# DERIVATIVES COMPLETE GUIDE

## Quant Finance

All Types • Pricing • Greeks • Models

From Vanilla to Exotic

# Derivative Universe

**Linear**

- Forwards
- Futures
- Swaps

**Vanilla Options**

- Calls/Puts
- European
- American

**Exotics**

- Barrier
- Asian
- Lookback
- Digital

**Structured**

- CDO/CLO
- Variance swaps
- Quanto
- Multi-asset

**Payoff =** $f(\text{Underlying}(t))$ at maturity or path

**Obligation to buy/sell at future date**

No optionality, symmetric payoff

**Forward Contract:**

$$\text{Payoff} = S_T - K$$

$$F_0 = S_0 e^{rT} \quad \text{(no dividends)}$$

$$F_0 = S_0 e^{(r-q)T} \quad \text{(continuous dividend } q\text{)}$$

**Futures vs Forwards:**

- Futures: exchange-traded, daily settlement (mark-to-market)
- Forwards: OTC, single settlement at maturity
- Convexity adjustment when rates stochastic: $F \neq \mathbb{E}[S_T]$

Forward price = no-arbitrage delivery price

**Interest Rate Swap (IRS):**

$$V_{\text{swap}} = \sum_{i=1}^{n} P(0, t_i) \left[ F(0, t_{i-1}, t_i) - K \right] \Delta t_i$$

Swap rate $K$ set s.t. $V_{\text{swap}}(0) = 0$

**Common Types:**

- **IRS**: Fixed vs floating rate
- **Currency swap**: Principal + interest in different currencies
- **CDS**: Credit protection, pay premium for default insurance
- **Total return swap**: Asset return vs financing rate

**Valuation:**

$$V = \text{PV(receive leg)} - \text{PV(pay leg)}$$

Use discount curve, forward rates, credit spreads

# Vanilla Options: Calls & Puts

**European Options:**

$$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

**Put-Call Parity:**

$$C - P = S_0 - Ke^{-rT}$$

**American Options:**

- Early exercise: $V_A \geq V_E$
- Call on non-dividend stock: never early exercise
- Put: may be optimal to exercise early
- Pricing: binomial trees, LSM, PDE methods

**First Order:**

**Delta:** $\Delta = \frac{\partial V}{\partial S}$

$$\Delta_{\text{call}} = \Phi(d_1), \quad \Delta_{\text{put}} = \Phi(d_1) - 1$$

**Vega:** $\mathcal{V} = \frac{\partial V}{\partial \sigma}$

$$\mathcal{V} = S_0 \phi(d_1)\sqrt{T}$$

**Theta:** $\Theta = \frac{\partial V}{\partial t}$ (time decay)

**Rho:** $\rho = \frac{\partial V}{\partial r}$ (rate sensitivity)

**Second Order:**

**Gamma:** $\Gamma = \frac{\partial^2 V}{\partial S^2}$

$$\Gamma = \frac{\phi(d_1)}{S_0 \sigma \sqrt{T}}$$

**Vanna:** $\frac{\partial^2 V}{\partial S \partial \sigma}$

**Volga:** $\frac{\partial^2 V}{\partial \sigma^2}$

**PDE:**

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

Delta-hedge: $\Pi = V - \Delta S$ is instantaneously riskless

**Knock-Out/Knock-In at barrier $H$:**

- **Down-and-Out Call**: dies if $S_t \leq H < K$

$$C_{\text{DO}} = C_{\text{vanilla}} - \left(\frac{S_0}{H}\right)^{2\lambda} C(S_0^2/H, K, \dots)$$

  where $\lambda = \frac{r+\sigma^2/2}{\sigma^2}$
- **Up-and-Out Put**: dies if $S_t \geq H > K$
- **Knock-In**: $V_{\text{KI}} + V_{\text{KO}} = V_{\text{vanilla}}$

**Double Barrier:** Two barriers $L < S_0 < U$, more complex reflection formula

**Rebate:** Pay $R$ if barrier hit

Cheaper than vanilla, hedging challenges near barrier

## Asian Options (Average):

$$\text{Payoff} = \max\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K, 0\right) \quad \text{or} \quad \max\left(S_T - \frac{1}{n}\sum_{i=1}^{n} S_{t_i}, 0\right)$$

- Arithmetic average: no closed form, MC or PDE
- Geometric average: closed form exists (log-normal)
- Lower volatility than vanilla $\Rightarrow$ cheaper

