Quantum I Assignment #5

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1.33 Starting with the translation operator applied to the expectation value for $\mathbf{x}$:

$$\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle$$

By equation 1.207 we know:

$$\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x} = d\mathbf{x}'$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{x}\right] &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}') - \mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}' \\
\mathbf{x} + d\mathbf{x}' - \mathbf{x} &= \mathscr{J}^\dagger(d\mathbf{x}')d\mathbf{x}'
\end{aligned}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{x}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{x}|\alpha\rangle + d\mathbf{x}'$.
Using the same process for $\mathbf{p}$: By equation 1.227 we know:

$$\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p} = 0$$

Since the translation operator is unitary we can apply $\mathscr{J}^\dagger(d\mathbf{x}')$ to both sides:

$$\begin{aligned}
\mathscr{J}^\dagger(d\mathbf{x}')\left[\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}(d\mathbf{x}')\mathbf{p}\right] &= 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathscr{J}^\dagger(d\mathbf{x}')\mathscr{J}(d\mathbf{x}')\mathbf{p} &= 0 \\
\mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}') - \mathbf{p} &= 0 \\
\mathbf{p} + d\mathbf{p}' - \mathbf{p} &= 0
\end{aligned}$$

This means that $\langle\alpha| \mathscr{J}^\dagger(d\mathbf{x}')\mathbf{p}\mathscr{J}(d\mathbf{x}')|\alpha\rangle \to \langle\alpha|\mathbf{p}|\alpha\rangle$.

1.34 Satisfies unitary property because $\mathbf{W}$ is hermitian:

$$\begin{aligned}
\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}') &= (1 - i\mathbf{W}\cdot d\mathbf{p})(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= (1 - i\mathbf{W}\cdot d\mathbf{p}^\dagger)(1 + i\mathbf{W}\cdot d\mathbf{p}) \\
&= 1 - i(\mathbf{W} - \mathbf{W}^\dagger) \\
&\simeq 1
\end{aligned}$$

Satisfies the associative property:

$$\begin{aligned}\mathscr{B}^\dagger(d\mathbf{p}')\mathscr{B}(d\mathbf{p}'') &= (1 + i\mathbf{W} \cdot d\mathbf{p}') \cdot (1 + i\mathbf{W} \cdot d\mathbf{p}'') \\ &\simeq 1 - i\mathbf{W} \cdot (d\mathbf{p}' d\mathbf{p}'') \\ &= \mathscr{B}(d\mathbf{p}' + d\mathbf{p}'')\end{aligned}$$

Satisfies the inverse property trivially:

$$\begin{gathered}\mathscr{B}(-d\mathbf{p}') = \mathscr{B}^{-1}(d\mathbf{p}') \\ 1 + i\mathbf{W} \cdot d\mathbf{p} = -(-1 - i\mathbf{W} \cdot d\mathbf{p})\end{gathered}$$

Since $d\mathbf{p}$ has units of $\frac{\mathrm{kg\ m}}{\mathrm{s}^2}$