

Math Methods Assignment #6

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1. (a) Starting with Ampere's Law:

$$\begin{aligned}\nabla \times \mathcal{B} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} \\ \nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c} \nabla \cdot \mathbf{J} + \frac{1}{c} \frac{\partial(\nabla \cdot \mathcal{E})}{\partial t} = 0 \quad \text{Using the identity: } \nabla \cdot (\nabla \times \mathbf{A}) = 0 \\ \nabla \cdot (\nabla \times \mathcal{B}) &= \frac{4\pi}{c} \nabla \cdot \mathbf{J} + \frac{4\pi}{c} \frac{\partial \rho}{\partial t} = 0 \quad \text{Using: } \nabla \cdot \mathcal{E} = 4\pi\rho \\ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} &= 0\end{aligned}$$

- (b) Starting with the curl of the electric field:

$$\begin{aligned}\nabla \times \mathcal{E} &= -\frac{1}{c} \frac{\partial \mathcal{B}}{\partial t} \\ \nabla \times \mathcal{E} &= -\frac{1}{c} \frac{\partial(\nabla \times \mathcal{A})}{\partial t} \\ \nabla \times \left(\mathcal{E} + \frac{1}{c} \frac{\partial \mathcal{A}}{\partial t} \right) &= 0 \\ \mathcal{E} &= -\nabla \phi - \frac{1}{c} \frac{\partial \mathcal{A}}{\partial t} \quad \text{Rewriting in terms of a scalar potential}\end{aligned}$$

- (c)

$$\begin{aligned}F &= \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix} \\ B &= \left[\frac{1}{2}(B_1 + B_3) \quad \frac{1}{2}(B_2 + B_3) \quad \frac{1}{2}(B_3 + B_1) \right]\end{aligned}$$

- (d) Starting with $\mathcal{B} = \nabla \times \mathbf{A}$:

$$\begin{aligned}F_{ij} &= \epsilon_{ijk}(\nabla \times \mathbf{A})_k \\ F_{ij} &= \epsilon_{ijk}\epsilon_{jlm}\partial_l A_m \quad \text{Rewriting the curl using Levi-Civita} \\ F_{ij} &= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_l A_m \\ F_{ij} &= \partial_i A_j - \partial_j A_i\end{aligned}$$

(e) Proving the first part:

$$F_{4j} = \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}$$

$$-\left(\frac{\partial A_4}{\partial x_j} - \frac{\partial A_j}{\partial x_4}\right) = \frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}$$

Assuming this 4th term is time, $\mathcal{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathcal{A}}{\partial t}$ and $\partial_t A_j = 0$ show that $F_{4j} = iE_j$.

(f) Starting with the definition given:

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

$$\partial_i(\partial_j A_k - \partial_k A_j) + \partial_j(\partial_k A_i - \partial_i A_k) + \partial_k(\partial_i A_j - \partial_j A_i) = 0$$

$$\partial_i \partial_j A_k - \partial_i \partial_k A_j + \partial_j \partial_k A_i - \partial_j \partial_i A_k + \partial_k \partial_i A_j - \partial_k \partial_j A_i = 0$$

Since A is a continuous function $\partial_i \partial_j A_k = \partial_j \partial_i A_k$ and the above expression equals zero.

(g) If any pair of indices is zero then that corresponds to a diagonal term in F which is zero.

(h)

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

$$\partial_t F_{jk} + \partial_j F_{k4} + \partial_k F_{4j} = 0$$

$$\partial_t B + \partial_j F_{k4} + \partial_k F_{4j} = 0 \quad \text{Since all } jk \text{ only terms are magnetic}$$

$$\partial_t B + \partial_j - \nabla \times \mathcal{E} = 0 \quad \text{Since the second term is equivalent to the curl}$$

This last expression is the thirds Maxwell's equation.

(i) In this case $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ becomes $\partial_i F_i + \partial_j F_j + \partial_k F_k = 0$ which is equivalent to $\nabla \cdot \mathcal{B} = 0$.

(j) To show that we get $\nabla \cdot \mathcal{E} = 4\pi\rho$ we treat the case where $k = 4$:

$$\partial_t E_l = \frac{4\pi}{c} J_l$$

If we integrate both sides we get $\nabla \cdot \mathcal{E} = 4\pi\rho$.