## Lookback Options:

$$\text{Call: } \max(S_{\max} - K, 0) \quad \text{Put: } \max(K - S_{\min}, 0)$$

$$\text{Floating: } S_T - S_{\min} \quad \text{or} \quad S_{\max} - S_T$$

- Closed forms available (complex)
- Most expensive path-dependent options

# Digital/Binary Options

**Cash-or-Nothing:**

$$\text{Call: } \begin{cases} Q & S_T > K \\ 0 & S_T \leq K \end{cases} \quad V = Qe^{-rT}\Phi(d_2)$$

**Asset-or-Nothing:**

$$\text{Call: } \begin{cases} S_T & S_T > K \\ 0 & S_T \leq K \end{cases} \quad V = S_0\Phi(d_1)$$

**Properties:**

- Vanilla $=$ Asset-or-Nothing $-$ $K\times$ Cash-or-Nothing
- Delta: $\Delta_{\text{digital}} = \frac{\phi(d_2)}{S_0\sigma\sqrt{T}} e^{-rT} Q$
- Gamma spike at strike near maturity
- Hard to hedge dynamically

Used in structured products, binary bets on events

**Quanto Options:** Foreign asset, domestic payout (no FX risk to holder)

$$C_{\text{quanto}} = S_0^f e^{(\rho\sigma_S\sigma_X - r_f)T}\Phi(d_1) - Ke^{-r_d T}\Phi(d_2)$$

where $\rho =$ correlation(asset, FX), adjustment in drift

**Composite/Rainbow Options:**

- **Best-of**: $\max(S_1, S_2, \ldots, S_n)$
- **Worst-of**: $\min(S_1, S_2, \ldots, S_n)$
- **Basket**: $\max\left(\sum w_i S_i - K, 0\right)$
- **Spread**: $\max(S_1 - S_2 - K, 0)$

Require correlation matrix, multi-dimensional integration or MC

## Variance Swap:

$$\text{Payoff} = N_{\text{var}}\left(\sigma^2_{\text{realized}} - K_{\text{var}}\right)$$

$$\sigma^2_{\text{realized}} = \frac{252}{n}\sum_{i=1}^{n} r_i^2, \quad r_i = \ln(S_i/S_{i-1})$$

Fair strike $K_{\text{var}}$: portfolio of OTM options

$$K_{\text{var}} = \frac{2e^{rT}}{T}\left[\int_0^{S_0}\frac{P(K)}{K^2}dK + \int_{S_0}^{\infty}\frac{C(K)}{K^2}dK\right]$$

## Volatility Swap:

$$\text{Payoff} = N_{\text{vol}}(\sigma_{\text{realized}} - K_{\text{vol}})$$

Convexity adjustment: $K_{\text{vol}} \approx \sqrt{K_{\text{var}}} - \frac{\text{skew}}{8K_{\text{var}}^{3/2}}$

Pure volatility exposure, model-independent replication

**Convertible Bond:**

$$V_{\text{CB}} = V_{\text{bond}} + V_{\text{equity option}} + V_{\text{credit}} - V_{\text{call}}$$

- Holder can convert to $n$ shares
- Hybrid: credit + equity vol + IR
- Pricing: PDE with free boundary or tree

**Callable/Putable Bonds:**

$$V_{\text{callable}} = V_{\text{straight}} - V_{\text{call option}}$$

$$V_{\text{putable}} = V_{\text{straight}} + V_{\text{put option}}$$

- Issuer call, investor put
- IR models (HW, BK) for embedded options
- OAS (option-adjusted spread) for relative value

**Credit Default Swap (CDS):**

Protection buyer pays premium $s$ until default or maturity

$$s \approx \frac{(1-R)\lambda}{1 + \lambda \cdot \text{risky duration}}$$

$R$ = recovery rate, $\lambda$ = hazard rate

**CDO (Collateralized Debt Obligation):**

- Pool of credit risks, tranched by seniority
- Equity $\to$ Mezzanine $\to$ Senior
- Pricing: copula models (Gaussian, Clayton), large pool approximation

**Total Return Swap:** Swap total return (coupon + capital gain) for floating + spread

Transfer credit risk, isolate default probability

**Swaption:** Option to enter swap at future date

$$V_{\text{payer}} = P(0, T_0)\left[S(0, T_0, T_n)\Phi(d_1) - K\Phi(d_2)\right] \cdot A$$

