STOCHASTIC MECHANICS in QUANT FINANCE

Random Processes & Mathematical Foundations

Modeling Uncertainty in Markets

Why Randomness?

Markets are Inherently Stochastic

Uncertainty drives prices, risk, and opportunity

Randomness

- Price shocks
- News arrivals
- Order flow

Continuous Time

- Asset evolution
- Risk dynamics
- Hedging

Modeling

- Derivatives pricing
- Risk management
- Portfolio theory

Wiener Process W_t

Continuous random walk with independent increments

Properties:

- $W_0 = 0$
- $W_t - W_s \sim \mathcal{N}(0, t - s)$ for $t > s$
- Independent increments
- Continuous paths, nowhere differentiable

Key Facts:

$$\mathbb{E}[W_t] = 0, \quad \mathrm{Var}(W_t) = t, \quad \mathbb{E}[W_t W_s] = \min(s, t)$$

Quadratic variation: $[W, W]_t = t$ (volatility accumulation)

SDEs: Modeling Dynamic Systems

Differential equations with random noise

General Form:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

Components:

- $\mu(X_t, t)$: Drift (deterministic trend)
- $\sigma(X_t, t)$: Diffusion (random volatility)
- dW_t: Brownian increment

Integral form: $X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s$

The Fundamental Theorem

How functions of stochastic processes evolve

For $f(X_t, t)$ where $dX_t = \mu dt + \sigma dW_t$:

$$\boxed{df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma\frac{\partial f}{\partial x} dW_t}$$

Key Insight:

- Second-order term $\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}$ arises from $(dW_t)^2 = dt$
- Essential for derivatives pricing and hedging

The Standard Stock Price Model

Log-normal asset price dynamics

SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Solution (via Itô):

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]$$

Properties:

- $S_t > 0$ always (log-normal)
- $\mathbb{E}[S_t] = S_0 e^{\mu t}$
- Returns $\log(S_t/S_0) \sim \mathcal{N}\left((\mu - \sigma^2/2)t, \sigma^2 t\right)$

No Arbitrage Condition

Expected future value equals current value

Definition:

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \quad \text{for } t \geq s$$

Examples:

- Brownian motion W_t is a martingale
- Under risk-neutral measure: $e^{-rt}S_t$ is a martingale
- Discounted portfolio value in arbitrage-free markets

Martingale property $\Leftrightarrow$ no arbitrage opportunities

Risk-Neutral Valuation

The Fundamental Pricing Formula

Derivative prices as discounted expectations

Risk-Neutral Measure $\mathbb{Q}$:

Under $\mathbb{Q}$, all assets grow at risk-free rate r:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$

Pricing Formula:

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\text{Payoff}(S_T)]$$

Key Principle:

- Drift becomes r (not physical μ)
- Volatility σ unchanged
- No investor risk preferences needed

The Nobel Prize Formula

PDE governing European option prices

Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

Boundary Conditions:

- Call: $V(S,T) = \max(S-K,0)$
- Put: $V(S,T) = \max(K-S,0)$

Derived via no-arbitrage replication and Itô's lemma

Black-Scholes Closed Form

Analytical Solution

Exact pricing for European calls and puts

Call Option:

$$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Put Option:

$$P = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

$\Phi(\cdot)$: standard normal CDF

Risk Measures

Derivatives of option price with respect to parameters

First Order:

- **Delta** $\Delta = \frac{\partial V}{\partial S}$
  Price sensitivity to stock
- **Vega** $\mathcal{V} = \frac{\partial V}{\partial \sigma}$
  Volatility sensitivity
- **Theta** $\Theta = \frac{\partial V}{\partial t}$
  Time decay
- **Rho** $\rho = \frac{\partial V}{\partial r}$
  Interest rate sensitivity

Second Order:

- **Gamma** $\Gamma = \frac{\partial^2 V}{\partial S^2}$
  Convexity, hedging error

