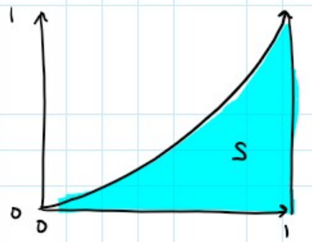
Lecture 2

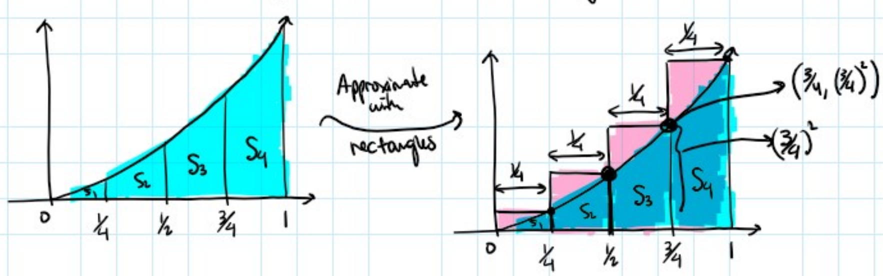
January 5, 2023 6:12 PM

LAST Time: Let's find the area of the region under the parabola from $x=0$ to $x=1$



Let A be the area of the region S.

To find A we approximate S with rectangles:

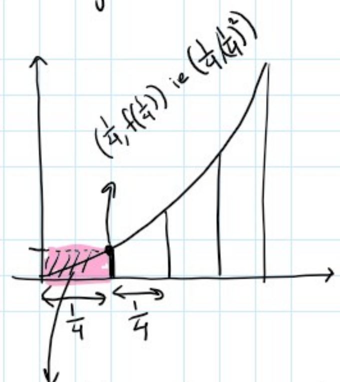


The height of each rectangle is the value of $f(x)=x^2$ at the right endpoint of each interval $[0,1/4], [1/4, 1/2], [1/2, 3/4], [3/4, 1]$

$$\text{Area} = R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

$$= \frac{15}{32} = 0.46875$$

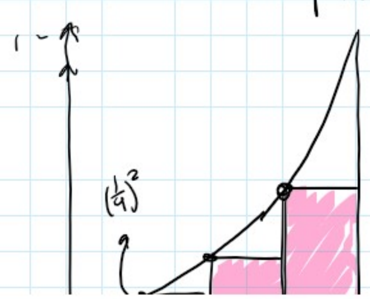
Lecture 2: My name is Gavin

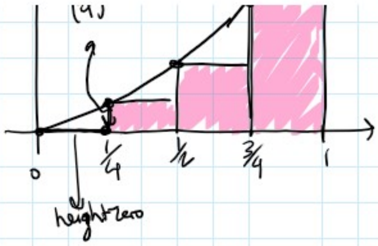


area of first rectangle is $= b \cdot h = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2$

therefore $A < R_4 = 0.46875$.

what if we used left endpoints?





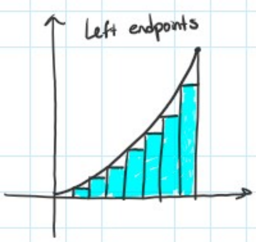
$[0, \tfrac{1}{4}], [\tfrac{1}{4}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{3}{4}], [\tfrac{3}{4}, 1]$

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$
$$= \frac{7}{32}$$
$$= 0.21875$$

we notice $L_4 < A$

So $0.21875 < A < 0.46875$

We can repeat this process with more and more rectangles



$L_8 = 0.2734375 < A < R_8 = 0.3984375$

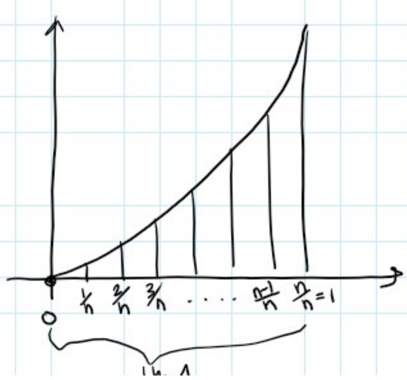
n	L_n	R_n
10	0.2850000	0.385000
20		
30		
50		
100		
1000	0.3328335	0.3338335

$$A \simeq \frac{L_{1000} + R_{1000}}{2} = 0.333335$$

we guess $A = 1/3$.

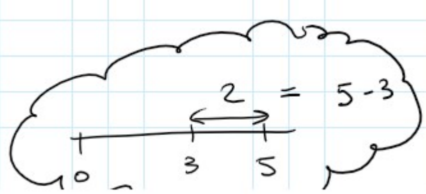
Claim: $\lim_{n \to \infty} R_n = 1/3$

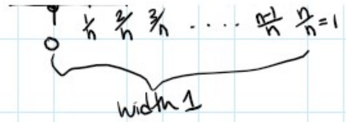
Proof:

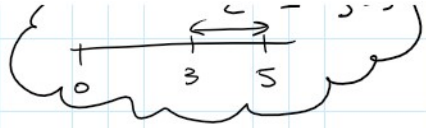


split the region into n rectangles, each of equal width

each will have width:
$$\frac{1-0}{n} = \frac{1}{n}$$







Our intervals are:

$$\left[0,\tfrac{1}{n}\right], \left[\tfrac{1}{n},\tfrac{2}{n}\right], \left[\tfrac{2}{n},\tfrac{3}{n}\right], \ldots, \left[\tfrac{n-1}{n},\tfrac{n}{n}\right]$$

we take the height of the rectangle as the value of $f(x)=x^2$ at the right endpoint

$$b \cdot h$$

$$R_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \qquad \boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{n(n+1)(2n+1)}{6n^3}$$