Looking at the remaining cases we get:

$$\partial_k(B_{lk} - E_l) = \frac{4\pi}{c} J_l$$

$$\partial_k B_{lk} = \frac{4\pi}{c} J_l + \partial_t E_l$$

$$\nabla \times \mathcal{B} = \frac{4\pi}{c} J_l + \partial_t \frac{1}{c} \mathcal{E}$$

(k)

$$L_{ij} \mathcal{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \alpha & i \sinh \alpha \\ 0 & 0 & -\sinh \alpha & \cosh \alpha \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ ic\rho_0(\vec{r}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -ic\rho_0(\vec{r}) \sinh \alpha \\ ic\rho_0(\vec{r}) \cosh \alpha \end{bmatrix} = \mathcal{J}'$$

Approximating for $v \ll c$, where $\cosh \alpha \approx 1$:

$$\mathcal{J}' = \begin{bmatrix} 0 \\ 0 \\ -c\rho_0(\vec{r}) \sinh \alpha \\ ic\rho_0(\vec{r}) \end{bmatrix}$$

(l)

$$L_{ij}F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \alpha & i \sinh \alpha \\ 0 & 0 & -\sinh \alpha & \cosh \alpha \end{bmatrix} \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

$$L_{ij}F = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ \cosh \alpha B_y - \sinh \alpha E_x & -\cosh \alpha B_x - \sinh \alpha E_y & -\sinh \alpha E_z & -i \cosh \alpha E_z \\ i \cosh \alpha E_x - i \sinh \alpha B_y & i \sinh \alpha B_x + i \cosh \alpha E_y & i \cosh \alpha E_z & -\sinh \alpha E_z \end{bmatrix}$$

