Classical Assignment #7

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1. From Goldstein page 153 we know that we can go from body coordinates to space axes using the following relation $\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}'$. Since $\mathbf{A} = \mathbf{BCD}$:

$$\mathbf{BCD} = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\psi)\sin(\phi) & \cos(\theta)\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) & \sin(\theta)\sin(\psi) \\ -\cos(\theta)\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\phi) & \cos(\theta)\cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) & \sin(\theta)\cos(\psi) \\ \sin(\theta)\sin(\phi) & \sin(\theta)(-\cos(\phi)) & \cos(\theta) \end{pmatrix}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\psi)\sin(\phi) & -\cos(\theta)\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\theta)\sin(\psi)\cos(\phi) + \cos(\psi)\sin(\phi) & \cos(\theta)\cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) & \sin(\theta)(-\cos(\phi)) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{pmatrix}$$

$$\omega_{bf} = \begin{pmatrix} \theta'\cos(\psi) + \sin(\theta)\sin(\psi)\phi' \\ \sin(\theta)\cos(\psi)\phi' - \theta'\sin(\psi) \\ \cos(\theta)\phi' + \psi' \end{pmatrix}$$

$$\mathbf{A}^{-1}\omega_{bf} = \begin{pmatrix} \theta'\cos(\phi) + \sin(\theta)\psi'\sin(\phi) \\ \theta'\sin(\phi) - \sin(\theta)\psi'\cos(\phi) \\ \cos(\theta)\psi' + \phi' \end{pmatrix}$$

