User Guide for “A Toolbox for Solving and Estimating Heterogeneous Agent Macro Models”*

Thomas Winberry†

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In this document I describe the details of implementing the computational method developed in “A Toolbox for Solving and Estimating Heterogeneous Agent Macro Models.” I frame the discussion in the context of the Krusell and Smith (1998) model, which has become the benchmark for testing solution methods in the heterogeneous agent literature. However, the method itself is general and the steps taken here can be used as a template for solving other heterogeneous agent models as well. `Matlab` and `Dynare` codes should be provided with this document, but if not they are available at `http://faculty.chicagobooth.edu/thomas.winberry`. Codes for solving the Khan and Thomas (2008) heterogeneous firm model from the main text are also available there.

1 Model

I keep my exposition brief since the model is well known.

Households There is a continuum of households indexed by $j \in [0, 1]$, each with preferences over consumption c_{jt} represented by the expected utility function

$$E \sum_{t=0}^{\infty} \beta^t \frac{c_{jt}^{1-\sigma} - 1}{1 - \sigma},$$

*Thanks to Jung Sakong and Alp Tuncay for fantastic research assistance.
†University of Chicago Booth School of Business; email: `Thomas.Winberry@chicagobooth.edu`.

where β is the subjective discount factor and $\frac{1}{\sigma}$ is the elasticity of intertemporal substitution. Each household supplies ε_{jt} efficiency units of labor to the labor market inelastically. ε_{jt} is distributed independently across households but within households follows a two-state Markov process $\varepsilon_{jt} \in \{\varepsilon_0 = 0, \varepsilon_1 = 1\}$ with transition probabilities $\pi(\varepsilon'|\varepsilon)$. Households with $\varepsilon_{jt} = 1$ receive after-tax labor earnings $w_t(1-\tau)$ where w_t is the real wage. Households with $\varepsilon_{jt} = 0$ receive unemployment benefits bw_t financed by the labor tax. The government's budget is balanced each period, implying that $\tau = \frac{b(1-L)}{L}$, where L is the mass of households with $\varepsilon_{jt} = 1$ (note that L is constant because the transition probabilities $\pi(\varepsilon'|\varepsilon)$ are constant over time).

Asset markets are incomplete; households can only trade in capital a_{jt+1} subject to the borrowing constraint $a_{jt+1} \geq \underline{a} = 0$. Capital pays real return r_t.

Firms There is a representative firm which produces output Y_t according to the production function

$$Y_t = e^{z_t} K_t^{\alpha} L^{1-\alpha},$$

where z_t is an aggregate productivity shock, K_t is the aggregate capital stock, L is aggregate labor, and α is the capital share. In equilibrium, factor prices are given by

$$\begin{aligned} r_t &= \alpha e^{z_t} K_t^{\alpha-1} L^{1-\alpha} - \delta \\ w_t &= (1-\alpha) e^{z_t} K_t^{\alpha} L^{-\alpha}, \end{aligned}$$

where δ is the depreciation rate of capital. Aggregate TFP follows the AR(1) process

$$z_{t+1} = \rho_z z_t + \sigma_z \omega_{t+1}, \text{ where } \omega_{t+1} \sim N(0,1).$$

Equilibrium The aggregate state of this economy is $\mathbf{s}_t = (z_t, \mu_t)$, where μ_t is the distribution of households over (ε, a)-pairs. A **recursive competitive equilibrium** is a list of functions $a'(\varepsilon, a; z, \mu)$, $r(z, \mu)$, $w(z, \mu)$, and $\mu'(z, \mu)$ such that

1. (Household optimization) Taking $r(z,\mu)$, $w(z,\mu)$, and $\mu'(z,\mu)$ as given, $a'(\varepsilon,a;z,\mu)$ satisfies

$$c(\varepsilon,a;z,\mu)^{-\sigma} \geq \beta E\left[\left(1+r\left(z',\mu'(z,\mu)\right)\right)c\left(\varepsilon',a'(\varepsilon,a;z,\mu);z',\mu'(z,\mu)\right)^{-\sigma}|\varepsilon,z,\mu\right],$$

with equality if $a'(\varepsilon,a;z,\mu) > \underline{a}$,

where $c(\varepsilon,a;z,\mu)=w(z,\mu)\left((1-\tau)\varepsilon+b(1-\varepsilon)\right)+(1+r(z,\mu))a-a'(\varepsilon,a;z,\mu)$ is optimal consumption.