$$F'_{23} = B_x \quad F'_{31} = \cosh \alpha B_y - \sinh \alpha E_x$$

2. $f[x] = 4x^3 - 32x^2 + 66x - 18;$

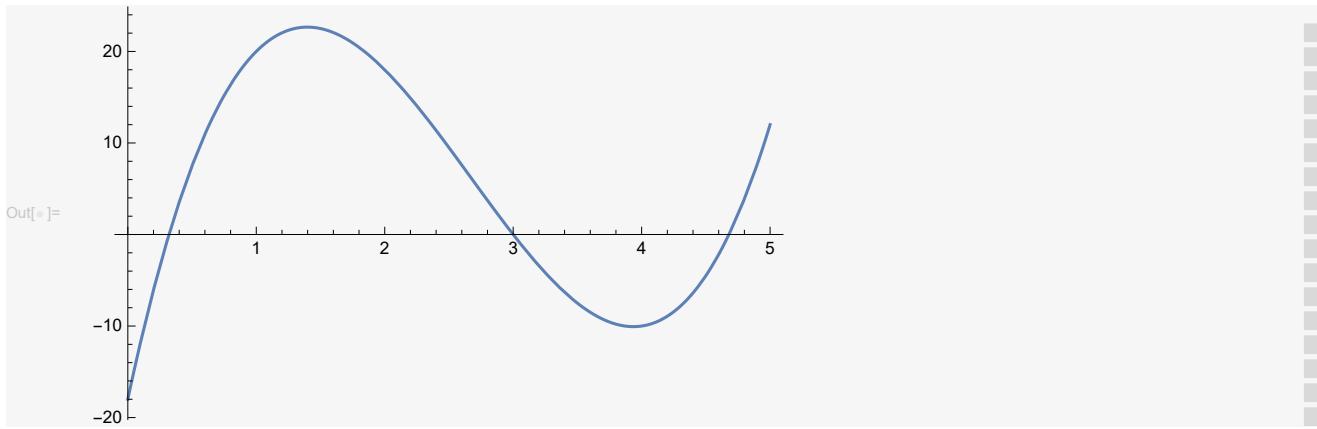
$Solve[f[x] == 0, x]$

$N[Solve[f[x] == 0, x]]$

$Plot[f[x], \{x, 0, 5\}]$

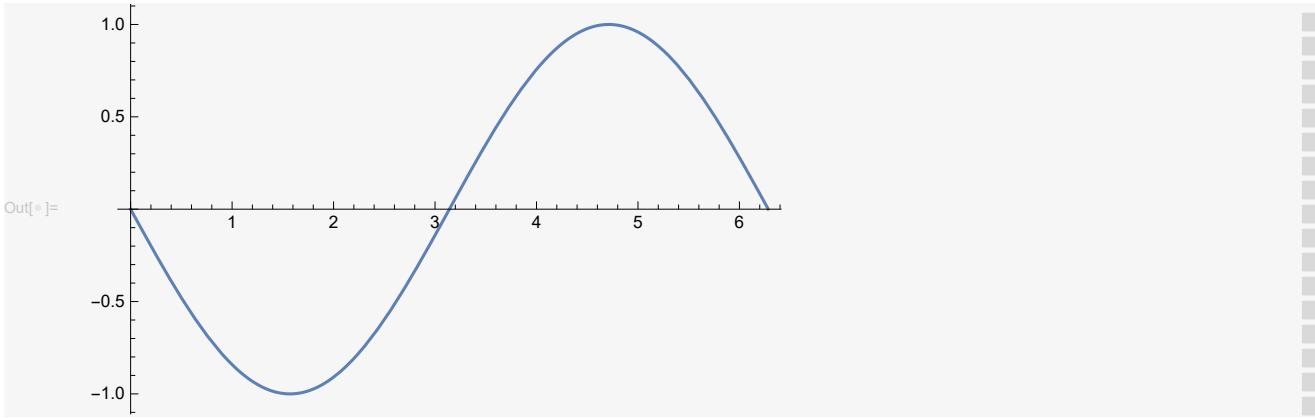
$$\{\{x \rightarrow 3\}, \{x \rightarrow \frac{1}{2}(5 - \sqrt{19})\}, \{x \rightarrow \frac{1}{2}(5 + \sqrt{19})\}\}$$

$$\{\{x \rightarrow 3.\}, \{x \rightarrow 0.320551\}, \{x \rightarrow 4.67945\}\}$$



3. $f[x_, u_] = Total[Table[D[Sin[u], \{u, i\}]/Factorial[i] x^i, \{i, 0, 20\}]]$
 $Plot[\{f[x, Pi]\}, \{x, 0, 2Pi\}]$

$$x \cos[u] - \frac{1}{6} x^3 \cos[u] + \frac{1}{120} x^5 \cos[u] - \frac{x^7 \cos[u]}{5040} + \frac{x^9 \cos[u]}{362880} - \frac{x^{11} \cos[u]}{39916800} + \frac{x^{13} \cos[u]}{6227020800} - \frac{x^{15} \cos[u]}{1307674368000} + \frac{x^{17} \cos[u]}{355687428096000} - \frac{x^{19} \cos[u]}{121645100408832000} + \sin[u] - \frac{1}{2} x^2 \sin[u] + \frac{1}{24} x^4 \sin[u] - \frac{1}{720} x^6 \sin[u] + \frac{x^8 \sin[u]}{40320} - \frac{x^{10} \sin[u]}{3628800} + \frac{x^{12} \sin[u]}{479001600} - \frac{x^{14} \sin[u]}{87178291200} + \frac{x^{16} \sin[u]}{20922789888000} - \frac{x^{18} \sin[u]}{6402373705728000} + \frac{x^{20} \sin[u]}{2432902008176640000}$$



4.

