Investigating the impact of exposure in probabilistic models for dynamical networks

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1. Methods

For temporal networks, we model exposure dynamically. We choose a simple setting in which the probability of exposure is *i.i.d.* in time, but varies between node pairs. For simplicity, all derivations are done undirected networks. However, this can easily be extended to directed networks if need be. Additionally, in previous work the exposure prior was quite weak as the propensity to be exposed for node i is the same for each node j it could be exposed to. Here, we increase the expressiveness of μ by making it a $\tilde{K}$-dimensional vector. As is the case with u and v, μ can be interpreted as a mixed membership community vector, *i.e.* we are now able to model exposure communities as well as communities due to affinity.

Resulting model These two changes result in the following prior:

$$\begin{aligned} p(\mathbf{Z}) &= \prod_t \prod_{i<j} p(Z_{i,j}^{(t)}) \\ &= \prod_t \prod_{i<j} \mu_i^T \mu_j^{Z_{i,j}^{(t)}} (1 - \mu_i^T \mu_j)^{1 - Z_{i,j}^{(t)}} \quad . \end{aligned} \tag{1}$$

We can see that the Bernulli prior on $\mathbf{Z}$ now depends on $\mu_i^T \mu_j$, that is the similarity between the exposure communities of i and j. Analogously to the static model, we choose the likelihood as:

$$p(\mathbf{A} \mid \theta, \mathbf{Z}) = \prod_t \prod_{i<j} Pois(A_{i,j}^{(t)}; \lambda_{i,j})^{Z_{i,j}^{(t)}} \delta(A_{i,j}^{(t)})^{(1 - Z_{i,j}^{(t)})} \quad . \tag{2}$$

2. Inference

We infer the optimal latent variables using *Maximum A Posteriori Estimation* (**MAP**), as we have prior information on $\mathbf{Z}$ that we can use.

$$\theta^* = \arg\max_\theta p(\theta \mid A) = \arg\max_\theta \log \sum_Z p(Z, \theta \mid A) \quad . \tag{3}$$

As the logarithm of a sum is hard to derive, we use Jensens inequality to approximate it:

$$\log \sum_Z p(Z, \theta \mid A) \geq \sum_Z q(Z) \log \frac{p(Z, \theta \mid A)}{q(Z)} := \mathcal{L}(q, \theta, \mu) \quad . \tag{4}$$

Here $q(\mathbf{Z})$ is any distribution satisfying $\sum_{\mathbf{Z}} q(\mathbf{Z}) = 1$.
However, exact equality is reached for

$$q(\mathbf{Z}) = \frac{p(\mathbf{Z}, \theta \mid \mathbf{A})}{\sum_{\mathbf{Z}} p(\mathbf{Z}, \theta \mid \mathbf{A})} \quad , \tag{5}$$

which is the posterior on $\mathbf{Z}$. Now we apply **EM** by iterating between updating $q(\mathbf{Z})$ according to equation **??** and updating θ and μ.

To obtain the parameter updates, we compute the derivative of $\mathcal{L}(q, \theta, \mu)$ *w.r.t.* θ and μ and set it to zero (**??**), resulting in the following update equations:

2.1. Updating θ

$\mathcal{L}(q, \theta, \mu)$ simplifies, as we only care about the parts which depend on θ:

$$\begin{aligned}
\mathcal{L}(q, \theta, \mu) &\propto \sum_{\mathbf{Z}} q(\mathbf{Z}) \log p(A \mid Z, \theta) \\
&= \sum_{\mathbf{Z}} q(\mathbf{Z}) \log \left(\prod_{t} \prod_{i<j} \delta(A_{i,j}^{(t)})^{1-Z_{i,j}^{(t)}} Pois(A_{i,j}^{(t)}; \lambda_{i,j})^{Z_{i,j}^{(t)}} \right) \\
&\propto \sum_{i<j} \sum_{t} \sum_{\mathbf{Z}} q(\mathbf{Z}) Z_{i,j}^{(t)} \log(Pois(A_{i,j}^{(t)}; \lambda_{i,j})) \\
&= \sum_{i<j} \sum_{t} Q_{i,j}^{(t)} \left(A_{i,j}^{(t)} \log(\lambda_{i,j}) - \lambda_{i,j} \right) \quad .
\end{aligned} \tag{6}$$

Now, we can take the derivative of the simplified $\mathcal{L}(q, \theta, \mu)$ and set it to zero to obtain the updates on θ. As $\log \lambda_{i,j}$ is difficult to derive, we use Jensens inequality [**?**] to estimate it.

$$\log(\lambda_{i,j}) = \log \left(\sum_{k,q} u_{i,k} u_{j,q} w_{k,q} \right) \geq \sum_{k,q} \rho_{ijkq} \log \left(\frac{u_{i,k} u_{j,q} w_{k,q}}{\rho_{ijkq}} \right) \quad . \tag{7}$$

For

$$\rho_{ijkq} = \frac{u_{i,k} u_{j,q} w_{k,q}}{\sum_{k,q} u_{i,k} u_{j,q} w_{k,q}} \quad , \tag{8}$$

exact equality is reached.
Now, we apply **EM** a second time by iterating between updating ρ_{ijkq} according to equation **??** and updating θ.

