

Quantum I Assignment #7

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1. (a)

$$\tilde{\mathbf{H}}|\alpha\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ \frac{1}{2}|u_2\rangle \\ \frac{1}{2}|u_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ |u_2\rangle \\ |u_3\rangle \end{bmatrix}$$

The values that can be found are $(\frac{1}{\sqrt{2}}, 1, 1)$. The probabilities are $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.
The mean value is:

$$\langle \tilde{\mathbf{H}} \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \frac{1}{3} = \frac{1}{2}$$

$$\Delta \tilde{\mathbf{H}} = \sqrt{\langle \tilde{\mathbf{H}}^2 \rangle + \langle \tilde{\mathbf{H}} \rangle^2}$$

$$\langle \tilde{\mathbf{H}}^2 \rangle = \langle \alpha | \tilde{\mathbf{H}}^2 | \alpha \rangle = \begin{bmatrix} \langle u_1 | \frac{1}{\sqrt{2}} & \langle u_2 | \frac{1}{2} & \langle u_3 | \frac{1}{2} | u_3 \rangle \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ \frac{1}{2}|u_2\rangle \\ \frac{1}{2}|u_3\rangle \end{bmatrix} = 2.5$$

$$\langle \tilde{\mathbf{H}} \rangle = \langle \alpha | \tilde{\mathbf{H}} | \alpha \rangle = \begin{bmatrix} \langle u_1 | \frac{1}{\sqrt{2}} & \langle u_2 | \frac{1}{2} & \langle u_3 | \frac{1}{2} | u_3 \rangle \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ \frac{1}{2}|u_2\rangle \\ \frac{1}{2}|u_3\rangle \end{bmatrix} = 1.5$$

$$\Delta \tilde{\mathbf{H}} = \sqrt{1.5^2 + 2.5}$$

(b)

$$\tilde{\mathbf{A}}|\alpha\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ \frac{1}{2}|u_2\rangle \\ \frac{1}{2}|u_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle \\ \frac{1}{2}|u_2\rangle \\ \frac{1}{2}|u_3\rangle \end{bmatrix}$$

The values that can be found are $(\frac{1}{\sqrt{2}}, 1/2, 1/2)$. The probabilities are $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The state vector is: $\begin{bmatrix} \frac{1}{\sqrt{2}}|u_1\rangle & \frac{1}{2}|u_2\rangle & \frac{1}{2}|u_3\rangle \end{bmatrix}$.

(c) The state vector is $\mathcal{U}(t)|\alpha\rangle = e^{-\frac{i}{\hbar}\frac{1}{\sqrt{2}}t}|u_1\rangle + e^{-\frac{it}{\hbar}}|u_2\rangle + e^{-\frac{it}{\hbar}}|u_3\rangle$

(d)

$$\langle \tilde{\mathbf{A}} \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}} \\ e^{\frac{-it}{\hbar}} \\ e^{\frac{-it}{\hbar}} \end{bmatrix} \frac{1}{3} = e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}} \frac{1}{\sqrt{2}} + e^{\frac{-it}{\hbar}} \frac{1}{2} + e^{\frac{-it}{\hbar}} \frac{1}{2}$$

$$\langle \tilde{\mathbf{B}} \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}} \\ e^{\frac{-it}{\hbar}} \\ e^{\frac{-it}{\hbar}} \end{bmatrix} \frac{1}{3} = e^{\frac{-i\frac{1}{\sqrt{2}}t}{\hbar}} \frac{1}{2} + e^{\frac{-it}{\hbar}} \frac{1}{\sqrt{2}} + e^{\frac{-it}{\hbar}} \frac{1}{2}$$

(e)

$$\tilde{\mathbf{A}} |\alpha\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} |u_1\rangle \\ \frac{1}{2} |u_2\rangle \\ \frac{1}{2} |u_3\rangle \end{bmatrix} = \begin{bmatrix} e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{\sqrt{2}} |u_1\rangle \\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{2} |u_2\rangle \\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{2} |u_3\rangle \end{bmatrix}$$

$$\tilde{\mathbf{B}} |\alpha\rangle = \begin{bmatrix} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} |u_2\rangle \\ \frac{1}{\sqrt{2}} |u_1\rangle \\ \frac{1}{2} |u_3\rangle \end{bmatrix} = \begin{bmatrix} e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{2} |u_1\rangle \\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{\sqrt{2}} |u_2\rangle \\ e^{\frac{-i\tilde{\mathbf{H}}t}{\hbar}} \frac{1}{2} |u_3\rangle \end{bmatrix}$$

