Quantum I Assignment #6

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Q-1 To satisfy the property that $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$:

$$\begin{aligned}\mathbb{1} &= (\lambda \left|\alpha\right\rangle + \lambda' \left|\beta\right\rangle)(\lambda^* \left\langle\alpha\right| + \lambda'^* \left\langle\beta\right|) \\ &= \lambda\lambda^* + \lambda\lambda'^* \langle\beta|\alpha\rangle + \lambda^*\lambda' \langle\alpha|\beta\rangle + \lambda'\lambda'^*\end{aligned}$$

This means they must satisfy the property $\lambda\lambda'^* = -\lambda^*\lambda'$.

2.2 H is not hermitian:

$$H^\dagger \neq H$$
$$H_{11}^* \left|1\right\rangle\left\langle 1\right| + H_{22}^* \left|2\right\rangle\left\langle 2\right| + H_{12}^* \left|1\right\rangle\left\langle 2\right| \neq H_{11} \left|1\right\rangle\left\langle 1\right| + H_{22} \left|2\right\rangle\left\langle 2\right| + H_{12} \left|1\right\rangle\left\langle 2\right|$$

This would violate the unitary property of the time evolution operator:

$$\mathcal{U}^\dagger(t)\mathcal{U}(t) = \exp\left(\frac{iH^\dagger t}{\hbar}\right)\exp\left(\frac{-iHt}{\hbar}\right) = \exp\left(\frac{i(H^\dagger - H)t}{\hbar}\right) \neq 1$$

2.3 (a) Starting with the definition of $\mathbf{S}\cdot\vec{\mathbf{n}}$:

$$\mathbf{S}\cdot\vec{\mathbf{n}} = \cos\left(\frac{\beta}{2}\right)\left|+\right\rangle + \sin\left(\frac{\beta}{2}\right)\left|-\right\rangle$$
$$\mathbf{S}_x = \frac{\hbar}{2}\left(\left|+\right\rangle + \left|-\right\rangle\right)$$

Applying the time evolution operator:

$$\mathcal{U}(t)\left|\mathbf{S}_n;+\right\rangle = \exp\left(\frac{-i\mathbf{H}t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right)\left|+\right\rangle + \exp\left(\frac{-i\mathbf{H}t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)\left|-\right\rangle$$
$$\mathcal{U}(t)\left|\mathbf{S}_n;+\right\rangle = \exp\left(\frac{-i\omega t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right)\left|+\right\rangle + \exp\left(\frac{-i\omega t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)\left|-\right\rangle$$
$$P(t) = \left|\left\langle\mathbf{S}_x;+|\mathbf{S}_n\right\rangle\right|^2 = \frac{1}{2}\left[\exp\left(\frac{-i\omega t}{\hbar}\right)\cos\left(\frac{\beta}{2}\right) + \exp\left(\frac{-i\omega t}{\hbar}\right)\sin\left(\frac{\beta}{2}\right)\right]^2$$
$$P(t) = \frac{1}{2}\left(1 + 2\sin\beta\cos\omega t\right)$$

(b) The probability of being in $|\mathbf{S}_n;-\rangle = 1 - \frac{1}{2}(1 + 2\sin\beta\cos\omega t)$

$$\langle \mathbf{S}_x \rangle = \frac{\hbar}{2}\left(\frac{1}{2}(1 + 2\sin\beta\cos\omega t)\right) - \frac{\hbar}{2}\left(1 - \frac{1}{2}(1 + 2\sin\beta\cos\omega t)\right)$$
$$\langle \mathbf{S}_x \rangle = \frac{\hbar}{4}\sin\beta\cos\omega t$$

(c)

$$\beta \to 0 \quad P(t) \to \frac{1}{2} \quad \langle \mathbf{S}_x \rangle \to 0$$
$$\beta \to \pi/2 \quad P(t) \to \frac{1}{2}(1 + 2\cos\omega t) \quad \langle \mathbf{S}_x \rangle \to \frac{\hbar}{4}\cos\omega t$$

2.4 Starting with the definitions (equations 2.63a-b):

$$|v_e\rangle = \cos\theta\,|v_1\rangle - \sin\theta\,|v_2\rangle$$
$$|v_\mu\rangle = \sin\theta\,|v_1\rangle + \cos\theta\,|v_2\rangle$$

Applying the time evolution operator:

$$\mathcal{U}(t)\,|v_e\rangle = \exp\left(\frac{-i\mathbf{H}t}{\hbar}\right)\cos\theta + \exp\left(\frac{i\mathbf{H}t}{\hbar}\right)\sin\theta$$
$$\mathcal{U}(t)\,|v_e\rangle = \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\cos\theta + \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\sin\theta$$
$$P(t) = |\langle v_e|v_e\rangle|^2 = \left(\exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\cos\theta + \exp\left(\frac{-ipc\left(1 + \frac{m^2c^2}{2p^2}\right)t}{\hbar}\right)\sin\theta\right)^2$$
$$P(v_e \to v_e) = 1 - \sin^2 2\theta\sin^2\left(\Delta m^2c^4\frac{L}{4E\hbar c}\right)$$

2.6 Starting with the commutation relation:

$$[[H,x],x] = [H,x]x - x[H,x] = (Hx - xH)x - x(Hx - xH)$$
$$[[H,x],x] = Hx^2 - xHx - xHx - x^2H = Hx^2 - x^2H$$
$$[[H,x],x] = \left(\frac{p^2}{2m} + V(x)\right)x^2 - x^2\left(\frac{p^2}{2m} + V(x)\right)$$

Applying the kets:

$$\langle a''|\,[[H,x],x]\,|a'\rangle = \langle a''|\left[\left(\frac{p^2}{2m} + V(x)\right)x^2 - x^2\left(\frac{p^2}{2m} + V(x)\right)\right]|a'\rangle$$
$$\langle a''|\,[[H,x],x]\,|a'\rangle = \langle a''|\left[\frac{p^2x^2}{2m} + V(x)x^2 - \frac{p^2x^2}{2m} + V(x)x^2\right]|a'\rangle$$

Taking the probability:

$$\sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'} - E_{a''}) = \frac{\hbar^2}{2m}$$