Rainbows

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5. April 2020

1 Geometrical Assumptions

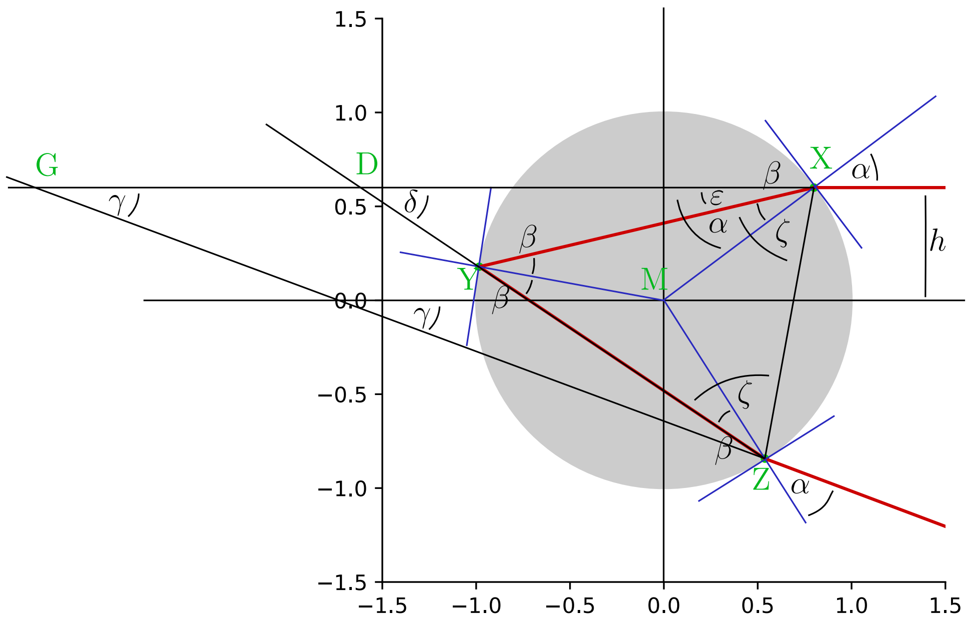

Abbildung 1: Principle scetch of the geometrical properties of a circular raindrop.

$$h = \sin(\alpha)$$

$$n_0 \sin(\alpha) = n_1 \sin(\beta)$$

$$\varepsilon = \alpha - \beta$$

$$\zeta = [180° - 2\beta]/2 = 90° - \beta$$

$$\delta = 180° - [[180° - 2\beta] + \varepsilon] = 2\beta - [\alpha - \beta] = 3\beta - \alpha$$

$$\gamma = 180° - 2[\zeta + \varepsilon] = 180° - 2[90° - \beta + \alpha - \beta] = 2[2\beta - \alpha]$$

$$\gamma(\alpha) = 2\left[2\arcsin\left(\frac{n_0}{n_1}\sin(\alpha)\right) - \alpha\right]$$

$$\gamma(h) = 2\left[2\arcsin\left(\frac{n_0}{n_1}h\right) - \arcsin(h)\right] \tag{1}$$

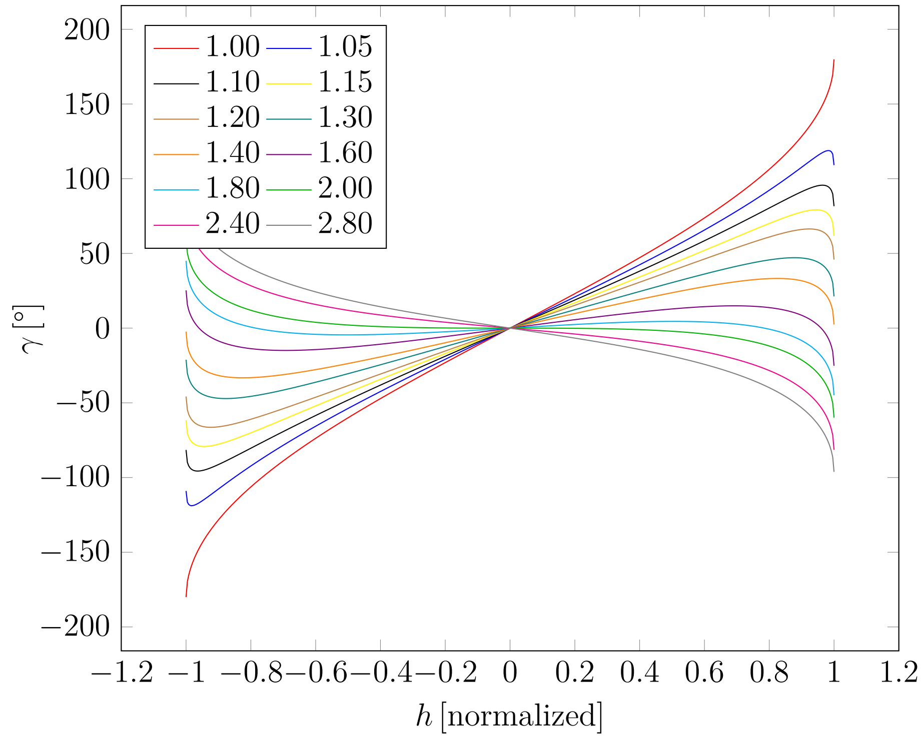

Abbildung 2: Excidence angle γ over incidence height h.