2. (Firm optimization and market clearing) Prices $r(z,\mu)$ and $w(z,\mu)$ satisfy

$$r(z,\mu) = \alpha e^z K^{\alpha-1}L^{1-\alpha}-\delta$$
$$w(z,\mu) = (1-\alpha)e^z K^{\alpha}L^{-\alpha},$$

where $K=\sum_\varepsilon \int a d\mu(\varepsilon,a)$ is aggregate capital.

3. (Evolution of distribution) For all measurable sets Δ_a,

$$\mu'(z,\mu)(\varepsilon,\Delta_a)=\sum_{\widetilde{\varepsilon}}\pi(\varepsilon|\widetilde{\varepsilon})\int 1\left\{a'(\widetilde{\varepsilon},a;z,\mu)\in\Delta_a\right\}\mu(\widetilde{\varepsilon},da).$$

2 The Solution Method

As described in the main text, the method follows three main steps:

1. Approximate the equilibrium objects using finite-dimensional objects.

2. Compute the stationary equilibrium of the approximated model without aggregate shocks (but still with idiosyncratic shocks).

3. Compute the aggregate dynamics of the approximated model using by perturbing it around steady state.

In this section I describe the details of how to perform each of these steps in the context of the Krusell and Smith (1998) model. Most of the work is in the first step; after approximating the equilibrium using finite-dimensional objects, the model is in the general form considered in the perturbation literature, so one can follow general steps. I therefore spend most attention in step 1, illustrating different choices that may be useful in alternative models as well.[1] This section assumes the reader is familiar with the main text.

2.1 Step 1: Approximate Equilibrium Using Finite-Dimensional Approximations

As in the main text, the goal of this section is to have a finite-dimensional representation of the model for a given time period t. However, the model contains two infinite-dimensional objects: the distribution of households and their decision rules. I approximate each of these in turn. Throughout, I follow the notational convention that an equilibrium object conditional on a realization of the aggregate state (z_t, μ_t) is simply represented by a time subcript $_t$.

Distribution Since the households' decisions involve the occasionally binding borrowing constraint $a_{jt+1} \geq \underline{a}$, the distribution of households features a positive mass at $\underline{a}$. To account for this, I separately approximate the mass of households at the constraint and the distribution of firms away from the constraint.

Mass at Constraint Denote the fraction of households with labor productivity ε at the borrowing constraint with a scalar $\widehat{m}_\varepsilon$. These scalars must follow the law of motion

$$\begin{aligned}\widehat{m}_{\varepsilon,t+1} \quad = \quad & \frac{1}{\pi\left(\varepsilon\right)}[\sum_{\widetilde{\varepsilon}}\left(1-\widehat{m}_{\widetilde{\varepsilon},t}\right)\pi\left(\widetilde{\varepsilon}\right)\pi\left(\varepsilon|\widetilde{\varepsilon}\right)\int 1\left\{a_t'\left(\widetilde{\varepsilon},a\right)=\underline{a}\right\}g_{\widetilde{\varepsilon},t}\left(a\right)da \\ & +\sum_{\widetilde{\varepsilon}}\widehat{m}_{\widetilde{\varepsilon},t}\pi\left(\varepsilon|\widetilde{\varepsilon}\right)\pi\left(\widetilde{\varepsilon}\right)1\left\{a_t'\left(\widetilde{\varepsilon},\underline{a}\right)=\underline{a}\right\}],\end{aligned}$$

where $\pi\left(\varepsilon\right)$ is the mass of households with productivity ε and $g_{\varepsilon,t}\left(a\right)$ is the p.d.f. of households with $a > \underline{a}$.

[1] In addition, the replication codes for the Khan and Thomas (2008) model show how to use the method to compute a model with two continuous individual states.

