Topological reconstruction from stochastic data with the help of Doi-Peliti-formalism

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Abstract

We investigate the relationship between the stochastic generator, the underlying topological space, and averaged observables of a stochastic process in both its discrete setting (Master-equation) and its continuous, field-theoretic description (Doi-Peliti)

1 Stochastic Processes

Definition 1 (Stochastic Process)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probability space, $(X, \Sigma(X))$ be a measurable space (wrt sigma-algebra). A stochastic process is a $\mathbb{R}$/time parametrised collection of continuous random variables $\{x_t(\omega) : T \times \Omega \to X\}$, where T is an index set (preferably $\mathbb{R}^+$). This pushes forward a probability measure $\mathcal{P}$ on X via $\mathcal{P}\left[x(t) \in \mathcal{A}\right] = \mathbb{P}\left[x_t^{-1}(\mathcal{A})\right]$ for all $\mathcal{A} \in \Sigma(X)$.

In statistical field theory, X almost always is a regular lattice, and almost as often as that a square lattice with regular spacing a. For time being, we therefore only consider $X = a\mathbb{Z}^d$ for some d. On this discrete space, the corresponding σ-algebra is a collection of points and therefore $\mathcal{P}$ is entirely described by its values on each lattice site, that is $p(\mathbf{x})$ where $\mathbf{x}$ is some element in X. We are now interested in the evolution of $p(\mathbf{x})$ over time. To avoid technical concerns, we assume a kind behaviour of p wrt whatever T is.

We assume for now that T is $\delta t \cdot \mathbb{Z}$. We demand some sort of map that takes $\cup_{t' \leq t} p_{t'}(\mathbf{x})$ and maps it to $p_{t+\delta t}(x)$, that is an entire knowledge of the past enables us to predict the evolution of the probability distribution. This is a quite generous demand that might be interesting later when we start discussing non-Markovian setups. For now, we restrict ourselves to processes where it is actually sufficient to know the present, that is we look for maps of the kind $M : p_t(x) \to p_{t+\delta t}(x)$. Also, we assume stationarity, i.e. the evolution does not explicitly depend on time.

Further, we are interested in local theories. A theory is local when the evolution of a state/probability at a point only depends on nearer neighbours for shorter times where "nearer" and "shorter" are to be understood in terms of X and T.

A discrete, markovian, stationary and local theory can be described by a Master-equation of the form

$$p_{t+\delta t}(x) - p_t(x) = \sum_{x' \in V(x)} M_{x',x} p_t(x') - p(x) \sum_{x' \in V(x)} M_{x,x'} \tag{1}$$

where $M_{x,y}$ is some positive number that is physically interpreted as the transition rate of particles to move from x to y, and where $V(x)$ is a vicinity of x that encapsules locality. This can be written as a linear operator $M : \mu_P(X) \to \mu_P(X)$ that maps the space of probability measures onto itself. We refer to M as the *local update rule*.

Equipped with a point-measure, we now construct a path-measure. Consider therefore Γ_X^ℓ the set of all paths on X of finite length. Each path has weight

$$\mathcal{P}[x(t)] = \prod_{t_i} p_{t_i}(x_{t_i}) = \prod_i (1+M)^i p_{t_0}(x_{t_i}) \tag{2}$$

Last we consider "measurements". For any sufficiently fast decaying function $f : X \to \mathbb{R}$ or $F : \Gamma_X^\ell \to \mathbb{R}$, we define $\langle f^n \rangle(t) = \sum_{x \in X} f(x)^n p_t(x)$ and $\langle F^n \rangle(t) = \sum_{x(t) \in \Gamma_X^l} F[x(t)]^n \mathcal{P}[x(t)]$. In a slightly deeper spirit, these observables are the only object we can measure and therefore the only positive real thing.

Within this framework, there are thus 3 components that are chosen by the "experimentator" / "observer" / nature: A topological space X, a local update rule M, and measurements of f.

Question: Given $\langle f^n \rangle$ and M, what is possible to say about X? What additional structures are sensible to impose?

2 Further

One section about the continuum limit, choices we face when deciding in the case of a Random walk whether $a/\delta t$ or $a^2/\delta t$ is kept finite.

Then, we also go from μ_X into $L^2(X)$ (reduction of possible probability measures, but Hilbert-structure gives much more back), Then M becomes $\mathcal{L}$, the Liouvillian or Fokker-Planck-operator, $\mathcal{L}$ has a spectrum, eigenfunctions etc. all observables can be expressed in eigenfunctions. Thus knowing $\mathcal{L}$ and measuring f, we may be able to reconstruct eigenfunctions, then we know things about X.