

Scaling equations

- Also known as using natural units or nondimensionalisation
- Can only represent limited range of numbers on a computer
- Want to avoid too large/small numbers in our codes
 - Choose sensible units
 - Better: scale away units, i.e. work with dimensionless variables.
 - Useful for debugging since it's easy to spot if results are suspiciously small/large
- Ex: simulation of the solar system

Silly choice of units: kg, m

$$\Rightarrow m_{\text{sun}} = 1.989 \times 10^{30} \text{ kg}$$

$$r \approx 1.496 \times 10^{11} \text{ m}$$

Clever choice of units: M_{\odot} , au

$$\Rightarrow m_{\text{sun}} = 1 M_{\odot}$$

$$r \approx 1 \text{ au}$$

- Ex: Exponential decay of radioactive nuclei

$$\frac{dN}{dt} = -\lambda N$$

Initial value: $N(t=0) = N_0$

$$\frac{dN}{dt} + \lambda N = 0$$

N : number of nuclei, $[N] = 1$

t : time, $[t] = s$

λ : decay constant, $[\lambda] = \frac{1}{s}$

- Define a scaled, dimensionless time variable (the indep. var.)

$$\hat{t} \equiv \lambda t$$

$$\Rightarrow \frac{d}{dt} = \frac{d\hat{t}}{dt} \frac{d}{d\hat{t}} = \lambda \frac{d}{d\hat{t}}$$

- Our diff. eq. becomes

$$\lambda \frac{dN}{d\hat{t}} + \lambda N = 0$$

$$\frac{dN}{d\hat{t}} + N = 0$$

- Since \hat{t} is dim.-less, we know that any step size $h \ll 1$ is a small step size, and that the time span we're interested in will typically be $\hat{t} \in [0, \text{a few}]$

- We can also scale the dependent variable (N)
- Sensible choice: use N_0

$$\hat{N} \equiv \frac{N}{N_0}, \text{ initial value: } \hat{N}_0 = \frac{N_0}{N_0} = 1$$

- Diff. eq. becomes

$$N_0 \frac{d\hat{N}}{d\hat{t}} + N_0 \hat{N} = 0$$

$$\boxed{\frac{d\hat{N}}{d\hat{t}} + \hat{N} = 0}$$

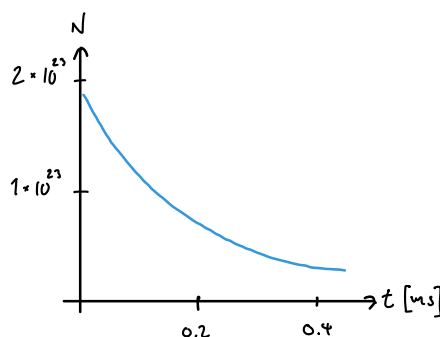
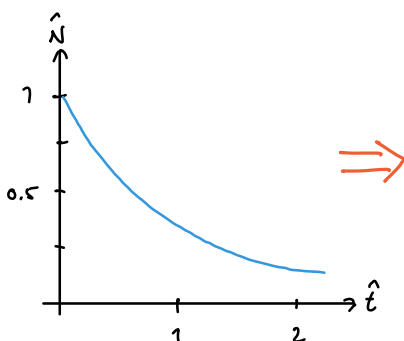
where we know that both \hat{N} and \hat{t} should be $O(1)$

Analytical solution:

$$\hat{N} = \hat{N}_0 e^{-\hat{t}} = e^{-\hat{t}} \quad \text{since } \hat{N}_0 = 1$$

$$\Rightarrow \frac{N}{N_0} = e^{-\lambda t} \Rightarrow \underline{N(t) = N_0 e^{-\lambda t}}$$

- Important! When presenting results, always use the original dimensionfull variables, or specify the natural units used



or

