Lecture

January 6, 2023 4:25 PM

Announcements:

- Tutorials start today
- Sid will teach next class (Wed 11^{th})

Quick Note:

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$= \underbrace{f(x_0)\Delta x}_{i=1} + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$$

$$= \sum_{i=0}^{n-1} f(x_i)\Delta x$$

§5.2: The definite integral.

Definition: Let f be a function on a domain $[a,b]$.
We divide $[a,b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.
Let $x_0, x_1, \dots, x_n$ be the endpoints of those subintervals, and let $x_1^*, \dots, x_n^*$ be any sample points, so that
x_i^* lies in the interval $[x_{i-1}, x_i]$.
Then, we define the definite interal of f from a to b as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that this limit exists independently of our choice of sample points.
If it does exist, we say that f is integrable on $[a,b]$.

A note on notation:

$$\int_a^b f(x)\,dx$$

Upper limit of integration (b)

function we're integrating: integrand ($f(x)$)

Integral sign looks like an S because integrals are limits of sums.

just a symbol for now. It lets us know that we're integrating with respect to the $\boxed{\text{dummy variable } x}$. It's like the Δx in the limit. (dx)

lower limit of integration (a)

"Dummy Variable":

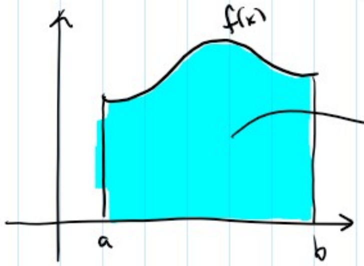
$\int^b f(x)dx$ is just a number (it doesn't depend on x)

$\int_a^b f(x)\,dx$ is just a number (it doesn't depend on x)

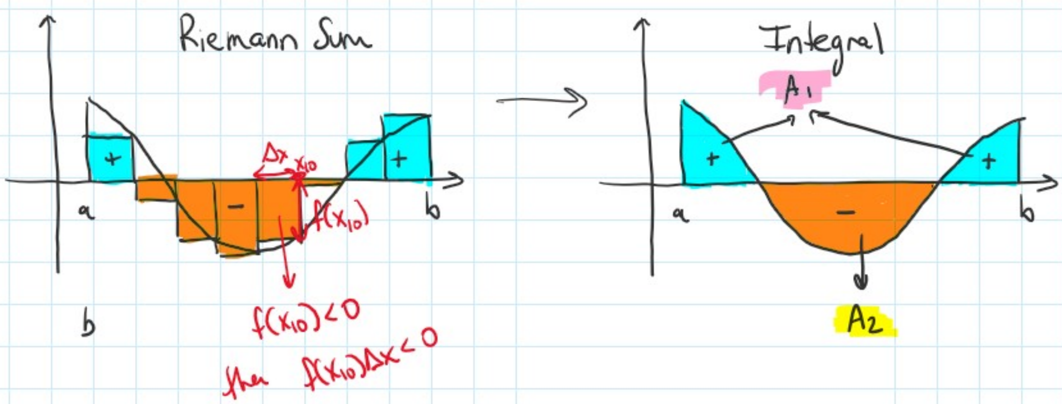
We can use any variable and it's still the same number

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt \qquad \text{But: } \int_a^x t^2\,dt \to \text{this depends on } x \text{ but not } t.$$

<u>Note</u>: If f is positive $\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ from a to b.



If f is negative in some places, we think of $\int_a^b f(x)\,dx$ as being the net area.



In that case $\int_a^b f(x)\,dx = A_1 - A_2$

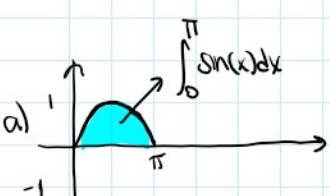
where A_1 is the area of the region above the x-axis and below f
and A_2 " " " " below the x-axis and above f

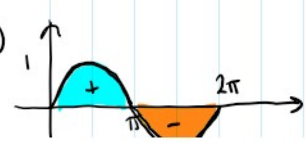
<u>Minute Math</u>:

a) Draw a picture of the area represented by $\int_0^{\pi} \sin(x)\,dx$

b) Do the same thing for $\int_0^{2\pi} \sin(x)\,dx$

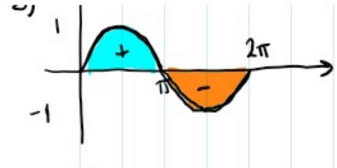
c) Use part b) to guess the value of $\int_0^{2\pi} \sin(x)\,dx$

a) 

b) 

c) Use part b) to guess the value of $\int_0 \sin(x)dx$

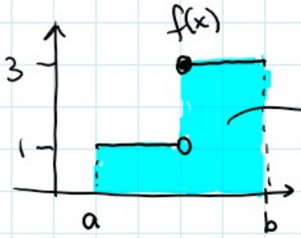
b)



c) $\int_0^{2\pi} \sin(x)dx = A_1 - A_2 = 0$

<u>Theorem</u>: If f is continuous on $[a,b]$, <u>or</u> if f has only finitely many jump discontinuities, then f is integrable on $[a,b]$; that is $\int_a^b f(x)dx$ exists.

<u>Ex</u>



$\int_a^b f(x)dx$ still makes sense

<u>Theorem</u> If f is continuous on $[a,b]$ then $\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

(meaning that when calculating we can ignore the theory of sample points)

<u>Ex</u> Evaluate $\int_0^3 (x^3-6x)dx$ using Rieman sums

<u>Sol</u>:

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$$

So
$$\int_0^3 (x^3-6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{27i^3}{n^3} - \frac{18i}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} \frac{27i^3}{n^3} - \sum_{i=1}^{n} \frac{18i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$\lim_{n\to\infty} \frac{n(n+1)}{n^2} = \lim_{n\to\infty} \frac{n}{n}\cdot\frac{n+1}{n}$$

$$= \lim_{n\to\infty} 1\cdot\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\left[\frac{n(n+1)}{2}\right]$$

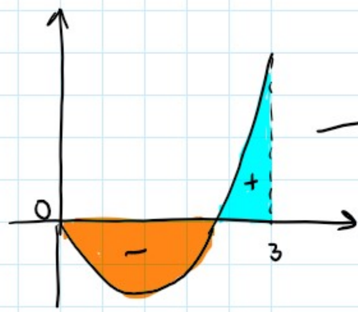
$$= \lim_{n\to\infty} \frac{81}{4}\cdot\frac{n(n+1)n(n+1)}{n^4} - \frac{54}{2}\cdot\frac{n(n+1)}{n^2}$$

$$= \lim_{n\to\infty} 1\cdot\left(1+\frac{1}{n}\right)$$

$$= 1$$

$$= \frac{81}{4}\cdot 1 - \frac{54}{2}\cdot 1$$

$$= -\frac{27}{4}$$



$\int_0^3 (x^3 - 6x)\,dx$ should be negative.

Our answer makes sense.