$A$ = annuity, Black's model with forward swap rate

**Cap/Floor:**

$$\text{Caplet: } \tau P(0, T_i)\max(L(T_{i-1}, T_i) - K, 0)$$

$$\text{Cap} = \sum \text{Caplets}, \quad \text{Floor} = \sum \text{Floorlets}$$

Black76 formula with forward LIBOR/SOFR

**Put-Call Parity:**

$$\text{Cap} - \text{Floor} = \text{Swap}$$

**Closed Form (when available):**

- Black-Scholes: vanilla Europeans
- Black76: futures options, caps/floors
- Bachelier: normal model for low/negative rates
- Heston: semi-closed (characteristic function + FFT)

**Fourier Methods:**

$$V(S_0) = e^{-rT}\mathbb{E}[\text{Payoff}] = e^{-rT}\int \text{Payoff}(S_T)p(S_T)dS_T$$

Use characteristic function $\varphi(\omega) = \mathbb{E}[e^{i\omega \ln S_T}]$
FFT for European options, Carr-Madan formula:

$$C(k) = \frac{e^{-\alpha k}}{\pi}\int_0^\infty e^{-i\omega k}\frac{\varphi(\omega - (\alpha+1)i)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha+1)\omega}d\omega$$

**Binomial Tree:**

$$S_{i,j} = S_0 u^j d^{i-j}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u$$

Risk-neutral probability: $p = \frac{e^{r\Delta t} - d}{u - d}$
Backward induction: $V_{i,j} = e^{-r\Delta t}[pV_{i+1,j+1} + (1-p)V_{i+1,j}]$

**Trinomial Tree:** Three branches (up, middle, down), more stable for barriers

**Advantages:**

- American options, early exercise
- Dividends, time-varying parameters
- Intuitive, easy to code

**Convergence:** $O(1/\sqrt{N})$ steps

**Black-Scholes PDE:**

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

**Finite Difference Methods:**

- **Explicit**: $V_i^{n+1} = f(V_{i-1}^n, V_i^n, V_{i+1}^n)$, conditionally stable
- **Implicit**: $f(V_{i-1}^{n+1}, V_i^{n+1}, V_{i+1}^{n+1}) = V_i^n$, unconditionally stable
- **Crank-Nicolson**: Average of explicit/implicit, $O(\Delta t^2, \Delta S^2)$

**Applications:** American options (free boundary), barriers, local vol, exotics

Greeks from finite differences: $\Delta \approx (V_{i+1} - V_{i-1})/(2\Delta S)$

**Algorithm:**

- Discretize: $S_{t+\Delta t} = S_t \exp\left[(r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z\right]$
- Simulate $N$ paths under $\mathbb{Q}$
- Compute payoffs: $V_i = g(S_T^{(i)})$ or $g(\text{path}_i)$
- Estimate: $V_0 = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} V_i$

**Variance Reduction:**

- Antithetic variates: $(Z, -Z)$
- Control variates: use correlated known price
- Importance sampling: shift distribution to relevant region
- Quasi-MC: low-discrepancy sequences (Sobol, Halton)

**LSM for American:** Longstaff-Schwartz, regression on basis functions

**Implied Vol $\sigma_{\text{imp}}$:** Market price inverted from BS: $C_{\text{market}} = BS(S, K, T, r, \sigma_{\text{imp}})$

**Volatility Smile/Skew:**

- $\sigma_{\text{imp}}(K, T)$ varies with strike and maturity
- Equity: skew (OTM puts expensive)
- FX: smile (OTM calls and puts expensive)
- Rates: humped smile

**Volatility Surface:** 2D function $\sigma(K, T)$, interpolate/extrapolate, arbitrage-free constraints:

- Calendar arbitrage: $\partial C / \partial T \geq 0$
- Butterfly arbitrage: $\partial^2 C / \partial K^2 \geq 0$

Local vol $\sigma_L(S, t)$ from surface via Dupire formula

**Local Volatility (Dupire):**

$$\sigma_L^2(K, T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

- Perfectly fits volatility surface
- $dS_t = rS_t dt + \sigma_L(S_t, t)S_t dW_t$
- Forward skew too flat (can't explain market dynamics)

**Stochastic Volatility (Heston):**

$$dS_t = rS_t dt + \sqrt{v_t}S_t dW_t^1$$

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^2, \quad dW^1 dW^2 = \rho dt$$

