VaR vs CVaR
Risk Metrics Compared

Value at Risk vs. Expected Shortfall

Quantitative Risk Comparison

A Comprehensive Technical Analysis

Overview

Foundations

- Risk metric definitions
- Mathematical formulations
- Core properties
- Probability frameworks

Comparative Analysis

- Key differences
- Strengths and weaknesses
- Coherence properties
- Tail risk treatment

Applications

- Regulatory compliance
- Portfolio optimization
- Computational methods
- Practical implementation

Quantifying Downside Risk in Financial Markets

Value at Risk (VaR)

Definition:

Maximum expected loss over a specific time horizon at a given confidence level.

Mathematical Formulation:

For confidence level $\alpha \in (0, 1)$ and loss distribution L:

$$\text{VaR}_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\}$$

Equivalently: $\text{VaR}_\alpha = F_L^{-1}(\alpha)$ where F_L is the cumulative distribution function.

Interpretation:

$\text{VaR}_{95\%} = \$10\text{M}$ means: 95% confidence that losses will not exceed \$10M over the specified period.

Quantile-based risk measure: VaR identifies the threshold loss value

VaR Properties and Limitations

Key Properties:

- Widely adopted in industry and regulation (Basel Accords)
- Intuitive interpretation as a percentile
- Computationally efficient for many distributions
- Single number summary of risk

Critical Limitations:

- **Not coherent**: fails subadditivity property
- **Ignores tail risk**: provides no information about losses beyond VaR threshold
- **Non-convex**: problematic for optimization
- **Discourages diversification**: $\text{VaR}(X+Y)$ can exceed $\text{VaR}(X)+\text{VaR}(Y)$

VaR only measures frequency, not severity of tail events

Expected Shortfall (CVaR)

Definition:
Expected loss given that the loss exceeds VaR. Also called Conditional VaR (CVaR) or Average VaR.

Mathematical Formulation:

$$\mathrm{ES}_\alpha = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\beta \, d\beta$$

For continuous distributions:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(p)\, dp$$

Interpretation:
$\mathrm{ES}_{95\%} = \$15\mathrm{M}$ means: given losses exceed the 95th percentile, the average loss is \$15M.

Tail-conditional expectation: ES quantifies average severity of tail losses

CVaR Properties and Advantages

Key Properties:

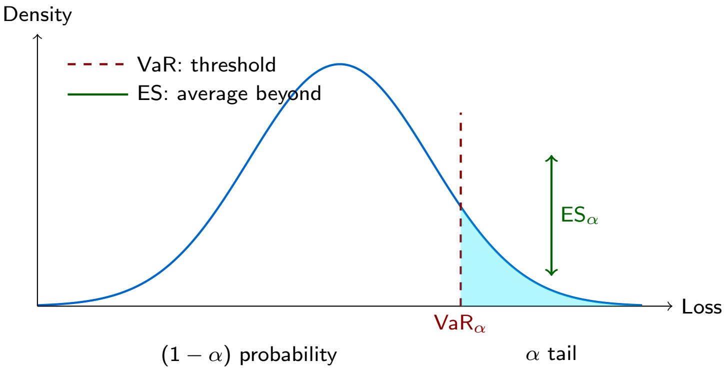
- **Coherent risk measure**: satisfies all four coherence axioms
- **Subadditive**: $\text{ES}(X + Y) \leq \text{ES}(X) + \text{ES}(Y)$
- **Convex**: suitable for optimization problems
- **Tail-sensitive**: accounts for severity of extreme losses
- **Monotonic**: if $X \leq Y$, then $\text{ES}(X) \leq \text{ES}(Y)$

Coherence Axioms:

- Translation invariance: $\rho(X + c) = \rho(X) + c$
- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$
- Monotonicity: $X \leq Y \implies \rho(X) \leq \rho(Y)$
- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

Note: VaR fails subadditivity, violating coherence.

Visual Comparison: Distribution Perspective



VaR marks the boundary; ES measures the average in the shaded tail region

Fundamental Inequality:

$$\mathrm{ES}_\alpha \geq \mathrm{VaR}_\alpha \quad \forall \alpha \in (0,1)$$

Equality holds only for deterministic losses beyond VaR.