Distribution Away from Constraint I assume that the distribution of households over assets $a > \underline{a}$ can be approximated using the p.d.f. $g_{\varepsilon,t}(a)$:

$$g_{\varepsilon,t}(a) \simeq g^0_{\varepsilon,t} \exp\left\{ g^1_{\varepsilon,t}\left(a - m^1_{\varepsilon,t}\right) + \sum_{i=2}^{n_g} g^i_{\varepsilon,t}\left[\left(a - m^1_{\varepsilon,t}\right)^i - m^i_{\varepsilon,t}\right]\right\}, \tag{1}$$

where n_g indexes the degree of approximation, $\left\{g^i_{\varepsilon,t}\right\}_{i=0}^{n_g}$ are parameters, and $\left\{m^i_{\varepsilon,t}\right\}_{i=1}^{n_g}$ are centralized moments of the distribution. The parameters $\mathbf{g}_{\varepsilon,t}$ and moments $\mathbf{m}_{\varepsilon,t}$ must be consistent with each other:

$$\begin{aligned} m^1_{\varepsilon,t} &= \int a g_{\varepsilon,t}(a)\, da \\ m^i_{\varepsilon,t} &= \int \left(a - m^1_{\varepsilon,t}\right)^i g_{\varepsilon,t}(a)\, da \text{ for } i = 2, ..., n_g. \end{aligned} \tag{2}$$

Through these consistency conditions (2), the moments $\mathbf{m}_{\varepsilon,t}$ completely determine the parameters of the distribution $\mathbf{g}_{\varepsilon,t}$.

Since the distribution is characterized by its moments $\mathbf{m}_t$, I approximate the law of motion by deriving a law of motion for these moments. The evolution of the moments is implied by the decision rules:

$$\begin{aligned} m^1_{\varepsilon,t+1} &= \frac{1}{\pi(\varepsilon)}[\sum_{\widetilde{\varepsilon}} \left(1 - \widehat{m}_{\widetilde{\varepsilon},t}\right) \pi(\widetilde{\varepsilon})\, \pi(\varepsilon|\widetilde{\varepsilon}) \int a'_t(\widetilde{\varepsilon}, a)\, g_{\widetilde{\varepsilon},t}(a)\, da \\ &\quad + \sum_{\widetilde{\varepsilon}} \widehat{m}_{\widetilde{\varepsilon},t} \pi(\widetilde{\varepsilon})\, \pi(\varepsilon|\widetilde{\varepsilon})\, a'_t(\widetilde{\varepsilon}, \underline{a})] \\ m^i_{\varepsilon,t+1} &= \frac{1}{\pi(\varepsilon)}[\sum_{\widetilde{\varepsilon}} \left(1 - \widehat{m}_{\widetilde{\varepsilon},t}\right) \pi(\widetilde{\varepsilon})\, \pi(\varepsilon|\widetilde{\varepsilon}) \int \left[a'_t(\widetilde{\varepsilon}, a) - m^1_{\varepsilon,t+1}\right]^i g_{\widetilde{\varepsilon},t}(a)\, da \\ &\quad + \sum_{\widetilde{\varepsilon}} \widehat{m}_{\widetilde{\varepsilon},t} \pi(\varepsilon|\widetilde{\varepsilon})\, \pi(\widetilde{\varepsilon}) \left[a'_t(\widetilde{\varepsilon}, \underline{a}) - m^1_{\varepsilon,t+1}\right]^i], \end{aligned}$$

for $i = 2, ..., n_g$.

I numerically approximate the integrals $\int a'_t(\widetilde{\varepsilon}, a)\, g_{\widetilde{\varepsilon},t}(a)\, da$ and $\int [a'_t(\widetilde{\varepsilon}, a) - m^1_{\varepsilon,t+1}]^i g_{\widetilde{\varepsilon},t}(a)\, da$ using Gauss-Legendre quadrature. This quadrature specifies nodes $\{a_j\}_{j=1}^{m_g}$ and weights $\{\omega_j\}_{j=1}^{m_g}$, where m_g is the order of the quadrature, and approximates the integrals with the finite sums $\sum_{j=1}^{m_g} \omega_j \times a'_t(\widetilde{\varepsilon}, a_j)\, g_{\widetilde{\varepsilon}}(a_j)$ and $\sum_{j=1}^{m_g} \omega_j \times \left[a'_t(\widetilde{\varepsilon}, a_j) - m^{1'}_{\varepsilon}\right]^i g_{\widetilde{\varepsilon},t}(a_j)$. I use the same quadrature to compute the aggregate capital stock from the distribution:

$$K_t = \sum_{\varepsilon} \pi(\varepsilon) \sum_{j=1}^{m_g} \omega_j a_j g_{\varepsilon,t}(a_j),$$

