Quantum I Assignment #3

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1.6 (a)

$$\mathrm{tr}(XY) = \sum_i \langle\psi_i| XY |\psi_i\rangle$$
$$\mathrm{tr}(XY) = \sum_{ij} \langle\psi_i| X |\psi_j\rangle \langle\psi_j| Y |\psi_i\rangle$$
$$\mathrm{tr}(XY) = \sum_{ij} \langle\psi_j| Y |\psi_i\rangle \langle\psi_i| X |\psi_j\rangle = \mathrm{tr}(YX)$$

(b) From Sakurai page 15:

$$XY|\alpha\rangle = X(Y|\alpha\rangle) \overset{\mathrm{DC}}{\leftrightarrow} (\langle\alpha| Y^\dagger)X^\dagger = \langle\alpha| Y^\dagger X^\dagger$$

(c) Using the Taylor series expansion:

$$\exp[if(A)] = \exp[if(a_0|\alpha_0\rangle)] + i\exp[if(a_0|\alpha_0\rangle)]a_1|\alpha_1\rangle f'(a_0|\alpha_0\rangle)\cdots$$

1.15 From Sakurai page 30:

$$|\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2$$

Replacing with our variables:

$$a' = s_z; +,\ b' = s_n; +,\ c' = s_z; -$$
$$P = |\langle s_z; -|s_n; +\rangle|^2 |\langle s_n; +|s_z; +\rangle|^2$$

Replacing with matrices:

$$S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad S_n = \frac{\hbar}{2}\begin{bmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix}$$
$$|+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |-\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$
$$P = |\begin{bmatrix}0\\1\end{bmatrix}^T \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^T \begin{bmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}|^2 |\begin{bmatrix}1\\0\end{bmatrix}^T \begin{bmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix}^T \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}|^2$$
$$P = |\begin{bmatrix}0 & -1\end{bmatrix} \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}|^2 |\begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}|^2$$
$$P = |-\sin\beta|^2 |\cos\beta|^2$$
$$P = \sin^2(\beta)\cos^2(\beta)$$

This maximum value is then $\beta = \frac{\pi}{4}$.

1.17 We can show that for any base ket $|\psi_n\rangle$, $AB|\psi_n\rangle = BA|\psi_n\rangle$:

$$AB|\psi_n\rangle = Ab_n|\psi_n\rangle = a_n b_n|\psi_n\rangle = BA|\psi_n\rangle$$

Since this is true for any $|\psi_n\rangle$ and we know that the simultaneous eigenkets form a complete orthonormal set of base kets $[A, B] = 0$

1.19 Assuming $|n\rangle$ is a nondegenerate energy state. Since $[A_1, H] = 0$, and $HA_1|n\rangle = A_1H|n\rangle = E_nA_1|n\rangle$ and since A_1 and H commute therefore $A_1|n\rangle = a_1|n\rangle$. In the same way $A_2|n\rangle = a_2|n\rangle$ but that means that $A_1A_2|n\rangle = a_1a_2|n\rangle$ which cannot be true if $A_1A_2 \neq A_2A_1$. The only way this could be true is if either a_1 or a_2 is 0.

1.24 Modeling the ice pick as an inverse pendulum we get the following equation of motion:[1]

$$\ddot{\theta} = \frac{g}{l}\sin\theta$$

Since we only care about minuscule movements around the equilibrium point we can make the small angle approximation:

$$\ddot{\theta} = \frac{g}{l}\theta$$
$$\theta(t) = c_1 e^{t\sqrt{g/l}} + c_2 e^{-t\sqrt{g/l}}$$

Solving for the starting conditions.

$$x_0 = \theta(0)l = (c_1 + c_2)l$$
$$p_0 = m\theta(0)l = ml\sqrt{\frac{g}{l}}(c_1 - c_2)$$

Apparently we can just ignore c_2 because it decays exponentially. We can now use x_0 and p_0 to get the uncertainty relation:

$$x_0 p_0 = \frac{\hbar^2}{4} = ml^2\sqrt{\frac{g}{l}}c_1^2$$
$$c_1 = \sqrt{\frac{\hbar^2}{4ml^2}\sqrt{\frac{l}{g}}}$$

If we say that the pick becomes unstable at about half a degree of rotation, or around 0.01 radians, a length of 1 m and a mass of 1 kg, we get the following results.

$$t = \log\left(\frac{\theta}{c_1}\right)\sqrt{\frac{l}{g}}$$
$$t = \log\left(\frac{0.01}{\sqrt{\frac{(1.05\cdot 10^{-34})^2}{4\cdot 1\cdot 1^2}\sqrt{\frac{1}{1}}}}\right)\sqrt{\frac{1}{1}} \approx 74 \text{ s}$$

[1]https://en.wikipedia.org/wiki/Inverted_pendulum