Math Methods Assignment #6

Johannes Byle

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1. (a) Starting with Ampere's Law:

$$\begin{aligned}
\nabla \times \mathcal{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} \\
\nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c}\nabla \cdot \boldsymbol{J} + \frac{1}{c}\frac{\partial(\nabla \cdot \mathcal{E})}{\partial t} = 0 \quad \text{Using the identity: } \nabla\,(\nabla \cdot \times A) = 0 \\
\nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c}\nabla \cdot \boldsymbol{J} + \frac{4\pi}{c}\frac{\partial \rho}{\partial t} = 0 \quad \text{Using: } \nabla \cdot \mathcal{E} = 4\pi\rho \\
\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} &= 0
\end{aligned}$$

(b) Starting with the curl of the electric field:

$$\begin{aligned}
\nabla \times \mathcal{E} &= -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t} \\
\nabla \times \mathcal{E} &= -\frac{1}{c}\frac{\partial(\nabla \times \mathcal{A})}{\partial t} \\
\nabla \times \left(\mathcal{E} + \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t}\right) &= 0 \\
\mathcal{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t} \quad \text{Rewriting in terms of a scalar potential}
\end{aligned}$$

(c)

$$F = \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{1}{2}(B_1 + B_1) & \frac{1}{2}(B_2 + B_2) & \frac{1}{2}(B_3 + B_3) \end{bmatrix}$$

(d) Starting with $\mathcal{B} = \nabla \times A$:

$$\begin{aligned}
F_{ij} &= \epsilon_{ijk}(\nabla \times A)_k \\
F_{ij} &= \epsilon_{ijk}\epsilon_{jlm}\partial_l A_m \quad \text{Rewriting the curl using Levi-Civita} \\
F_{ij} &= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_l A_m \\
F_{ij} &= \partial_i A_j - \partial_j A_i
\end{aligned}$$

(e) Proving the first part:

$$\begin{aligned}F_{4j} &= \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j} \\ -(\frac{\partial A_4}{\partial x_j} - \frac{\partial A_j}{\partial x_4}) &= \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}\end{aligned}$$

Assuming this 4th term is time, $\mathcal{E} = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}$ and $\partial_t A_j = 0$ show that $F_{4j} = iE_j$.

(f) Starting with the definition given:

$$\begin{aligned}\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} &= 0 \\ \partial_i(\partial_j A_k - \partial_k A_j) + \partial_j(\partial_k A_i - \partial_i A_k) + \partial_k(\partial_i A_j - \partial_j A_i) &= 0 \\ \partial_i\partial_j A_k - \partial_i\partial_k A_j + \partial_j\partial_k A_i - \partial_j\partial_i A_k + \partial_k\partial_i A_j - \partial_k\partial_j A_i &= 0\end{aligned}$$

Since A is a continuous function $\partial_i\partial_j A_k = \partial_j\partial_i A_k$ and the above expression equals zero.

(g) If any pair of indices is zero then that corresponds to a diagonal term in F which is zero.

(h)

$$\begin{aligned}\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} &= 0 \\ \partial_t F_{jk} + \partial_j F_{k4} + \partial_k F_{4j} &= 0 \\ \partial_t B + \partial_j F_{k4} + \partial_k F_{4j} &= 0 \quad \text{Since all } jk \text{ only terms are magnetic} \\ \partial_t B + \partial_j - \nabla \times \mathcal{E} &= 0 \quad \text{Since the second term is equivalent to the curl}\end{aligned}$$

This last expression is the thirds Maxwell's equation.

(i) In this case $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ becomes $\partial_i F_i + \partial_j F_j + \partial_k F_k = 0$ which is equivalent to $\nabla \cdot \mathcal{B} = 0$.