PDE Relation:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

Delta hedging: dynamic replication strategy

Volatility is Random

σ itself follows a stochastic process

Heston Model:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^2$$

Correlation: $dW_t^1 dW_t^2 = \rho dt$

Features:

- Mean reversion in variance
- Skew and smile in implied volatility
- Leverage effect ($\rho < 0$)

Discontinuous Price Movements

Rare events and tail risk

Merton Jump-Diffusion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_t dJ_t$$

where J_t is a compound Poisson process

Components:

- Diffusion: σdW_t (normal fluctuations)
- Jumps: dJ_t (sudden shocks)
- Jump size: log-normal $\sim \mathcal{N}(\mu_J, \sigma_J^2)$
- Jump intensity: λ (arrivals per unit time)

General Jump Framework

Stationary independent increments, càdlàg paths

Lévy-Khintchine Formula:

$$\mathbb{E}[e^{iuX_t}] = \exp\left[t\left(i\gamma u - \frac{\sigma^2 u^2}{2} + \int (e^{iux} - 1 - iux\mathbb{1}_{|x|<1})\nu(dx)\right)\right]$$

Popular Models:

- Variance Gamma (VG)
- Normal Inverse Gaussian (NIG)
- CGMY process

Captures skewness, excess kurtosis, and heavy tails

PDE $\leftrightarrow$ Expectation

Bridge between differential equations and probability

Theorem:
If $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ and $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0$$

with terminal condition $u(x,T) = g(x)$, then

$$u(x,t) = \mathbb{E}^x\left[e^{-r(T-t)} g(X_T)\right]$$

Enables Monte Carlo pricing from PDEs

Numerical Pricing

Simulate paths, average payoffs

Algorithm:

- Discretize SDE: $S_{t+\Delta t} = S_t + \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} Z$, $Z \sim \mathcal{N}(0,1)$
- Generate N paths under risk-neutral measure
- Compute payoff for each path: $V_i = g(S_T^{(i)})$
- Average and discount: $V_0 = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} V_i$

Advantages:

- Path-dependent options (Asian, lookback, barrier)
- High-dimensional problems
- Flexible models

Convergence: $O(1/\sqrt{N})$, variance reduction techniques

Change of Probability Measure

Transform drift without changing volatility

Theorem:

Under measure $\mathbb{Q}$ defined by Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \theta_s^2 ds - \int_0^T \theta_s dW_s\right)$$

the process $\tilde{W}_t = W_t + \int_0^t \theta_s ds$ is a $\mathbb{Q}$-Brownian motion

Application: Transform physical measure $\mathbb{P}$ to risk-neutral $\mathbb{Q}$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \Rightarrow \quad dS_t = r S_t dt + \sigma S_t d\tilde{W}_t$$

Term Structure Dynamics

Modeling evolution of the yield curve

Short Rate Models:

- **Vasicek**: $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$
- **CIR**: $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$
- **Hull-White**: Time-dependent mean reversion

Forward Rate Models:

- **HJM Framework**: Model entire forward curve
- No-arbitrage drift condition

Zero-coupon bond: $P(t, T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]$

Optimal Filtering

Estimate hidden states from noisy observations

Linear State-Space Model:

$$X_{t+1} = AX_t + w_t, \quad w_t \sim \mathcal{N}(0, Q)$$
$$Y_t = HX_t + v_t, \quad v_t \sim \mathcal{N}(0, R)$$

Kalman Filter:

- Prediction: $\hat{X}_{t|t-1} = A\hat{X}_{t-1|t-1}$
- Update: $\hat{X}_{t|t} = \hat{X}_{t|t-1} + K_t(Y_t - H\hat{X}_{t|t-1})$
- Optimal for linear-Gaussian systems

Stochastic Calculus Powers Finance

Core Principles:

- Brownian motion is fundamental
- Itô's lemma enables derivatives
- Martingales imply no arbitrage
- Risk-neutral pricing works
- Models balance reality vs tractability
- Calibration requires data & math

**"Uncertainty is not risk to be avoided,"
but opportunity to be modeled"**