CALCULUS ESSENTIALS

for Quantitative Finance

Core Mathematical Foundations

Derivatives · Integrals · Differential Equations

Derivatives: Core Concepts

Definition

The derivative measures the instantaneous rate of change of a function.

Basic Definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Essential Rules:

Power Rule: $\frac{d}{dx} x^n = n x^{n-1}$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Exponential: $\frac{d}{dx} e^x = e^x$

Logarithm: $\frac{d}{dx} \ln(x) = \frac{1}{x}$

Derivatives: Applications in Finance

Option Greeks:

Delta: $\Delta = \frac{\partial V}{\partial S}$ measures price sensitivity to underlying asset

Gamma: $\Gamma = \frac{\partial^2 V}{\partial S^2}$ measures rate of change of delta

Vega: $\mathcal{V} = \frac{\partial V}{\partial \sigma}$ measures sensitivity to volatility

Theta: $\Theta = \frac{\partial V}{\partial t}$ measures time decay

Rho: $\rho = \frac{\partial V}{\partial r}$ measures interest rate sensitivity

Portfolio Optimization:

Finding maximum Sharpe ratio requires taking derivatives of:

$$SR = \frac{\mu_p - r_f}{\sigma_p}$$

Set gradient to zero to find optimal portfolio weights.

Integrals: Core Concepts

Definition

Integration computes the accumulated change or area under a curve.

Definite Integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Fundamental Theorem of Calculus:

If $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Integrals: Applications in Finance

Present Value Calculations:

Continuous cash flows with rate $c(t)$ and discount rate r:

$$PV = \int_0^T c(t)e^{-rt}\,dt$$

Expected Values:

For continuous random variable with density $f(x)$:

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$

Variance Calculation:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Ordinary Differential Equations: Fundamentals

Definition

ODEs relate a function to its derivatives, describing dynamic systems.

First Order Linear ODE:

$$\frac{dy}{dt} + p(t)y = q(t)$$

Solution uses integrating factor $\mu(t) = e^{\int p(t)\,dt}$

Second Order Linear ODE:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

Characteristic equation: $ar^2 + br + c = 0$ determines solution structure.

Separable ODEs:

$$\frac{dy}{dt} = g(t)h(y) \quad \Rightarrow \quad \int \frac{1}{h(y)}\,dy = \int g(t)\,dt$$

Mean Reversion Models:

Vasicek interest rate model:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

The term $a(b - r_t)$ represents mean reversion to level b at speed a.

Bond Pricing Equation:

For zero-coupon bond price $P(t, T)$:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 P}{\partial r^2} + \mu r \frac{\partial P}{\partial r} - rP = 0$$

Portfolio Dynamics:

Wealth process under continuous trading:

$$\frac{dW_t}{dt} = rW_t + \pi_t(\mu - r)$$

where π_t is the amount invested in risky assets.

Partial Differential Equations: Fundamentals

Definition

PDEs involve functions of multiple variables and their partial derivatives.

Black-Scholes PDE:

For option value $V(S, t)$ with underlying price S and time t:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Heat Equation:

General form:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Black-Scholes can be transformed into heat equation via change of variables.

Partial Differential Equations: Solution Methods

Boundary Conditions:

European call option at expiry:

$$V(S,T) = \max(S - K, 0)$$

At boundaries:

$$V(0,t) = 0, \quad V(S,t) \to S - Ke^{-r(T-t)} \text{ as } S \to \infty$$

Numerical Methods:

Finite difference discretization:

$$\frac{V_{i,j+1} - V_{i,j}}{\Delta t} = \alpha \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta S)^2}$$

Monte Carlo simulation: Generate random paths, calculate payoffs, discount back.

Fourier methods: Transform PDE to algebraic equation, solve, inverse transform.

Stochastic Differential Equations: Core Theory

Definition

SDEs model systems with both deterministic drift and random fluctuations.

General Form:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

where W_t is a Brownian motion (Wiener process).

Geometric Brownian Motion:

Stock price model:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

Solution:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]$$

Stochastic Differential Equations: Ito Calculus

Ito's Lemma:

For function $f(X_t, t)$ where $dX_t = \mu\, dt + \sigma\, dW_t$:

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x}\, dW_t$$

Key Application:

Deriving Black-Scholes equation from GBM dynamics.

For portfolio $\Pi = V - \Delta S$:

$$d\Pi = dV - \Delta\, dS$$

Applying Ito's lemma to $V(S, t)$ and requiring $d\Pi = r\Pi\, dt$ yields Black-Scholes PDE.

This risk-neutral framework underpins modern derivatives pricing.

Key Connections

Integration of Concepts

Modern quantitative finance relies on the interplay between these calculus tools.

Derivatives:
Sensitivity analysis for risk management and hedging strategies.

Integrals:
Pricing continuous cash flows and computing expected values under probability distributions.

ODEs:
Modeling deterministic dynamics in interest rates, portfolio wealth, and term structure.

PDEs:
Option pricing under diffusion processes, solved analytically or numerically.

SDEs:
Capturing market randomness, enabling Monte Carlo methods and derivative valuation.