This implies that the factor prices can be written

$$\begin{aligned} r_t &= \alpha e^{z_t} K_t^{\alpha-1} L^{1-\alpha} - \delta \\ w_t &= (1-\alpha) e^{z_t} K_t^{\alpha} L^{-\alpha}. \end{aligned}$$

Household Decision Rules The main challenge in approximating households' decision rules is that they face the occasionally binding borrowing constraint $\underline{a}$. Nonconvexities such as this are common in the heterogeneous agent literature, so in the codes I provide two options for approximating the decision rules: either approximate the savings decisions with linear splines, which directly take the constraint into account, or approximate the conditional expectation of future consumption with polynomials, which smooths over the kink. In this particular model I have found the polynomial approximation of the conditional expectation to be more efficient so I describe that here. However, in other models it may be necessary to use splines; for details, see the codes.[2]

Approximate Conditional Expectation with Chebyshev Polynomials Define the conditional expectation function:

$$\psi_t(\varepsilon, a) = E\left[\beta (1 + r_{t+1}) c_{t+1} \left(\varepsilon', a_t'(\varepsilon, a)\right)^{-\sigma}\right], \tag{3}$$

The advantage of this formulation is that the expectation operator smooths over kinks in the savings or consumption policies. The savings and consumption policies can be derived from the conditional expectation through the conditions

$$\begin{aligned} a_t'(\varepsilon, a) &= \max\left\{\underline{a}, w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - \psi_t(\varepsilon, a)^{-\frac{1}{\sigma}}\right\} \\ c_t(\varepsilon, a) &= w_t((1-\tau)\varepsilon + b(1-\varepsilon)) + (1+r_t)a - a_t'(\varepsilon, a), \end{aligned}$$

[2] On some older computers, people have found that the spline approximation is actually faster than the polynomial approximation, presumably because the Dynare pre-processor has less complicated equations to parse.

where the first line uses the fact that, if the borrowing constraint is not binding, the optimal decision follows the Euler Equation

$$\left(w_t\left(\left(1-\tau\right)\varepsilon+b\left(1-\varepsilon\right)\right)+\left(1+r_t\right)a-a_t'\left(\varepsilon,a\right)\right)^{-\sigma}=\psi_t\left(\varepsilon,a\right).$$

I approximate the conditional expectation function using Chebyshev polynomials:

$$\widehat{\psi}_t\left(\varepsilon,a\right)\simeq\exp\left\{\sum_{i=1}^{n_\psi}\theta_{\varepsilon i,t}T_i\left(\xi\left(a\right)\right)\right\},\tag{4}$$

where n_ψ is the order of approximation, T_i is the i^{th} order Chebyshev polynomial, and $\xi\left(a\right)=2\frac{a-\underline{a}}{\overline{a}-\underline{a}}-1$ transforms the interval $[\underline{a},\overline{a}]$ to $[-1,1]$ (on which the Chebyshev polynomials are defined), and the θs are coefficients.

Given this approximation of the conditional expectation function, I approximate the household's optimality conditions using collocation, which forces the optimality condition to hold exactly on a set of nodes $\{a_j\}_{j=1}^{n_\psi}$:

$$\exp\left\{\sum_{i=1}^{n_\psi}\theta_{\varepsilon i,t}T_i\left(\xi\left(a_j\right)\right)\right\}=E\left[\beta\left(1+r_{t+1}\right)\sum_{\varepsilon'}\pi\left(\varepsilon'|\varepsilon\right)\widehat{c}_t\left(\varepsilon',\widehat{a}_t'\left(\varepsilon,a_j\right)\right)^{-\sigma}\right],$$

where

$$\begin{aligned}\widehat{a}_t'\left(\varepsilon,a_j\right)&=\max\left\{\begin{array}{c}\underline{a},w_t\left(\left(1-\tau\right)\varepsilon+b\left(1-\varepsilon\right)\right)\\+\left(1+r_t\right)a_j-\left(\exp\left\{\sum_{i=1}^{n_\psi}\theta_{\varepsilon i,t}T_i\left(\xi\left(a_j\right)\right)\right\}\right)^{-\frac{1}{\sigma}}\end{array}\right\}\\\widehat{c}_t\left(\varepsilon,a_j\right)&=w_t\left(\left(1-\tau\right)\varepsilon+b\left(1-\varepsilon\right)\right)+\left(1+r_t\right)a_j-\widehat{a}_t'\left(\varepsilon,a_j\right).\end{aligned}$$

