March 10, 2023 10:52 AM

Minute Math: Determine whether the following integral is convergent or divergent. Evaluate it if it is convergent:

$$\int_1^\infty 2^{-x}\,dx \qquad\qquad 2^{-x} = \left(e^{\ln(2)}\right)^{-x} = e^{-\ln(2)x}$$

Sol: $$\int_1^\infty 2^{-x}\,dx = \lim_{t\to\infty}\int_1^t 2^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-\ln(2)x}\,dx = \lim_{t\to\infty}\left[\frac{e^{-\ln(2)x}}{-\ln(2)}\right]_1^t = \frac{1}{-\ln(2)}\lim_{t\to\infty}\left[2^{-x}\right]_1^t$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k}$$

$$= \frac{1}{-\ln(2)}\lim_{t\to\infty}\left(\cancelto{0}{2^{-t}} - 2^{-1}\right)$$

$$= \frac{1}{\ln(2)}\cdot\frac{1}{2}$$

Ex 4: For what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

$\int_1^\infty \frac{1}{x^2}\,dx$ conv.

$\int_1^\infty \frac{1}{x}\,dx$ diverges

Sol: we know from example 1 that when $p=1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty \frac{1}{x}\,dx \quad \text{DIV.}$$

$$\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \left(\frac{1}{1-p}\right)\left[x^{-p+1}\right]_1^t = \left(\frac{1}{1-p}\right)\left(t^{-p+1}-1\right) = \left(\frac{1}{1-p}\right)\left(\frac{1}{t^{p-1}}-1\right)$$

Therefore we focus on the case where $p\neq 1$. When $p\neq 1$,

$$\circledast \quad \int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left(\frac{1}{1-p}\right)\left(\frac{1}{t^{p-1}}-1\right)$$

If $p>1$ then $p-1>0$, thus $\frac{1}{t^{p-1}}\to 0$ $\quad$ ($t\to\infty$, so $t^{p-1}\to\infty$, so $\frac{1}{t^{p-1}}\to 0$)

thus for $p>1$, $$\circledast \quad \int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left(\frac{1}{1-p}\right)\left(\cancelto{0}{\frac{1}{t^{p-1}}}-1\right) = \left(\frac{1}{1-p}\right)(-1) = \frac{1}{p-1}$$

Otherwise, if $p<1$ then $p-1<0$, ie $1-p>0$, so $\frac{1}{t^{p-1}} = t^{1-p}\to\infty$

thus, for $p<1$: $$\circledast \quad \int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left(\frac{1}{1-p}\right)\left(\underbrace{\frac{1}{t^{p-1}}}_{\to\infty}-1\right)$$ thus the integral does not converge.

We summarize the result of Ex4:

$\boxed{\text{p-test}}$: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p\leq 1$

Type 2: Discontinuous integrands

Say f is cont. on $[a,b)$ but has vertical asymptote at $x=b$:

Setting $A(t) = \int_a^t f(x)dx$

it is natural to define:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

<u>Recall</u>: "$\lim\limits_{t\to b^-}$" means a limit as $t \to b$ but only for $t<b$. "Left hand limit"

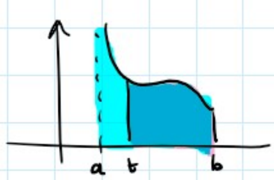
<u>Definition 2</u>: Improper Integral of Type 2 (Discontinuous Integrand)

a) If f is cont. on $[a,b)$ and is discont. at b, then:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx \quad \left(\text{If it exists as a finite number}\right)$$

b) If f is cont. on $(a,b]$ and is discont. at a, then:

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx \quad \left(\text{If it exists as a finite number}\right)$$