Parameters: $\kappa$ (mean reversion), $\theta$ (long-run var), $\xi$ (vol-of-vol), $\rho$ (leverage)

# Jump-Diffusion & Lévy Models

**Merton Jump-Diffusion:**

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_t dJ_t$$

$J_t$ = compound Poisson, jumps $\sim \mathcal{N}(\mu_J, \sigma_J^2)$, intensity $\lambda$
Jump component (Mean Jump Size): $\mathbb{E}[e^{\text{Jump Size}} - 1] = e^{\mu_J + \sigma_J^2/2} - 1$

**Lévy Processes:**

- **Variance Gamma**: infinite activity, pure jump
- **CGMY**: $Y < 2$ controls tail behavior
- **NIG** (Normal Inverse Gaussian): semi-heavy tails

**Lévy-Khintchine:**

$$\mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}, \quad \psi(u) = iub - \frac{\sigma^2 u^2}{2} + \int (e^{iux} - 1 - iux\mathbb{1}_{|x|<1})\nu(dx)$$

Captures jumps, fat tails, skewness; FFT pricing

## Short Rate Models:

**Vasicek:** $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$

- Mean-reverting, Gaussian (can go negative)
- Closed form for bonds: $P(t, T) = A(t, T)e^{-B(t,T)r_t}$

**CIR:** $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$

- Non-central chi-squared, stays positive if $2\kappa\theta > \sigma^2$

**Hull-White:** $dr_t = (\theta(t) - \kappa r_t)dt + \sigma dW_t$

- Time-dependent $\theta(t)$ fits initial yield curve
- Trinomial tree implementation

Price bonds, swaptions, caps via PDE or tree

**Heath-Jarrow-Morton (HJM):** Model entire forward curve $f(t, T)$:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t$$

No-arbitrage drift condition: $\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s)ds$

**LIBOR Market Model (LMM/BGM):** Model discretely compounded forward rates $L_i(t)$:

$$dL_i(t) = \mu_i(t)L_i(t)dt + \sigma_i(t)L_i(t)dW_i(t)$$

Under forward measure $\mathbb{Q}^{T_i}$:

$$dL_i(t) = \sigma_i(t)L_i(t)dW_i^{T_i}(t)$$

- Prices caps/floors using Black76
- MC simulation for exotic IR derivatives
- Calibrate to cap/swaption volatilities

# SABR Model

**Stochastic Alpha Beta Rho:**

$$dF_t = \sigma_t F_t^{\beta} dW_t^1$$

$$d\sigma_t = \alpha \sigma_t dW_t^2, \quad dW^1 dW^2 = \rho dt$$

**Implied Vol Approximation:**

$$\sigma_B(K,F) \approx \frac{\alpha}{(FK)^{(1-\beta)/2}[1+\ldots]} \left\{ \frac{z}{x(z)} \right\} \left[ 1 + \left( \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(FK)^{1-\beta}} + \frac{\rho\beta\alpha}{4(FK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} \right) T \right]$$

where $z = \frac{\alpha}{\sigma(FK)^{(1-\beta)/2}} \ln(F/K)$

**Applications:**

- FX options: $\beta = 0.5$ or 1
- Interest rates: swaptions, caps
- Fits market smile, explicit formulas

**Delta Hedging:**

$$\Pi = V - \Delta S \Rightarrow d\Pi = \Theta dt + \frac{1}{2}\Gamma(dS)^2$$

Rebalance to maintain $\Delta$-neutral

**Greeks Hedging:**

- **Vega**: hedge with other options
- **Gamma**: scalp, manage convexity
- **Rho**: interest rate swaps

**VaR & CVaR:**

$$\mathrm{VaR}_\alpha = -\inf\{x : P(\Delta\Pi < x) \leq 1 - \alpha\}$$

$$\mathrm{CVaR}_\alpha = \mathbb{E}[\Delta\Pi | \Delta\Pi < -\mathrm{VaR}_\alpha]$$

**XVA:** CVA (credit), DVA (debit), FVA (funding), KVA (capital), MVA (margin)

# Model Calibration

**Objective:** Fit model parameters $\theta$ to market prices $P_i^{\text{mkt}}$:

$$\theta^* = \arg\min_{\theta} \sum_{i=1}^{n} w_i \left( P_i^{\text{model}}(\theta) - P_i^{\text{mkt}} \right)^2$$

