Classical Assignment #7

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1. From Goldstein page 153 we know that we can go from body coordinates to space axes using the following relation $\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}'$. Since $\mathbf{A} = \mathbf{BCD}$:

$$\mathbf{BCD} = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\psi)\sin(\phi) & \cos(\theta)\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) & \sin(\theta)\sin(\psi) \\ -\cos(\theta)\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\phi) & \cos(\theta)\cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) & \sin(\theta)\cos(\psi) \\ \sin(\theta)\sin(\phi) & \sin(\theta)(-\cos(\phi)) & \cos(\theta) \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\psi)\sin(\phi) & -\cos(\theta)\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\theta)\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) & \cos(\theta)\cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) & \sin(\theta)(-\cos(\phi)) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{pmatrix}$$

$$\omega_{bf} = \begin{pmatrix} \theta'\cos(\psi) + \sin(\theta)\sin(\psi)\phi' \\ \sin(\theta)\cos(\psi)\phi' - \theta'\sin(\psi) \\ \cos(\theta)\phi' + \psi' \end{pmatrix}$$

$$\mathbf{A}^{-1}\omega_{bf} = \begin{pmatrix} \theta'\cos(\phi) + \sin(\theta)\psi'\sin(\phi) \\ \theta'\sin(\phi) - \sin(\theta)\psi'\cos(\phi) \\ \cos(\theta)\psi' + \phi' \end{pmatrix}$$

2. (a) Simply plugging in equation (5.22) from Goldstein into Mathematica:

```
Clear[Global`*]
x = {r Cos[θ], rSin[θ], z};
f[j_, k_]:=(x[[1]]^2 + x[[2]]^2 + z^2)KroneckerDelta[j, k] − x[[j]]x[[k]];
i[j_, k_, hi_, hf_]:=MIntegrate[Integrate[Integrate[f[j, k]r, {z, ((hf − hi)/R)r + hi, hf}], {r, 0, R}], {θ, 0, 2Pi}]
ITensor[hi_, hf_]:=Table[FullSimplify[i[j, k, hi, hf]], {j, 1, 3}, {k, 1, 3}];
com = ITensor[−(3/4)h, (1/4)h];
origin = ITensor[0, h];
MatrixForm[com]
MatrixForm[origin]
RVec = {0, 0, −3/4};
steiner[j_, k_]:=(origin[[j, k]] − M(Norm[RVec]^2KroneckerDelta[j, k] − RVec[[j]]RVec[[k]]))
MatrixForm[Table[TrueQ[com[[j, k]]==steiner[j, k]], {j, 1, 3}, {k, 1, 3}]]
```

i.
$$\begin{pmatrix} \frac{1}{80}hM\pi R^2\left(h^2 + 4R^2\right) & 0 & 0 \\ 0 & \frac{1}{80}hM\pi R^2\left(h^2 + 4R^2\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi R^4 \end{pmatrix}$$

ii.
$$\begin{pmatrix} \frac{1}{20}hM\pi R^2\left(4h^2 + R^2\right) & 0 & 0 \\ 0 & \frac{1}{20}hM\pi R^2\left(4h^2 + R^2\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi R^4 \end{pmatrix}$$

(b) This array should be all True, but I think I've made a mistake in my definition of the Steiner equation.

$$\begin{pmatrix} \text{False} & \text{True} & \text{True} \\ \text{True} & \text{False} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

3. (a) The moment of inertia tensor in the body frame is:

ClearAll[Global*]
spacexm$_1$ = {0, bSin[θ], bCos[θ]};
spacexm$_2$ = {0, -bSin[θ], -bCos[θ]};
bodyxm$_1$ = {b, 0, 0};
bodyxm$_2$ = {-b, 0, 0};
M = {m_1, m_2};
x = {bodyxm$_1$, bodyxm$_2$};
com = Total[Table[x[[i, j]]M[[i]], {i, 1, Length[M]}, {j, 1, 3}]]/Total[M];
inertia[j_, k_, r_]:=FullSimplify[(r − com).(r − com)KroneckerDelta[j, k] − r[[j]]r[[k]]];
iTensor = Table[Total[Table[inertia[j, k, x[[i]]], {i, 1, Length[x]}]], {j, 1, 3}, {k, 1, 3}];
MatrixForm[iTensor]
MatrixForm [FullSimplify [iTensor, $m_1 == m_2$]]

$$\begin{pmatrix} b^2\left(\frac{4m_1^2}{(m_1+m_2)^2}-1\right)+b^2\left(\frac{4m_2^2}{(m_1+m_2)^2}-1\right) & 0 & 0 \\ 0 & \frac{4b^2 m_1^2}{(m_1+m_2)^2}+\frac{4b^2 m_2^2}{(m_1+m_2)^2} & 0 \\ 0 & 0 & \frac{4b^2 m_1^2}{(m_1+m_2)^2}+\frac{4b^2 m_2^2}{(m_1+m_2)^2} \end{pmatrix}$$

If we assume $m_1 = m_2$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b^2 & 0 \\ 0 & 0 & 2b^2 \end{pmatrix}$$

