= \int dx' x' \langle p'|x'\rangle \langle x'|\alpha\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx' x' \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle \quad \text{using (1.264)} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \frac{\partial}{\partial p'} i\hbar \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle \\
&= i\hbar\frac{\partial}{\partial p'} \int dx' \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{-ip'x'}{\hbar}\right] \langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle \\
\langle p'|x|\alpha\rangle &= i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle
\end{aligned}$$

ii. This holds true from the previous result:

$$\begin{aligned}
\langle \beta|x|\alpha\rangle &= \int dp' \langle \beta|p'\rangle \langle p'|x|\alpha\rangle = \int dp' x \langle \beta|p'\rangle \langle p'\alpha\rangle \\
&= \int dp' \langle \beta|p'\rangle i\hbar\frac{\partial}{\partial p'} \langle p'\alpha\rangle \quad \text{from the previous part} \\
\langle \beta|x|\alpha\rangle &= \int dp' \phi_\beta(p') i\hbar\frac{\partial}{\partial p'} \phi_\alpha(p')
\end{aligned}$$

(b) It translates momentum:

$$\exp\left(\frac{ix\Xi}{\hbar}\right) |p\rangle = c\,|p + dp\rangle$$