2. (a) Simply plugging in equation (5.22) from Goldstein into Mathematica:

```
Clear[Global`*]
x = {r Cos[θ], rSin[θ], z};
f[j_, k_]:=(x[[1]]^2 + x[[2]]^2 + z^2)KroneckerDelta[j, k] − x[[j]]x[[k]];
i[j_, k_, hi_, hf_]:=MIntegrate[Integrate[Integrate[f[j, k]r, {z, ((hf − hi)/R)r + hi, hf}], {r, 0, R}], {θ, 0, 2Pi}]
ITensor[hi_, hf_]:=Table[FullSimplify[i[j, k, hi, hf]], {j, 1, 3}, {k, 1, 3}];
com = ITensor[−(3/4)h, (1/4)h];
origin = ITensor[0, h];
MatrixForm[com]
MatrixForm[origin]
RVec = {0, 0, −3/4};
steiner[j_, k_]:=(origin[[j, k]] − M(Norm[RVec]^2KroneckerDelta[j, k] − RVec[[j]]RVec[[k]]))
MatrixForm[Table[TrueQ[com[[j, k]]==steiner[j, k]], {j, 1, 3}, {k, 1, 3}]]
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i. $$\begin{pmatrix} \frac{1}{80}hM\pi R^2\left(h^2 + 4R^2\right) & 0 & 0 \\ 0 & \frac{1}{80}hM\pi R^2\left(h^2 + 4R^2\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi R^4 \end{pmatrix}$$

ii. $$\begin{pmatrix} \frac{1}{20}hM\pi R^2\left(4h^2 + R^2\right) & 0 & 0 \\ 0 & \frac{1}{20}hM\pi R^2\left(4h^2 + R^2\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi R^4 \end{pmatrix}$$

(b) This array should be all True, but I think I've made a mistake in my definition of the Steiner equation.

$$\begin{pmatrix} \text{False} & \text{True} & \text{True} \\ \text{True} & \text{False} & \text{True} \\ \text{True} & \text{True} & \text{True} \end{pmatrix}$$

3. (a) The moment of inertia tensor in the body frame is:

ClearAll[Global`*]
spacexm$_1$ = $\{0, b\text{Sin}[\theta], b\text{Cos}[\theta]\}$;
spacexm$_2$ = $\{0, -b\text{Sin}[\theta], -b\text{Cos}[\theta]\}$;
bodyxm$_1$ = $\{b, 0, 0\}$;
bodyxm$_2$ = $\{-b, 0, 0\}$;
$M = \{m_1, m_2\}$;
$x = \{\text{bodyxm}_1, \text{bodyxm}_2\}$;
com = Total[Table[$x[[i, j]]M[[i]]$, $\{i, 1, \text{Length}[M]\}$, $\{j, 1, 3\}$]]/Total[M];
inertia[j_, k_, r_]:=FullSimplify[$(r - \text{com}).(r - \text{com})$KroneckerDelta[$j, k$] $- r[[j]]r[[k]]$];
iTensor = Table[Total[Table[inertia[$j, k, x[[i]]$], $\{i, 1, \text{Length}[x]\}$]], $\{j, 1, 3\}$, $\{k, 1, 3\}$];
MatrixForm[iTensor]
MatrixForm [FullSimplify [iTensor, $m_1 == m_2$]]

$$\begin{pmatrix} b^2\left(\frac{4m_1^2}{(m_1+m_2)^2} - 1\right) + b^2\left(\frac{4m_2^2}{(m_1+m_2)^2} - 1\right) & 0 & 0 \\ 0 & \frac{4b^2m_1^2}{(m_1+m_2)^2} + \frac{4b^2m_2^2}{(m_1+m_2)^2} & 0 \\ 0 & 0 & \frac{4b^2m_1^2}{(m_1+m_2)^2} + \frac{4b^2m_2^2}{(m_1+m_2)^2} \end{pmatrix}$$

If we assume $m_1 = m_2$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b^2 & 0 \\ 0 & 0 & 2b^2 \end{pmatrix}$$

We can then calculate L using the A matrix we used in quesion 1:

d = $\{\{\text{Cos}[\phi], \text{Sin}[\phi], 0\}, \{-\text{Sin}[\phi], \text{Cos}[\phi], 0\}, \{0, 0, 1\}\}$;
$c = \{\{1, 0, 0\}, \{0, \text{Cos}[\theta], \text{Sin}[\theta]\}, \{0, -\text{Sin}[\theta], \text{Cos}[\theta]\}\}$;
$B = \{\{\text{Cos}[\psi], \text{Sin}[\psi], 0\}, \{-\text{Sin}[\psi], \text{Cos}[\psi], 0\}, \{0, 0, 1\}\}$;
$a = B.c.d$;
$\phi = 0$;
$\psi = 0$;
ωTotal = $a.\{0, 0, \omega\}$;
L = iTensor.ωTotal;
SetAttributes[$\{m, b, \omega\}$, Constant];
FullSimplify[MatrixForm[L]]
FullSimplify[MatrixForm[Dt[L]]]

$$L = \left\{0, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega\sin(\theta)}{(m_1 + m_2)^2}, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega\cos(\theta)}{(m_1 + m_2)^2}\right\}$$

Since torque equals $\frac{dL}{dt}$:

$$\tau = \left\{0, \frac{4b^2\left(m_1^2 + m_2^2\right)\omega d\theta\cos(\theta)}{(m_1 + m_2)^2}, -\frac{4b^2\left(m_1^2 + m_2^2\right)\omega d\theta\sin(\theta)}{(m_1 + m_2)^2}\right\}$$

(b) Calculating the moment of inertia using the code from above in space fixed coordinates:

$$\begin{pmatrix} \frac{4b^2\left(m_1^2+m_2^2\right)}{(m_1+m_2)^2} & 0 & 0 \\ 0 & b^2\left(\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right) & -2b^2\sin(\theta)\cos(\theta) \\ 0 & -2b^2\sin(\theta)\cos(\theta) & b^2\left(-\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right) \end{pmatrix}$$

Assuming $m_1 = m_2$:

$$\begin{pmatrix} 2b^2 & 0 & 0 \\ 0 & 2b^2\cos^2(\theta) & -2b^2\sin(\theta)\cos(\theta) \\ 0 & -2b^2\sin(\theta)\cos(\theta) & 2b^2\sin^2(\theta) \end{pmatrix}$$

Repeating the same code from above, except without multiplying by A:

$$L = \left\{0, -2b^2\omega\sin(\theta)\cos(\theta), b^2\omega\left(-\cos(2\theta)-\frac{8m_1m_2}{(m_1+m_2)^2}+3\right)\right\}$$

$$\frac{dL}{dt} = \left\{0, -2b^2\omega d\theta\cos(2\theta), 4b^2\omega d\theta\sin(\theta)\cos(\theta)\right\}$$

4. The kinetic energy will have a term from the translation, which will be a rotation around the apex of the cone. Defining θ as the angle between the cone and the x-axis (the cone is resting on the x-y plane) we get the following translational kinetic energy:

$$V = \dot{\theta}\frac{3}{4}h\cos\alpha$$

$$T = \frac{9}{32}M\dot{\theta}^2h^2\cos^2\alpha$$

For the rotational kinetic energy we need to find the angular velocity which we can define from $\dot{\theta}$:

$$\omega = \frac{4V}{3h\sin\alpha} = \frac{\dot{\theta}\cos\alpha}{\sin\alpha} = \dot{\theta}\cot\alpha$$

Using the moment of inertia tensor derived in the previous problem:

$$I = \begin{bmatrix} \frac{1}{20}hM\pi R^2\left(4h^2+R^2\right) & 0 & 0 \\ 0 & \frac{1}{20}hM\pi R^2\left(4h^2+R^2\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi R^4 \end{bmatrix}$$

Substituting $h\sin\alpha = R$:

$$I = \begin{bmatrix} \frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) & 0 & 0 \\ 0 & \frac{1}{20}hM\pi h\sin^2\alpha\left(4h^2+h\sin^2\alpha\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi h\sin^4\alpha \end{bmatrix}$$

Finding the rotational kinetic energy with $T = \omega\cdot I\cdot\omega$:

$$\omega\cdot I\cdot\omega = \begin{bmatrix} \dot{\theta} \\ 0 \\ \dot{\theta}\cot\alpha \end{bmatrix} \begin{bmatrix} \frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) & 0 & 0 \\ 0 & \frac{1}{20}hM\pi h\sin^2\alpha\left(4h^2+h\sin^2\alpha\right) & 0 \\ 0 & 0 & \frac{1}{10}hM\pi h\sin^4\alpha \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ \dot{\theta}\cot\alpha \end{bmatrix}$$

$$\omega\cdot I\cdot\omega = \dot{\theta}^2\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) + \dot{\theta}^2\cot^2\alpha\frac{1}{10}hM\pi h\sin^4\alpha$$

The final kinetic energy is then:

$$T = \frac{9}{32}M\dot{\theta}^2h^2\cos^2\alpha + \frac{1}{2}\left[\dot{\theta}^2\frac{1}{20}hM\pi R^2\left(4h^2+h\sin^2\alpha\right) + \dot{\theta}^2\cot^2\alpha\frac{1}{10}hM\pi h\sin^4\alpha\right]$$