(j) To show that we get $\nabla \cdot \mathcal{E} = 4\pi\rho$ we treat the case where $k = 4$:

$$\partial_t E_l = \frac{4\pi}{c} J_l$$

If we integrate both sides we get $\nabla \cdot \mathcal{E} = 4\pi\rho$.
Looking at the remaining cases we get:

$$\begin{aligned}\partial_k(B_{lk} - E_l) &= \frac{4\pi}{c} J_l \\ \partial_k B_{lk} &= \frac{4\pi}{c} J_l + \partial_t E_l \\ \nabla \times \mathcal{B} &= \frac{4\pi}{c} J_l + \partial_t \frac{1}{c}\mathcal{E}\end{aligned}$$

(k)

$$L_{ij}\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha & i\sinh\alpha \\ 0 & 0 & -\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ ic\rho_0(\vec{r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c\rho_0(\vec{r})\sinh\alpha \\ ic\rho_0(\vec{r})\cosh\alpha \end{bmatrix} = \mathcal{J}'$$

Approximating for $v \ll c$, where $\cosh\alpha \approxeq 1$:

$$\mathcal{J}' = \begin{bmatrix} 0 \\ 0 \\ -c\rho_0(\vec{r})\sinh\alpha \\ ic\rho_0(\vec{r}) \end{bmatrix}$$

(1)

$$L_{ij}F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha & i\sinh\alpha \\ 0 & 0 & -\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

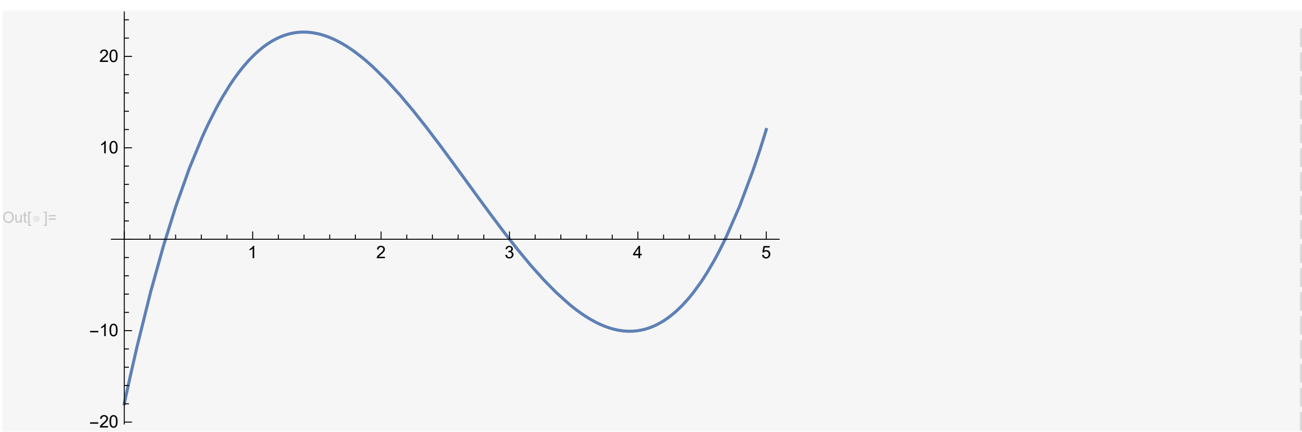
$$L_{ij}F = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ \cosh\alpha B_y - \sinh\alpha E_x & -\cosh\alpha B_x - \sinh\alpha E_y & -\sinh\alpha E_z & -i\cosh\alpha E_z \\ i\cosh\alpha E_x - i\sinh\alpha B_y & i\sinh\alpha B_x + i\cosh\alpha E_y & i\cosh\alpha E_z & -\sinh\alpha E_z \end{bmatrix}$$

$$F'_{23} = B_x \quad F'_{31} = \cosh\alpha B_y - \sinh\alpha E_x$$

2. f[x]=4x^3 $-\ 32x$^2 $+\ 66x - 18$;
   Solve$[f[x] == 0, x]$
   N[Solve$[f[x] == 0, x]$]
   Plot$[f[x], \{x, 0, 5\}]$