Spectral Representation:

VaR as special case of weighted average:

$$\mathrm{VaR}_\alpha = \int_\alpha^1 F_L^{-1}(p)\delta(p-\alpha)\, dp$$

ES as uniform weighted average:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(p)\, dp$$

Representation as Optimization:

VaR: $\mathrm{VaR}_\alpha = \min\{l : P(L \leq l) \geq \alpha\}$

ES: $\mathrm{ES}_\alpha = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-\alpha}\mathbb{E}[(L-z)^+] \right\}$

The ES formulation is convex in z, enabling efficient optimization.

Direct Comparison Matrix

Property	VaR	CVaR/ES
Coherence	**No** (fails subadditivity)	**Yes** (all axioms)
Tail sensitivity	**Low** (ignores tail)	**High** (averages tail)
Convexity	**No**	**Yes**
Computation	Fast (quantile)	Moderate (integration)
Interpretation	Simple threshold	Conditional average
Optimization	Non-convex	Convex
Diversification	May penalize	Always rewards
Regulatory use	Basel II/III	Basel IV (planned)
Back-testing	Straightforward	Complex

ES addresses VaR's theoretical deficiencies at the cost of complexity

Numerical Example

Scenario: Portfolio returns follow normal distribution $\mathcal{N}(0, \sigma^2)$

Parameters:

- Initial portfolio value: \$100M
- Daily volatility: $\sigma = 2\%$
- Confidence level: $\alpha = 95\%$

VaR Calculation:

$$\text{VaR}_{95\%} = -\mu + \sigma\Phi^{-1}(0.95) = 0 + 0.02 \times 1.645 = 3.29\%$$
$$\text{VaR}_{95\%} = \$100M \times 0.0329 = \$3.29M$$

ES Calculation:
For normal distribution: $\text{ES}_\alpha = \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$

$$\text{ES}_{95\%} = 0.02 \times \frac{\phi(1.645)}{0.05} = 0.02 \times 2.06 = 4.12\%$$
$$\text{ES}_{95\%} = \$100M \times 0.0412 = \$4.12M$$

ES is 25% higher than VaR, capturing tail severity.

Fat-Tailed Distributions:

For Student's t-distribution with ν degrees of freedom:

- As $\nu \to \infty$: approaches normal (VaR $\approx$ ES)
- Small ν: fat tails (ES $\gg$ VaR)
- Ratio $\frac{\text{ES}}{\text{VaR}}$ increases with tail thickness

Comparison Table ($\alpha = 95\%$):

Distribution	VaR	ES	Ratio (ES/VaR)
Normal	1.64σ	2.06σ	1.26
t(5 df)	2.02σ	2.87σ	1.42
t(3 df)	2.35σ	3.71σ	1.58

Heavier tails amplify the difference between VaR and ES

VaR Estimation:

1. **Historical Simulation:**

$$\text{VaR}_\alpha = \text{empirical } \alpha\text{-quantile of historical losses}$$

2. **Parametric (Variance-Covariance):**

$$\text{VaR}_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$$

3. **Monte Carlo Simulation:** Generate N scenarios, sort, take $\lceil \alpha N \rceil$-th largest loss.

ES Estimation:

1. **Historical:**

$$\text{ES}_\alpha = \text{mean of losses exceeding VaR}_\alpha$$

2. **Parametric:** Closed-form for specific distributions (e.g., normal, t).
3. **Monte Carlo:**

$$\text{ES}_\alpha \approx \frac{1}{N(1-\alpha)} \sum_{i: L_i \geq \text{VaR}_\alpha} L_i$$

Complexity: ES requires averaging tail scenarios, VaR only threshold identification.

VaR Optimization:

$$\min_{\mathbf{w}} \mathrm{VaR}_\alpha(\mathbf{w}^T\mathbf{R}) \quad \text{s.t.} \quad \sum_i w_i = 1,\ \mathbf{w} \geq 0$$

Issues:

- Non-convex objective function
- Multiple local minima
- Sensitive to small data changes
- May not reward diversification

CVaR Optimization:

$$\min_{\mathbf{w}} \mathrm{ES}_\alpha(\mathbf{w}^T\mathbf{R}) \quad \text{s.t.} \quad \sum_i w_i = 1,\ \mathbf{w} \geq 0$$

Advantages:

- Convex optimization problem
- Unique global minimum
- Standard solvers applicable (LP, QP)
- Naturally promotes diversification

Rockafellar-Uryasev (2000) reformulation enables efficient CVaR optimization.