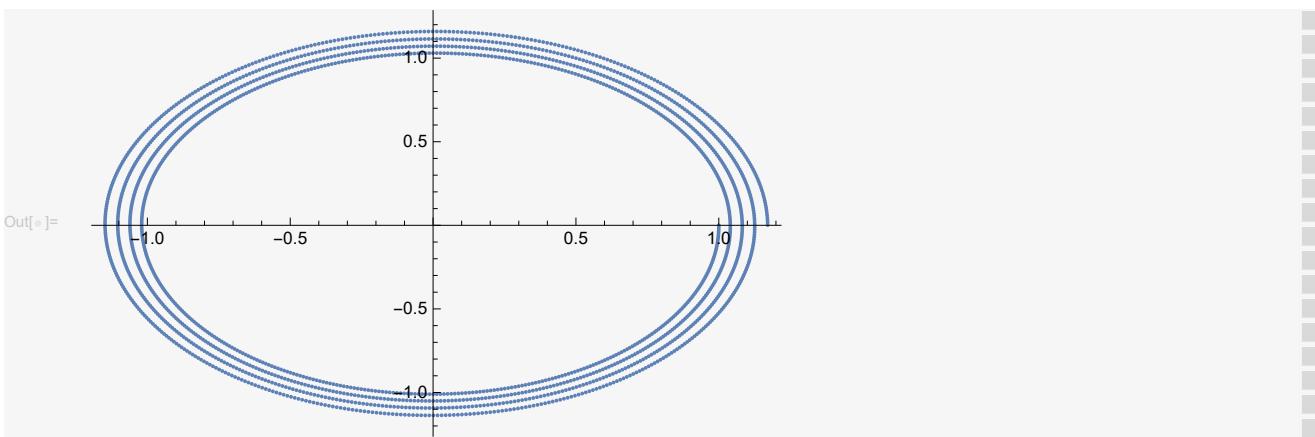
```
A=Table[Table[Sin[i j], {j,10}], {i, 10}];  
b=Table[i,{i,10}];  
LinearSolve[N[A],N[b]]
```

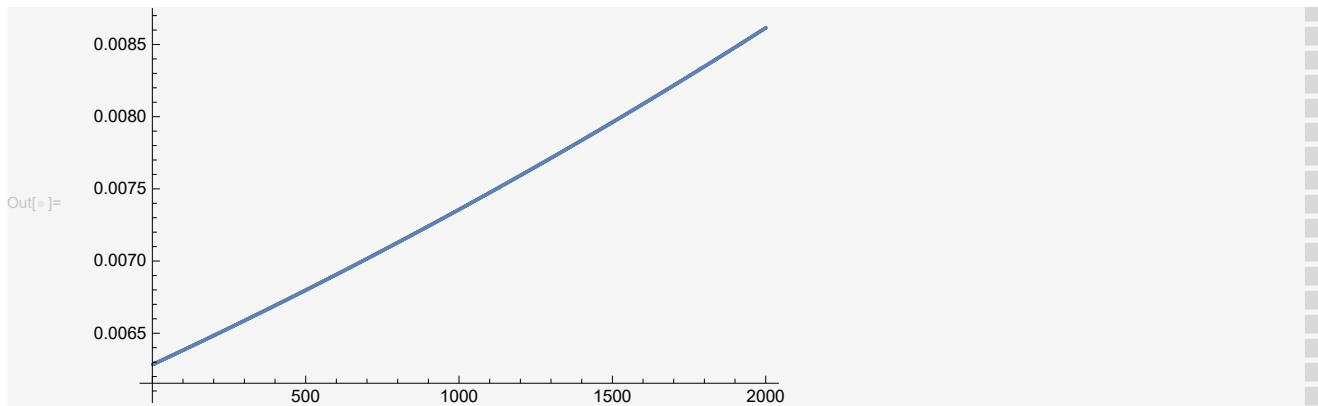
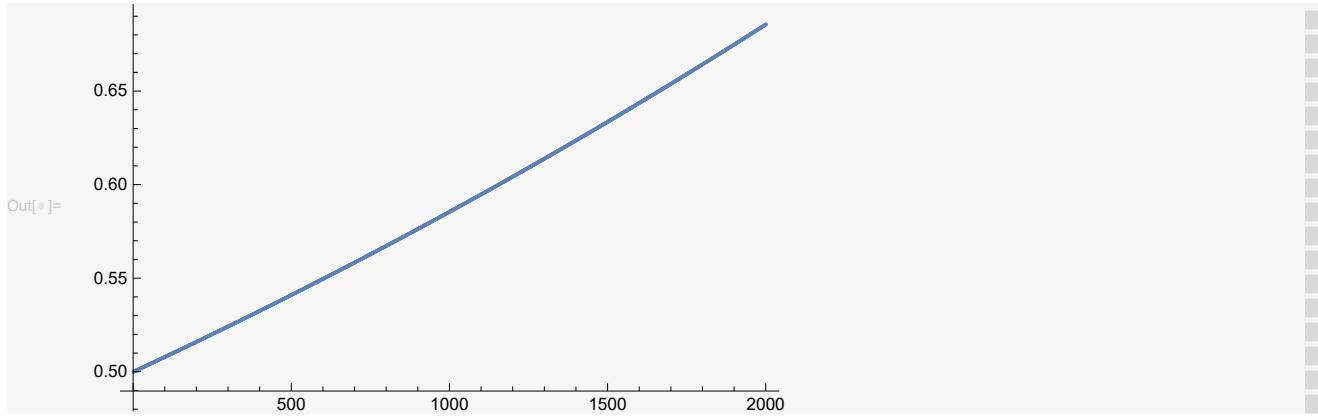
$\{2.83492, -5.65071, -16.77, -9.82246, 2.24527, -5.75988, -2.63877, 2.96037, 25.6627, 23.0544\}$

