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12.1

$$\left(\frac{1}{\sqrt{2}}\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle-\frac{1}{\sqrt{2}}\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left\langle\mathbf{r}_1,\mathbf{r}_2|\psi_A\right\rangle-\frac{1}{\sqrt{2}}\left\langle\mathbf{r}_2,\mathbf{r}_1|\psi_A\right\rangle\right)$$

$$\frac{1}{2}\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left\langle\mathbf{r}_1,\mathbf{r}_2|\psi_A\right\rangle-\frac{1}{2}\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left\langle\mathbf{r}_2,\mathbf{r}_1|\psi_A\right\rangle-\frac{1}{2}\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\left\langle\mathbf{r}_1,\mathbf{r}_2|\psi_A\right\rangle+\frac{1}{2}\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\left\langle\mathbf{r}_2,\mathbf{r}_1|\psi_A\right\rangle$$

$$\left\langle\mathbf{r}_1,\mathbf{r}_2|\psi_S\right\rangle=-\left\langle\mathbf{r}_2,\mathbf{r}_1|\psi_S\right\rangle$$

$$\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left|\psi_A\right\rangle+\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\left|\psi_A\right\rangle$$

Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are arbitrary they can be switched:

$$\left(\frac{1}{\sqrt{2}}\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle-\frac{1}{\sqrt{2}}\left|\mathbf{r}_2,\mathbf{r}_1\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left\langle\mathbf{r}_1,\mathbf{r}_2|\psi_A\right\rangle-\frac{1}{\sqrt{2}}\left\langle\mathbf{r}_2,\mathbf{r}_1|\psi_A\right\rangle\right)=2\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left|\mathbf{r}_1,\mathbf{r}_2\right\rangle\left|\psi_A\right\rangle$$

12.2

a)

$$E\left(n_1,n_2\right)=\left(n_1+n_2+1\right)\hbar\omega$$

$$E_0=\hbar\omega$$

$$E_1=2\hbar\omega$$

$$\left(\frac{1}{\sqrt{2}}|1,0\rangle+\frac{1}{\sqrt{2}}|0,1\rangle\right)|0,0\rangle$$

$$\left(\frac{1}{\sqrt{2}}|1,0\rangle-\frac{1}{\sqrt{2}}|0,1\rangle\right)|1,0\rangle$$

$$\left(\frac{1}{\sqrt{2}}|1,0\rangle-\frac{1}{\sqrt{2}}|0,1\rangle\right)|1,-1\rangle$$

$$\left(\frac{1}{\sqrt{2}}|1,0\rangle-\frac{1}{\sqrt{2}}|0,1\rangle\right)|1,1\rangle$$

b) The energy is lowest when the particles are close together, the particles that can come closest are particles that are in the ground state, thus the spin-0 state will be lowered more than the spin-1 state.

12.4

$$\psi=e^{-cx^2}$$

$$\hat{\mathbf{H}}\psi=\frac{-\hbar^2}{2m}\frac{\delta^2\psi}{\delta x^2}+\lambda x^4\psi=\frac{-\hbar^2}{2m}\left(2ce^{-cx^2}\right)\left(2cx^2-1\right)+\lambda x^2e^{-cx^2}$$

$$\int_{-\infty}^{\infty}\psi\hat{\mathbf{H}}\psi dx=\int_{-\infty}^{\infty}e^{-2cx^2}\left(\lambda\psi^4-\frac{2\hbar^2c^2}{m}x^2+\frac{\hbar^2c}{m}\right)dx=\sqrt{\frac{\pi}{2c}}\left(\frac{3\lambda}{16}\frac{1}{c^2}+\frac{\hbar^2c}{2m}\right)$$

$$\int_{-\infty}^{\infty}e^{-2cx^2}dx=\sqrt{\frac{\pi}{2c}}$$

$$E=\frac{3\lambda}{16c^2}+\frac{\hbar^2c}{2m}$$

$$\frac{dE}{dc}=\frac{6\lambda}{16c^3}+\frac{\hbar^2}{2m}=0$$

$$c=\sqrt[3]{\frac{3\lambda m}{4\hbar^2}}$$

$$E=\left(\frac{3\sqrt[3]{3}}{4}\right)\lambda^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3}\approx 1.082\lambda^{1/3}\left(\frac{\hbar^2}{2m}\right)^{2/3}$$