We show the derivations for $u_{m,n}$. However, $w_{m,n}$ can be obtained in a similar way:

$$
\begin{aligned}
\frac{\partial}{\partial u_{m,n}}\mathcal{L}(q,\theta,\mu) &= \sum_{i<j}\sum_{t} Q_{i,j}^{(t)}\left(A_{i,j}^{(t)}\frac{\partial}{\partial u_{m,n}}\log(\lambda_{i,j}) - \frac{\partial}{\partial u_{m,n}}\lambda_{i,j}\right) \\
&= \sum_{j,t} Q_{m,j}^{(t)}\left(A_{m,j}^{(t)}\frac{\sum_q \rho_{mjnq}}{u_{mn}} - \sum_q v_{jq}w_{nq}\right) \overset{!}{=} 0 \qquad (9)\\
\Leftrightarrow u_{m,n} &= \frac{\sum_j \sum_q \rho_{mjnq}\sum_t Q_{m,j}^{(t)}A_{mj}^{(t)}}{\sum_j\sum_q v_{jq}w_{nq}\sum_t Q_{m,j}^{(t)}} \quad ,
\end{aligned}
$$

$$
w_{m,n} = \frac{\sum_{ij}\rho_{ijmn}\sum_t Q_{i,j}^{(t)} t A_{ij}^{(t)}}{\sum_{ij} u_{im}v_{jn}\sum_t Q_{i,j}^{(t)}} \quad . \tag{10}
$$

2.2. Updating μ

$$
\begin{aligned}
\mathcal{L}(q,\theta,\mu) &\propto \sum_{\mathbf{Z}} q(\mathbf{Z})\log p(Z) \\
&= \sum_{\mathbf{Z}} q(\mathbf{Z})\sum_t\sum_{i<j} Z_{i,j}^{(t)}\log(\mu_i^T\mu_j) + (1-Z_{i,j}^{(t)})\log(1-\mu_i^T\mu_j) \\
&= \sum_{i<j}\log(\mu_i^T\mu_j)\left(\sum_t Q_{i,j}^{(t)}\right) + \log(1-\mu_i^T\mu_j)\left(T - \sum_t Q_{i,j}^{(t)}\right) \quad .
\end{aligned} \tag{11}
$$

Again, the parameter updates on μ can be obtained by computing the derivative with respect to μ and setting it to zero:

$$
\begin{aligned}
\frac{\partial}{\partial \mu_{k,l}}\mathcal{L}(q,\theta,\mu) &= \sum_{i<j}\left(\sum_t Q_{i,j}^{(t)}\right)\frac{\delta_{k,i}\mu_{j,l}+\delta_{k,j}\mu_{i,l}}{\mu_i^T\mu_j} \\
&\quad - \left(T - \sum_t Q_{i,j}^{(t)}\right)\frac{\delta_{k,i}\mu_{j,l}+\delta_{k,j}\mu_{i,l}}{1-\mu_i^T\mu_j} \qquad (12)\\
&= \sum_{i\neq k}\mu_{il}\left(\frac{\sum_t Q_{i,k}^{(t)}}{\mu_i^T\mu_k} - \frac{T-\sum_t Q_{i,k}^{(t)}}{1-\mu_i^T\mu_k}\right) \overset{!}{=} 0 \quad .
\end{aligned}
$$

2.3. Updating $Q_{i,j}^{(t)}$

$$
\begin{aligned}
Q_{i,j}^{(t)} &= p(Z_{i,j}^{(t)} = 1 \,|\, \theta, \mathbf{A}) = p(Z_{i,j}^{(t)} = 1 \,|\, \theta, A_{i,j}^{(t)}) \\
&= \frac{p(Z_{i,j}^{(t)} = 1) p(A_{i,j}^{(t)} \,|\, Z_{i,j}^{(t)} = 1, \theta)}{\sum_{\mathbf{Z}} p(Z_{i,j}^{(t)}) p(A_{i,j}^{(t)} \,|\, Z_{i,j}^{(t)}, \theta)} \\
&= \frac{\mu_i^T \mu_j \mathrm{Pois}(A_{i,j}^{(t)}; \lambda_{i,j})}{\mu_i^T \mu_j \mathrm{Pois}(A_{i,j}^{(t)}; \lambda_{i,j}) + (1 - \mu_i^T \mu_j) \delta(A_{i,j}^{(t)})}
\end{aligned}
\tag{13}
$$

The complete inference algorithm is summarized below (**??**)

Algorithm 1 EM

Input: network $A \in \mathbb{N}^{N\times N\times T}$
number of affinity communities K
number of exposure communities $\tilde{K}$

Output: memberships $u, v \in \mathbb{R}^{N\times K}$
network affinity matrix $w \in \mathbb{R}^{K\times K}$
exposure memberships $\mu \in \mathbb{R}^{N\times\tilde{K}}$
posterior estimate $Q \in \mathbb{R}^{N\times N}$ for Z

Initialize $\theta : (u, v, w), \mu$ at random.

Repeat until $\mathcal{L}(q, \theta, \mu)$ converges:

1. Calculate ρ and Q (E-step):

$$\rho_{ijkq} = \frac{u_{i,k}v_{j,q}w_{k,q}}{\sum_{k,q} u_{i,k}v_{j,q}w_{k,q}} ,$$

update $Q^{(t)}_{i,j}$ *acc.* to **??**,

2. Update parameters θ (M-step):

i) for each node i and community k update memberships:
update u_{ik}(*acc.* to **??**)
update v_{ik}(*acc.* to **??**)

ii) for each pair of communities k, q update affinity matrix:

update w_{kq}(*acc.* to **??**)

iii) update prior on exposure indicator for each node i and exposure community k:

update μ_{ik}(*acc.* to **??**)