**Instruments:**

- **Equity**: vanilla options across strikes/maturities
- **FX**: 25-delta RR, 25-delta BF, ATM vol
- **Rates**: caps, swaptions (diagonal, co-terminal)

**Methods:**

- Levenberg-Marquardt, global optimization
- Regularization for stability
- Out-of-sample validation

Liquid instruments, relevant to portfolio, recalibrate regularly

# Numerical Considerations

### Performance:

- **Closed form**: microseconds, instant Greeks
- **Trees/PDE**: milliseconds to seconds
- **Monte Carlo**: seconds to minutes, parallel on GPU
- **FFT**: sub-second for vanilla, fast recalc

### Accuracy vs Speed:

- MC: $O(1/\sqrt{N})$ convergence, variance reduction critical
- Trees: oscillation, Richardson extrapolation
- PDE: boundary conditions, stability (CFL condition)

**Adjoint Differentiation:** Compute all Greeks in one pass: $\nabla_\theta V$ for $n$ parameters in $O(1)$ cost

Trade-off: accuracy, speed, flexibility for exotic payoffs

**Basel III/IV:**

- CVA capital charge: $K_{\text{CVA}} = 2.33\sqrt{\text{UL}_{\text{CVA}}}$
- SA-CCR: standardized approach for counterparty credit risk
- FRTB: fundamental review of trading book

**IFRS 9 / FASB:**

- Fair value: Level 1 (quoted), Level 2 (observable), Level 3 (model)
- Hedge accounting: cash flow, fair value, net investment hedges
- ECL: expected credit loss provisioning

**Dodd-Frank / EMIR:** Central clearing, margin requirements, trade reporting

Independent valuation, model validation, PnL attribution

# Market Microstructure & Trading

**Market Making:**

- Quote bid-ask, earn spread, manage inventory
- Delta-hedge, vol arbitrage
- Order flow toxicity, adverse selection

**Liquidity:**

- Bid-ask spread, market depth
- Illiquidity premium in exotic pricing
- Transaction costs, slippage

**Volatility Trading:**

- Long gamma: profit from realized vol $>$ implied
- Short gamma: collect theta, risk of jumps
- Dispersion: trade index vs single-stock vol

Greeks management, dynamic hedging, P&L sources

**Machine Learning:**

- Neural networks for pricing (fast approximation)
- Reinforcement learning for optimal hedging
- GANs for scenario generation

**Rough Volatility:**

$$v_t = v_0 + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \xi \sqrt{v_s} dW_s, \quad H < 1/2$$

Fractional Brownian motion, matches realized vol scaling

**Quantum Computing:** Amplitude estimation for MC, exponential speedup (theoretical)

**Climate Derivatives:** Weather options, temperature futures, carbon credits

# Summary: Key Takeaways

**Instruments:**

- Linear: forwards, futures, swaps
- Vanilla: calls, puts, Europeans, Americans
- Exotics: barriers, Asians, lookbacks, digitals
- Structured: CDOs, variance swaps, quantos

**Greeks:**

- $\Delta, \Gamma$: directional, convexity
- $\mathcal{V}$: vol sensitivity
- $\Theta$: time decay
- $\rho$: rates

**Models:**

- Black-Scholes: vanilla baseline
- Local vol: fit smile
- Stochastic vol: Heston, SABR
- Jump-diffusion: fat tails
- IR models: HW, LMM

**Pricing:**

- Closed form when available
- Trees for American
- PDE for exotics
- MC for path-dependent
- FFT for efficiency

**Risk management + model calibration + numerical methods = derivatives toolkit**

# Essential References


**Core Texts:**

- Hull, J. *Options, Futures, and Other Derivatives* (10e)
- Shreve, S. *Stochastic Calculus for Finance I & II*
- Joshi, M. *The Concepts and Practice of Mathematical Finance*
- Brigo, D. & Mercurio, F. *Interest Rate Models: Theory and Practice*

**Advanced:**

- Gatheral, J. *The Volatility Surface*
- Andersen, L. & Piterbarg, V. *Interest Rate Modeling* (3 vols)
- Cont, R. & Tankov, P. *Financial Modelling with Jump Processes*

**Numerical:**

- Glasserman, P. *Monte Carlo Methods in Financial Engineering*
- Duffy, D. *Finite Difference Methods in Financial Engineering*

# THANK YOU

## Questions?

Master derivatives pricing, hedging & risk management

From theory to practice