Quantum I Assignment #8

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1. (a) Since $\psi(x,0)$ must be normalized:

$$\int_{-\infty}^{\infty} \psi(x,0)dx = \frac{\sqrt{\pi}A\left(\sin(\beta)+\sqrt{2}\cos(\beta)\right)}{\sqrt{\alpha^2}} = 1$$
$$A = -\frac{\sqrt{\alpha^2}}{\sqrt{\pi}(-\sin(\beta)) + \left(-\sqrt{2\pi}\right)\cos(\beta)}$$

(b) Since $E_n = (n+\frac{1}{2})\hbar\omega$:

$$E = \frac{1}{2}\hbar\omega,\ \frac{5}{2}\hbar\omega$$

(c)

$$\int \psi(x,0)x\psi(x,0) = \frac{A^2 e^{\alpha^2(-x^2)}\left(2\sqrt{2}\sin(2\beta)\left(2\alpha^2x^2+1\right) - \cos(2\beta)\left(4\alpha^4x^4+4\alpha^2x^2+3\right) + 4\alpha^4x^4 + 4\alpha^2x^2 + 7\right)}{8\alpha^2}$$
$$\int_{-\infty}^{\infty} \psi(x,0)x\psi(x,0) = 0$$

2.15 [1]

(a) Since $\psi(x) = \langle x|e^{\frac{-ipa}{\hbar}}|0\rangle$ which is the spatial translation operator where $e^{\frac{-ipa}{\hbar}}|x\rangle = |x+a\rangle$:

$$\psi(x) = \int \langle x|x'+a\rangle\langle x'|0\rangle\, dx'$$
$$\psi(x) = \int \delta(x'-(x-a))\langle x'|0\rangle\, dx'$$
$$\psi(x) = \langle x-a|0\rangle$$

Plugging this into the equation given:

$$\psi(x) = \langle x-a|0\rangle = \pi^{-1/4}x_0^{-1/2}\exp\left[-\frac{1}{2}\left(\frac{x-a}{x_0}\right)^2\right]$$

(b) The probability is:

$$\text{Probability} = \left|\int \langle x'|0\rangle * \langle x'|0\rangle\, dx'\right|^2$$
$$\text{Probability} = \left|\int\left(\pi^{-1/4}x_0^{-1/2}\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]\right)^2\right|^2$$
$$\text{Probability} = -\frac{he^{-\frac{mx\omega}{h}}}{\sqrt{\pi}m\omega\sqrt{\frac{h}{m\omega}}}$$

Which doesn not change for $t > 0$

[1] https://www3.nd.edu/ bjanko/p70007/qm1hw4answers.pdf

2.16 (a) Starting by solving for x and p:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$
$$p = -\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Solving for $\langle m|x|n\rangle$:

$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle m|a + a^\dagger|n\rangle$$
$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle m|\rangle\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right)$$
$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right)$$

Solving for $\langle m|p|n\rangle$:

$$\langle m|x|n\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\langle m|a - a^\dagger|n\rangle$$
$$\langle m|x|n\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\langle m|\rangle\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)$$
$$\langle m|x|n\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\delta_{m,n-1} - \sqrt{n+1}\delta_{m,n+1}\right)$$

Solving for $\langle m|\{x,p\}|n\rangle$, using $\{x,p\} = i\hbar(a^{\dagger 2} - a^2)$:

$$\langle m|x|n\rangle = i\hbar\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2}\right)$$

Solving for $\langle m|x^2|n\rangle$:

$$\langle m|x^2|n\rangle = \langle m|\left(\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\right)^2|n\rangle$$

Solving for $\langle m|p^2|n\rangle$:

$$\langle m|p^2|n\rangle = \langle m|\left(\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)\right)^2|n\rangle$$

2.19 i.
ii.
iii.