Rainbows

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1 Geometrical Assumptions

A raindrop is assumed to be perfectly circular, its radius normalized to 1 and the inside and outisde described via refractive inidices n_1 and n_0.

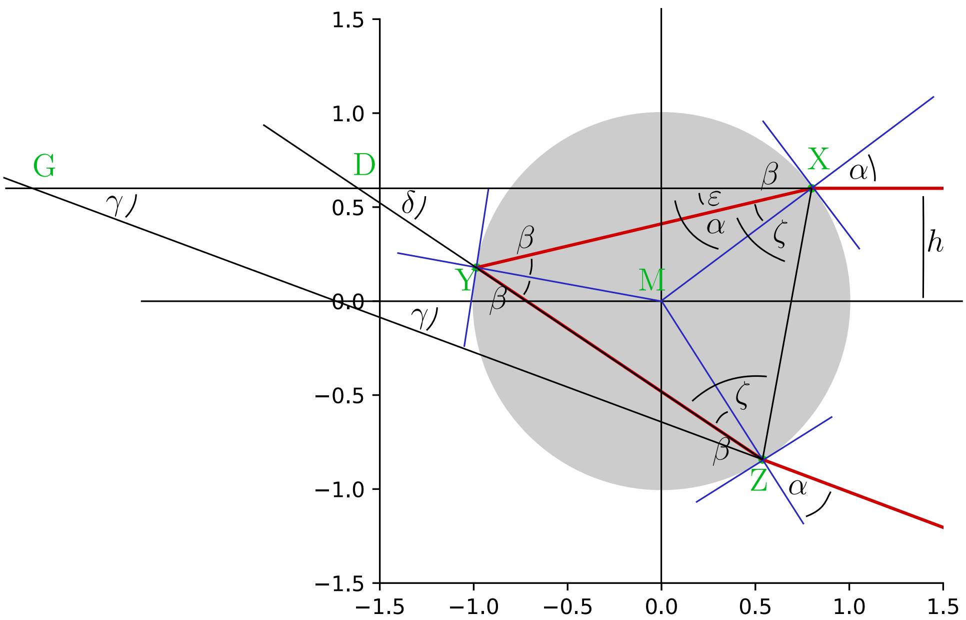

Figure 1: Principle scetch of the geometrical properties of a circular raindrop.

From this simple sketch the following relations can be deduced:

$$h = \sin(\alpha),$$

where Snell's law states:

$$n_0 \sin(\alpha) = n_1 \sin(\beta),$$

$$\varepsilon = \alpha - \beta,$$

$$\zeta = [180° - 2\beta]/2 = 90° - \beta,$$

$$\delta = 180° - [[180° - 2\beta] + \varepsilon] = 2\beta - [\alpha - \beta] = 3\beta - \alpha,$$

$$\gamma = 180° - 2[\zeta + \varepsilon] = 180° - 2[90° - \beta + \alpha - \beta] = 2[2\beta - \alpha],$$

$$\gamma(\alpha) = 2\left[2\arcsin\left(\frac{n_0}{n_1}\sin(\alpha)\right) - \alpha\right],$$

$$\gamma(h) = 2\left[2\arcsin\left(\frac{n_0}{n_1}h\right) - \arcsin(h)\right]. \tag{1}$$

These relations are shown in figure 2. Please note that the plot for $\frac{n_0}{n_1} = 1$ shall only be understood as a border-case $[\lim_{\frac{n_0}{n_1}\to 1}\gamma(h)]$ - physically there would be no such effect for homogenous refractive indices.