We can then calculate L using the A matrix we used in quesion 1:

d = {{Cos[ϕ], Sin[ϕ], 0}, {−Sin[ϕ], Cos[ϕ], 0}, {0, 0, 1}};
c = {{1, 0, 0}, {0, Cos[θ], Sin[θ]}, {0, −Sin[θ], Cos[θ]}};
B = {{Cos[ψ], Sin[ψ], 0}, {−Sin[ψ], Cos[ψ], 0}, {0, 0, 1}};
a = B.c.d;
ϕ = 0;
ψ = 0;
ωTotal = a.{0, 0, ω};
L = iTensor.ωTotal;
SetAttributes[{m, b, ω}, Constant];
FullSimplify[MatrixForm[L]]
FullSimplify[MatrixForm[Dt[L]]]

$$L = \left\{0, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega\sin(\theta)}{(m_1 + m_2)^2}, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega\cos(\theta)}{(m_1 + m_2)^2}\right\}$$

Since torque equals $\frac{dL}{dt}$:

$$\tau = \left\{0, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega d\theta\cos(\theta)}{(m_1 + m_2)^2}, -\frac{4b^2\left(m_1^2 + m_2^2\right)\omega d\theta\sin(\theta)}{(m_1 + m_2)^2}\right\}$$

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(b) Calculating the moment of inertia using the code from above in space fixed coordinates:

$$
\begin{pmatrix}
\frac{4b^2\left(m_1^2+m_2^2\right)}{(m_1+m_2)^2} & 0 & 0 \\
0 & b^2\left(\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right) & -2b^2\sin(\theta)\cos(\theta) \\
0 & -2b^2\sin(\theta)\cos(\theta) & b^2\left(-\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right)
\end{pmatrix}
$$

Assuming $m_1 = m_2$:

$$
\begin{pmatrix}
2b^2 & 0 & 0 \\
0 & 2b^2\cos^2(\theta) & -2b^2\sin(\theta)\cos(\theta) \\
0 & -2b^2\sin(\theta)\cos(\theta) & 2b^2\sin^2(\theta)
\end{pmatrix}
$$

Repeating the same code from above, except without multiplying by A:

$$
L = \left\{0, -2b^2\omega\sin(\theta)\cos(\theta), b^2\omega\left(-\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right)\right\}
$$

$$
\frac{dL}{dt} = \left\{0, -2b^2\omega d\theta\cos(2\theta), 4b^2\omega d\theta\sin(\theta)\cos(\theta)\right\}
$$

4. The kinetic energy will have a term from the translation, which will be a rotation around the apex of the cone. Defining θ as the angle between the cone and the x-axis (the cone is resting on the x-y plane) we get the following translational kinetic energy:

$$
V = \dot{\theta}\frac{3}{4}h\cos\alpha
$$

$$
T = \frac{9}{32}M\dot{\theta}^2 h^2\cos^2\alpha
$$

For the rotational kinetic energy we need to find the angular velocity which we can define from $\dot{\theta}$:

$$
\omega = \frac{4V}{3h\sin\alpha} = \frac{\dot{\theta}\cos\alpha}{\sin\alpha} = \dot{\theta}\cot\alpha
$$

Using the moment of inertia tensor derived in the previous problem:

$$
I = \begin{bmatrix}
\frac{1}{20}hM\pi R^2\left(4h^2+R^2\right) & 0 & 0 \\
0 & \frac{1}{20}hM\pi R^2\left(4h^2+R^2\right) & 0 \\
0 & 0 & \frac{1}{10}hM\pi R^4
\end{bmatrix}
$$

Substituting $h\sin\alpha = R$:

$$
I = \begin{bmatrix}
\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) & 0 & 0 \\
0 & \frac{1}{20}hM\pi h\sin^2\alpha\left(4h^2+h\sin^2\alpha\right) & 0 \\
0 & 0 & \frac{1}{10}hM\pi h\sin^4\alpha
\end{bmatrix}
$$

Finding the rotational kinetic energy with $T = \omega\cdot I\cdot\omega$:

$$
\omega\cdot I\cdot\omega = \begin{bmatrix}\dot{\theta}\\0\\\dot{\theta}\cot\alpha\end{bmatrix}\begin{bmatrix}
\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) & 0 & 0 \\
0 & \frac{1}{20}hM\pi h\sin^2\alpha\left(4h^2+h\sin^2\alpha\right) & 0 \\
0 & 0 & \frac{1}{10}hM\pi h\sin^4\alpha
\end{bmatrix}\begin{bmatrix}\dot{\theta}\\0\\\dot{\theta}\cot\alpha\end{bmatrix}
$$

$$
\omega\cdot I\cdot\omega = \dot{\theta}^2\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right)+\dot{\theta}^2\cot^2\alpha\frac{1}{10}hM\pi h\sin^4\alpha
$$

The final kinetic energy is then:

$$
T = \frac{9}{32}M\dot{\theta}^2 h^2\cos^2\alpha + \frac{1}{2}\left[\dot{\theta}^2\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right)+\dot{\theta}^2\cot^2\alpha\frac{1}{10}hM\pi h\sin^4\alpha\right]
$$

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