5. (a)

(b-d)

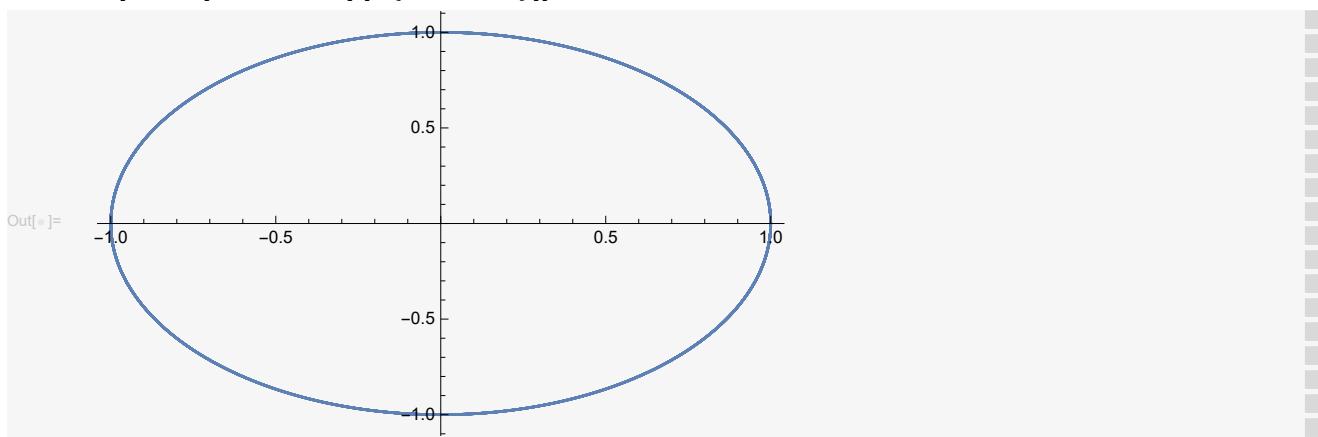
```
Clear[x,v,energy,dotenergy,ndiv,dt];  
ndiv=2000;  
dt=N[(8 Pi)/ndiv];  
v[n_]:=v[n]=v[n-1]-x[n-1] dt;  
v[0]=0;  
x[n_]:=x[n]=x[n-1]+v[n-1] dt;  
x[0]=1;  
ListPlot[Table[{x[t],v[t]}, {t,0,ndiv}]]  
energy[n_]:=energy[n]=(x[n]^2 + v[n]^2)/2  
ListPlot[Table[energy[t], {t, 0, ndiv}]]  
dotenergy[n_]:=dotenergy[n] = (energy[n + 1] - energy[n])/dt  
ListPlot[Table[dotenergy[t], {t, 0, ndiv}]]
```

