Forward Euler

- Look at a single first-order diff. eq. $\boxed{\frac{dy}{dt} = f(t,y)}$
- Init. value prob, we know $y(t_0)$
  want to find the corresponding solution $y(t)$

Algo

$$\boxed{y_{i+1} = y_i + hf_i}$$

- Taylor exp of $y(t+h)$, h small

$$y(t+h) = y(t) + y'(t)h + O(h^2)$$

$$y(t+h) = y(t) + f(t,y)h + O(h^2)$$

- Discretize

$$\Rightarrow y_{i+1} = y_i + f_i h + O(h^2)$$

- Truncate

$$y_{i+1} \approx y_i + f_i h$$, truncation error $O(h^2)$

- Algo. for approximating $y_1, y_2, y_3 \ldots$, starting from y_0

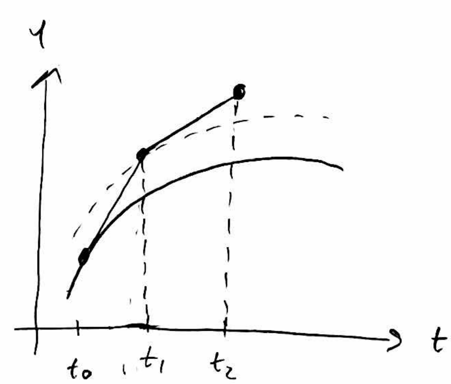
$$\boxed{y_{i+1} = y_i + hf_i}$$

  - Local truncation error $O(h^2)$ at each step
  - A total of $n \propto \frac{1}{h}$ steps

$\Rightarrow$ Global error $O(nh^2) = O(h)$ ← "First order method" which is not good

+ very simple
+ basic element in many other algos.

- Single-step method. We only need the current point y_i to find the next.



$$\frac{d^2x}{dT} = f(t, x, \frac{dx}{dT})$$

Euler, coupled

- $v_{i+1} = v_i + h f_i$
- $x_{i+1} = x_i + h v_i$

$$\left[v_i' = a_i = f_i \right]$$

$$\dot{v} = f$$

$$\dot{x} = v$$

Euler-Cromer

- $v_{i+1} = v_i + h f_i$
- $x_{i+1} = x_i + h v_{i+1}$

Predictor-Corrector method

- Simple improvement to Euler
- Also a single-step method (only requires knowing Y_i)

Algorithm

1) Predict: $$Y^*_{i+1} = Y_i + h f_i$$
$$f^*_{i+1} = f(t_{i+1}, Y^*_{i+1})$$

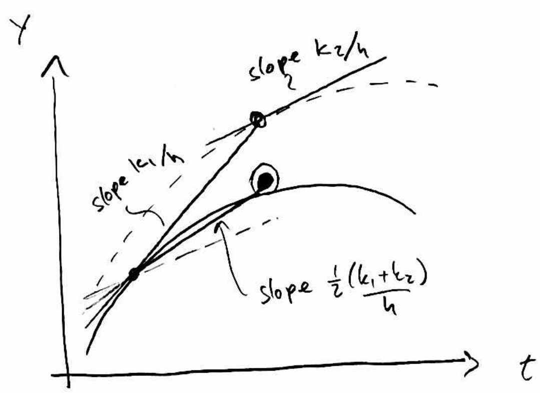
2) Correct: $$Y_{i+1} = Y_i + h \frac{f^*_{i+1} + f_i}{2}$$

Alt. notation:

$$k_1 \equiv h f_i = h f(t_i, Y_i)$$
$$k_2 \equiv h f^*_{i+1} = h f(t_{i+1}, Y^*_{i+1})$$

$$\Rightarrow \quad Y_{i+1} = Y_i + \frac{\cancel{h}\,1}{2}(k_1 + k_2)$$

- Euler uses gradient at a single point (f_i) to predict next point.
- Could improve by using <u>average</u> gradient between the two points t_i and t_{i+1}
- We <u>want</u> $$Y_{i+1} = Y_i + h\left(\frac{f_i + f_{i+1}}{2}\right)$$
  but we don't know f_{i+1}. But we can <u>predict it</u> using a simple forward Euler step.



(Predictor - Corrector cont.)

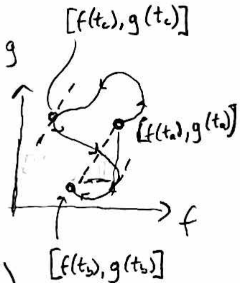
- Local error (trunc.) is $\mathcal{O}(h^3)$ $\Rightarrow$ Global error $\mathcal{O}(h^2)$

- One order better than FE, but also req. one extra evaluation of f (at $f(t_{i+1}, y^*_{i+1})$)

Proof:

$$Y(t+h) = Y(t) + h\,Y'(t) + \tfrac{1}{2}h^2 Y''(t) + \underbrace{R_2(t)}_{\mathcal{O}(h^3)}$$

Remainder



- Know from Mean Value Theorem (Cauchy's MVT, a.k.a. extended MVT) that there exist a $\xi \in [t, t+h)$ such that

$$Y(t+h) = Y(t) + h\,Y'(t) + \tfrac{1}{2}h^2 Y''(t) + \tfrac{1}{3!}h^3 Y'''(\xi)$$

exact!

