Math Methods Assignment #5

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1. (a) From the definition of orthogonality given by Arfken (page 105), to show that they are orthogonal we must show that $g_{ij} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} = 0$ for all $i \neq j$:

$$\frac{\partial \boldsymbol{r}}{\partial q_\eta} \cdot \frac{\partial \boldsymbol{r}}{\partial q_\phi} = \left(\sinh\eta\cos\phi\hat{i} + \cosh\eta\sin\phi\hat{j}\right) \cdot \left(-\cosh\eta\sin\phi\hat{i} + \sinh\eta\cos\phi\hat{j}\right) = 0$$
$$\frac{\partial \boldsymbol{r}}{\partial q_\eta} \cdot \frac{\partial \boldsymbol{r}}{\partial q_z} = \left(\sinh\eta\cos\phi\hat{i} + \cosh\eta\sin\phi\hat{j}\right) \cdot \left(\hat{k}\right) = 0$$
$$\frac{\partial \boldsymbol{r}}{\partial q_\phi} \cdot \frac{\partial \boldsymbol{r}}{\partial q_z} = \left(-\cosh\eta\sin\phi\hat{i} + \sinh\eta\cos\phi\hat{j}\right) \cdot \left(\hat{k}\right) = 0$$

   (b) The metric tensor is defined as $ds^2 = g_{ij}dq_idq_j$. Since we know the coordinate system is orthogonal we only need to calculate the diagonal terms:

$$ds^2 = \begin{bmatrix} d\eta & d\phi & dz \end{bmatrix} \begin{bmatrix} \sinh^2\eta\cos^2\phi + \cosh^2\eta\sin^2\phi & 0 & 0 \\ 0 & \cosh^2\eta\sin^2\phi + \sinh^2\eta\cos^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\eta \\ d\phi \\ dz \end{bmatrix}$$

   (c) Defining $g_{ii} = h_i^2$ we can use the relation from Arfken: $\nabla f = \hat{q}_i \frac{1}{h_i}\frac{\partial f}{\partial q_i}$:

$$\nabla f = \frac{\hat{\eta}}{\sinh\eta\cos\phi + \cosh\eta\sin\phi}\frac{\partial f}{\partial \eta} + \frac{\hat{\phi}}{\cosh\eta\sin\phi + \sinh\eta\cos\phi}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

   (d) Using equation (2.21) from Arfken:

$$\nabla \cdot \mathcal{E} = \frac{\frac{\partial}{\partial \eta}\mathcal{E}_\eta + \frac{\partial}{\partial \phi}\mathcal{E}_\phi + \frac{\partial}{\partial z}\mathcal{E}_z(\cosh\eta\sin\phi + \sinh\eta\cos\phi)}{(\cosh\eta\sin\phi + \sinh\eta\cos\phi)}$$

   (e) Using equation (2.22) from Arfken:

$$\nabla^2 \cdot \mathcal{E} = \frac{\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial}{\partial z}\left[(\cosh\eta\sin\phi + \sinh\eta\cos\phi)^2\frac{\partial f}{\partial z}\right]}{(\cosh\eta\sin\phi + \sinh\eta\cos\phi)^2}$$

2.

$$\nabla^2\phi = \frac{\epsilon^4 - 6\epsilon^2r^2 + r^4}{r(\epsilon^2 + r^2)^3}$$

(a) Looking at the limits in steps:

$$\lim_{\epsilon \to 0} \nabla^2 \phi(r_0) = \frac{r_0^4}{r_0^6} = \frac{1}{r_0^2}$$

This diverges as r_0 approaches 0:

$$\lim_{r_0 \to 0} \frac{1}{r_0^2} \to \infty$$

(b)