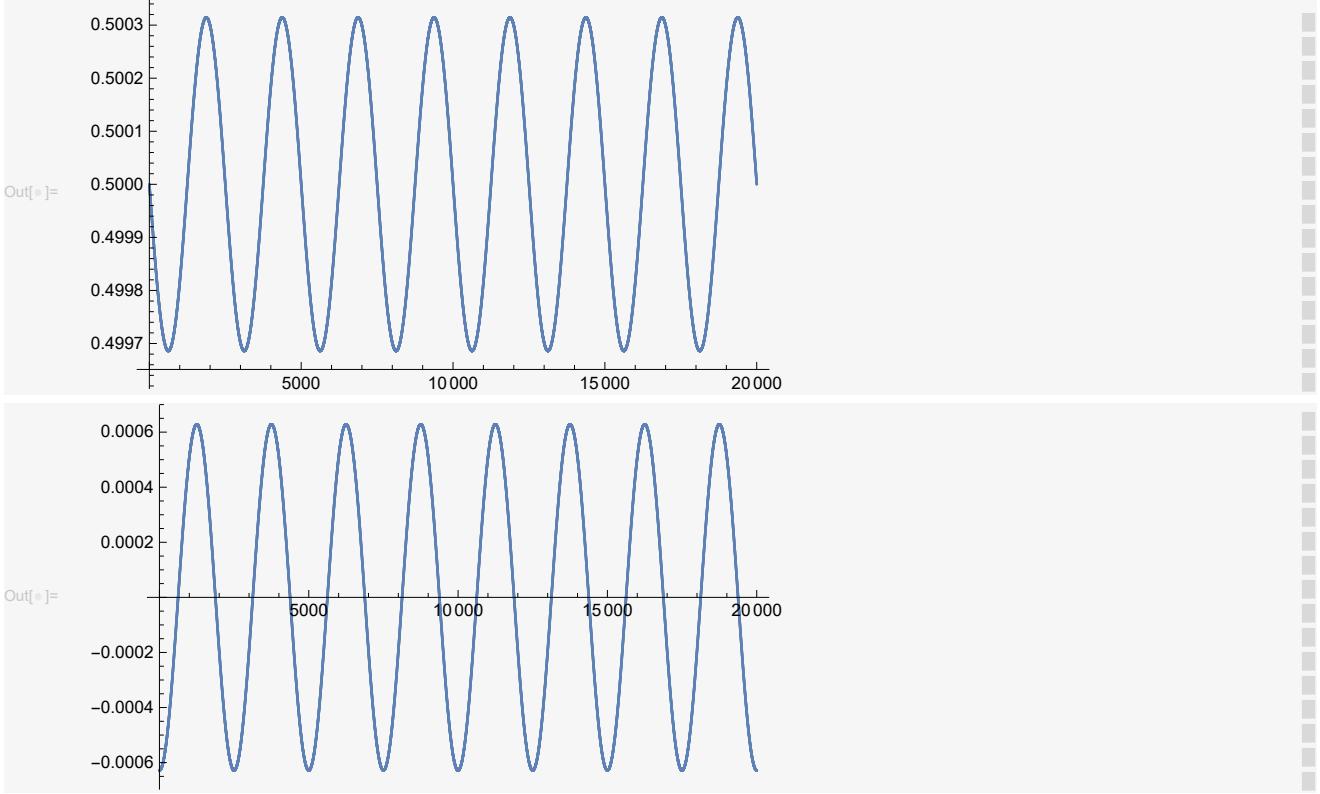




```

Clear[x,v,energy,dotenergy,ndiv,dt];
ndiv=2000;
dt=N[(8 Pi)/ndiv];
v[n_]:=v[n]=v[n-1]-x[n-1] dt;
v[0]=0;
x[n_]:=x[n]=x[n-1]+v[n] dt;
x[0]=1;
ListPlot[Table[{x[t],v[t]}, {t,0,ndiv}]];
energy[n_]:=energy[n]=(x[n]^2 + v[n]^2)/2
ListPlot[Table[energy[t], {t, 0, ndiv}]]
dotenergy[n_]:=dotenergy[n] = (energy[n+1] - energy[n])/dt
ListPlot[Table[dotenergy[t], {t, 0, ndiv}]]
```





(a) Redefining $x \rightarrow Lx$:

$$-\frac{\hbar^2}{2mL^2} \frac{\partial^2\psi}{\partial x^2} - e\mathcal{E}Lx\psi = E\psi \quad (1)$$

Dividing out one of the terms:

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{2e\mathcal{E}mL^3}{\hbar^2}x\psi = \frac{EmL^2}{\hbar^2}\psi \quad (2)$$

(b) The most logical unit for $E \rightarrow \epsilon E_0$ is $E_0 = \frac{\hbar^2}{mL^2}$:

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{2e\mathcal{E}mL^3}{\hbar^2}x\psi = \epsilon\psi$$

(c) Defining our “control knob” $\gamma = \frac{2e\mathcal{E}mL^3}{\hbar^2}$ gives us:

$$-\frac{\partial^2\psi}{\partial x^2} - \gamma x\psi = \epsilon\psi$$

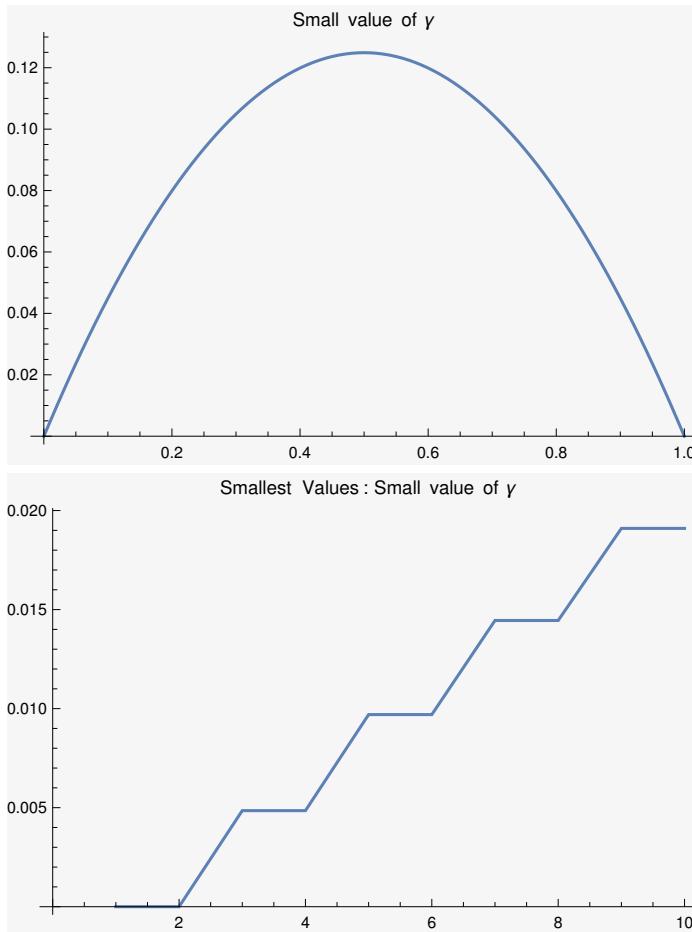
The control knob controls the ratio of kinetic to “spring” potential energy.

(d) Clear[Global*]
eigenstates[γ_- , ϵ_- , ndiv $_-$, l $_$]:=
Module[{dx, npts, mat, bvec, phi},
dx = N[l/ndiv];
npts = ndiv - 1;
mat = Table[If[i == j, (-2/dx^2) - (γ_- dx), 0] + If[Abs[i - j] == 1, 1/dx^2, 0], {i, 1, npts}, {j, 1, npts}];