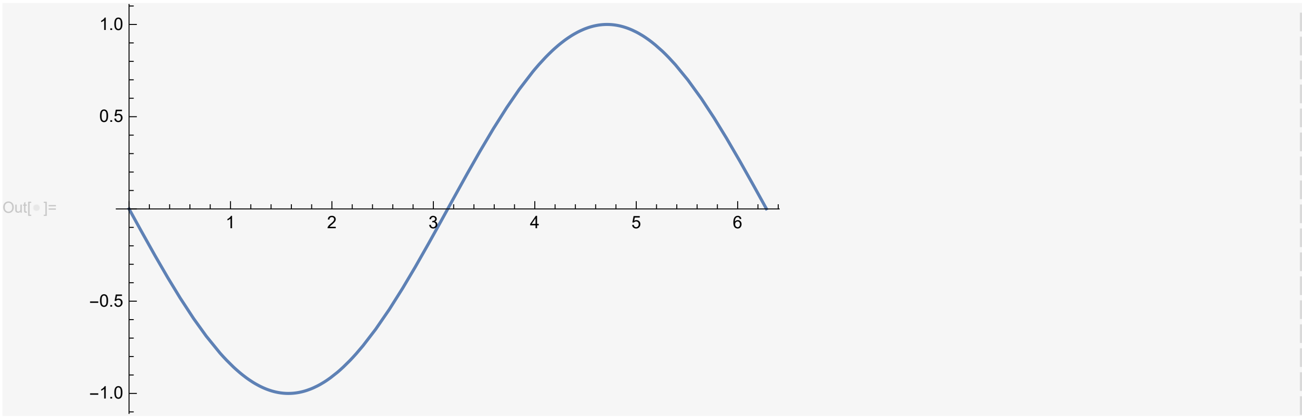
$\{\{x \to 3\}, \{x \to \frac{1}{2}(5 - \sqrt{19})\}, \{x \to \frac{1}{2}(5 + \sqrt{19})\}\}$

$\{\{x \to 3.\}, \{x \to 0.320551\}, \{x \to 4.67945\}\}$



3. f[x_,u_]=Total[Table[D[Sin[u],{u,i}]/Factorial[i] x^$i, \{i, 0, 20\}$]]
   Plot$[\{f[x, \text{Pi}]\}, \{x, 0, 2\text{Pi}\}]$

$x\text{Cos}[u] - \frac{1}{6}x^3\text{Cos}[u] + \frac{1}{120}x^5\text{Cos}[u] - \frac{x^7\text{Cos}[u]}{5040} + \frac{x^9\text{Cos}[u]}{362880} - \frac{x^{11}\text{Cos}[u]}{39916800} + \frac{x^{13}\text{Cos}[u]}{6227020800} - \frac{x^{15}\text{Cos}[u]}{1307674368000} + \frac{x^{17}\text{Cos}[u]}{355687428096000} - \frac{x^{19}\text{Cos}[u]}{121645100408832000} + \text{Sin}[u] - \frac{1}{2}x^2\text{Sin}[u] + \frac{1}{24}x^4\text{Sin}[u] - \frac{1}{720}x^6\text{Sin}[u] + \frac{x^8\text{Sin}[u]}{40320} - \frac{x^{10}\text{Sin}[u]}{3628800} + \frac{x^{12}\text{Sin}[u]}{479001600} - \frac{x^{14}\text{Sin}[u]}{87178291200} + \frac{x^{16}\text{Sin}[u]}{20922789888000} - \frac{x^{18}\text{Sin}[u]}{6402373705728000} + \frac{x^{20}\text{Sin}[u]}{2432902008176640000}$