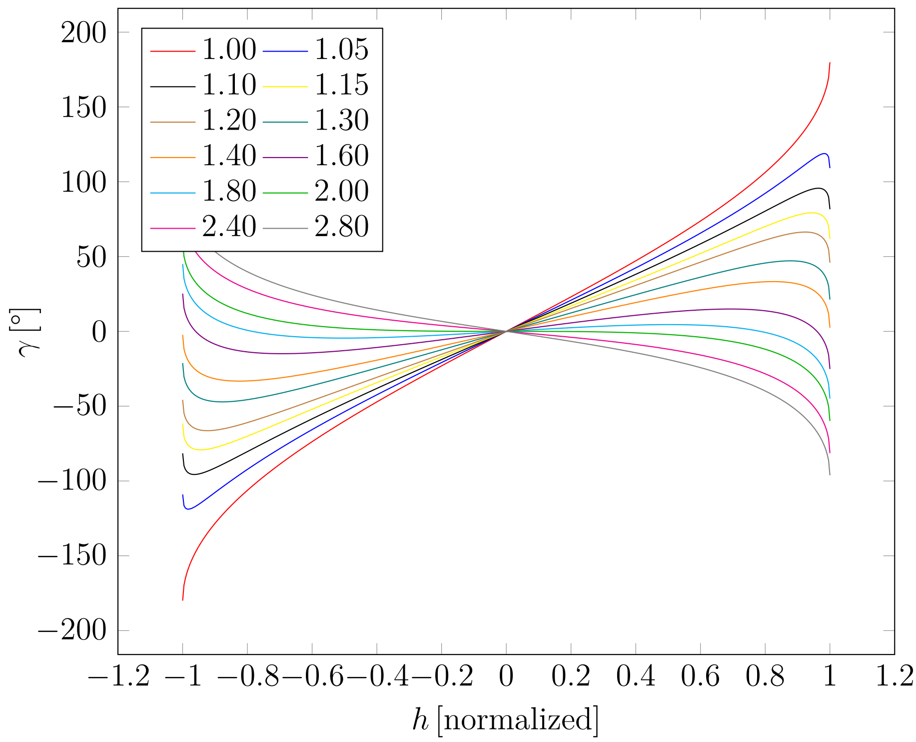

Figure 2: Excidence angle γ over incidence height h.

2 Extrema

The derivate is [using $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$]:

$$\frac{d}{dh}\gamma(h) = 2\left[2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h]^2}} - \frac{1}{\sqrt{1-h^2}}\right].$$

Thus the local extrema are:

$$\begin{aligned}
0 &\stackrel{!}{=} \frac{d}{dh}\gamma(h_{\text{extr.}}) && (2)\\
&= 2\left[2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h_{\text{extr.}}]^2}} - \frac{1}{\sqrt{1-h_{\text{extr.}}^2}}\right],\\
\frac{1}{\sqrt{1-h_{\text{extr.}}^2}} &= 2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h_{\text{extr.}}]^2}},\\
1-\left[\frac{n_0}{n_1}h_{\text{extr.}}\right]^2 &= \left[2\frac{n_0}{n_1}\right]^2\left[1-h_{\text{extr.}}^2\right],\\
\left[\frac{n_1}{2n_0}\right]^2 - \left[\frac{1}{2}h_{\text{extr.}}\right]^2 &= 1-h_{\text{extr.}}^2,\\
\frac{3}{4}h_{\text{extr.}}^2 &= 1-\left[\frac{n_1}{2n_0}\right]^2,\\
h_{\text{extr.}}^{\pm} &= \pm\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]}. && (3)
\end{aligned}$$

From this one finds:

$$1 < \frac{n_1}{n_0} \leq 2. \tag{4}$$

And the angle values at these local extreme are:

$$\gamma_{\text{extr.}}^{\pm} = \gamma(h_{\text{extr.}}^{\pm}) = \pm 2\left[2\arcsin\left(\frac{n_0}{n_1}\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]}\right) - \arcsin\left(\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]}\right)\right]. \tag{5}$$