Approximate Equilibrium Conditions With all of the approximations, the equilibrium becomes computable, replacing the true aggregate state (z,μ) with $(z,\mathbf{m})$, the true distribution with the parametric family (1), and the true conditional expectation with the Chebyshev approximation (4). To collect these conditions into the general form of Schmitt-Grohe and Uribe (2004), define the state vector $\mathbf{x}_t=(z_t,\mathbf{m}_t)'$, which contains the predetermined variables, and the control vector $\mathbf{y}_t=(\boldsymbol{\theta}_t,\mathbf{g}_t,r_t,w_t)'$, which contains the non-predetermined variables. Then the equilibrium is characterized by the function

$$f\left(\mathbf{y}_t,\mathbf{y}_{t+1},\mathbf{x}_t,\mathbf{x}_{t+1};\chi\right)=\begin{bmatrix} m^{1\prime}_{\varepsilon,t}-\frac{1}{\pi(\varepsilon)}[\sum_{\widetilde{\varepsilon}}\left(1-\widehat{m}_{\widetilde{\varepsilon},t}\right)\pi\left(\widetilde{\varepsilon}\right)\pi\left(\varepsilon|\widetilde{\varepsilon}\right)\sum_j\omega_j a'_t\left(\widetilde{\varepsilon},a_j\right)g_{\widetilde{\varepsilon},t}\left(a_j\right) \\ +\sum_{\widetilde{\varepsilon}}\widehat{m}_{\widetilde{\varepsilon},t}\pi\left(\widetilde{\varepsilon}\right)\pi\left(\varepsilon|\widetilde{\varepsilon}\right)a'_t\left(\widetilde{\varepsilon},\underline{a}\right)] \\ m^{i\prime}_{\varepsilon,t}-\frac{1}{\pi(\varepsilon)}[\sum_{\widetilde{\varepsilon}}\left(1-\widehat{m}_{\widetilde{\varepsilon},t}\right)\pi\left(\widetilde{\varepsilon}\right)\pi\left(\varepsilon|\widetilde{\varepsilon}\right)\sum_j\left[a'_t\left(\widetilde{\varepsilon},a_j\right)-m^{1\prime}_{\varepsilon,t}\right]^i g_{\widetilde{\varepsilon},t}\left(a_j\right) \\ +\sum_{\widetilde{\varepsilon}}\widehat{m}_{\widetilde{\varepsilon},t}\pi\left(\widetilde{\varepsilon}\right)\pi\left(\varepsilon|\widetilde{\varepsilon}\right)\left[a'_t\left(\widetilde{\varepsilon},\underline{a}\right)-m^{1\prime}_{\varepsilon}\right]^i] \\ \exp\left\{\sum_{i=1}^{n_\psi}\theta_{\varepsilon i,t}T_i\left(\xi\left(a_j\right)\right)\right\}-\beta\left(1+r_{t+1}\right)\sum_{\varepsilon'}\pi\left(\varepsilon'|\varepsilon\right)\widehat{c}_t\left(\varepsilon',\widehat{a}'_t\left(\varepsilon,a_j\right)\right)^{-\sigma} \\ m^1_{\varepsilon,t}-\sum_j\omega_j a_j g_{\varepsilon,t}\left(a_j\right) \\ m^i_{\varepsilon,t}-\sum_j\omega_j\left(a_j-m^1_{\varepsilon,t}\right)^i g_{\varepsilon,t}\left(a_j\right) \\ r_t-\left(\alpha e^{z_t}K_t^{\alpha-1}L^{1-\alpha}-\delta\right) \\ w_t-\left(1-\alpha\right)e^{z_t}K_t^{\alpha}L^{-\alpha} \\ z_{t+1}-\rho_z z_t-\chi\times\sigma_z\omega_{t+1} \end{bmatrix}$$

such that

$$E_t\left[f\left(\mathbf{y}_t,\mathbf{y}_{t+1},\mathbf{x}_t,\mathbf{x}_{t+1};\chi\right)\right]=\mathbf{0}, \tag{5}$$

where χ is the perturbation parameter.