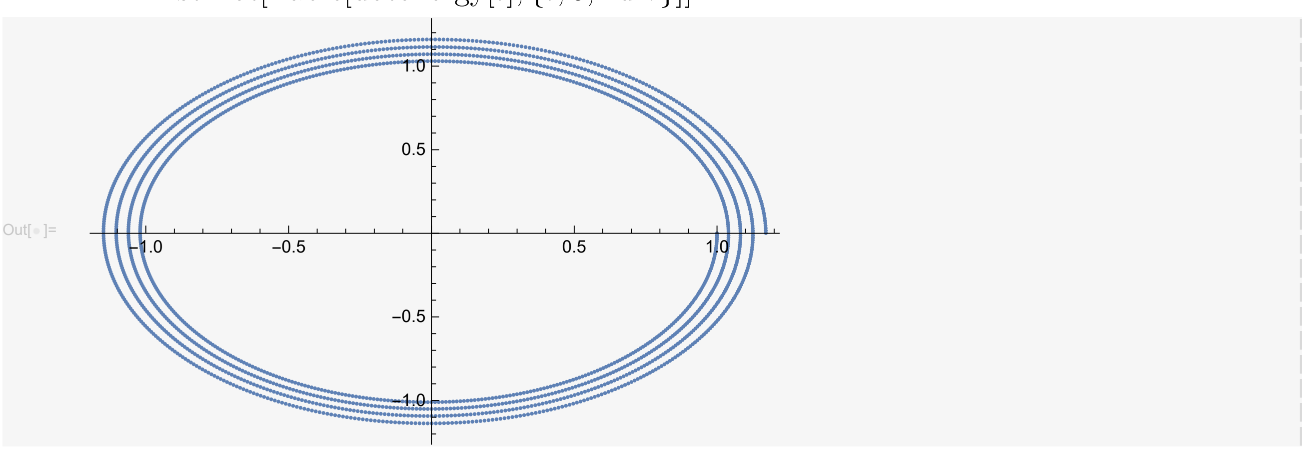


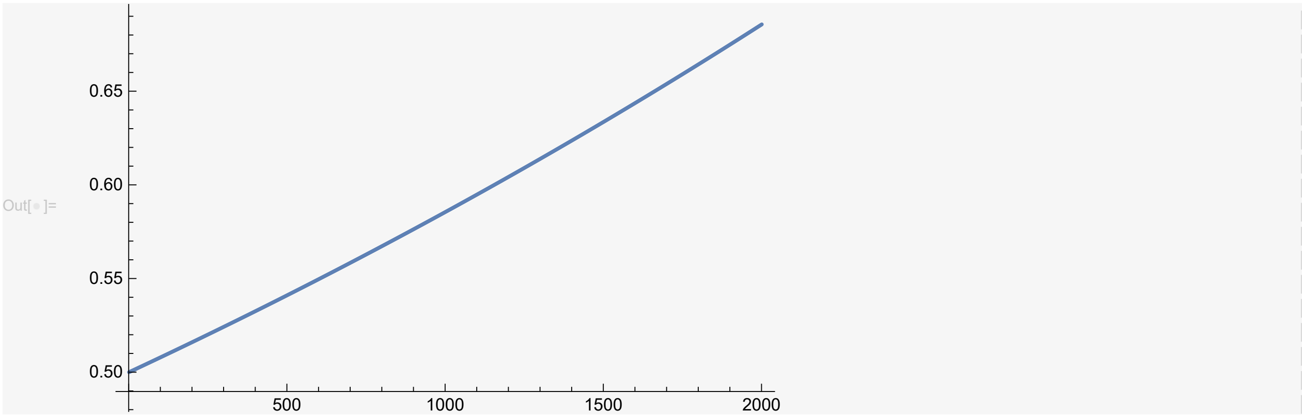
4.
```
A=Table[Table[Sin[i j], {j,10}], {i, 10}];
b=Table[i,{i,10}];
LinearSolve[N[A],N[b]]
```

