

# Classical Mechanics Assignment #1

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1. (a) Starting with the forces, simplifying, and substituting variables:

$$\begin{aligned}F_{\text{total}} &= F_{\text{spring}} + F_{\text{damping}} + F_{\text{piston}} \\m\ddot{x} &= -kx - m\nu\dot{x} + kX(t) \\\ddot{x} + \frac{k}{m}x + \nu\dot{x} &= \frac{k}{m}(t) \\\ddot{x} + \nu\dot{x} + \omega_0^2x &= F_0(t)\end{aligned}$$

- (b) Complementary solution:

$$x(t) = e^{-\beta t} \left[ A_1 e^{-i\sqrt{\omega_0^2 - \beta^2}t} + A_2 e^{i\sqrt{\omega_0^2 - \beta^2}t} \right]$$

Starting the particular solution by finding the derivatives of  $X(t)$ :

$$\begin{aligned}X(t) &= X_0 e^{\alpha t} \cos(\omega t + \delta) \\\dot{X}(t) &= e^{\alpha t} X_0 (\alpha \cos(\omega t + \delta) - \omega \sin(\omega t + \delta)) \\\ddot{X}(t) &= X_0 e^{\alpha t} ((\alpha^2 - \omega^2) \cos(\omega t + \delta) - 2\alpha\omega \sin(\omega t + \delta))\end{aligned}$$

Substituting back into equation:

$$\begin{aligned}X_0 e^{\alpha t} ((\alpha^2 - \omega^2) \cos(\omega t) - 2\alpha\omega \sin(\omega t)) + \\\nu e^{\alpha t} X_0 (\alpha \cos(\omega t) - \omega \sin(\omega t)) + \\\omega_0^2 X_0 e^{\alpha t} \cos(\omega t) &= \omega_0^2 e^{\alpha t} \cos(\omega t)\end{aligned}$$

Solving for cos terms:

$$\begin{aligned}X_0 e^{\alpha t} (\alpha^2 - \omega^2) + \nu e^{\alpha t} X_0 \alpha + \omega_0^2 X_0 e^{\alpha t} &= \omega_0^2 e^{\alpha t} \\X_0 [e^{\alpha t} (\alpha^2 - \omega^2) + \nu e^{\alpha t} \alpha + \omega_0^2 e^{\alpha t}] &= \omega_0^2 e^{\alpha t} \\X_0 &= \frac{\omega_0^2 e^{\alpha t}}{e^{\alpha t} (\alpha^2 - \omega^2) + \nu e^{\alpha t} \alpha + \omega_0^2 e^{\alpha t}} \\X_0 &= \frac{\omega_0^2}{\alpha^2 + \alpha\nu - \omega^2 + \omega_0^2}\end{aligned}$$

Particular solution:

$$e^{-\beta t} \left[ A_1 e^{-i\sqrt{\omega_0^2 - \beta^2}t} + A_2 e^{i\sqrt{\omega_0^2 - \beta^2}t} \right] = \frac{\omega_0^2}{\alpha^2 + \alpha\nu - \omega^2 + \omega_0^2} e^{\alpha t} \cos(\omega t)$$

- (c) At the steady state only the forcing function matters, and the amplitude of the steady state function will be maximum when:

$$\omega = \sqrt{\alpha^2 + \alpha\nu + \omega_0^2} = \omega_R$$

This does depend on the sign of  $\alpha$ , as if  $\alpha$  is negative as  $t \rightarrow \infty$  it will tend toward 0.

2. (a) Assuming the "ripples" can be modeled by a sinusoidal function, the motion of the car can be described by the driven damped oscillator discussed in class. In this case the equation for the resonance frequency:

$$\begin{aligned}\omega_r &= \sqrt{\omega_0^2 + 2\beta^2} \\ \frac{v}{x_{\text{spacing}}} &= \sqrt{\omega_0^2 + 2\beta^2} \\ v &= x_{\text{spacing}} \sqrt{\omega_0^2 + 2\beta^2}\end{aligned}$$

Since  $\omega_0 = \sqrt{\frac{k}{m}}$  and  $\beta = \frac{b}{2m}$ :

$$\begin{aligned}\omega_r &= \sqrt{\omega_0^2 + 2\beta^2} \\ \frac{v}{x_{\text{spacing}}} &= \sqrt{\omega_0^2 + 2\beta^2} \\ v &= x_{\text{spacing}} \sqrt{\frac{k}{m} + \frac{b^2}{4m^2}}\end{aligned}$$

Plugging in some reasonable values we can check whether or not our equations make physical sense. I used the mass of my car  $m = 1565$  kg, and damping  $b = 2000$  N s/m and stiffness  $k = 22,000$  N/m parameters from a research paper.<sup>1</sup>

$$v = 2\sqrt{\frac{22000}{1565} + \frac{2000^2}{4 \cdot 1565^2}} \approx 7.6 \text{ m/s} \approx 17 \text{ mph}$$