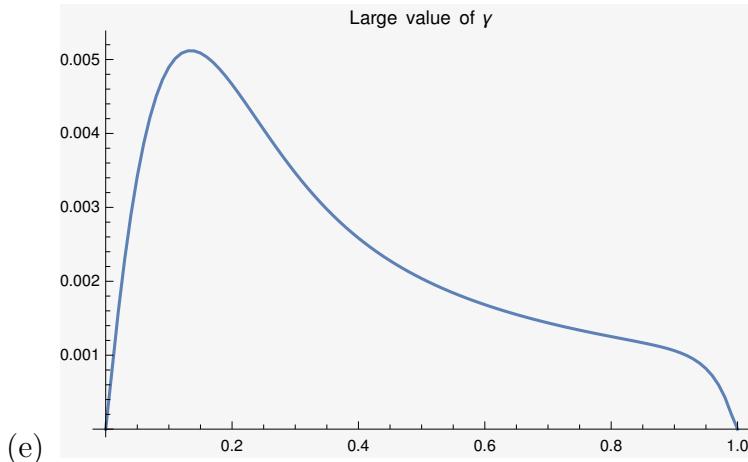
```

bvec = Table[- $\epsilon$ , {j, 1, npts}];
bvec[[1]] = 0;
bvec[[npts]] = 0;
phi = LinearSolve[mat, bvec];
phiTot = Join[{0}, phi, {0}];
phidat = Table[{(j - 1) * dx, phiTot[[j]]}, {j, 1, ndiv + 1}]
];
 $\gamma_{\text{large}} = 10^3;$ 
 $\gamma_{\text{small}} = 10^{(-3)};$ 
 $\epsilon = 1;$ 
l = 1;
ndiv = 100;
small = eigenstates [ $\gamma_{\text{small}}, \epsilon, \text{ndiv}, l$ ] ;
large = eigenstates [ $\gamma_{\text{large}}, \epsilon, \text{ndiv}, l$ ] ;
ListPlot[small, Joined → True, PlotLabel → Small value of  $\gamma$ ]
ListPlot[TakeSmallest[small[[All, 2]], 10], Joined → True, PlotLabel → Smallest Values: Small value]
ListPlot[large, Joined → True, PlotLabel → Large value of  $\gamma$ ]
ListPlot[TakeSmallest[large[[All, 2]], 10], Joined → True, PlotLabel → Smallest Values: Large value]
lowestEigenstates = Table[TakeSmallest[Select[eigenstates[10 * j,  $\epsilon$ , ndiv, l][[All, 2]], # > 0&], 2], {j, 0, 100}];
ListPlot[Transpose[lowestEigenstates], Joined → False, PlotLabel → Lowest eigenstates]

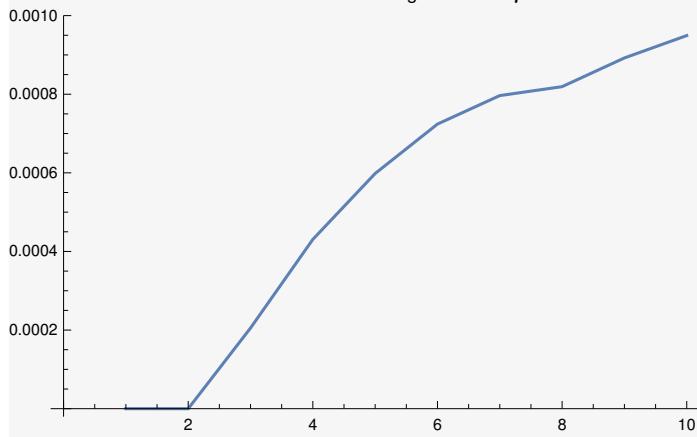
```



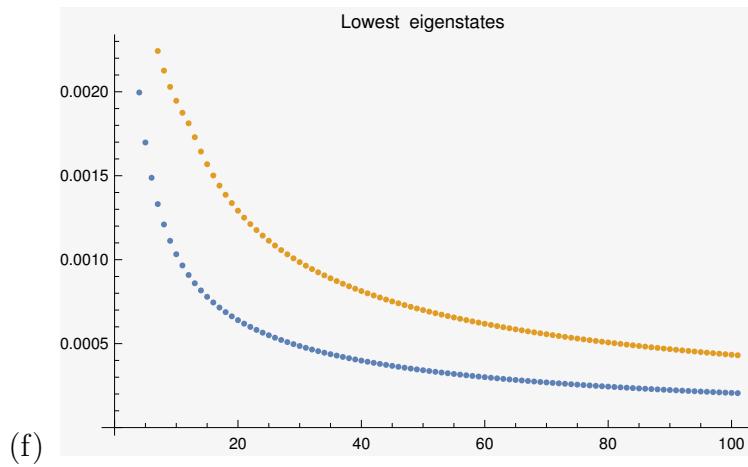
The lowest eigenstates vary linearly.



Smallest Values : Large value of γ



The lowest eigenstates vary parabolically.



I'm not sure about the critical values.