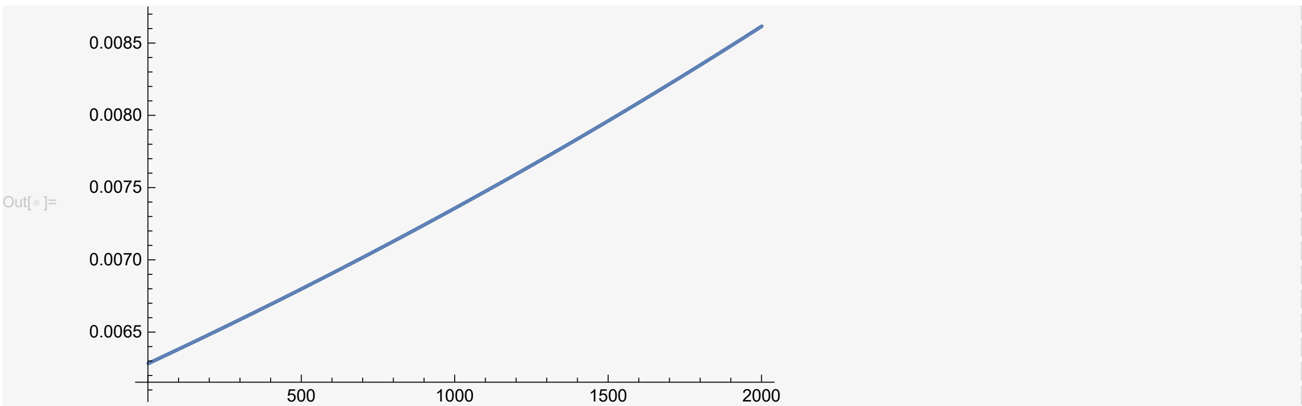
{2.83492, −5.65071, −16.77, −9.82246, 2.24527, −5.75988, −2.63877, 2.96037, 25.6627, 23.0544}

5. (a)

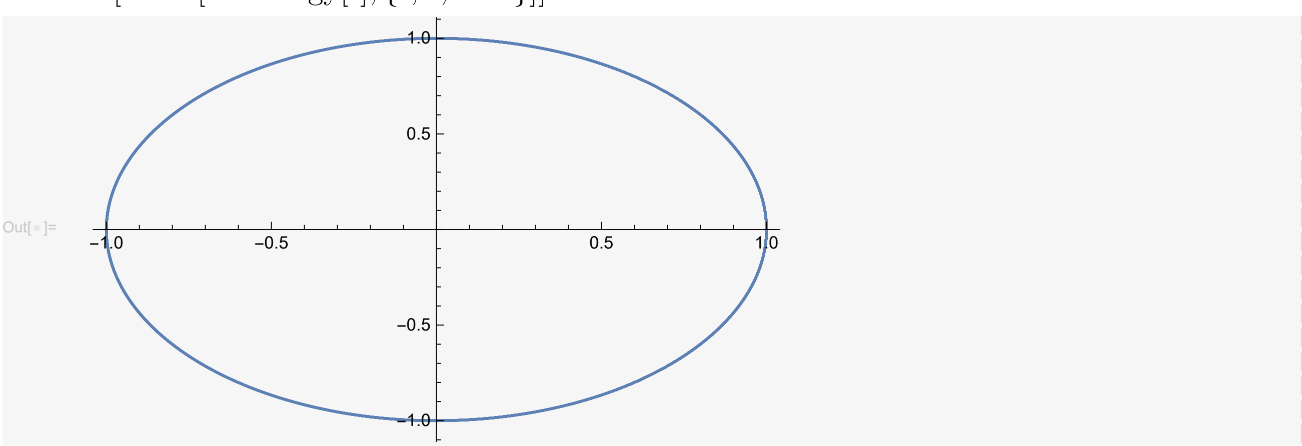
(b-d)
```
Clear[x,v,energy,dotenergy,ndiv,dt];
ndiv=2000;
dt=N[(8 Pi)/ndiv];
v[n_]:=v[n]=v[n-1]-x[n-1] dt;
v[0]=0;
x[n_]:=x[n]=x[n-1]+v[n-1] dt;
x[0]=1;
ListPlot[Table[{x[t],v[t]},{t,0,ndiv}]]
energy[n_]:=energy[n]=(x[n]^2 + v[n]^2)/2
ListPlot[Table[energy[t], {t, 0, ndiv}]]
dotenergy[n_]:=dotenergy[n] = (energy[n + 1] − energy[n])/dt
ListPlot[Table[dotenergy[t], {t, 0, ndiv}]]
```