13.2

$$\mathbf{j}_{sc}=\frac{\hbar}{2\mu i}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)$$

$$\mathbf{j}_{sc}\xrightarrow[r\to\infty]{}\frac{\hbar}{2\mu i}\left(Af(\theta,\phi)\frac{e^{-ikr}}{r}\nabla Af(\theta,\phi)\frac{e^{ikr}}{r}-Af(\theta,\phi)\frac{e^{ikr}}{r}\nabla Af(\theta,\phi)\frac{e^{-ikr}}{r}\right)$$

$$\mathbf{j}_{sc} \xrightarrow[r\to\infty]{} \frac{\hbar A^2}{2\mu i} f(\theta,\phi)\left(\frac{e^{-ikr}}{r}\nabla f(\theta,\phi)\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}\nabla f(\theta,\phi)\frac{e^{-ikr}}{r}\right)$$

$$\nabla f(\theta,\phi)\frac{e^{ikr}}{r} = \left(\frac{\delta}{\delta r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\delta}{\delta\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}\hat{\phi}\right) f(\theta,\phi)\frac{e^{ikr}}{r}$$

$$\nabla f(\theta,\phi)\frac{e^{ikr}}{r} = f(\theta,\phi)\frac{ie^{ikr}(kr+i)}{r^2}\hat{\mathbf{r}} + \frac{e^{ikr}}{r}\left(\frac{1}{r}\frac{\delta}{\delta\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}\hat{\phi}\right) f(\theta,\phi)$$

$$\nabla f(\theta,\phi)\frac{e^{-ikr}}{r} = \left(\frac{\delta}{\delta r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\delta}{\delta\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}\hat{\phi}\right) f(\theta,\phi)\frac{e^{-ikr}}{r}$$

$$\nabla f(\theta,\phi)\frac{e^{-ikr}}{r} = f(\theta,\phi)\frac{e^{-ikr}(-1-ikr)}{r^2}\hat{\mathbf{r}} + \frac{e^{-ikr}}{r}\left(\frac{1}{r}\frac{\delta}{\delta\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\delta}{\delta\phi}\hat{\phi}\right) f(\theta,\phi)$$

$$\mathbf{j}_{sc} \xrightarrow[r\to\infty]{} \frac{\hbar A^2}{2\mu i} f(\theta,\phi)\left(f(\theta,\phi)\frac{kr+i}{r^3} - f(\theta,\phi)\frac{-1-ikr}{r^3}\right)$$

$$\mathbf{j}_{sc} \xrightarrow[r\to\infty]{} \frac{\hbar A^2}{2\mu i} f(\theta,\phi)^2\mathbf{u}_r$$

13.4 a)

$$\left(\nabla^2 + k^2\right)\psi(x) = \frac{2m}{\hbar^2}V(x)\psi(x)$$

$$\left(\nabla^2 + k^2\right)G(x) = \frac{\delta^2}{\delta x^2}G(x,x') + k^2G(x,x') = \delta(x-x')$$

$$\psi(x) = \int \frac{2m}{\hbar^2}V(x')\psi(x')\delta(x-x')dx'$$

In order to get the general solution:

$$\psi(x) = Ae^{ikx} + \int \frac{2m}{\hbar^2}V(x')\psi(x')G(x,x')dx'$$

b)

$$\frac{\delta^2}{\delta x^2}G(x,x') + k^2G(x,x') = \delta(x-x')$$

$$\int_{x'+\varepsilon}^{x'-\varepsilon}\frac{\delta^2}{\delta x^2}G(x,x')dx = \int_{x'+\varepsilon}^{x'-\varepsilon}\delta(x-x') - k^2G(x,x')dx$$

$$\frac{\delta}{\delta x}G(x,x') = 1 - \int_{x'+\varepsilon}^{x'-\varepsilon}k^2G(x,x')dx$$

$$\left(\frac{\delta G}{\delta x}\right)_{x=x'_+} - \left(\frac{\delta G}{\delta x}\right)_{x=x'_-} = 1$$

$$\left(\frac{\delta}{\delta x}\left(\frac{1}{2ik}e^{ik(x-x')}\right)\right)_{x=x'_+} - \left(\frac{\delta}{\delta x}\left(\frac{1}{2ik}e^{-ik(x-x')}\right)\right)_{x=x'_-} = 1$$

c)

$$\psi(x) = Ae^{ikx} + \int \frac{2m}{\hbar^2}V(x')\psi(x')\frac{1}{2ik}e^{-ik(x-x')}dx'$$

$$\psi(x) = Ae^{ikx} + Ae^{-ikx'}\int \frac{2m}{\hbar^2}V(x')\frac{1}{2ik}e^{-ik(x-x')}dx'$$

$$\psi \xrightarrow[r\to-\infty]{} Ae^{ikx} + Ae^{-ikx'}\int_{-\infty}^{\infty}\frac{2m}{\hbar^2}V(x')\frac{1}{2ik}e^{-ik(x-x')}dx'$$

$$R = \left|\frac{m}{ik\hbar^2}\int_{-\infty}^{\infty}e^{2ikx'}V(x')dx'\right|^2$$

d)

$$\left|\frac{m}{ik\hbar^2}\int_{-\infty}^{\infty} e^{2ikx'}V(x')dx'\right|^2 = 1-\left(1+\left[V_0^2/4E(E-V_0)\right]\sin^2\sqrt{2m/\hbar)(E-V_0)a}\right)^{-1}$$

