Stability analysis

(Matrix method.
Von Neuman next week?)

- Go back to ODE: $\boxed{\frac{du}{dt} = f(t,u)}$

- System of ODEs:

$$\frac{du_1}{dt} = f_1(t,u_1) \qquad\qquad \frac{du_1}{dt} = f_1(t,u_1,u_2,\dots)$$

$$\frac{du_2}{dt} = f_2(t,u_2) \qquad\qquad \frac{du_2}{dt} = f_2(t,u_1,u_2,\dots)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

<u>decoupled</u>   <u>coupled</u>

- Coupled system:

$$\frac{d}{dt}\bar{u} = \bar{f}(t,\bar{u})$$

- Coupled system of <u>linear</u> ODEs:

$$\boxed{\frac{d}{dt}\bar{u} = L\bar{u}}$$

Example: $\frac{dx}{dt} = v$

$\frac{dv}{dt} = -\frac{k}{m}x$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

- Say we solve this with Forward Euler:
  - One eq: $u^{n+1} = u^n + h f^n$ ← time index
  - Syst. of eqs.: $\bar{u}^{n+1} = \bar{u}^n + h L \bar{u}^n$

$$(*) \qquad \boxed{\bar{u}^{n+1} = T \bar{u}^n}$$

[with $T = \mathbb{1} + hL$ for Forward Euler]

- T: the update matrix

- Matrix method for stability analysis:
  - Let $\bar{u}$ be our approx. to the true solution $\bar{u}_{true}$

$$\bar{u}^n = \bar{u}^n_{true} + \underbrace{\bar{\varepsilon}^n}_{\text{error}}$$

  - Insert into (*)

$$\bar{u}^{n+1}_{true} + \bar{\varepsilon}^{n+1} = T \bar{u}^n_{true} + T \bar{\varepsilon}^n$$

  - Taylor expand $\bar{u}^{n+1}_{true}$ about $\bar{u}^n_{true}$:

$$\bar{u}^{n+1}_{true} = \bar{u}^n_{true} + \bar{f}^n h + O(h^2) = (\mathbb{1} + hL)\bar{u}^n_{true} + O(h^2)$$

$$\approx T \bar{u}^n_{true}$$

$$\Rightarrow \quad \boxed{\bar{\varepsilon}^{n+1} = T \bar{\varepsilon}^n}$$

[Matrix eq. for error propagation]

$$\bar{\varepsilon}^{n+1} = T \bar{\varepsilon}^n + O(h^2)$$

- Condition for stability of scheme:

All eigenvalues λ_i of T must satisfy

[Ended lecture here]

$$\boxed{|\lambda_i| \leq 1}$$

[$\rho(T) \leq 1$

spectral radius of T

$\rho(T) = \max\{|\lambda_1|, |\lambda_2|, \dots\}$]

- In general: To determine if scheme is stable, identify update matrix T and check eigenvalues.

[Cont. here]

- Why req. $|\lambda_i| \leq 1$?

What we need is $\dfrac{|T\bar{\varepsilon}^n|}{|\bar{\varepsilon}^n|} \leq 1$

- Start from $\bar{\varepsilon}^{n+1} \approx T\bar{\varepsilon}^n$ $\longleftrightarrow$ I.e. leaving out the $O(h^2)$ term that gives rise to global error.

and diagonalise T:

$$T = R D R^{-1}$$

, where R has eigenvectors of T as columns, and D has eigenvalues along diagonal

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \dots)$$

$$\Rightarrow \quad \bar{\varepsilon}^{n+1} = R D R^{-1} \bar{\varepsilon}^n$$

- Left-multiply by rotation matrix R^{-1}

$$R^{-1}\bar{\varepsilon}^{n+1} = D R^{-1}\bar{\varepsilon}^{n}$$

Define : $R^{-1}\bar{\varepsilon} = \bar{\eta}$

Error vector in rotated basis

$$\bar{\eta}^{n+1} = D\bar{\eta}^{n}$$

- Error propagation in a <u>decoupled</u> basis $(D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots))$
- Can consider each error component individually

$$\eta_1^{n+1} = \lambda_1 \eta_1^{n}$$

$$\eta_2^{n+1} = \lambda_2 \eta_2^{n}$$

$$\vdots$$

- For the method to be stable, <u>none</u> of the errors can grow at every step. Must have

$$\boxed{g_i = \left|\frac{\eta_i^{n+1}}{\eta_i^{n}}\right| \leq 1} \quad \text{for all } i = 1, 2, \ldots$$

which is the same as $\boxed{|\lambda_i| \leq 1}$ for all $i = 1, 2, \ldots$

- The scheme will still <u>accumulate a global error</u>, (can be seen from the terms we left out in Taylor exp. and from inhomog. part of ODE.) but it won't blow up. (We have looked at the error <u>propagation</u> that could blow up.)