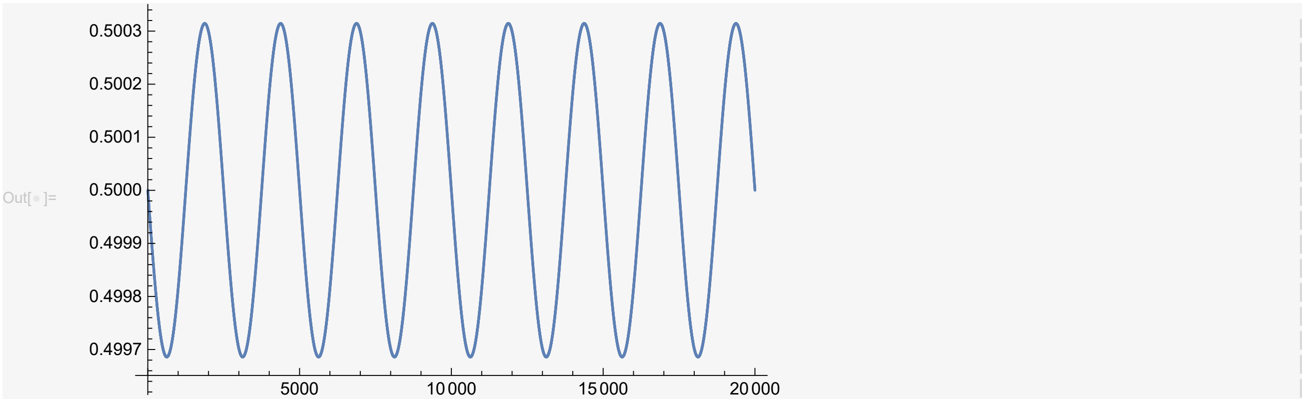


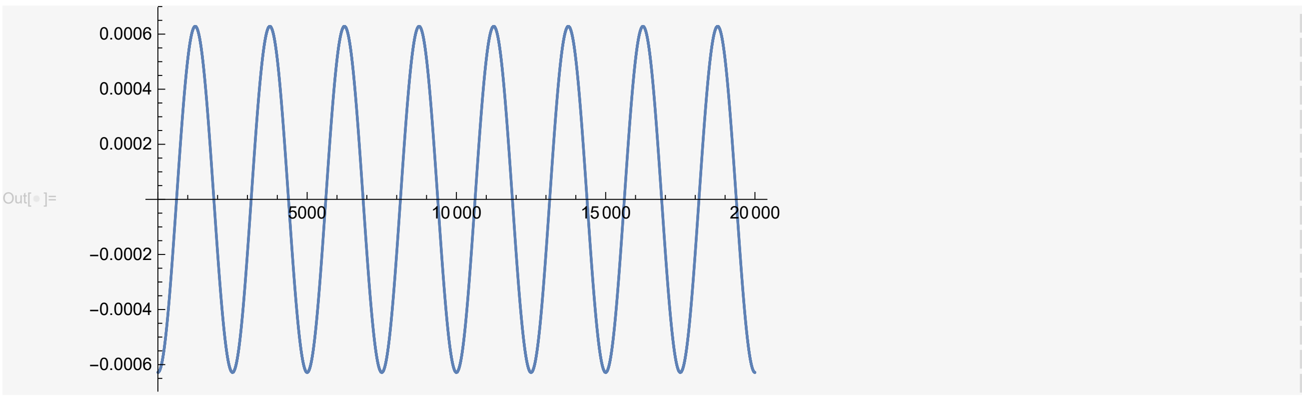





```
Clear[x,v,energy,dotenergy,ndiv,dt];
ndiv=2000;
dt=N[(8 Pi)/ndiv];
v[n_]:=v[n]=v[n-1]-x[n-1] dt;
v[0]=0;
x[n_]:=x[n]=x[n-1]+v[n] dt;
x[0]=1;
ListPlot[Table[{x[t],v[t]},{t,0,ndiv}]]
energy[n_]:=energy[n]=(x[n]^2 + v[n]^2)/2
ListPlot[Table[energy[t], {t, 0, ndiv}]]
dotenergy[n_]:=dotenergy[n] = (energy[n + 1] − energy[n])/dt
ListPlot[Table[dotenergy[t], {t, 0, ndiv}]]
```