This is a reasonable number and the units make sense:

$$\begin{aligned}v &= m \sqrt{\frac{N}{m \cdot kg} + \frac{N^2 \cdot s^2}{m^2 \cdot kg^2}} \\ v &= m \sqrt{\frac{kg \cdot m}{m \cdot kg \cdot s^2} + \frac{kg^2 \cdot m^2 \cdot s^2}{m^2 \cdot kg^2 \cdot s^4}} \\ v &= m \sqrt{\frac{1}{s^2} + \frac{1}{s^2}} = \frac{m}{s}\end{aligned}$$

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<sup>1</sup><https://journals.sagepub.com/doi/full/10.1177/1687814016648638> Analysis of suspension with variable stiffness and variable damping force for automotive applications

- (b) So that the suspension doesn't fall apart I probably want a damping constant that only allows resonances at very high speeds, say above 100 mph:

$$\sqrt{4m^2 \left( \frac{v^2}{x_{\text{spacing}}^2} - \frac{k}{m} \right)} = b$$

$$\sqrt{4 \cdot 1565^2 \left( \frac{45}{2} - \frac{22000}{1565} \right)} \approx 9100 \text{ N s/m}$$

3. (a) Since  $F = \frac{dU}{ds}$  the change in energy of the system is simply:

$$m\ddot{x} = F = \dot{E} = -(x^2 + \dot{x}^2 - 1) \dot{x} - x$$

- (b) By the definitions that  $x = r \cos \theta$  and  $\dot{x} = r \sin \theta$ :

$$r^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$r^2 = x^2 + \dot{x}^2$$

Taking a time derivative and solving for  $\ddot{x}$ :

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + \dot{x}^2)$$

$$2r\dot{r} = 2x\dot{x} + 2\dot{x}\ddot{x}$$

$$\ddot{x} = \frac{r\dot{r}}{\dot{x}} - x$$

Substituting back in for  $x$  and  $\dot{x}$

$$\ddot{x} = \frac{r\dot{r}}{r \sin \theta} - r \cos \theta$$

$$\ddot{x} = \frac{\dot{r}}{\sin \theta} - r \cos \theta$$

Plugging it back into the original equation of motion:

$$\frac{\dot{r}}{\sin \theta} - r \cos \theta = -(x^2 + \dot{x}^2 - 1) \dot{x} - x$$

$$\frac{\dot{r}}{\sin \theta} - r \cos \theta = -((r \cos \theta)^2 + (r \sin \theta)^2 - 1) r \sin \theta - r \cos \theta$$

$$\frac{\dot{r}}{\sin \theta} - r \cos \theta = -(r^2 - 1) r \sin \theta - r \cos \theta$$

$$\dot{r} = -(r^2 - 1) r \sin^2 \theta$$

$$\dot{r} = r(1 - r^2) \sin^2 \theta$$

Repeating the same process to find  $\dot{\theta}$ :

By the definitions that  $x = r \cos \theta$  and  $\dot{x} = r \sin \theta$ :

$$\frac{\dot{x}}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

Taking a time derivative and solving for  $\dot{\theta}$ :

$$\begin{aligned}\frac{d}{dt} \tan \theta &= \frac{d}{dt} \frac{\dot{x}}{x} \\ \frac{\dot{\theta}}{\cos^2 \theta} &= \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} \\ \dot{\theta} &= \cos^2 \theta \left( \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} \right)\end{aligned}$$

Substituting back in the equation of motion from part (a) for  $\ddot{x}$ :

$$\dot{\theta} = \cos^2 \theta \left( \frac{-(x^2 + \dot{x}^2 - 1)\dot{x} - x}{x} - \frac{\dot{x}^2}{x^2} \right)$$

Substituting back in for  $x$  and  $\dot{x}$  and simplifying:

$$\begin{aligned}\dot{\theta} &= \cos^2 \theta \left( \frac{-((r \cos \theta)^2 + (r \sin \theta)^2 - 1)r \sin \theta - r \cos \theta}{r \cos \theta} - \frac{(r \sin \theta)^2}{(r \cos \theta)^2} \right) \\ \dot{\theta} &= \cos^2 \theta \left( \frac{-(r^2 - 1) \sin \theta - \cos \theta}{\cos \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \right) \\ \dot{\theta} &= \cos^2 \theta \left( \frac{-(r^2 - 1) \sin \theta}{\cos \theta} - \left[ 1 + \frac{\sin^2 \theta}{\cos^2 \theta} \right] \right) \\ \dot{\theta} &= \cos^2 \theta \left( \frac{-(r^2 - 1) \sin \theta}{\cos \theta} - \sec^2 \theta \right) \\ \dot{\theta} &= -(r^2 - 1) \sin \theta \cos \theta - 1\end{aligned}$$