2.1 Starting with the expression $\frac{d\tilde{\mathbf{A}}}{d= i\hbar} \frac{1}{i\hbar} [\tilde{\mathbf{A}}, \tilde{\mathbf{H}}]$:

$$\frac{d\tilde{\mathbf{S}}_z}{dt} = \frac{1}{i\hbar} [\tilde{\mathbf{S}}_z, \omega\tilde{\mathbf{S}}_z] = \frac{1}{i\hbar} 0 = 0$$

$$\frac{d\tilde{\mathbf{S}}_x}{dt} = \frac{1}{i\hbar} [\tilde{\mathbf{S}}_x, \omega\tilde{\mathbf{S}}_z] = \frac{\omega}{i\hbar} i\hbar S_y = -\omega S_y$$

$$\frac{d\tilde{\mathbf{S}}_y}{dt} = \frac{1}{i\hbar} [\tilde{\mathbf{S}}_y, \omega\tilde{\mathbf{S}}_z] = \frac{\omega}{i\hbar} i\hbar S_x = \omega S_x$$

Repeating to get the second derivative:

$$\frac{d^2\tilde{\mathbf{S}}_z}{dt^2} = \frac{d}{dt} \frac{1}{i\hbar} 0 = 0$$

$$\frac{d^2\tilde{\mathbf{S}}_x}{dt^2} = \frac{d}{dt} \frac{\omega}{i\hbar} i\hbar S_y = -\omega^2 S_y$$

$$\frac{d^2\tilde{\mathbf{S}}_y}{dt^2} = \frac{d}{dt} \frac{\omega}{i\hbar} i\hbar S_x = -\omega^2 S_x$$

These two equations are basic differential equations, so it is clear that $\tilde{\mathbf{S}}_x = \alpha e^{-i\omega t}$ and $\tilde{\mathbf{S}}_y = \beta e^{-i\omega t}$

2.3 (a) Starting with the definition of $\tilde{\mathbf{S}} \cdot \vec{\mathbf{n}}$:

$$\begin{aligned}\tilde{\mathbf{S}} \cdot \vec{\mathbf{n}} &= \cos\left(\frac{\beta}{2}\right) |+\rangle + \sin\left(\frac{\beta}{2}\right) |-\rangle \\ \tilde{\mathbf{S}}_x &= \frac{\hbar}{2} (|+\rangle + |-\rangle)\end{aligned}$$

Applying the time evolution operator:

$$\begin{aligned}\mathcal{U}(t) |\tilde{\mathbf{S}}_n; +\rangle &= \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right) \cos\left(\frac{\beta}{2}\right) |+\rangle + \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right) \sin\left(\frac{\beta}{2}\right) |-\rangle \\ \mathcal{U}(t) |\tilde{\mathbf{S}}_n; +\rangle &= \exp\left(\frac{-i\omega t}{\hbar}\right) \cos\left(\frac{\beta}{2}\right) |+\rangle + \exp\left(\frac{-i\omega t}{\hbar}\right) \sin\left(\frac{\beta}{2}\right) |-\rangle \\ P(t) &= |\langle \tilde{\mathbf{S}}_x; + | \tilde{\mathbf{S}}_n \rangle|^2 = \frac{1}{2} \left[\exp\left(\frac{-i\omega t}{\hbar}\right) \cos\left(\frac{\beta}{2}\right) + \exp\left(\frac{-i\omega t}{\hbar}\right) \sin\left(\frac{\beta}{2}\right) \right]^2 \\ P(t) &= \frac{1}{2} (1 + 2 \sin \beta \cos \omega t)\end{aligned}$$