(a) Redefining $x \to Lx$:

$$-\frac{\hbar^2}{2mL^2}\frac{\partial^2\psi}{\partial x^2} - e\mathcal{E}Lx\psi = E\psi \tag{1}$$

Dividing out one of the terms:

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{2e\mathcal{E}mL^3}{\hbar^2}x\psi = \frac{EmL^2}{\hbar^2}\psi \tag{2}$$

(b) The most logical unit for $E \to \epsilon E_0$ is $E_0 = \frac{\hbar^2}{mL^2}$:

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{2e\mathcal{E}mL^3}{\hbar^2}x\psi = \epsilon\psi$$

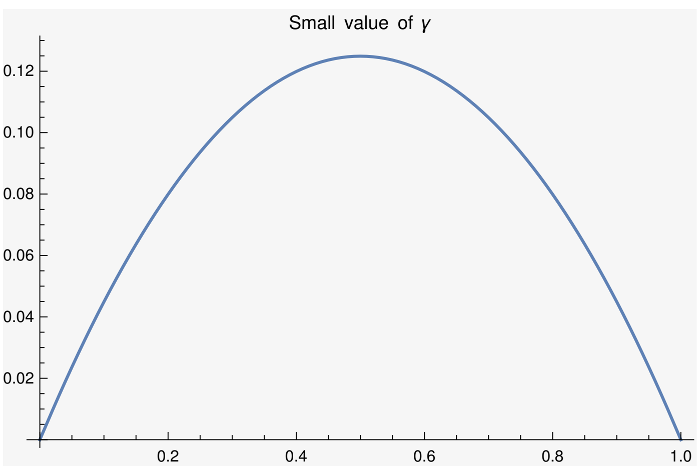
(c) Defining our "control knob" $\gamma = \frac{2e\mathcal{E}mL^3}{\hbar^2}$ gives us:

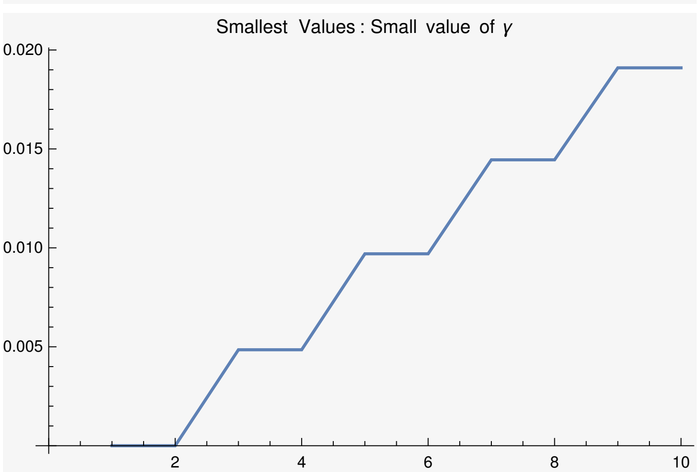
$$-\frac{\partial^2\psi}{\partial x^2} - \gamma x\psi = \epsilon\psi$$

The control knob controls the ratio of kinetic to "spring" potential energy.

(d) Clear[Global*]
eigenstates[γ_, ϵ_, ndiv_, l_]:=
Module[{dx, npts, mat, bvec, phi},
dx = N[l/ndiv];
npts = ndiv − 1;
mat = Table[If[i==j, (−2/dx^2)−(γjdx), 0]+If[Abs[i−j]==1, 1/dx^2, 0], {i, 1, npts}, {j, 1, npts}];

