

# Math Methods Assignment #7

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1. It satisfies associativity:

$$\begin{aligned} e_k &= e^{\frac{2ik\pi}{n}} \\ (e_a \cdot e_b)e_c &= e_a(e_b \cdot e_c) \\ (e^{\frac{2ia\pi}{n}} \cdot e^{\frac{2ib\pi}{n}})e^{\frac{2ic\pi}{n}} &= e^{\frac{2ia\pi}{n}}(e^{\frac{2ib\pi}{n}} \cdot e^{\frac{2ic\pi}{n}}) \\ e^{\frac{2ia\pi}{n} + \frac{2ib\pi}{n} + \frac{2ic\pi}{n}} &= e^{\frac{2ia\pi}{n} + \frac{2ib\pi}{n} + \frac{2ic\pi}{n}} \end{aligned}$$

There is an identity element:  $e^0$

Each element has an inverse:  $e^{\frac{2i(n-k)\pi}{n}}$

The group is abelian:

$$\begin{aligned} e_a \cdot e_b &= e_b \cdot e_a \\ e^{\frac{2ia\pi}{n}} \cdot e^{\frac{2ib\pi}{n}} &= e^{\frac{2ib\pi}{n}} \cdot e^{\frac{2ia\pi}{n}} \\ e^{\frac{2i\pi(a+b)}{n}} &= e^{\frac{2i\pi(a+b)}{n}} \end{aligned}$$

2. (a)  $AB - BA = -(BA - AB)$

(b)

$$\begin{aligned} A[B, C] - [A, C]B &= A(BC - CB) - (AC - CA)B \\ &= ABC - ACB - ACB - CAB \\ [AB, C] &= ABC - CAB \end{aligned}$$

(c)

$$\begin{aligned} A\{B, C\} - \{A, C\}B &= A(BC + CB) - (AC + CA)B \\ &= ABC + ACB - ACB - CAB \\ [AB, C] &= ABC - CAB \end{aligned}$$

(d)

$$\begin{aligned} [A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= 0 \\ [A, (BC - CB)] + [B, (CA - AC)] + [C, (AB - BA)] &= 0 \\ A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C &= 0 \\ ABC - ACB - BCA + CBA + BCA - BAC - CAB + CAB + ACB - CBA - ABC + BAC &= 0 \\ 0 &= 0 \end{aligned}$$

3. Clear[Global\*]

$a_1 = \{\{0, 1\}, \{1, 0\}\};$

$a_2 = \{\{0, -I\}, \{I, 0\}\};$

$a_3 = \{\{1, 0\}, \{0, -1\}\};$

MatrixForm[Table[MatrixForm[ $a_i.a_j + a_j.a_i$ ], {j, 1, 3}, {i, 1, 3}]]

MatrixForm[Table[ $a_i.a_j + a_j.a_i == 2\text{KroneckerDelta}[j, i]\text{IdentityMatrix}[2]$ , {j, 1, 3}, {i, 1, 3}]]

MatrixForm[Table[MatrixForm[ $a_i.a_j - a_j.a_i$ ], {j, 1, 3}, {i, 1, 3}]]

MatrixForm[Table[ $a_i.a_j - a_j.a_i == 2I\text{Total}[\text{Table}[\text{LeviCivitaTensor}[3][[i, j, k]]a_k, \{k, 1, 3\}]]$ , {j, 1, 3}, {i, 1, 3}]]

$$\begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\ \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

4. Clear[Global\*]

$a = \{\{\text{Cos}[\theta], \text{Sin}[\theta]\}, \{-\text{Sin}[\theta], \text{Cos}[\theta]\}\};$

values = Eigenvalues[a];

vectors = Eigenvectors[a];

p = Normalize/@Transpose[vectors];

MatrixForm[values]

MatrixForm[vectors]

MatrixForm[p]  
MatrixForm[FullSimplify[Inverse[p].a.p]]

$$\begin{pmatrix} \cos[\theta] - i\sin[\theta] \\ \cos[\theta] + i\sin[\theta] \end{pmatrix}$$

$$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

5. Because  $AA^{-1} = \mathbb{1}$ :

$$\begin{aligned} \mathbb{1} &= (C^{-1} - C^{-1}D(C + D)^{-1})(C + D) \\ &= C^{-1}(C + D) - C^{-1}D(C + D)^{-1}(C + D) \\ &= C^{-1}(C + D) - C^{-1}D\mathbb{1} \\ &= C^{-1}C + C^{-1}D - C^{-1}D \\ &= C^{-1}C \\ &= \mathbb{1} \end{aligned}$$

6. Starting from the definition of the derivative of a polynomial as  $\frac{\partial^i}{\partial t^i} f^{(n)}(x) = \frac{n!}{(n-i)!} x^{n-i}$ <sup>1</sup>:

$$\begin{aligned} Tf(t) &= \sum_{in} \frac{(x \frac{\partial}{\partial t})^n}{n!} f^{(i)}(t) = \sum_{ni} \frac{x^n}{n!} \frac{\partial^n}{\partial t^n} f^{(i)}(t) \\ Tf(t) &= \sum_{ni} \frac{x^n}{n!} \frac{n!}{(n-i)!} t^{n-i} \\ Tf(t) &= \sum_{ni} \binom{n}{i} t^{n-i} x^n \end{aligned}$$

Since this last terms is simply the binomial expansion we have:

$$Tf(t) = \sum_i (t + x)^i = f(t + x)$$

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<sup>1</sup>[https://en.wikipedia.org/wiki/Binomial\\_coefficient#Binomial\\_coefficients\\_as\\_polynomials](https://en.wikipedia.org/wiki/Binomial_coefficient#Binomial_coefficients_as_polynomials)