We want $\lim_{n\to\infty} R_n = \frac{1}{3}$, so we calculate:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6}\lim_{n\to\infty}\frac{n}{n}\cdot\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}$$

$$= \frac{1}{6}\lim_{n\to\infty} 1\cdot\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}\cdot 1\cdot 1\cdot 2$$

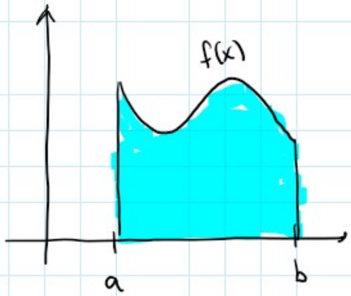
$$= \frac{1}{3}$$

Exercise for home: Show $\lim_{n\to\infty} L_n = 1/3$

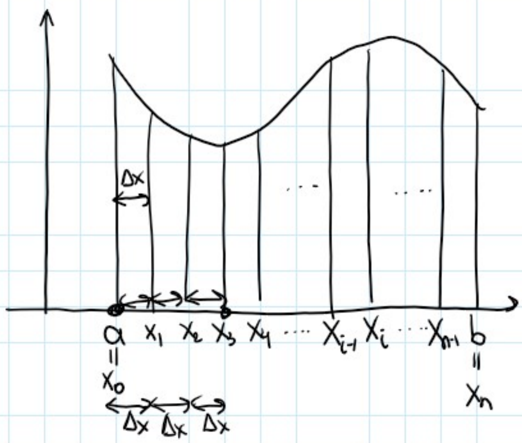
So we conclude $\underbrace{\lim_{n\to\infty} L_n}_{=1/3} \le A \le \underbrace{\lim_{n\to\infty} R_n}_{=1/3}$

so $A = 1/3$.

How can we do this approximation procedure in general?



Area under the curve of $f(x)$ from $x=a$ to $x=b$.



we split the interval into n strips of equal width.

$$\frac{b-a}{n} = \Delta x$$ ← the distance between interval endpoints

So we divide the interval $[a,b]$ into subinterval

$$[x_0,x_1],\ [x_1,x_2],\ \dots,\ [x_{i-1},x_i],\ \dots,\ [x_{n-1},x_n]$$

where

$$\begin{aligned}
x_0 &= a \\
x_1 &= a+\Delta x \\
x_2 &= a+2\Delta x \\
x_3 &= a+3\Delta x \\
&\vdots \\
x_i &= a+i\Delta x \\
&\vdots \\
x_n &= a+n\Delta x
\end{aligned}$$

let's double check that x_n is truely b:

$$x_n = a+n\Delta x = a+\cancel{n}\left(\frac{b-a}{\cancel{n}}\right) = \cancel{a}+b-\cancel{a} = b$$

$$[x_0,x_1],\ [x_1,x_2],\ \dots,\ [x_{i-1},x_i],\ \dots,\ [x_{n-1},x_n]$$

then

$$\begin{aligned}
R_n &= \Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_i) + \cdots + \Delta x f(x_n) \\
&= \sum_{i=1}^{n} f(x_i)\Delta x
\end{aligned}$$

$\Delta x = \frac{b-a}{n}$ and $x_i = a+i\Delta x$

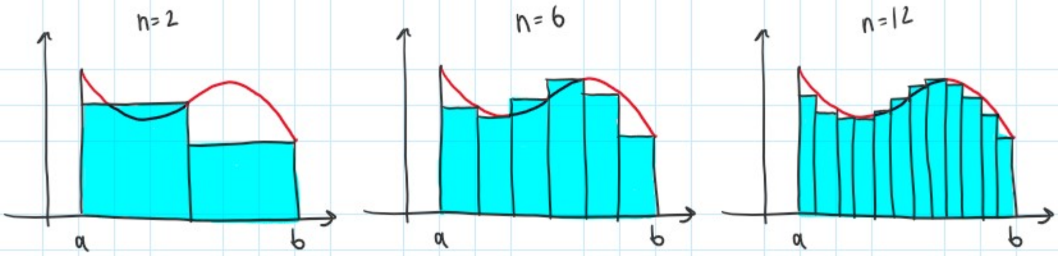
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

$\Delta x = \frac{b-a}{n}$

and $x_i = a + i\Delta x$

This is called a right-Riemann sum.

For a continuous function, these approximations get better as n increases.



So, the area of the region is $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

In fact, it doesn't matter if we use left or right endpoints for a continuous function

subinterval: $[x_{i-1}, x_i]$ then $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

So we could also write $A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$

<u>and</u> in fact, we can make the height of our rectangles be the value of f at <u>any</u> point in the subinterval $[x_{i-1}, x_i]$

We write x_i^* to be a point in the interval $[x_{i-1}, x_i]$, we call such a point a <u>sample point</u>.

then $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.

<u>Examples</u>

if $x_i^* = x_i$ we're using right endpoints

if $x_i^* = x_{i-1}$ " " left "

if $x_i^* = \overline{x}_i = \frac{x_i + x_{i-1}}{2}$ we're using the midpoint of the interval as a sample point.

$$A = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$$

"M" for midpoint