Basel II/III (Current):

- Market risk: 10-day VaR at 99% confidence
- Capital requirement: $\max(\text{VaR}_{t-1}, k \times \text{avg VaR}_{\text{60days}})$
- Multiplication factor $k \geq 3$ based on backtesting
- Criticized for ignoring tail risk severity

Basel IV (Proposed):

- Shift from VaR to ES for market risk
- ES at 97.5% confidence level
- Better captures tail risk
- More conservative capital requirements

Other Regulations:

- Solvency II (insurance): Uses $\text{VaR}_{99.5\%}$ for 1-year horizon
- EMIR (derivatives): Both VaR and ES reporting
- FRTB (Fundamental Review): ES-based framework

Backtesting and Validation

VaR Backtesting:

Binomial test: Count exceptions where $L_t > \text{VaR}_\alpha$

Expected exceptions: $N(1-\alpha)$

Basel traffic light:

- Green: ≤ 4 exceptions (250 days, 99%)
- Yellow: 5-9 exceptions
- Red: ≥ 10 exceptions (model inadequate)

ES Backtesting:

More challenging due to conditional nature:

- Compare realized average tail loss to predicted ES
- Tests: Acerbi-Szekely test, McNeil-Frey test
- Requires sufficient tail observations
- Less statistical power than VaR tests

VaR backtesting is simpler; ES validation requires more sophisticated methods

Practical Implementation Considerations

Data Requirements:

Aspect	VaR	ES
Sample size	Moderate	Large (for tail)
Historical depth	1-3 years	3-5 years
Frequency	Daily/Intraday	Daily
Quality	Standard	High (tail accuracy)

Computational Cost:

- VaR: $O(N \log N)$ for sorting historical data
- ES: $O(N)$ additional for tail averaging
- Monte Carlo: ES adds minimal overhead beyond VaR
- Large portfolios: dimension affects both equally

Model Risk:

- VaR: Sensitive to quantile estimation
- ES: More stable, but biased by tail model assumptions
- Both require careful distribution choice for parametric methods

Motivation:

Both VaR and ES focus on tail behavior. EVT provides rigorous tail modeling.

Generalized Pareto Distribution (GPD):

For excesses over threshold u:

$$F_u(y) = 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi}$$

where ξ is shape parameter, β is scale parameter.

EVT-based VaR:

$$\text{VaR}_\alpha = u + \frac{\beta}{\xi}\left[\left(\frac{n}{N_u}(1-\alpha)\right)^{-\xi} - 1\right]$$

EVT-based ES:

$$\text{ES}_\alpha = \frac{\text{VaR}_\alpha}{1-\xi} + \frac{\beta - \xi u}{1-\xi}$$

EVT improves both metrics' performance in fat-tailed distributions.

Portfolio Context:
For portfolio return $R_p = \mathbf{w}^T \mathbf{R}$ where $\mathbf{R}$ is multivariate:

Marginal VaR:

$$\text{MVaR}_i = \frac{\partial \text{VaR}_\alpha(R_p)}{\partial w_i}$$

Measures incremental risk contribution of asset i.

Marginal ES:

$$\text{MES}_i = \frac{\partial \text{ES}_\alpha(R_p)}{\partial w_i} = \mathbb{E}[R_i \mid R_p \geq \text{VaR}_\alpha]$$

Component Decomposition:
CVaR admits Euler allocation:

$$\text{ES}_\alpha(R_p) = \sum_{i=1}^{n} w_i \cdot \text{MES}_i$$

This property enables risk budgeting and attribution. VaR lacks this clean decomposition.

Advantage: ES provides better risk allocation framework.

Dynamic Risk Measures:
Time-varying VaR/ES using:

- GARCH models: $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$
- Exponentially weighted moving average (EWMA)
- Regime-switching models

Spectral Risk Measures:
Generalization with risk aversion function ϕ: $\rho_\phi(L) = \int_0^1 F_L^{-1}(p)\phi(p)\,dp$
VaR: $\phi(p) = \delta(p - \alpha)$ (point mass)
ES: $\phi(p) = \frac{1}{1-\alpha}\mathbb{I}_{p \geq \alpha}$ (uniform weight on tail)

Distortion Risk Measures:
Apply distortion function g to distribution: $\rho_g(L) = \int_0^\infty g(P(L > x))\,dx$
Framework unifies VaR, ES, and other risk measures.