13.6

$$f(\theta,\phi) = -\frac{\mu C}{2\pi\hbar^2}\int_0^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta\sin\theta e^{iqr\cos\theta}$$

$$f(\theta,\phi) = -\frac{\mu C}{\hbar^2}\int_0^\infty dr\int_0^\pi d\theta\sin\theta e^{iqr\cos\theta}$$

$$f(\theta,\phi) = -\frac{2\mu C}{\hbar^2}\int_0^\infty dr\frac{\sin(qr)}{qr}$$

$$f(\theta,\phi) = -\frac{\mu C}{\hbar^2}\frac{\pi r}{q|r|}$$

$$\frac{d\sigma}{d\Omega} = \left|\frac{\mu C}{\hbar^2}\frac{\pi r}{q|r|}\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2C^2\pi^2}{\hbar^4q^2}$$

13.8 a)

Because the function of ψ is independent of ϕ the plane waves only include $Y_{l,0}$'s, and because the plane wave is a free particle the radial functions must be spherical Bessel functions.

b)

$$c_lj_l(kr) = \frac{1}{\sqrt{(2l)!}}\int d\Omega Y_{l,l}^*\left\{\left[\frac{1}{i}e^{i\phi}\left(i\frac{\delta}{\delta\theta}-\cot\theta\frac{\delta}{\delta\phi}\right)\right]^l e^{ikr\cos\theta}\right\}$$

$$\left(\frac{\delta}{\delta\phi}\frac{1}{i}e^{i\phi}\right)^l = e^{i\phi l}$$

$$c_lj_l(kr) = \frac{1}{\sqrt{(2l)!}}\int d\Omega Y_{l,l}^*\left[(-1)^le^{i\phi l}\sin^l\theta\frac{d^l}{d(\cos\theta)^l}e^{ikr\cos\theta}\right]$$

c)

$$Y_{l,l} = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l\theta$$

$$c_lj_l(kr) = \frac{1}{\sqrt{(2l)!}}\int d\Omega\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{-il\phi}\sin^l\theta\left[(-1)^le^{i\phi l}\sin^l\theta\frac{d^l}{d(\cos\theta)^l}e^{ikr\cos\theta}\right]$$

$$c_lj_l(kr) = (ikr)^l\frac{2^l l!}{\sqrt{(2l)!}}\sqrt{\frac{4\pi}{(2l+1)!}}\int d\Omega\left|Y_{l,l}(\theta,\phi)\right|^2e^{ikr\cos\theta}$$

d)

$$c_l\frac{(kr)^l}{(2l+1)!!}\xrightarrow[r\to 0]{}(ikr)^l\frac{2^l l!}{\sqrt{(2l)!}}\sqrt{\frac{4\pi}{(2l+1)!}}\int d\Omega\left|Y_{l,l}(\theta,\phi)\right|^2e^{ikr\cos\theta}$$

$$c_l\xrightarrow[r\to 0]{}\frac{(2l+1)!!}{(kr)^l}(ikr)^l\frac{2^l l!}{\sqrt{(2l)!}}\sqrt{\frac{4\pi}{(2l+1)!}}\int d\Omega\left|Y_{l,l}(\theta,\phi)\right|^2e^{ikr\cos\theta}$$

$$c_l\xrightarrow[r\to 0]{}i^l\sqrt{4\pi(2l+1)}$$

13.10

$$\psi(\mathbf{r})\xrightarrow[r\to\infty]{}\sum_l\left[A_lj_l(kr)-B_l\eta_l(kr)\right]P_l(\cos\theta)$$

$$\tan\delta_l = \frac{j_l(ka)}{\eta_l(ka)}$$

For small x, $\tan x = x$

$$j_l(\rho) \propto \rho^l$$

$$\eta_l(\rho) \propto -\rho^{-(l+1)}$$

$$\delta_1 \approx -\frac{(ka)^3}{3}$$