Math Methods Assignment #2

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1. (a)

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{l}\cos\theta)^2 + \dot{l}^2\sin^2\theta) \tag{1}$$

(b) Since M is confined to the x axis only m is dependent on gravity:

$$L = T - V \tag{2}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m((\dot{x} + \dot{l}\cos\theta)^2 + \dot{l}^2\sin^2\theta) + mgl\sin\theta \tag{3}$$

Expanding the terms:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2\cos^2\theta + \dot{l}^2\sin^2\theta) + mgl\sin\theta \tag{4}$$

Simplifying with trig identities:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2) + mgl\sin\theta \tag{5}$$

(c) Solving for $\ddot{x}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(M+m)\dot{x} + m\dot{l}\cos\theta\right] = (M+m)\ddot{x} + m\ddot{l}\cos\theta \tag{7}$$

$$\frac{\partial L}{\partial x} = 0 \tag{8}$$

$$(M+m)\ddot{x} + m\ddot{l}\cos\theta = 0 \tag{9}$$

$$\ddot{x} = -\mu\ddot{l}\cos\theta \tag{10}$$

Solving for $\ddot{l}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) - \frac{\partial L}{\partial l} = 0 \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = \frac{d}{dt}\left[m(\dot{x}\cos\theta + \dot{l})\right] = m(\ddot{x}\cos\theta + \ddot{l}) \tag{12}$$

$$\frac{\partial L}{\partial l} = mg\sin\theta \tag{13}$$

$$m(\ddot{x}\cos\theta + \ddot{l}) - mg\sin\theta = 0 \tag{14}$$

$$\ddot{l} = g\sin\theta - \ddot{x}\cos\theta \tag{15}$$

De-coupling the equations, starting with $\ddot{x}$:

$$\ddot{x} = -\mu\left(g\sin\theta - \ddot{x}\cos\theta\right)\cos\theta \tag{16}$$
$$\ddot{x} = -\mu g\cos\theta\sin\theta + \mu\ddot{x}\cos^2\theta \tag{17}$$
$$\ddot{x}\left(1-\mu\cos^2\theta\right) = -\mu g\cos\theta\sin\theta \tag{18}$$
$$\ddot{x} = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta} \tag{19}$$

De-coupling $\ddot{l}$:

$$\ddot{l} = g\sin\theta + \mu\ddot{l}\cos^2\theta \tag{20}$$
$$\ddot{l}\left(1-\mu\cos^2\theta\right) = g\sin\theta + \tag{21}$$
$$\ddot{l} = \frac{g\sin\theta}{1-\mu\cos^2\theta} \tag{22}$$

(d) No, $\ddot{l}$ will never be negative because μ will always be less than 1 and that means that equation (22) will always be positive.

(e) Integrating $\ddot{x}$:

$$\int -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}dt = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t + \dot{x}_0 \tag{23}$$
$$\int\left[-\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t_1 + c_1\right]dt = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 + \dot{x}_0 t + x_0 \tag{24}$$

Since both the wedge and the block start at rest:

$$x(t) = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 \tag{25}$$

Solving for when $\Delta x = \frac{h}{\tan\theta}$:

$$x(t) = -\frac{\mu g\cos\theta\sin\theta}{1-\mu\cos^2\theta}t^2 = \frac{h}{\tan\theta} \tag{26}$$
$$t = -\sqrt{\frac{h}{\tan\theta}\frac{1-\mu\cos^2\theta}{\mu g\cos\theta\sin\theta}} \tag{27}$$

Repeating the same process for Δl results in:

$$l(t) = \frac{g\sin\theta}{1-\mu\cos^2\theta}t^2 = \frac{h}{\sin\theta} \tag{28}$$
$$t = -\sqrt{\frac{h}{\sin\theta}\frac{1-\mu\cos^2\theta}{g\sin\theta}} \tag{29}$$

(f) If $M \to \infty$ then $\mu \to 0$. This means that $\Delta x = 0$ because $t = -\sqrt{\frac{h}{\tan\theta}\frac{1-\mu\cos^2\theta}{\mu g\cos\theta\sin\theta}} \to \infty$, which makes sense since the wedge is infinitely heavy and wont move. Δl on the other hand goes to $t = -\sqrt{\frac{h}{\sin\theta}\frac{1-\mu\cos^2\theta}{g\sin\theta}} \to -\sqrt{\frac{h}{g\sin^2\theta}}$

(g) It doesn't.

