Math Methods Assignment #5

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October 8, 2021

1. (a) From the definition of orthogonality given by Arfken (page 105), to show that they are orthogonal we must show that $g_{ij} = \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} = 0$ for all $i \neq j$:

$$\frac{\partial \boldsymbol{r}}{\partial q_\eta} \cdot \frac{\partial \boldsymbol{r}}{\partial q_\phi} = \left(\sinh\eta\cos\phi\hat{i} + \cosh\eta\sin\phi\hat{j}\right) \cdot \left(-\cosh\eta\sin\phi\hat{i} + \sinh\eta\cos\phi\hat{j}\right) = 0$$
$$\frac{\partial \boldsymbol{r}}{\partial q_\eta} \cdot \frac{\partial \boldsymbol{r}}{\partial q_z} = \left(\sinh\eta\cos\phi\hat{i} + \cosh\eta\sin\phi\hat{j}\right) \cdot \left(\hat{k}\right) = 0$$
$$\frac{\partial \boldsymbol{r}}{\partial q_\phi} \cdot \frac{\partial \boldsymbol{r}}{\partial q_z} = \left(-\cosh\eta\sin\phi\hat{i} + \sinh\eta\cos\phi\hat{j}\right) \cdot \left(\hat{k}\right) = 0$$

(b) The metric tensor is defined as $ds^2 = g_{ij}dq_i dq_j$. Since we know the coordinate system is orthogonal we only need to calculate the diagonal terms:

$$ds^2 = \begin{bmatrix} d\eta & d\phi & dz \end{bmatrix} \begin{bmatrix} \sinh^2\eta\cos^2\phi + \cosh^2\eta\sin^2\phi & 0 & 0 \\ 0 & \cosh^2\eta\sin^2\phi + \sinh^2\eta\cos^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\eta \\ d\phi \\ dz \end{bmatrix}$$

(c) Defining $g_{ii} = h_i^2$ we can use the relation from Arfken: $\nabla f = \hat{q}_i \frac{1}{h_i}\frac{\partial f}{\partial q_i}$:

$$\nabla f = \frac{\hat{\eta}}{\sinh\eta\cos\phi + \cosh\eta\sin\phi}\frac{\partial f}{\partial \eta} + \frac{\hat{\phi}}{\cosh\eta\sin\phi + \sinh\eta\cos\phi}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

(d) Using equation (2.21) from Arfken:

$$\nabla \cdot \mathcal{E} = \frac{\frac{\partial}{\partial \eta}\mathcal{E}_\eta + \frac{\partial}{\partial \phi}\mathcal{E}_\phi + \frac{\partial}{\partial z}\mathcal{E}_z(\cosh\eta\sin\phi + \sinh\eta\cos\phi)}{(\cosh\eta\sin\phi + \sinh\eta\cos\phi)}$$

(e) Using equation (2.22) from Arfken:

$$\nabla^2 \cdot \mathcal{E} = \frac{\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial}{\partial z}\left[(\cosh\eta\sin\phi + \sinh\eta\cos\phi)^2\frac{\partial f}{\partial z}\right]}{(\cosh\eta\sin\phi + \sinh\eta\cos\phi)^2}$$

2.

$$\nabla^2\phi = \frac{2\epsilon^4 - 6r^2\epsilon^2}{r\left(r^2+\epsilon^2\right)^3}$$

(a) Looking at the limits in steps:

$$\lim_{\epsilon\to 0} \nabla^2\phi(r_0) = \frac{0}{r_0\left(r_0^2\right)^3}$$

$$\lim_{r_0\to 0} 0 \to 0$$

(b) Again, looking at the limits in steps:

$$\lim_{\epsilon\to 0} \nabla^2\phi(r_0) = \frac{2\epsilon^4}{0} \to \infty$$

$$\lim_{r_0\to 0} \infty \to \infty$$

(c) At all points other than zero we know from part (a) that when $\epsilon \to 0$ $\phi(r) \to \frac{1}{r}$ the limit of the laplacian approaches 0. But for the point at zero, we can treat ϵ as a number infinitely close to the origin, since we have already evaluated $\phi(r)$ at $r = r_0 \to \infty$ which as we approach the origin ($\epsilon \to 0$). Since $\nabla^2\frac{1}{r}$ is zero except for the origin, it is a function that has the same behaviour as the delta function, and thus $\int f(r)\nabla^2\frac{1}{r}d^3r = f(0)$.

3. (a) Starting from the definition of polar coordinates:

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$

$$\nabla f = \frac{\hat{\varphi}}{\frac{\partial \boldsymbol{r}}{\partial\varphi}\cdot\frac{\partial \boldsymbol{r}}{\partial\varphi}}\frac{\partial f}{\partial\varphi} + \frac{\hat{\rho}}{\frac{\partial \boldsymbol{r}}{\partial\rho}\cdot\frac{\partial \boldsymbol{r}}{\partial\rho}}\frac{\partial f}{\partial\rho}$$
$$\nabla f = \frac{\hat{\varphi}}{\rho\left(\cos^2\varphi + \sin^2\varphi\right)}\frac{\partial f}{\partial\varphi} + \frac{\hat{\rho}}{\cos^2\varphi + \sin^2\varphi}\frac{\partial f}{\partial\rho}$$
$$\nabla f = \frac{\hat{\varphi}}{\rho}\frac{\partial f}{\partial\varphi} + \hat{\rho}\frac{\partial f}{\partial\rho}$$

(b)

$$\nabla\times\vec{g} = \frac{\partial}{\partial\rho}\frac{\hat{\varphi}}{\rho}\frac{\partial f}{\partial\varphi} + \frac{\partial}{\partial\varphi}\hat{\rho}\frac{\partial f}{\partial\rho}$$
$$\nabla\times\vec{g} = \left(-\frac{1}{\rho^2}\frac{\partial f}{\partial\varphi} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{\partial f}{\partial\varphi}\right)\hat{\varphi} + \hat{\rho}\frac{\partial}{\partial\varphi}\frac{\partial f}{\partial\rho}$$

(c) Since the gradient of scalar field is conservative, the line integral will be zero, since it's a loop and will start and end in the same points.[1]

(d) This agrees with the question statement.

[1] https://math.stackexchange.com/questions/1435044/line-integral-of-conservative-field-in-polar-coordinates