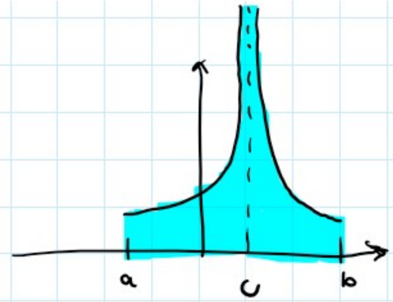
the integrals in a) and b) are called Convergent (or (CONV) if the limit exists as a finite # and divergent or (DIV) if the limit does not.

c) If f has a discontinuity at c, where $a<c<b$ and <u>both</u> $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ Converge, then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

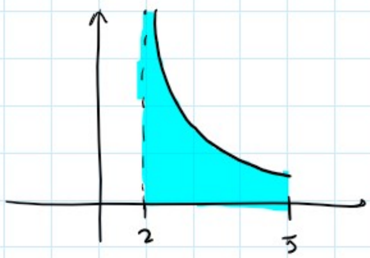
converge, then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



Ex 5: Find $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

Sol: First, we notice that this integral is improper because $\frac{1}{\sqrt{x-2}}$ has a vertical asymptote at $x=2$.



we use definition 2.b).

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \qquad \int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$$

$$= \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5$$

$$= \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right)$$

$$= 2(\sqrt{3} - \sqrt{0})$$

$$= 2\sqrt{3}$$

Ex 6: Skipped, in textbook.

Ex 7: Evaluate $\int_0^3 \frac{1}{x-1}dx$ if possible.

Sol: vertical asymptote at $x=1$.

$$\int_0^3 \frac{1}{x-1}dx = \int_0^1 \frac{1}{x-1}dx + \int_1^3 \frac{1}{x-1}dx$$

Now: $\int_0^1 \frac{1}{x-1}dx \underset{\text{Def 2.a)}}{=} \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \left[\ln|x-1|\right]_0^t$

$$= \lim_{t\to 1^-} \left(\ln|t-1| - \cancel{\ln|0-1|}\right) \qquad = \ln(1) = 0$$

$$= \lim_{t\to 1^-} \ln(1-t) \qquad \text{when } t\to 1^- \quad 1-t\to 0^+$$

$$= \lim_{y\to 0^+} \ln(y)$$

$$= -\infty$$

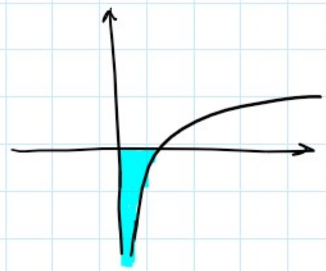
So $\int_0^1 \frac{1}{x-1}\,dx$ is divergent and so $\int_0^3 \frac{1}{x-1}\,dx$ is divergent.

<u>WARNING</u>: If we had not noticed the asymptote at $x=1$, we might have calculated (incorrectly):

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln|3-1| - \cancel{\ln|-1|}^{\,0} = \ln(2)$$

WRONG

<u>Ex 8</u>: $\int_0^1 \ln x\,dx$



<u>Sol</u>: vertical asymptote at $x=0$.

$$\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+} \int_t^1 \ln(x)\,dx$$

$$= \lim_{t\to 0^+} \Big[x\ln(x) - x\Big]_t^1$$

$$= \lim_{t\to 0^+} \Big((1\cancel{\ln(1)}^{\,0} - 1) - (t\ln(t) - t)\Big)$$

$$= \lim_{t\to 0^+} -t\ln(t) \cancel{+t} - 1$$

$$= -\Big(\lim_{t\to 0^+} t\ln(t)\Big) - 1$$

$$\lim_{t\to 0^+} t\ln(t) = \lim_{t\to 0^+} \underbrace{\frac{\ln(t)}{(1/t)}}_{\frac{\infty}{\infty}\text{ form}} \overset{L'H}{=} \lim_{t\to 0^+} \frac{(1/t)}{-(1/t^2)} = \lim_{t\to 0^+} -t = 0$$

$$\int_0^1 \ln(x)\,dx = -1$$