

Math Methods Assignment #1

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1. (a)

$$\begin{aligned}E &= \frac{1}{2}mv^2 \\P_0 &= \frac{E}{\Delta t} = \frac{\Delta mv^2}{2\Delta t} \\v &= \sqrt{\frac{2P_0\Delta t}{\Delta m}} \\p &= \Delta m \sqrt{\frac{2P_0\Delta t}{\Delta m}} = \sqrt{2P_0\Delta m\Delta t}\end{aligned}$$

(b) Tsiolkovsky rocket equation from wikipedia:

$$\Delta v = v_e \ln \frac{m_0}{m_f}$$

Substituting in the answer from part (a):

$$v_f = \sqrt{2P_0 \frac{\Delta t}{\Delta m}} \ln \left(\frac{m}{m - \Delta m} \right)$$

(c) From the same wikipedia article, substituting in the answer from part (a) and converting Δ to d :

$$\begin{aligned}mv' &= -v_e m' \\mv' + v_e m' &= 0 \\v' &= \sqrt{2P_0 \frac{dt}{dm}} \frac{m'}{m}\end{aligned}$$

Assuming $\frac{dt}{dm} = (m')^{-1}$:

$$\begin{aligned}v' &= \sqrt{\frac{2P_0}{m'}} \frac{m'}{m} \\v &= \sqrt{2P_0} \int_0^{t_f} \sqrt{\frac{1}{m'}} \frac{m'}{m} dt\end{aligned}$$

Using the Euler-Lagrange equation:

$$\begin{aligned}
m' \frac{\delta f}{\delta m'} - f &= c \\
m' \frac{\delta}{\delta m'} \left(\sqrt{\frac{1}{m'} \frac{m'}{m}} \right) - \sqrt{\frac{1}{m'} \frac{m'}{m}} &= c \\
m' \frac{1}{2m} \sqrt{\frac{1}{m'}} - \sqrt{\frac{1}{m'} \frac{m'}{m}} &= c \\
-\frac{m'}{2m} \sqrt{\frac{1}{m'}} &= c \\
m' &= 4c^2 m^2
\end{aligned}$$

(d) Final velocity in part (c), using the equation for v derived in the first part of (c):

$$\begin{aligned}
v &= \sqrt{2P_0} \int_0^{t_f} \sqrt{\frac{1}{m'} \frac{m'}{m}} dt \\
v &= \sqrt{2P_0} \int_0^{t_f} \sqrt{\frac{1}{4c^2 m^2} \frac{4c^2 m^2}{m}} dt \\
v &= \sqrt{2P_0} \int_0^{t_f} 2c^2 m \sqrt{\frac{1}{c^2 m^2}} dt \\
v &= 2c^2 m \sqrt{\frac{2P_0}{c^2 m^2}} t_f
\end{aligned}$$

If c is positive:

$$v = 2ct_f \sqrt{2P_0}$$

This differs from the answer in part (b) because it does not depend on m' or m ?

2. (a) Using the Euler-Lagrange equation:

$$\begin{aligned}
y' \frac{\delta f}{\delta y'} - f &= c \\
y' \frac{\delta}{\delta y'} \left[\left(\frac{\delta y}{\delta x} \right)^2 + \alpha y \frac{\delta y}{\delta x} \right] - \left[\left(\frac{\delta y}{\delta x} \right)^2 + \alpha y \frac{\delta y}{\delta x} \right] &= c \\
y' \frac{\delta}{\delta y'} \left[(y')^2 + \alpha y y' \right] - \left[(y')^2 + \alpha y y' \right] &= c \\
y' \cdot (2y' + \alpha y) - \left[(y')^2 + \alpha y y' \right] &= c \\
y' &= \sqrt{c}
\end{aligned}$$

Since y' is constant:

$$y = Ax + B$$

Since $y(0) = 0$ and $y(x_f) = y_f$:

$$\begin{aligned}
B &= 0 \\
y_f &= A \cdot x_f A = \frac{y_f}{x_f}
\end{aligned}$$

(b) It doesn't because $\alpha y \frac{\delta y}{\delta x}$ is a total derivative and thus is path independent.

3. Differentiating $I(\epsilon)$ with respect to ϵ :

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B} \left[\frac{\delta f}{\delta y} \frac{dy}{d\epsilon} + \frac{\delta f}{\delta y'} \frac{dy'}{d\epsilon} + \frac{\delta f}{\delta y''} \frac{dy''}{d\epsilon} \right]$$

Since y' endpoints are prescribed, the integration by parts trick on page 46 will work on both the $\frac{\delta f}{\delta y'} \frac{dy'}{d\epsilon}$ and $\frac{\delta f}{\delta y''} \frac{dy''}{d\epsilon}$ terms.

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B} \left[\frac{\delta f}{\delta y} - \frac{d}{dx} \left(\frac{\delta f}{\delta y'} \right) - \frac{d^2}{dx^2} \left(\frac{\delta f}{\delta y'} \right) \right] \frac{dy}{d\epsilon} dx$$

This requires that $y(x, \epsilon)$ and all its derivatives through third order are continuous functions of x and ϵ

4. (a) From the definition of ds :

$$ds = \sqrt{1 + y'^2} dx$$

Using the Euler-Lagrange equation:

$$\begin{aligned} f &= \frac{\sqrt{1 + y'^2}}{u} \\ y' \frac{\delta}{\delta y'} \left[\frac{\sqrt{1 + y'^2}}{u} \right] - \frac{\sqrt{1 + y'^2}}{u} &= \text{const} \\ \frac{y'^2}{u \sqrt{y'^2 + 1}} - \frac{\sqrt{1 + y'^2}}{u} &= \frac{y'^2 - (1 + y'^2)}{u \sqrt{y'^2 + 1}} = \text{const} \\ \frac{1}{u \sqrt{y'^2 + 1}} &= \frac{1}{u \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} = \frac{1}{u \sqrt{\cot^2 \phi + 1}} = \text{const} \\ \frac{1}{u \sqrt{\csc^2 \phi}} &= \frac{\sin \phi}{u} = \text{const} \end{aligned}$$

(b) Since we assumed u above was a function of y we can simply take our result from part (a):

$$\begin{aligned} \frac{\sin \phi}{\alpha y} &= \text{const} \\ \sin \phi &= \alpha y \\ \sin(\arctan y') &= \alpha y \end{aligned}$$

Using WolframAlpha to solve this differential equation gives:

$$\alpha x + x_0 = \sqrt{1 - \alpha^2 y^2} - \tanh^{-1} \left(\sqrt{1 - \alpha^2 y^2} \right)$$

Assuming the \tanh^{-1} can be ignored as a phase or something similar, it's clear that $\alpha x + x_0 = \sqrt{1 - \alpha^2 y^2}$ is the equation of a circle centered on the x axis.