

Quantum I Assignment #2

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Q.1 (a)

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{i}{2} \frac{-i}{2} + \left(\frac{1}{2}\right)^2 = 1$$

$|\psi_0\rangle$ is normalized

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{i}{\sqrt{3}} \frac{-i}{\sqrt{3}} = \frac{2}{3}$$

$|\psi_1\rangle$ is not normalized

(b)

$$\tilde{\mathbf{P}}_0 = |\psi_0\rangle \langle \psi_0|$$

Representing this as a matrix:

$$\begin{bmatrix} \frac{1}{2} |u_1\rangle \langle u_1| & \frac{-i}{2\sqrt{2}} |u_1\rangle \langle u_2| & \frac{1}{2\sqrt{2}} |u_1\rangle \langle u_3| \\ \frac{i}{2\sqrt{2}} |u_2\rangle \langle u_1| & \frac{1}{4} |u_2\rangle \langle u_2| & \frac{i}{4} |u_2\rangle \langle u_3| \\ \frac{1}{2\sqrt{2}} |u_3\rangle \langle u_1| & \frac{-i}{4} |u_3\rangle \langle u_2| & \frac{1}{4} |u_3\rangle \langle u_3| \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix}$$

This is hermitian because it is equal to its complex conjugate transpose.

$$\tilde{\mathbf{P}}_1 = |\psi_1\rangle \langle \psi_1|$$

Representing this as a matrix:

$$\begin{bmatrix} \frac{1}{3} |u_1\rangle \langle u_1| & \frac{-i}{3} |u_1\rangle \langle u_2| \\ \frac{i}{3} |u_2\rangle \langle u_1| & \frac{1}{3} |u_2\rangle \langle u_2| \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{3} & \frac{-i}{3} \\ \frac{i}{3} & \frac{1}{3} \end{bmatrix}$$

This is hermitian because it is equal to its complex conjugate transpose.

1.9 (a)

$$\prod_{a_i} (A - a_i) = (A - a_1)(A - a_2) \cdots (A - a_i)$$

When this operator operates on some eigenket of A since its eigenvalue will be in the series, one of the terms will go to zero, making the whole expression zero.

(b)

$$\prod_{a'' \neq a'} (A - a') = \frac{(a' - a'') (a''' - a'')}{(a' - a'') (a' - a'')} \dots \frac{(a_n - a'')}{(a' - a'')}$$

(c)

$$S_z = \frac{\hbar}{2} [(|+\rangle \langle +|) - (|-\rangle \langle -|)]$$

$$\prod_{a_i} S_z |+\rangle = \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right) \left(\frac{\hbar}{2} - 0\right) = 0$$

1.11

$$\tilde{\mathbf{S}} = (S_x, S_y, S_z)$$

$$\tilde{\mathbf{n}} = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$$

$$\tilde{\mathbf{S}} \cdot \tilde{\mathbf{n}} = (S_x \cos \alpha \sin \beta, S_y \sin \alpha \sin \beta, S_z \cos \beta)$$

$$\tilde{\mathbf{S}} \cdot \tilde{\mathbf{n}} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cos \alpha \sin \beta + \frac{\hbar}{2i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sin \alpha \sin \beta + \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cos \beta$$

$$\tilde{\mathbf{S}} \cdot \tilde{\mathbf{n}} = \frac{\hbar}{2} \left(\begin{bmatrix} 0 & \cos \alpha \sin \beta \\ \cos \alpha \sin \beta & 0 \end{bmatrix} + \begin{bmatrix} 0 & i \sin \alpha \sin \beta \\ -i \sin \alpha \cos \beta & 0 \end{bmatrix} + \begin{bmatrix} \cos \beta & 0 \\ 0 & -\cos \beta \end{bmatrix} \right)$$

$$\tilde{\mathbf{S}} \cdot \tilde{\mathbf{n}} = \begin{bmatrix} \cos \beta & \cos \alpha \sin \beta + i \sin \alpha \sin \beta \\ \cos \alpha \sin \beta - i \sin \alpha \sin \beta & -\cos \beta \end{bmatrix}$$

Solving for $|+\rangle$ as $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$|\tilde{\mathbf{S}} \cdot \tilde{\mathbf{n}}; +\rangle = \begin{bmatrix} \cos \beta & \cos \alpha \sin \beta + i \sin \alpha \sin \beta \\ \cos \alpha \sin \beta - i \sin \alpha \sin \beta & -\cos \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} \cos \beta & e^{i\alpha} \sin \beta \\ e^{-i\alpha} \sin \beta & -\cos \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} a \\ b \end{bmatrix}$$

As a system of equations:

$$a \cos \beta + b e^{i\alpha} \sin \beta = \frac{\hbar}{2} a$$

$$a e^{-i\alpha} \sin \beta - b \cos \beta = \frac{\hbar}{2} b$$

Solving for a :

$$b = \frac{a \sin \beta e^{-i\alpha}}{(\cos \beta + \frac{\hbar}{2})}$$

$$a \left(\cos \beta - \frac{\hbar}{2} \right) + \frac{a \sin \beta e^{-i\alpha}}{(\cos \beta + \frac{\hbar}{2})} (\sin \beta e^{i\alpha})$$

1.17 We can show that for any base ket $|\psi_n\rangle$, $AB|\psi_n\rangle = BA|\psi_n\rangle$:

$$AB|\psi_n\rangle = Ab_n|\psi_n\rangle = a_nb_n|\psi_n\rangle = BA|\psi_n\rangle$$

Since this is true for any $|\psi_n\rangle$ and we know that the simultaneous eigenkets form a complete orthonormal set of base kets $[A, B] = 0$.¹

1.18 If $|\psi\rangle$ is a eigenket of both A and B then:

$$\begin{aligned} A|\psi\rangle &= a|\psi\rangle \\ B|\psi\rangle &= b|\psi\rangle \\ (AB + BA)|\psi\rangle &= 2ab|\psi\rangle \end{aligned}$$

The only way this is true is if either a or b is 0.²

¹https://en.wikipedia.org/wiki/Complete_set_of_commuting_observables

²<http://peeterjoot.com/2015/09/28/can-anticommuting-operators-have-a-simulaneous-eigenket/>