A solution to the model (5) is a function g which sets the control variable as a function of states, $\mathbf{y}_t=g\left(\mathbf{x}_t;\chi\right)$, and a function h which controls the law of motion for states, $\mathbf{x}_{t+1}=h\left(\mathbf{x}_t;\chi\right)+\chi\times\eta\omega_{t+1}$, where $\eta=\left(1,\mathbf{0}_{2n_g}\right)'$. I will now approximate these h and g in the remaining steps. Next, in Step 2, I compute the stationary equilibrium without aggregate shocks. Then, in Step 3, I compute the aggregate dynamics by perturbing the system around this stationary equilibrium.

2.2 Step 2: Solve for Stationary Equilibrium

In terms of the notation defined above, the stationary equilibrium of the model is represented by two vectors, $\mathbf{x}^*$ and $\mathbf{y}^*$, such that $f\left(\mathbf{y}^*,\mathbf{y}^*,\mathbf{x}^*,\mathbf{x}^*;0\right)=0$. In principle this is just a non-linear system of equations which could be solved using standard numerical techniques. However, in practice the system is large, so numerical algorithms are unstable. Instead, I stationary equilibrium in terms of a single non-linear equation in the aggregate capital stock K, which must solve the following root-finding problem as in Aiyagari (1994):

1. Compute factor prices $r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta$ and $w = (1-\alpha) K^{\alpha} L^{-\alpha}$.
2. Solve for the conditional expectation function $\boldsymbol{\theta}$.
3. Using the implied decision rules, solve for the invariant distribution $\mathbf{m}$ and implied parameters $\mathbf{g}$.
4. Update aggregate capital $K' = \sum_{\varepsilon} \pi(\varepsilon) \sum_{j=1}^{m_g} \omega_j \times a'(\varepsilon, a_j) g_{\varepsilon}(a)$.
5. Return $K' - K$ and solve for a zero of this equation.

For more details of the individual steps, see the included code `steadystate.m`.

2.3 Step 3: Solve for Aggregate Dynamics

Given the value of $\mathbf{x}^*$ and $\mathbf{y}^*$ in the stationary equilibrium, all that we need to do is import the model equations in `Dynare`; `Dynare` will then differentiate these equations, evaluate them at their stationary values, and solve the resulting system. The main challenge in importing the model into `Dynare` is that `Dynare` does not accept matrix expressions, which are used heavily in the steady state code. I therefore re-write these matrix expressions as loops over scalar variables using `Dynare`'s macro-processor.

The `Dynare` code has four main parts; for full details, see the code `dynamics.mod`. First, I declare the parameters of the model, which include not only the economic parameters but also approximation parameters, such as the grids and polynomial terms. Second, I declare the variables of the model, which are the contents of the vectors $\mathbf{x}$ and $\mathbf{y}$. Third, I specify the model equations, relying heavily on the macro-processor loops. Finally, I instruct `Dynare` to take a first order expansion of the model and perform its default analysis.

Table 1: Parameterization

Parameter	Description	Value	Parameter	Description	Value
β	Discount factor	.96	ε_0	Unemployed productivity	0
σ	Utility curvature	1	ε_1	Employed productivity	1
$\underline{a}$	Borrowing constraint	0	$\pi(\varepsilon_0 \rightarrow \varepsilon_1)$	U to E transition	.5
α	Capital share	.36	$\pi(\varepsilon_1 \rightarrow \varepsilon_0)$	E to U transition	.038
δ	Capital depreciation	.10	ρ_z	Aggregate TFP AR(1)	.859
b	UI replacement rate	.15	σ_z	Aggregate TFP AR(1)	.014

Notes: Annual parameterization, chosen for illustrative purposes.

Higher Order Approximations In principle, `Dynare` will compute a higher order expansion of the aggregate dynamics by changing the option `order=1` to `order=2` or `order=3`. Unfortunately, this will typically cause `Dynare` to crash in evaluating and storing the derivatives of the model's equilibrium conditions; in order to evaluate these derivatives, `Dynare` creates many temporary variables, which causes `Matlab`'s workspace to run out of space. However, this problem can be avoided by using the `use_dll` option in `Dynare`'s model block. `use_dll` instructs `Dynare` to evaluate the derivatives in a compiled `.mex` function, which is evaluated outside of `Matlab`, and therefore does not run into the workspace limit. Unfortunately, this option significantly increases the time it takes for `Dynare` to pre-process the model. At this point there is no way around this issue, but recall that this pre-processing time is a fixed cost which does not need to be re-performed in each iteration of an estimation exercise.

