TEN NON-NEGOTIABLE QUANTITATIVE FINANCE CONCEPTS

The Mathematical Foundations of Modern Finance

1. Feynman-Kac Theorem

Concept

A foundational theorem linking Partial Differential Equations (PDEs) to expected values of stochastic processes.

Significance

- Provides the formal link between the Black-Scholes PDE and Risk-Neutral Valuation.
- Allows option pricing PDEs to be solved using Monte Carlo simulation.

Mathematical Link The value V of a contingent claim at time t is:

$$V(t,x) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} f(X_T) \mid X_t = x\right]$$

Where $\mathbb{Q}$ is the risk-neutral measure and $f(X_T)$ is the payoff.

2. Girsanov's Theorem

Concept
A theorem that specifies how the drift of a stochastic process changes when the underlying probability measure is changed.

Significance

- It mathematically justifies the shift from the real-world measure ($\mathbb{P}$) to the risk-neutral measure ($\mathbb{Q}$).
- It confirms that the random term (volatility) remains unchanged during this measure switch.

Measure Change The differential of the Brownian motion changes from $\mathbb{P}$ to $\mathbb{Q}$ by introducing the market price of risk λ_t:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda_t dt$$

3. Local Volatility Model (Dupire)

Concept

A class of models that assumes volatility is a deterministic function of both the asset price (S) and time (t).

Significance

- It can perfectly match the entire observed market volatility surface (skew and smile).
- Used for pricing complex derivatives that rely on the market's implied volatility structure.

The Dupire Equation The local variance $\sigma^2_{Loc}(K, T)$ is determined by the cross-derivatives of the market option price $C(K, T)$:

$$\sigma^2_{Loc}(K, T) = 2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

4. ARCH/GARCH Models

Concept

Time series models used to model volatility clustering, where high volatility periods are typically followed by high volatility.

Significance

- Provides a more accurate forecast of future variance than simple historical methods.
- Essential for modern risk management and volatility trading strategies.

GARCH(1,1) Variance Equation The current conditional variance σ_t^2 depends on a long-run variance ω, past squared error ϵ_{t-1}^2, and past variance σ_{t-1}^2:

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

5. Cointegration

Concept

A statistical property where two or more non-stationary time series have a stationary (mean-reverting) linear combination.

Significance

- Identifies stable, long-term relationships between asset prices or interest rates.
- It is the key statistical requirement for statistical arbitrage and pairs trading.

Cointegration Relationship If X_t and Y_t are non-stationary, but their spread Z_t is stationary (mean-reverting):

$$Z_t = Y_t - \beta X_t$$

The parameter β is the cointegrating vector.

6. Expected Shortfall (ES)

Concept

A coherent risk measure that quantifies the expected loss in the worst α percent of outcomes. It is also known as Conditional Value at Risk (CVaR).

Significance

- Addresses the weaknesses of VaR by explicitly measuring tail risk (extreme losses).
- Used in regulatory frameworks like Basel III (as ES).

Definition The Expected Shortfall at confidence level α is the expected loss L given that the loss exceeds the VaR threshold:

$$ES_\alpha = E[L \mid L \geq VaR_\alpha(L)]$$

7. Merton Model of Corporate Debt

Concept

A structural model that treats a firm's equity as a call option on the firm's total assets, where the strike price is the face value of the firm's debt.

Significance

- Provides a theoretical link between a firm's market capitalization and its default risk.
- Used to derive the Distance-to-Default (DtD) metric for credit analysis.

Equity Value (as a Call Option) The equity value E based on the asset value V_A and debt face value F:

$$E = V_A N(d_1) - Fe^{-rT} N(d_2)$$

Where N is the cumulative standard normal distribution.

8. Monte Carlo Simulation

Concept

A numerical method that uses repeated random sampling to approximate the expectation of a complex payoff function.

Significance

- The primary method for pricing path-dependent options (like Asian or Barrier options).
- Necessary when analytical or lattice methods are infeasible due to complexity or high dimensionality.

Expected Value Approximation The expected value $E[f(X)]$ is approximated by the average of N simulated outcomes $X^{(i)}$:

$$E[f(X)] \approx \frac{1}{N} \sum_{i=1}^{N} f(X^{(i)})$$

9. Finite Difference Methods (FDM)

Concept
A numerical technique used to solve Partial Differential Equations (PDEs) by approximating derivatives with discrete differences over a grid.

Significance

- Used extensively for solving option pricing PDEs like Black-Scholes.
- Particularly suited for American options to find the early exercise boundary.

Approximation of Time Derivative A forward difference approximation for the time derivative:

$$\frac{\partial V}{\partial t} \approx \frac{V_{i,j+1} - V_{i,j}}{\Delta t}$$

Where $V_{i,j}$ is the option value at price $i\Delta S$ and time $j\Delta t$.

10. Principal Component Analysis (PCA)

Concept

A statistical procedure that uses an orthogonal transformation to convert a set of possibly correlated variables into a set of linearly uncorrelated variables.

Significance

- Essential for dimensionality reduction in risk management and portfolio construction.
- Commonly used to model the term structure of interest rates with 2 to 3 components (level, slope, curvature).

Covariance Matrix Decomposition PCA relies on the eigendecomposition of the covariance matrix Σ:

$$\Sigma = P \Lambda P^T$$

Where P contains the principal components (eigenvectors) and Λ is the diagonal matrix of eigenvalues (variances).