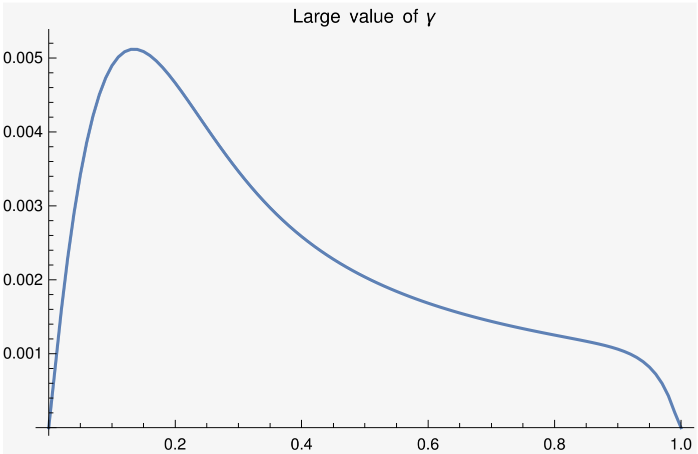
```
bvec = Table[−ϵ, {j, 1, npts}];
bvec[[1]] = 0;
bvec[[npts]] = 0;
phi = LinearSolve[mat, bvec];
phiTot = Join[{0}, phi, {0}];
phidat = Table[{(j − 1) ∗ dx, phiTot[[j]]}, {j, 1, ndiv + 1}]
];
γ_large = 10^3;
γ_small = 10^(−3);
ϵ = 1;
l = 1;
ndiv = 100;
small = eigenstates [γ_small, ϵ, ndiv, l] ;
large = eigenstates [γ_large, ϵ, ndiv, l] ;
ListPlot[small, Joined → True, PlotLabel → Small value of γ]
ListPlot[TakeSmallest[small[[All, 2]], 10], Joined → True, PlotLabel → Smallest Values: Small value
ListPlot[large, Joined → True, PlotLabel → Large value of γ]
ListPlot[TakeSmallest[large[[All, 2]], 10], Joined → True, PlotLabel → Smallest Values: Large value
lowestEigenstates = Table[TakeSmallest[Select[eigenstates[10 ∗ j, ϵ, ndiv, l][[All, 2]], # >
0&], 2], {j, 0, 100}];
ListPlot[Transpose[lowestEigenstates], Joined → False, PlotLabel → Lowest eigenstates]
```

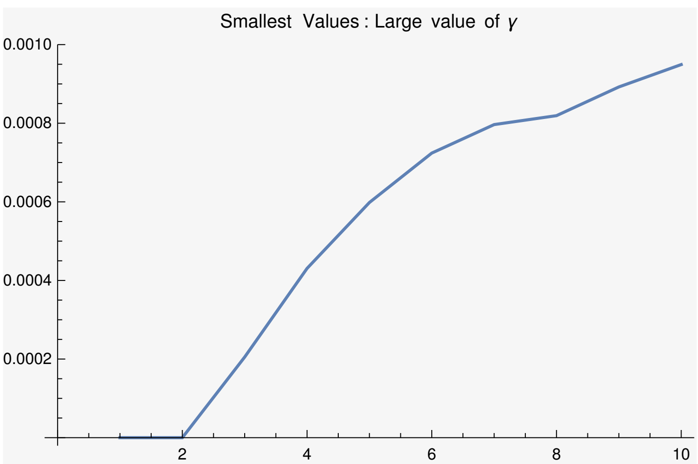




The lowest eigenstates vary linearly.

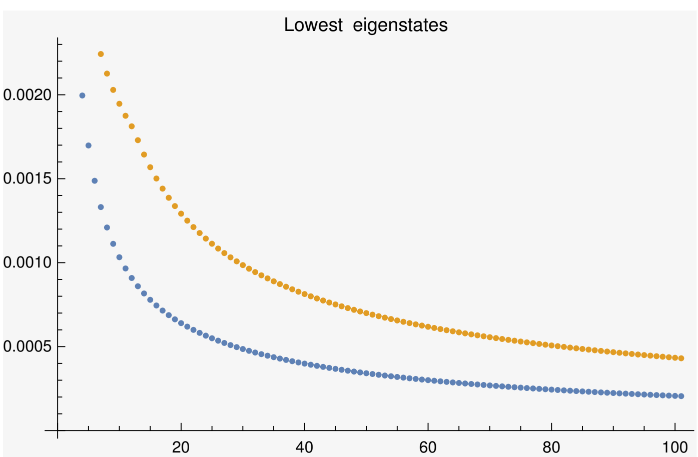
(e)





The lowest eigenstates vary parabolically.

(f)



I'm not sure about the critical values.