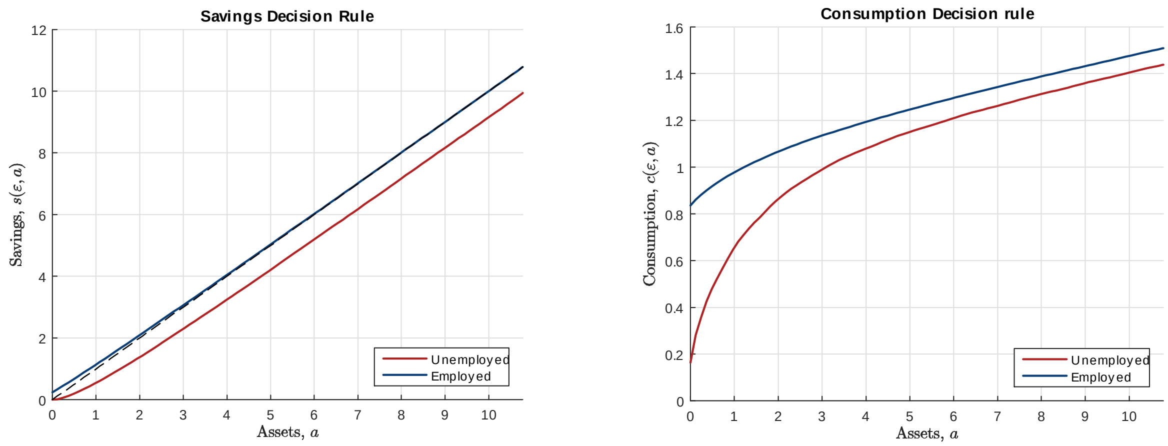
3 Results

The parameters of the model are contained in Table 1. The model frequency is one year. The transition matrix for idiosyncratic shocks implies an aggregate employment rate of .93 and a mean duration of unemployment of 1 year.

The steady state household decision rules are plotted in Figure 1. Thee savings function is approximately linear for all of the state space except very close to the borrowing constraint.

Figure 1: Individual Decision Rules in Steady State

(a) Savings policy (b) Consumption policy



Notes: Solution to individual problem in steady state. (a) plots the expectation of next period's marginal utility of consumption, conditional on the current state. (b) plots the savings function implied by the conditional expectation function. (c) plots the implied consumption function.

Similarly, the consumption function is approximately linear away from the borrowing constraint, but curved close to it.

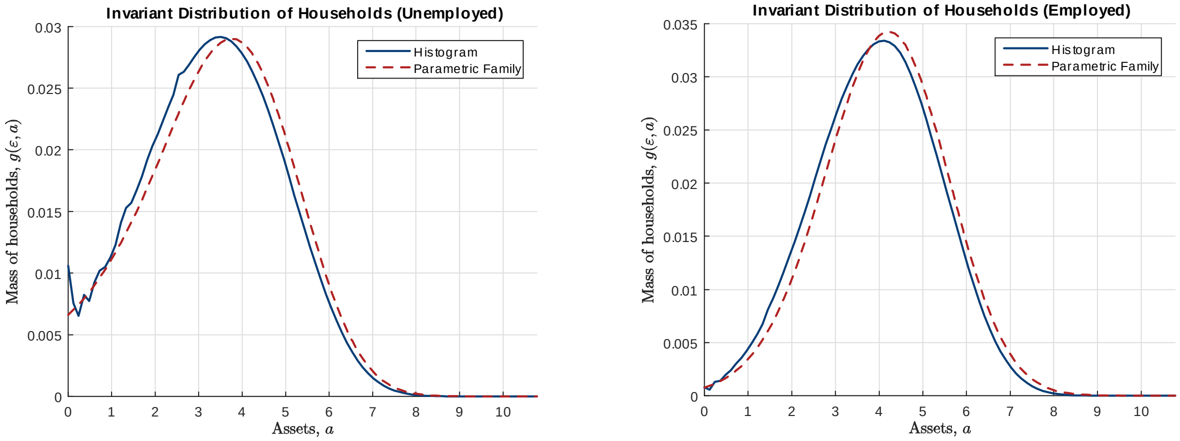
The resulting stationary distribution of households is plotted in Figure 2. The solid lines plot a nonparametric histogram approximation of the distribution and the dashed lines plot the approximation using the parametric family (1). Employed households hold on average more wealth than the unemployed, and in particular have almost no mass at the borrowing constraint. In contrast, the unemployed have a mass point at the borrowing constraint, which can be seen in the histogram. The parametric family provides an acceptable fit to the distribution, although less well than in the main text due to the presence of the borrowing constraint.[3] Since so few households are at the borrowing constraint, I ignore the borrowing constraint, although as discussed in Section 2.1 the method can be extended to incorporate this mass point.