Cauchy's MVT:
There exists a t_c such that
$$\frac{g'(t_c)}{f'(t_c)} = \frac{g(t_b) - g(t_a)}{f(t_b) - f(t_a)}$$

- Use $Y'(t) = f(t)$:

$$Y(t+h) = Y(t) + h\,f(t) + \tfrac{1}{2}h^2 f'(t) + \tfrac{1}{3!}h^3 f''(\xi)$$

- Replace $f'(t)$ with a forward diff. + remainder (MVT again...)

$$f'(t) = \frac{f(t+h) - f(t)}{h} - \tfrac{1}{2}h\,f''(\eta)$$

$$\Rightarrow \quad Y(t+h) = Y(t) + h\,f(t) + \tfrac{1}{2}h^2\left[\frac{f(t+h) - f(t)}{h} - \tfrac{1}{2}h\,f''(\eta)\right] + \tfrac{1}{3!}h^3 f''(\xi)$$

$$= Y(t) + h\,f(t) + \tfrac{1}{2}h\left[f(t+h) - f(t)\right] \underbrace{- \tfrac{1}{4}h^3 f''(\eta) + \tfrac{1}{3!}h^3 f''(\xi)}_{\mathcal{O}(h^3)}$$

$$\boxed{Y(t+h) = Y(t) + h\,\frac{f(t+h) - f(t)}{2} + \mathcal{O}(h^3)}$$

$$\boxed{Y_{i+1} = Y_i + h\,\frac{f_{i+1} - f_i}{2} + \mathcal{O}(h^3)}$$

- So, the local (trunc.) error is $\mathcal{O}(h^3)$, giving a global error $\mathcal{O}(h^2)$

$$\boxed{\frac{dy}{dt} = f(t, y)}$$

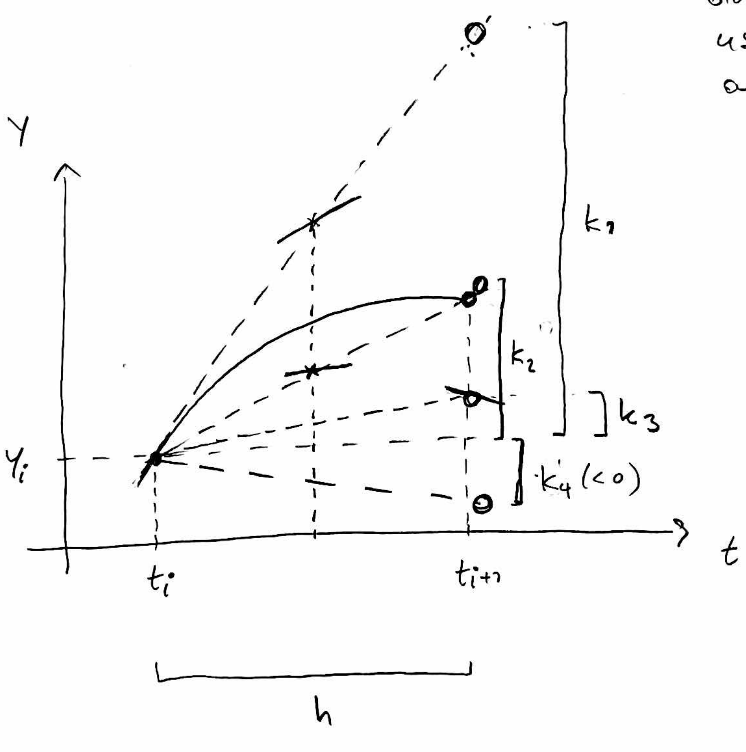
Runge-Kutta

- More "sophisticated" version of Predictor-Corrector
- An m-th order RK method uses m estimates of the gradient on the interval $t_i \le t \le t_{i+1}$ to determine Y_{i+1}
- Local error $\mathcal{O}(h^{m+1}) \Rightarrow$ global error $\mathcal{O}(h^m)$
- Classic choice : RK4

Algorithm :

$$\boxed{\begin{array}{lll}
1. & k_1 = & h f(t_i, Y_i) \\
2. & k_2 = & h f(t_i + \frac{1}{2}h, Y_i + \frac{1}{2}k_1) \\
3. & k_3 = & h f(t_i + \frac{1}{2}h, Y_i + \frac{1}{2}k_2) \\
4. & k_4 = & h f(t_i + h, Y_i + k_3) \\
5. & Y_{i+1} = & Y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]
\end{array}}$$

- 4 evaluations of f
- Global error $\mathcal{O}(h^4)$, so can use larger h than F.E. and P.C. $\Rightarrow$ More efficient



$$Y_{i+1} = Y_i + \frac{1}{6}k_1 + \frac{2}{6}k_2 + \frac{2}{6}k_3 + \frac{1}{6}k_4$$

RK4

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} \frac{dy}{dt}\, dt$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} \frac{dy}{dt}\, dt$$

$$\boxed{y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)\, dt}$$

Solve with Simpson's rule

Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{b-a}{6}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Two midpoint