2 Extrema

Extrema $[\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}]$

$$\frac{d}{dh}\gamma(h) = 2\left[2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h]^2}} - \frac{1}{\sqrt{1-h^2}}\right]$$

$$\begin{aligned}
0 &\overset{!}{=} \frac{d}{dh}\gamma(h_{\text{extr.}}) \qquad (2)\\
&= 2\left[2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h]^2}} - \frac{1}{\sqrt{1-h^2}}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\sqrt{1-h^2}} &= 2\frac{n_0}{n_1}\frac{1}{\sqrt{1-[\frac{n_0}{n_1}h_{\text{extr.}}]^2}}\\
1-\left[\frac{n_0}{n_1}h_{\text{extr.}}\right]^2 &= \left[2\frac{n_0}{n_1}\right]^2\left[1-h^2_{\text{extr.}}\right]\\
\left[\frac{n_1}{2n_0}\right]^2 - \left[\frac{1}{2}h_{\text{extr.}}\right]^2 &= 1-h^2_{\text{extr.}}\\
\frac{3}{4}h^2_{\text{extr.}} &= 1-\left[\frac{n_1}{2n_0}\right]^2\\
h^{\pm}_{\text{extr.}} &= \pm\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]} \qquad (3)
\end{aligned}$$

$$1 < \frac{n_1}{n_0} \leq 2 \qquad (4)$$

$$\gamma^{\pm}_{\text{extr.}} = \gamma(h^{\pm}_{\text{extr.}}) = \pm 2\left[2\arcsin\left(\frac{n_0}{n_1}\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]}\right) - \arcsin\left(\sqrt{\frac{1}{3}\left[4-\left[\frac{n_1}{n_0}\right]^2\right]}\right)\right] \qquad (5)$$

For water at 20 °C and light of 550 nm [$n_0 = 1$, $n_1 = 1.3347$] this yields an angle of $\gamma^{+}_{\text{extr.}} = 41.83°$

3 Power Density Superelevation

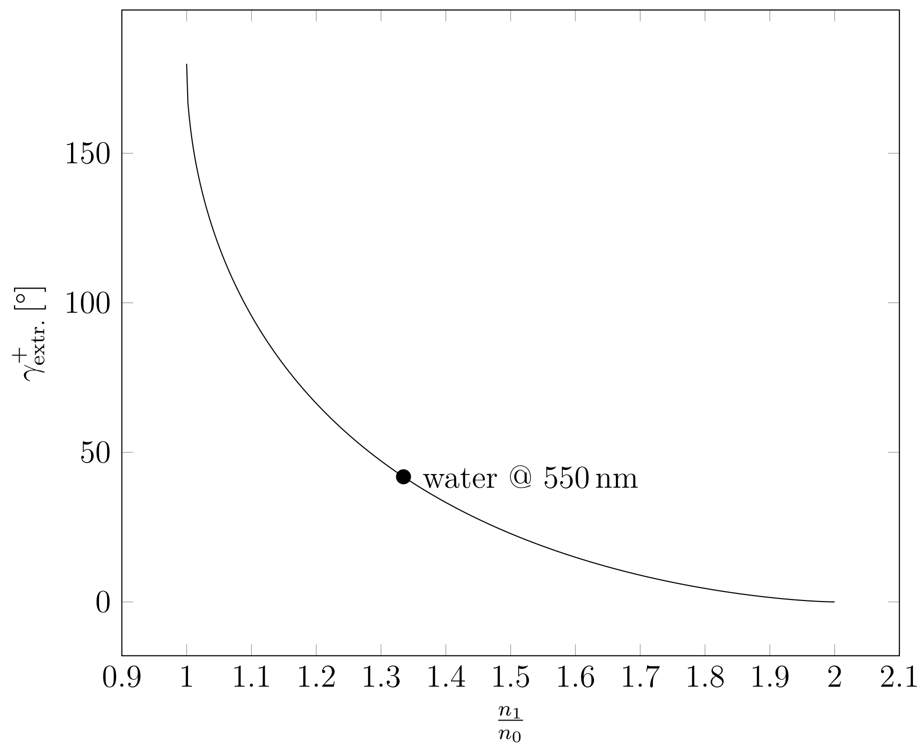

Abbildung 3: Excidence angle $\gamma+$ over incidence height h.