(b) The probability of being in $|\tilde{\mathbf{S}}_n; -\rangle = 1 - \frac{1}{2} (1 + 2 \sin \beta \cos \omega t)$

$$\begin{aligned}\langle \tilde{\mathbf{S}}_x \rangle &= \frac{\hbar}{2} \left(\frac{1}{2} (1 + 2 \sin \beta \cos \omega t) \right) - \frac{\hbar}{2} \left(1 - \frac{1}{2} (1 + 2 \sin \beta \cos \omega t) \right) \\ \langle \tilde{\mathbf{S}}_x \rangle &= \frac{\hbar}{4} \sin \beta \cos \omega t\end{aligned}$$

(c)

$$\begin{aligned}\beta \rightarrow 0 \quad P(t) &\rightarrow \frac{1}{2} \quad \langle \tilde{\mathbf{S}}_x \rangle \rightarrow 0 \\ \beta \rightarrow \pi/2 \quad P(t) &\rightarrow \frac{1}{2} (1 + 2 \cos \omega t) \quad \langle \tilde{\mathbf{S}}_x \rangle \rightarrow \frac{\hbar}{4} \cos \omega t\end{aligned}$$

2.4 Starting with the definitions (equations 2.63a-b):

$$\begin{aligned}|v_e\rangle &= \cos \theta |v_1\rangle - \sin \theta |v_2\rangle \\ |v_\mu\rangle &= \sin \theta |v_1\rangle + \cos \theta |v_2\rangle\end{aligned}$$

Applying the time evolution operator:

$$\begin{aligned}\mathcal{U}(t) |v_e\rangle &= \exp\left(\frac{-i\tilde{\mathbf{H}}t}{\hbar}\right) \cos \theta + \exp\left(\frac{i\tilde{\mathbf{H}}t}{\hbar}\right) \sin \theta \\ \mathcal{U}(t) |v_e\rangle &= \exp\left(\frac{-ipc \left(1 + \frac{m^2 c^2}{2p^2}\right) t}{\hbar}\right) \cos \theta + \exp\left(\frac{-ipc \left(1 + \frac{m^2 c^2}{2p^2}\right) t}{\hbar}\right) \sin \theta \\ P(t) &= |\langle v_e | v_e \rangle|^2 = \left(\exp\left(\frac{-ipc \left(1 + \frac{m^2 c^2}{2p^2}\right) t}{\hbar}\right) \cos \theta + \exp\left(\frac{-ipc \left(1 + \frac{m^2 c^2}{2p^2}\right) t}{\hbar}\right) \sin \theta \right)^2 \\ P(v_e \rightarrow v_e) &= 1 - \sin^2 2\theta \sin^2 \left(\Delta m^2 c^4 \frac{L}{4E\hbar c} \right)\end{aligned}$$

2.9 (a)

$$\int_{-a}^a A^2(x' - a)^2(x' + a)^{2e^{-ikx'}}(x' - a)^2(x' + a)^{2e^{ikx'}} dx' = \frac{256a^9 A^2}{315} = 1$$

$$A = \pm \frac{3\sqrt{35}}{16a^{9/2}}$$

(b)

$$\langle x \rangle = \int_{-a}^a A^2(x' - a)^2(x' + a)^{2e^{-ikx'}} x(x' - a)^2(x' + a)^{2e^{ikx'}} dx' = 0$$

$$\langle p \rangle = -i\hbar \int_{-a}^a A^2(x' - a)^2(x' + a)^{2e^{-ikx'}} \frac{\partial}{\partial x'} (x' - a)^2(x' + a)^{2e^{ikx'}} dx' = -\hbar k$$

$$\langle x^2 \rangle = \int_{-a}^a A^2(x' - a)^2(x' + a)^{2e^{-ikx'}} x^2(x' - a)^2(x' + a)^{2e^{ikx'}} dx' = \frac{a^2}{11}$$

$$\langle p^2 \rangle = \int_{-a}^a A^2(x' - a)^2(x' + a)^{2e^{-ikx'}} \frac{\partial^2}{\partial x'^2} (x' - a)^2(x' + a)^{2e^{ikx'}} dx' = \hbar \frac{3}{a^2} + k^2$$

(c)