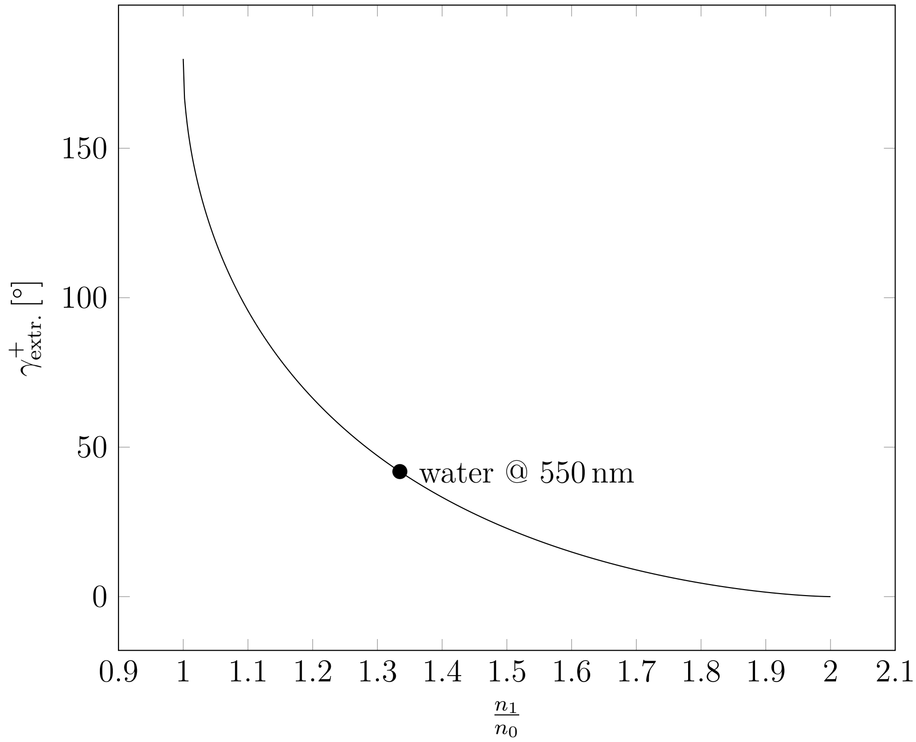

Figure 3: Excidence angle $\gamma+$ over incidence height h.

For water at 20 °C and light of 550 nm [$n_0 = 1$, $n_1 = 1.3347$] this yields an angle of $\gamma^+_{\text{extr.}} = 41.83°$.

Because of what will be shown in the next section, this also corresponds to the angles of maximum power density excidence.

3 Power Density Superelevation

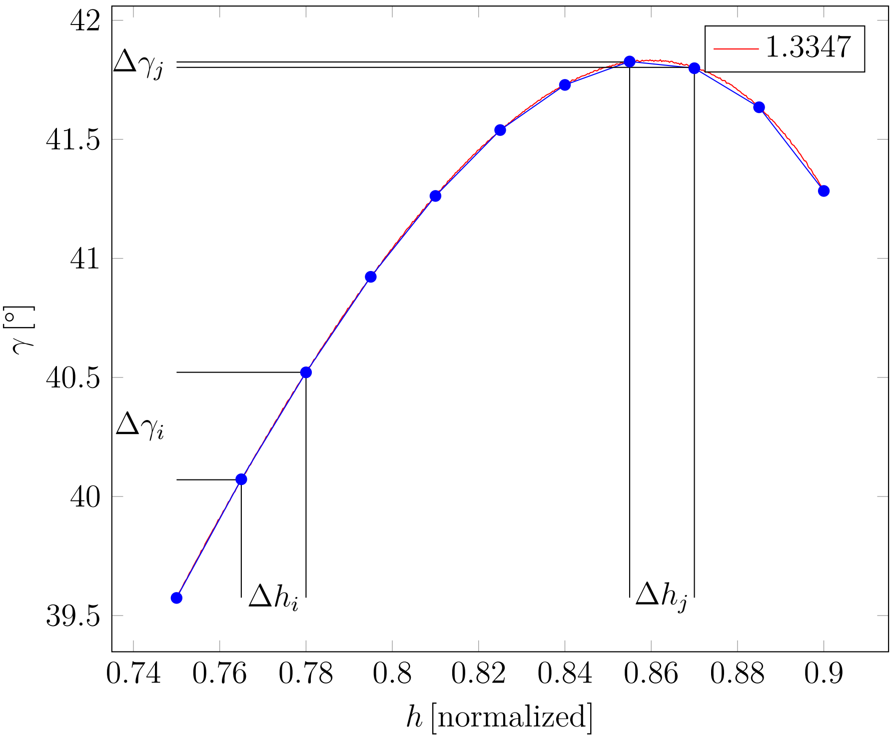

Figure 4: Excidence angle γ over incidence height h.

As the incident power density p_h is written in terms of h [where the total power is $P = \int_{\mathbb{R}} p_h(x)dx$] it has to be transformed when changing the underlying coordinates. This is accomplished as followes:

$$p_\gamma(x) = \sum_{y \ | \ x=\gamma(y)} p_h(y) \left| \frac{1}{\left.\frac{d\gamma(z)}{dz}\right|_{z=y}} \right|$$

which ensures that $P = \int_{\mathbb{R}} p_\gamma(x)dx$.