2. (a) Starting with the equation for the center of mass $y_{cm} = \frac{1}{M}\sum_i m_i y_i$ and using L_r for the length of the right side and L_l for the left side:

$$L_r = \frac{L+y}{2} \quad L_l = \frac{L-y}{2} \tag{30}$$

$$y_{cm} = \frac{1}{\rho L}\left[(\rho L_r)\left(\frac{L_r}{2}\right) + (\rho L_l)\left(\frac{L_l}{2} + y\right)\right] \tag{31}$$

$$y_{cm} = \frac{1}{L}\left[\left(\frac{L_r^2}{2}\right) + \left(\frac{L_l^2}{2} + yL_L\right)\right] \tag{32}$$

$$y_{cm} = \frac{1}{L}\left[\left(\frac{\left(\frac{L+y}{2}\right)^2}{2}\right) + \left(\frac{\left(\frac{L-y}{2}\right)^2}{2} + y\frac{L-y}{2}\right)\right] \tag{33}$$

$$y_{cm} = \frac{1}{2L}\left[\frac{L^2+2yL+y^2}{4} + \frac{L^2-2yL+y^2}{4} + yL - y^2\right] \tag{34}$$

$$y_{cm} = \frac{1}{2L}\left[\frac{L^2-y^2}{2} + yL\right] \tag{35}$$

$$y_{cm} = \frac{L^2+2yL-y^2}{4L} \tag{36}$$

(b)

$$L = \frac{1}{2}m\dot{y}_{cm}^2 - mgy_{cm} \tag{37}$$

$$L = \frac{1}{2}m\left[\frac{d}{dt}\left(\frac{L^2+2yL-y^2}{4L}\right)\right]^2 - mg\left[\frac{L^2+2yL-y^2}{4L}\right] \tag{38}$$

$$L = \frac{1}{2}m\left[\frac{2\dot{y}L - 2y\dot{y}}{4L}\right]^2 - mg\left[\frac{L^2+2yL-y^2}{4L}\right] \tag{39}$$

$$L = m\frac{\dot{y}^2L^2 - Ly\dot{y}^2 + y^2\dot{y}^2}{8L^2} - mg\left[\frac{L^2+2yL-y^2}{4L}\right] \tag{40}$$

(c) Since we know the initial conditions we can solve this using conservation of energy:

$$\frac{1}{2}m\dot{y}_{cm}^2 + mgy_{cm} = mg\frac{L}{4} \tag{41}$$

$$m\frac{\dot{y}^2L^2 - Ly\dot{y}^2 + y^2\dot{y}^2}{8L^2} + mg\left[\frac{L^2+2yL-y^2}{4L}\right] = mg\frac{L}{4} \tag{42}$$

$$\dot{y}^2 = \frac{8gL^2\left(\frac{L}{4} - \left[\frac{L^2+2yL-y^2}{4L}\right]\right)}{L^2 - Ly + y^2} \tag{43}$$

$$\dot{y} = \sqrt{\frac{8gL^2\left(\frac{L}{4} - \left[\frac{L^2+2yL-y^2}{4L}\right]\right)}{L^2 - Ly + y^2}} \tag{44}$$

(d) We can differentiate the answer to the previous section to find the acceleration:

$$\ddot{y} = \frac{d}{dt}\left(\sqrt{\frac{8gL^2\left(\frac{L}{4} - \left[\frac{L^2+2yL-y^2}{4L}\right]\right)}{L^2 - Ly + y^2}}\right) \tag{45}$$

(e) This answer is greater than g because the entire system is gaining potential energy, yet only a smaller and smaller section of rope is accelerating, thus for energy to be conserved that section of the rope must gain velocity faster than the acceleration due to gravity.

3. (a)

$$V = \frac{1}{2}k(s-a)^2 - mgs \tag{46}$$

(b)

$$T = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}M\dot{s}^2 = m\dot{s}^2 \tag{47}$$

$$L = T - V = m\dot{s}^2 - \frac{1}{2}k(s-a)^2 + mgs \tag{48}$$

$$L = m\dot{s}^2 - \frac{1}{2}k(s^2 - 2sa + a^2) + mgs \tag{49}$$

(c)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \tag{50}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}(2m\dot{s}) = 2m\ddot{s} \tag{51}$$

$$\frac{\partial L}{\partial s} = -\frac{1}{2}k(2s - 2a) + mg \tag{52}$$

$$2m\ddot{s} + k(s-a) - mg = 0 \tag{53}$$

$$s(t) = a + c_1 \sin\left(\sqrt{\frac{k}{2}}t\right) + c_2 \cos\left(\sqrt{\frac{k}{2}}t\right) + \frac{gm}{k} \tag{54}$$

(d) The equilibrium position where $\ddot{s} = 0$:

$$\ddot{s} = \frac{mg - k(s-a)}{2m} \tag{55}$$

(e) The frequency is clear from the equations of motion:

$$\omega = \frac{k}{2} \tag{56}$$