Figure 3 plots the impulse responses of key aggregate variables to an aggregate TFP shock, as directly output by `Dynare` using a first-order approximation of the dynamics. Higher aggregate TFP increases the marginal product of labor, which increases the real wage and therefore labor income and consumption. Higher TFP also increases the marginal product of capital, which

[3]Theoretically, with the borrowing constraint the distribution is characterized by a finite (but large) number of mass points, which I am approximating with one mass point plus a density.

Figure 2: Invariant Distribution of Households

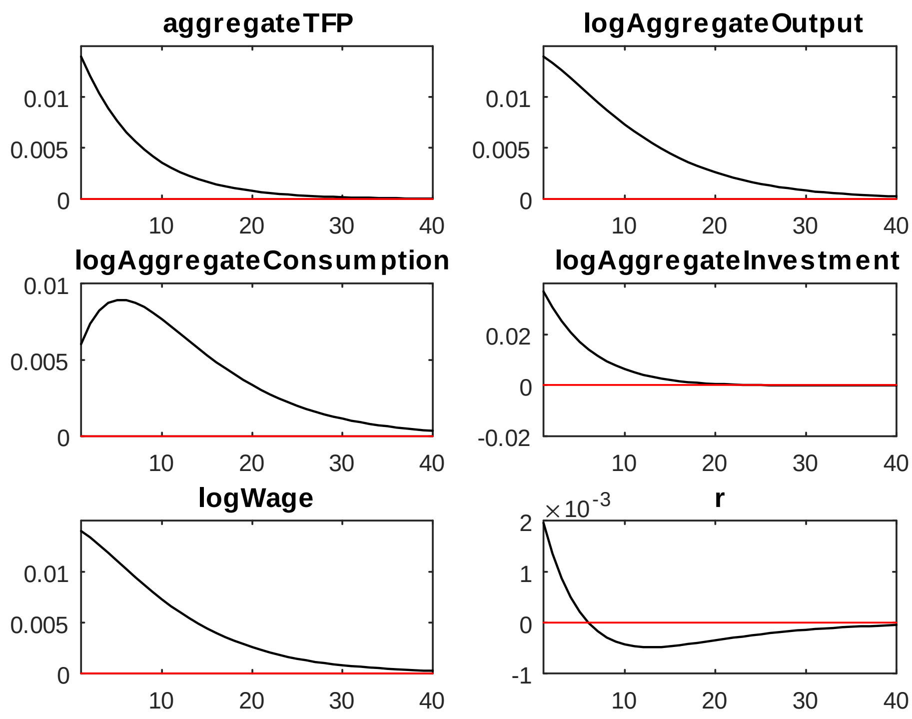
(a) Unemployed households (b) Employed households



Notes: Invariant distribution of households conditional on idiosyncratic shock. "Histogram" refers to approximating the distribution with a nonparametric histogram, as in Young (2010). "Parametric family" refers to approximating the distribution with the parametric family introduced in the main text.

increases the real interest rate and therefore the incentive to accumulate capital. The resulting business cycle statistics are recorded in Table 2. As usual, consumption is less volatile than output, which is less volatile than investment. All series are highly correlated given that fluctuations are driven by only aggregate TFP shocks.

Figure 3: Impulse Responses



Notes: Impulse responses to an aggregate TFP shock, as visualized by Dynare.

Table 2: Business Cycle Statistics

SD (relative to output)		**Correlation with Output**	
Output	(1.32%)	×	×
Consumption	0.5	Consumption	.912
Investment	2.651	Investment	.975
Real wage	1	Real wage	1
Real interest rate	0.15	Real interest rate	.898

Notes: Standard deviation of aggregate variables. All variables are HP-filtered with smoothing parameter $\lambda = 100$ and, with the exception of the real interest rate, have been logged. Standard deviations for variables other than output are expressed relative to that of output.