Stress Testing Framework:
Standard VaR/ES capture normal market conditions. Stress testing addresses extreme scenarios.

Approaches:

1. **Historical Stress:**
   - Apply past crisis scenarios (2008, COVID-19)
   - Calculate VaR/ES under historical stress
   - ES better captures cascading effects
2. **Hypothetical Scenarios:**
   - Designer scenarios: rate shocks, credit events
   - Compute losses under specified conditions
   - Both metrics applicable
3. **Reverse Stress Testing:**
   - Identify scenarios causing ES to exceed capital
   - ES provides better threshold for solvency

ES inherently incorporates stress via tail averaging; VaR requires explicit stress overlay

Liquidity Risk Integration

Liquidity-Adjusted VaR (LVaR): $\text{LVaR} = \text{VaR} + \text{Liquidity Cost}$
Liquidity cost from bid-ask spread and price impact: $\text{LC} = \frac{1}{2}\sum_i |w_i| V_i s_i + \sum_i \lambda_i |w_i| V_i$
where s_i is spread, λ_i is price impact coefficient.

Liquidity-Adjusted ES (LES):
$\text{LES}_\alpha = \mathbb{E}[\text{Loss} + \text{Liquidation Cost} \mid \text{Loss} \geq \text{VaR}_\alpha]$

Time Horizon Effects:

Horizon	VaR Scaling	ES Scaling
1-day	Base	Base
10-day	$\sqrt{10}\times$ Base	$\sqrt{10}\times$ Base
With liquidity	Non-linear	Non-linear

Both metrics require liquidity adjustments for realistic risk assessment.

Credit Risk Applications

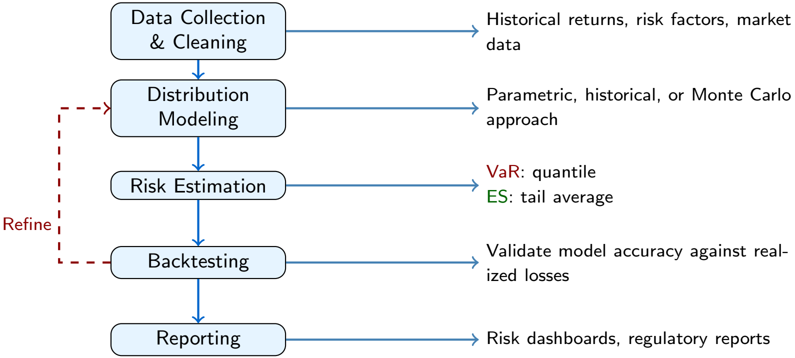
Credit VaR:
Loss distribution from defaults and migrations: $L = \sum_{i=1}^{n} \text{EAD}_i \times \text{LGD}_i \times \mathbb{I}_{\text{default}_i}$
VaR_α: Maximum credit loss at confidence α

Credit CVaR:
Average loss in tail scenarios: $\text{CVaR}_\alpha = \mathbb{E}[L \mid L \geq \text{VaR}_\alpha]$
Critical for:

- Counterparty credit risk
- Credit portfolio management
- Economic capital allocation

Wrong-Way Risk:
Adverse dependence between exposure and counterparty credit quality.
ES better captures correlation in joint tail events than VaR.

Credit concentrations amplify tail losses, making ES more informative.


Data Collection & Cleaning
Historical returns, risk factors, market data
Distribution Modeling
Parametric, historical, or Monte Carlo approach
Risk Estimation
VaR: quantile
ES: tail average
Refine
Backtesting
Validate model accuracy against realized losses
Reporting
Risk dashboards, regulatory reports

Python Libraries:

- `numpy/scipy`: Statistical functions, quantile calculations
- `arch`: GARCH models for dynamic volatility
- `riskfolio-lib`: Portfolio optimization with CVaR
- `pymc`: Bayesian modeling for uncertainty quantification

R Packages:

- `PerformanceAnalytics`: VaR/ES calculation and charting
- `rugarch`: GARCH model estimation
- `copula`: Multivariate dependence modeling
- `QRM`: Quantitative risk management toolkit

Commercial Platforms:

- RiskMetrics, Barra, Axioma: Enterprise risk systems
- MATLAB Financial Toolbox: Comprehensive risk analytics
- Bloomberg PORT, FactSet: Integrated market data + risk

VaR vs CVaR: Decision Framework

Risk Measurement Need

Use VaR

When:

- Regulatory requirement
- Simple communication
- Fast computation needed
- Backtesting priority

Use CVaR/ES

When:

- Portfolio optimization
- Tail risk critical
- Coherence required
- Risk decomposition needed
- Fat-tailed distributions

Recommendation: Use both metrics for comprehensive risk assessment

Machine Learning Integration:

- Neural networks for non-parametric VaR/ES estimation
- Random forests for conditional quantile regression
- LSTM networks for time series volatility forecasting
- Attention mechanisms for factor importance

Elicitability Research:

VaR: Elicitable (optimal forecast identifiable via loss function)
ES: Not elicitable individually, but jointly elicitable with VaR
Impact: Statistical testing and model selection framework

Climate Risk:

- Extending VaR/ES to climate scenarios
- Long-horizon tail risk assessment
- Physical and transition risk quantification
- ES preferred for capturing extreme climate events

Limitations and Criticisms

Both Metrics:

- Backward-looking (rely on historical data)
- Model-dependent (distributional assumptions)
- Ignore extreme tail beyond confidence level
- Cannot capture systemic risk fully
- Sensitive to estimation error with limited data

VaR-Specific:

- Procyclical (encourages selling in downturns)
- Gaming potential (portfolio manipulation)
- Ignores magnitude of extreme losses

ES-Specific:

- Higher estimation error (requires tail data)
- More complex to communicate
- Computationally intensive for large portfolios
- Backtesting challenges

No single metric captures all dimensions of risk.

Beyond VaR and ES:

Maximum Drawdown: $\text{MDD} = \max_{t\in[0,T]} \left(\max_{s\in[0,t]} V_s - V_t\right)$ Captures worst peak-to-trough decline.

Volatility: $\sigma = \sqrt{\mathbb{E}[(R-\mu)^2]}$ Symmetric risk measure, no tail focus.

Semi-Variance: $\text{SV} = \mathbb{E}[(R-\mu)^2 \mathbb{I}_{R<\mu}]$ Downside risk only.

Omega Ratio: $\Omega(\tau) = \frac{\int_\tau^\infty [1-F(r)]dr}{\int_{-\infty}^\tau F(r)dr}$ Ratio of upside to downside potential.

Use multiple metrics for robust risk management framework

Summary

VaR

- Quantile-based threshold
- Simple, widely adopted
- Regulatory standard
- Fast computation
- Not coherent
- Ignores tail severity
- Non-convex

Best For:

- Compliance reporting
- Quick risk snapshots
- Standard scenarios

CVaR/ES

- Tail-conditional average
- Theoretically superior
- Coherent measure
- Tail-sensitive
- Convex for optimization
- More complex
- Higher data requirements

Best For:

- Portfolio optimization
- Risk budgeting
- Fat-tailed distributions

Key Insight: ES $\geq$ VaR always. ES provides fuller picture of tail risk.

Quick Reference: Key Formulas

Value at Risk (VaR): $\text{VaR}_\alpha = F_L^{-1}(\alpha) = \inf\{l : P(L \leq l) \geq \alpha\}$

Expected Shortfall (ES/CVaR): $\text{ES}_\alpha = \mathbb{E}[L \mid L \geq \text{VaR}_\alpha] = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_\beta \, d\beta$

Normal Distribution: $\text{VaR}_\alpha = \mu + \sigma \Phi^{-1}(\alpha), \quad \text{ES}_\alpha = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$

Relationship: $\text{ES}_\alpha \geq \text{VaR}_\alpha, \quad \lim_{\alpha \to 1} \frac{\text{ES}_\alpha}{\text{VaR}_\alpha} \geq 1$

Portfolio Scaling (i.i.d. returns): $\text{VaR}_T = \sqrt{T} \cdot \text{VaR}_1, \quad \text{ES}_T = \sqrt{T} \cdot \text{ES}_1$