This can be seen in figure 4, where the power incident e.g. in Δh_j [$= \int_{h_j}^{h_{j+1}} p_h(x)dx$] exits in a much smaller angle region $\Delta\gamma_j$ compared to the power incident in Δh_i — which in return means that the power density p_γ has to be much higher for γ_j than for γ_i.

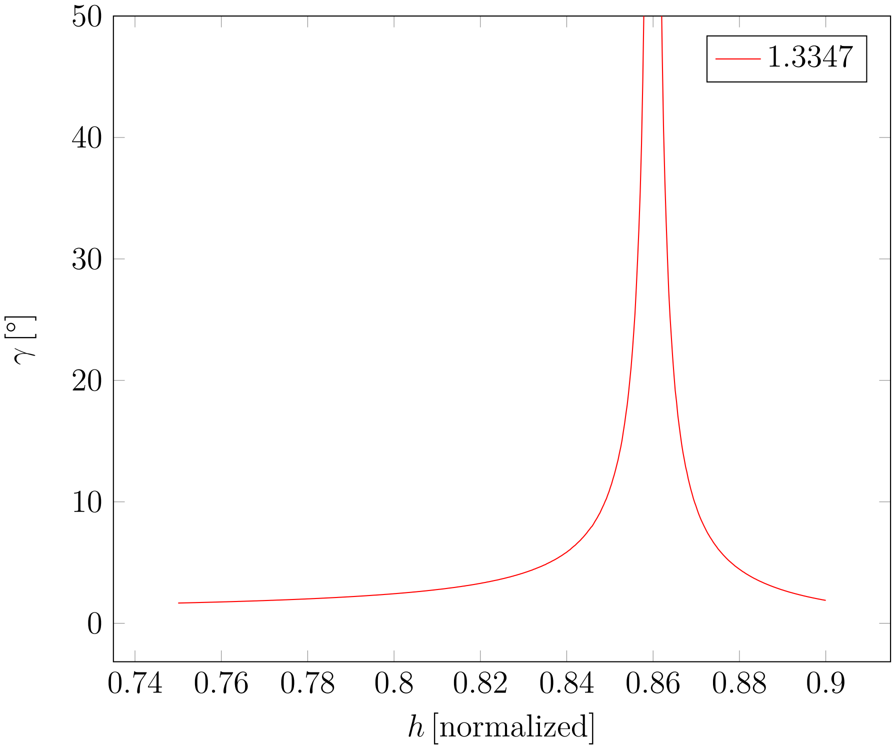

Figure 5: Power density scaling $\left|\frac{1}{\left.\frac{d\gamma(z)}{dz}\right|_{z=y}}\right|$ just because of the geometry.

Because $\gamma(h)$ exhibits a local extremum the derivative becomes 0 and thus the absolute value of its inverse ∞ at that point. Regarding numerical calculations this is a major inconvenience. It however justifies to only consider the geometrical relations for this investigation.

This however requires, that no other physcial effects annihilate all power transmitted at that extremum. I checked Fresnel for water and found the following:

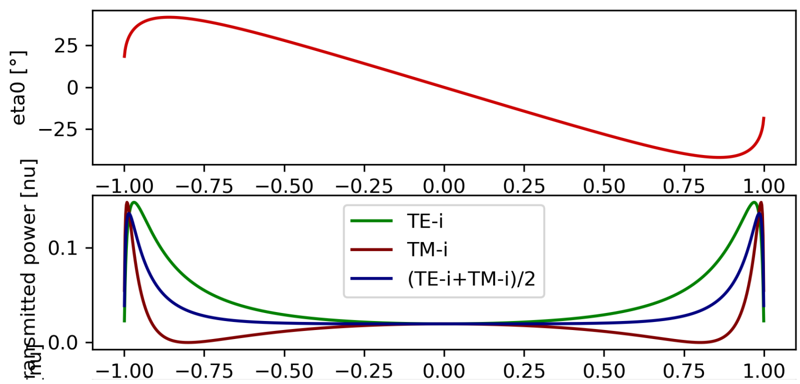

Figure 6: Angle and transmitted power in terms of Fresnel over h [eta0 $= -\gamma$].