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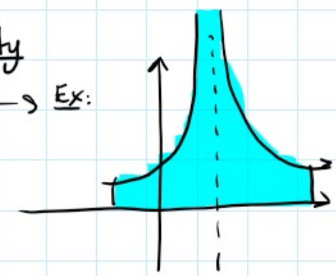
§7.8 Improper Integrals

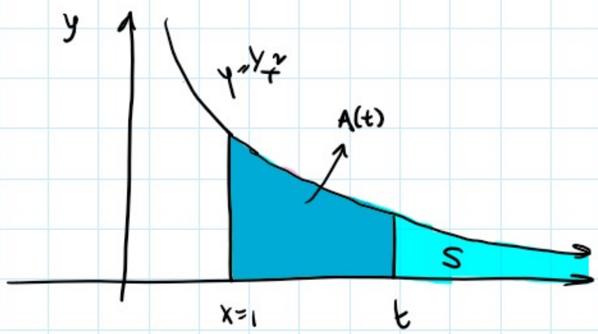
Today we will extend the notion of a definite integral $\int_a^b f(x)\,dx$ to infinite intervals and also to the case where f has an infinite discontinuity → Ex:



In either case, the integral is called improper.

Type 1: Infinite intervals

Motivating example: Consider the region S that lies under $y = 1/x^2$, above the x-axis, and to the left of $x = 1$.



S is an unbounded region

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

Definition 1 Improper integrals of type 1 (Infinite intervals)

a) If $\int_a^t f(x)\,dx$ exists for all $t \geq a$. then,

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$ (Provided this limit exists as a finite number)

b) If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$ (Provided this limit exists as a finite number)

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called Convergent (CONV) if the corresponding limit exists and is called divergent (DIV) if the limit doesn't exists.

c) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are CONV, we define

c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are CONV, we define

$$\longrightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^\infty f(x)dx,$$

where a can be any real Number. If there is some a for which either $\int_a^\infty f(x)dx$ <u>or</u> $\int_{-\infty}^a f(x)dx$ are DIV we say $\int_{-\infty}^\infty f(x)dx$ is <u>divergent</u>.

⚠ <u>WARNING</u>: Be careful with how we defined $\int_{-\infty}^{\infty} f(x)dx$. In particular, we did <u>not</u> define it as $\lim_{t\to\infty}\int_{-t}^{t} f(x)dx$.

To see why, let $f(x) = x$.

Then, for all $t \geq 0$ $\quad \int_{-t}^{t} x\,dx = 0 \quad$ (because $f(x)=x$ is odd)

So therefore $\quad \lim_{t\to\infty}\int_{-t}^{t} x\,dx = \lim_{t\to\infty} 0 = 0.$

However, for any a, $\quad \int_a^\infty x\,dx = \lim_{t\to\infty}\int_a^t x\,dx = \lim_{t\to\infty} \frac{t^2}{2} - \frac{a^2}{2} = \infty$

and so $\int_a^\infty x\,dx$ diverges, and so $\int_{-\infty}^{\infty} x\,dx$ is a divergent integral.

<u>Ex 1</u> Determine if $\int_1^\infty \frac{1}{x}dx$ is CONV or DIV

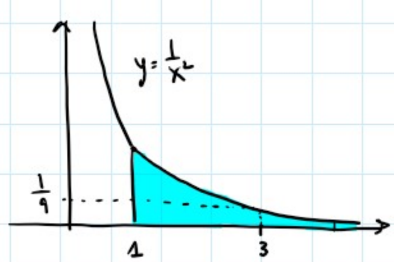
<u>Sol</u>: $\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}\left[\ln|x|\right]_1^t = \lim_{t\to\infty} \ln(t) - \underbrace{\cancel{\ln(1)}}^{0} = \lim_{t\to\infty}\ln(t) = \infty$

Thus $\int_1^\infty \frac{1}{x}dx$ is divergent.

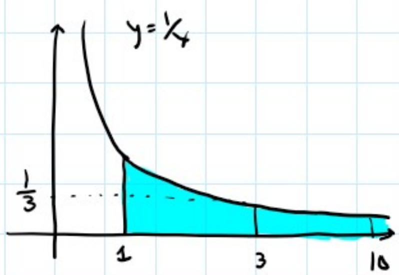
If we compare our result from Ex1 to the motivating example, we see:

$$\int_1^\infty \frac{1}{x^2}dx \text{ converges, but } \int_1^\infty \frac{1}{x}dx \text{ diverges.}$$

Geometrically:



$y = \frac{1}{x^2}$ decays fast enough to



$y = \frac{1}{x}$ doesn't decay fast enough

$y = \frac{1}{x^2}$ decays fast enough to make the area finite

$y = \frac{1}{x}$ doesn't decay fast enough

Ex 2: Evaluate $\displaystyle\int_{-\infty}^{0} xe^x\,dx$

Sol: By definition 1b), $\displaystyle\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$

$$\int_t^0 xe^x\,dx = \left[xe^x\right]_t^0 - \int_t^0 e^x\,dx = \left[xe^x - e^x\right]_t^0$$

$\left[F(x)+G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= (0-1) - (te^t - e^t)$$
$$= -te^t - 1 + e^t$$

So
$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\left(-te^t - 1 + e^t\right)$$
$$= \lim_{t\to-\infty}(-te^t) + \lim_{t\to-\infty}(-1) + \cancelto{0}{\lim_{t\to-\infty} e^t}$$
$$= -\Big(\cancelto{0}{\lim_{t\to-\infty} te^t}\Big) - 1$$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\underbrace{\frac{t}{e^{-t}}}_{\frac{-\infty}{\infty}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty} -e^t = 0$$

use L'H.

thus $\displaystyle\int_{-\infty}^{0} xe^x\,dx = -1$

__L'Hopital's Rule:__ if $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ has one of the following indeterminate form: $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$

then $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

(here a can be ∞ or $-\infty$, or any number)

Ex Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$

Sol: In definition 1c) we pick $a=0$ because it is the most convenient.

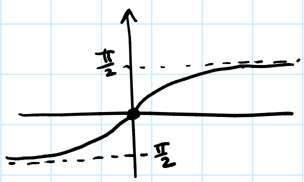
So $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx$ so long as __both__ converge

$\displaystyle\int^{\infty}\frac{1}{\ }\ \ldots = \lim \int^{t}\frac{1}{\ }\ \ldots = \lim\left[\ \ldots\ \right]^t$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \left[\arctan(x)\right]_0^t$$

$$= \lim_{t\to\infty} \arctan(t) - \cancelto{0}{\arctan(0)}$$

$$= \frac{\pi}{2}$$



Similarly $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \left[\arctan(x)\right]_t^0 = \lim_{t\to-\infty} -\arctan(t)$

$$= -\left(-\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

Thus $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

<u>Minute Math</u>: Determine whether $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$ converges.

Notice that for all $t \geq 1$, $\displaystyle\int_1^t \frac{\ln(x)}{x}\,dx = \int_0^{\ln(t)} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln(t)} = \frac{(\ln(t))^2}{2}$

$u = \ln(x)$
$du = \frac{1}{x}dx$

Thus $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \frac{(\ln(t))^2}{2} = \infty$

this diverges.