Project:
Algorithmic complexity and graphs

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Table of Contents

4. Line graphs, hamiltonian paths, and eulerian paths

4.1)

Eulerian Path

An eulerian path is a route within a given graph that visits every edge between two nodes exactly once. An eulerian path may only exist within a given graph if all nodes are of an even degree or two or fewer nodes are of an odd degree. An even and odd degree refers to the number of edges attached to a given node. There may only be a maximum of two odd degrees as any node accessed by an edge, must be left via another edge. Thus, odd-degree nodes mark dead ends for an eulerian path, and can conversely only be the start and end points of a given path.

Hamiltonian Path

A hamiltonian path is a route within a given graph that visits each node exactly once. By its definitions, the hamiltonian path visits each edge it travels on only once. However, it does not need to visit every edge within a given graph.

Connection in a line graph

Given graph G that contains an eulerian path, we can assert that its line graph L(g) contains a hamiltonian path. We know this because in an eulerian path every edge is visited exactly once by the path. In a line graph, edges become nodes and thus the eulerian path of edges has become a hamiltonian path of nodes.

The same is not necessarily true in reverse. A hamiltonian path only visits nodes once but doesn't pass every edge, so a graph that gets transformed into its line graph may contain more than two nodes of an odd degree, making it impossible to generate an eulerian path within them.

4.2)

We start by generating a graph G that contains an eulerian path. This graph is displayed in figure 1. Following the nodes, we can prove that graph G is eulerian as the stepping order of [0, 7, 1, 8, 3, 5, 1, 2, 4, 6, 9, 0] visits every edge exactly once and thus creates an eulerian cycle. This cycle can turn into an eulerian path by removing the connection between 0 and 9.

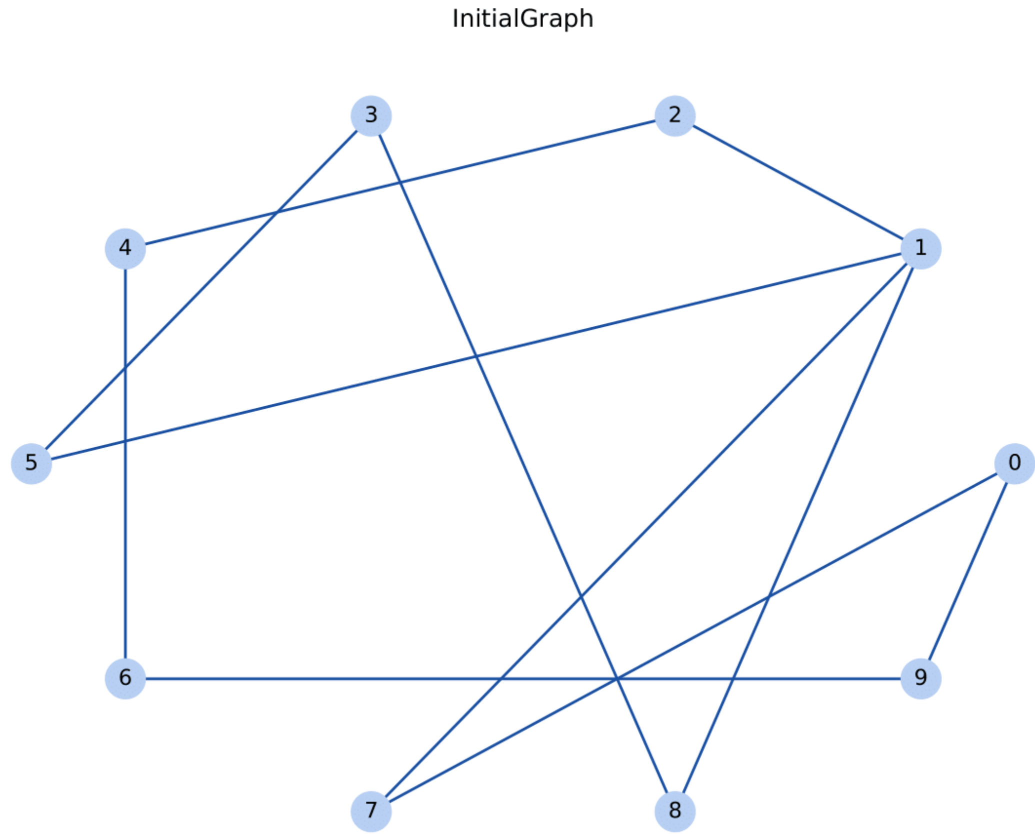

Figure 1. Visualization of graph G containing an eulerian cycle

Next, we will generate a line graph L(g) from graph G. The line graph L(g) is displayed in figure 2.