- Note: A stable method will still accumulate a global error

  - Say situation at timestep n is

$$\bar{u}^n = \bar{u}^n_{true} + \bar{\varepsilon}^n$$

  - Approx for next timestep will be

$$\bar{u}^{n+1} = T\bar{u}^n$$

$$= T\bar{u}^n_{true} + T\bar{\varepsilon}^n$$

$$= \left[\text{Next approx. } \underline{\text{if}} \text{ starting from } \underline{\text{true}}\ u^n \text{ value}\right] + \left[\text{Propagation of old error}\right]$$

$$\frac{d}{dt}\bar{u} = \bar{f}(t,\bar{u})$$

FE:
$$T\bar{u}^n_{true} = \bar{u}^n_{true} + h\frac{d}{dt}\bar{u}^n_{true}$$
$$= \bar{u}^n_{true} + h\bar{f}(t,\bar{u}^n)$$
$$= \left[1 + hL\right]\bar{u}^n_{true}$$
$$= \bar{u}^{n+1}_{true} + O(h^2)$$

$$\bar{u}^{n+1} = \Big[\bar{u}^{n+1}_{true} + \underbrace{O(h^2)}_{\text{New local error}}\Big] + \underbrace{\Big[T\bar{\varepsilon}^n\Big]}_{\text{Propagated old error}}$$

  - Can write this as

$$\underline{\bar{u}^{n+1} = \bar{u}^{n+1}_{true} + \bar{\varepsilon}^{n+1}} \quad \text{with} \quad \underline{\bar{\varepsilon}^{n+1} = T\bar{\varepsilon}^n + O(h^2)}$$

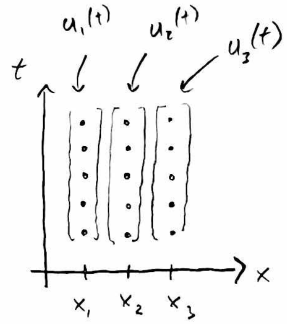
  - Stability: What happens to the propagated error

$$\text{If } |T\bar{\varepsilon}^n| > |\bar{\varepsilon}^n| \implies \underline{\text{unstable}}$$

  - Global error (Accuracy): The accumulation of all the $O(h^2)$ contributions.

- So far we've talked about <u>systems of coupled ODEs</u>

- Can view discretized PDE as system of coupled ODEs, where the coupled variables are the u's at each spatial grid point, i.e. $u_1, u_2, u_3, \ldots$



- Stability of <u>FD scheme</u> for <u>1+1 dim diffusion eq.</u> :

$$\bar{u}^{n+1} = A\,\bar{u}^{n} \longrightarrow \text{(Already on the form } \bar{u}^{n+1} = T\bar{u}^{n}\text{)}$$

$$\bar{u}^{n+1} = (\mathbb{1} - \alpha B)\,\bar{u}^{n} \qquad B = \text{tridiag}(-1, 2, -1)$$

- Reminder from Proj. 2 :

Eigenvalues of tridiag(a, d, a) : $\lambda_i = d + 2a\cos\left(\frac{i\pi}{N+1}\right)$

$\Rightarrow$ Eigenvalues of $(\mathbb{1} - \alpha B)$: $\lambda_i = 1 - \alpha\left(2 - 2\cos\left(\frac{i\pi}{N+1}\right)\right)$

- The requirement $|\lambda_i| \leq 1$

implies

$$-1 \leq 1 - 2\alpha\left(1 - \cos\left(\frac{i\pi}{N+1}\right)\right) \leq 1$$

$$-2 \leq -2\alpha\left(1 - \cos(\ldots)\right) \leq 0$$

$$0 \leq \alpha\left(1 - \cos(\ldots)\right) \leq 1$$

$$\alpha \leq \frac{1}{1 - \cos(\ldots)} \qquad \cos(\ldots) \geq -1 \;\Downarrow\; \frac{1}{1-\cos(\ldots)} \geq \frac{1}{2}$$

$$\Rightarrow \boxed{\alpha \leq \frac{1}{2}} \qquad (\Delta t \leq 2\Delta x^2)$$

- Stability of <u>BD scheme</u> for <u>1+1 dim. diffusion eq</u>:

$$A\,\bar{u}^{n+1} = \bar{u}^n$$

$$\left[\mathbb{1} + \alpha B\right]\bar{u}^{n+1} = \bar{u}^n$$

$$\Rightarrow \quad \bar{u}^{n+1} = \underbrace{\left[\mathbb{1} + \alpha B\right]^{-1}}_{\text{update matrix}}\bar{u}^n$$

- Eigenvalues of $[\mathbb{1} + \alpha B]$: $\lambda_i = 1 + \alpha(2 - 2\cos\theta_i)$

$$= 1 + 2\alpha(1 - \cos\theta_i)$$

$\Rightarrow$ Eigenvalues of $[\mathbb{1} + \alpha B]^{-1}$:

$$\theta_i = \frac{i\pi}{N+1}$$

$$\lambda_i = \frac{1}{1 + 2\alpha(1 - \cos\theta_i)}$$

- Stability req. $\quad |\lambda_i| \leq 1$

$$-1 \leq \frac{1}{1 + 2\alpha(1 - \cos\theta_i)} \leq 1$$

$$\Big|\; 1 - \cos\theta_i \geq 0$$

$$\frac{1}{1 + 2\alpha(1 - \cos\theta_i)} \leq 1$$

$$\alpha = \frac{\Delta t}{\Delta x^2}$$

<u>for all $\alpha \geq 0$</u> (any $\Delta t, \Delta x$)

So BD scheme is <u>unconditionally</u> stable.