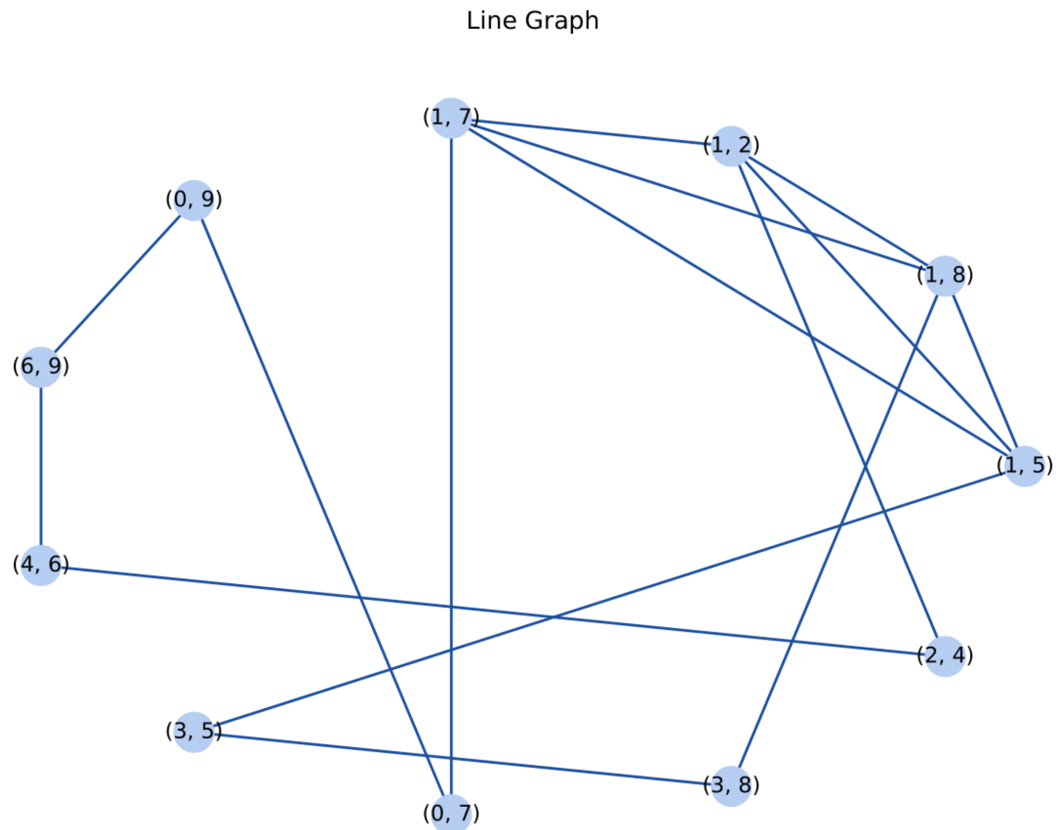

Figure 2. Visualization of the line graph L(g) of graph G

We can prove that the eulerian path within graph G is the hamiltonian path within its line graph L(g). For this, we will take the initial step array containing the eulerian path and create a new array, where each node contains the step taken in the eulerian path. For example [0, 7, 1] will turn into [[0,7], [7,1]]. Applying this to the entire step array creates the path array [[0, 7], [7, 1], [1, 8], [8, 3], [3, 5], [5, 1], [1, 2], [2, 4], [4, 6], [6, 9], [9, 0]]. This path is hamiltonian as it visits each node within the line graph exactly once, as displayed seen in figure 3.

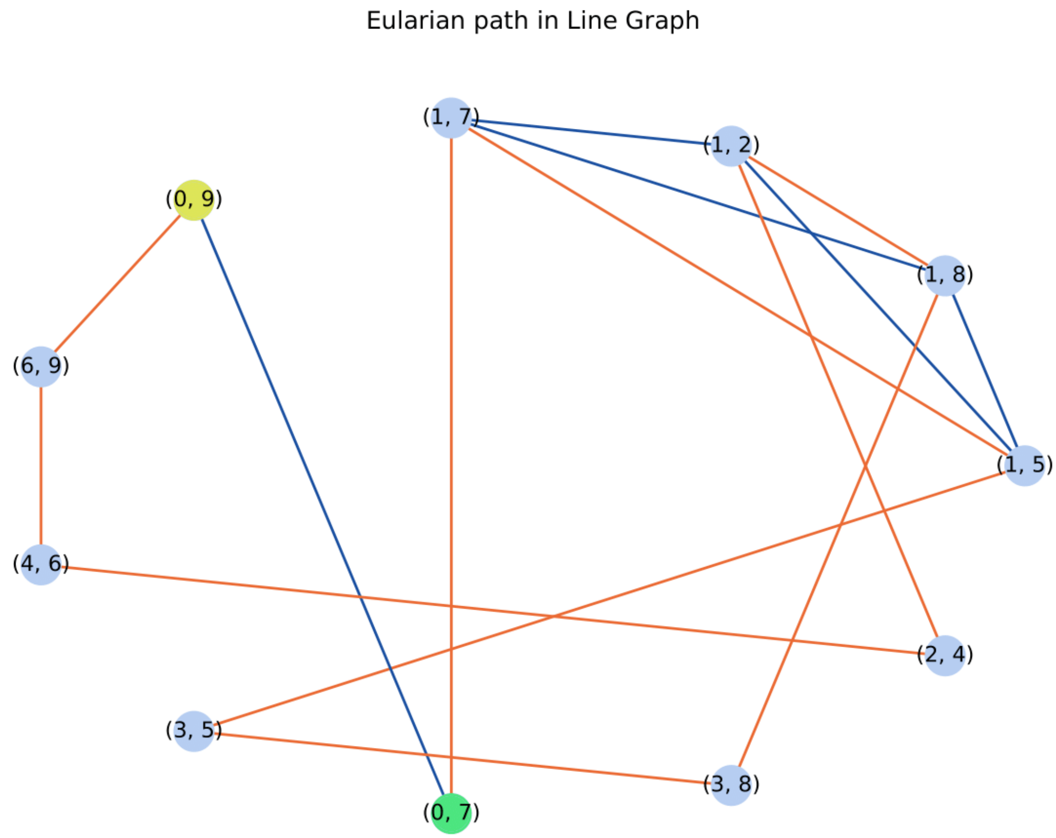

Figure 3. The eulerian cycle of graph G applied to the line graph L(g) is a hamiltonian path.

4.3)

The “harder” problem between the eulerian and hamiltonian path from an algorithmic view is the hamiltonian path. That is because the eulerian path can be determined and verified within polynomial time, while the hamiltonian path cannot be determined within polynomial time. The eulerian path problem following the Hierholzer algorithm is of linear complexity O(e), where e is the number of edges within the graph. The Hamiltonian path problem is an NP problem, meaning that there currently exists no solution to the problem that runs within polynomial time.