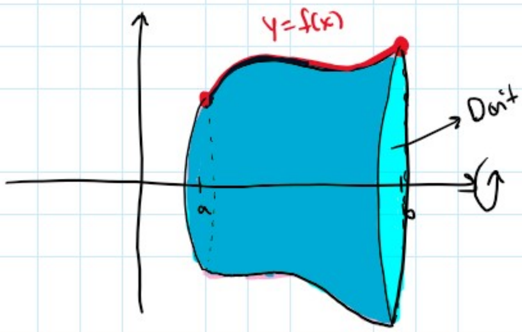
February 24, 2023 10:47 AM

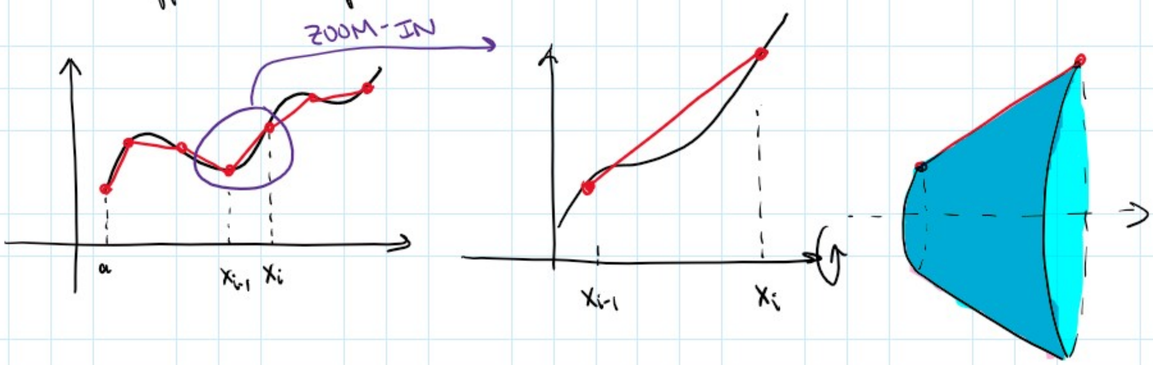
Last time:

LAST TIME:

What is the surface area of a solid of revolution?

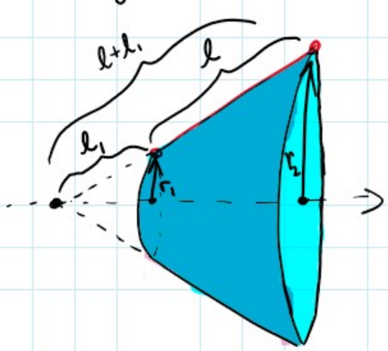


How do we approximate complicated surface areas?



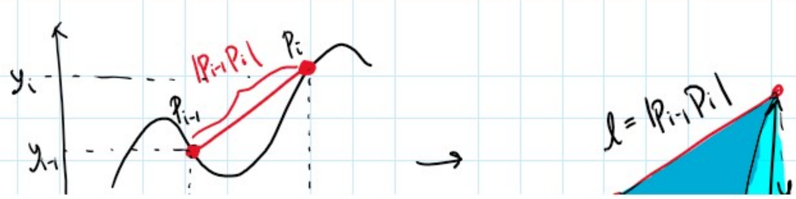
A rotated line segment looks like this, which we will call a "band"

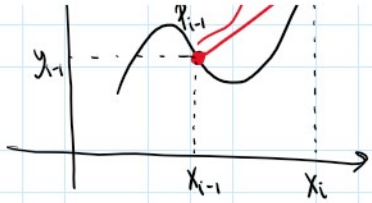
Let's get the surface area of the "band" shape.

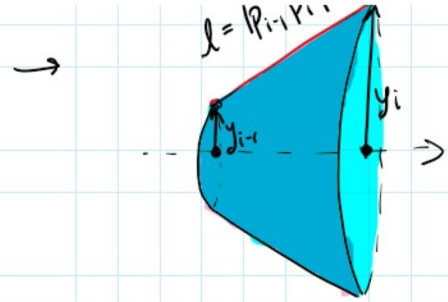


$$A = 2\pi l\left(\frac{r_1+r_2}{2}\right)$$

What does this mean for our approximation?







So, the surface area of the i^{th} $= 2\pi \, |P_{i-1}P_i| \left(\frac{y_{i-1}+y_i}{2}\right)$

From our work on §8.1, we saw:

$$|P_{i-1}P_i| = \sqrt{1+[f'(x_i^*)]^2}\,\Delta x, \quad \text{for some } x_i^* \in [x_{i-1}, x_i]$$

Also, when Δx is small, the interval $[x_{i-1}, x_i]$ will be a small interval meaning that x_{i-1}, x_i, and x_i^* will all be close to eachother,
So, because f is continuous,

$$y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$$
$$y_i = f(x_i) \approx f(x_i^*)$$

therefore: $\left(\frac{y_{i-1}+y_i}{2}\right) \approx \left(\frac{f(x_i^*)+f(x_i^*)}{2}\right) = f(x_i^*)$

So, the i^{th} band has surface area approximately equal to:

$$2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$

So, our surface area

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$$
$$= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

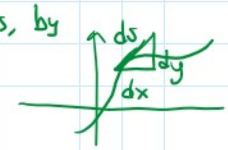
In, summary, the surface area A of a surface of revolution obtained by rotating the curve $y=f(x)$, $a \le x \le b$, about the x-axis, is

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

So, in Liebniz notation, $y=f(x)$

$$\circledast \quad S = \int_a^b 2\pi y \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{ds}$$

we remember the formula for ds, by
$(ds)^2 = (dx)^2 + (dy)^2$



Similarly, for a curve $x=g(y)$, $c \le y \le d$, rotated around the y-axis

Similarly, for a curve $x = g(y)$, $c \le y \le d$, rotated around the y-axis

$$(**) \quad S = \int_c^d 2\pi x \underbrace{\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}_{ds}$$

Now, using the formula for ds:

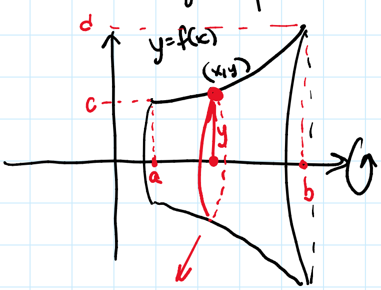
$$(*): \quad S = \int_a^b 2\pi y \, ds \qquad\qquad (**): \quad S = \int_c^d 2\pi x \, ds$$

where you can use either:

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad\qquad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

(which we obtain by manipulating $(ds)^2 = (dx)^2 + (dy)^2$)

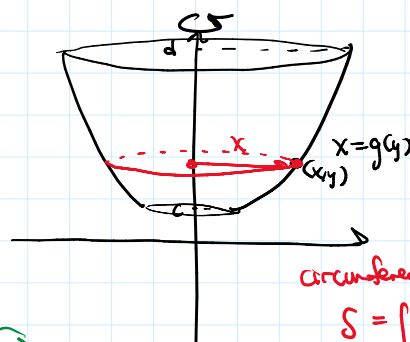
To remember these formulas, think of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by a point (x,y) on the curve as it is rotated about an axis:



circumference $2\pi y$

$$S = \int_a^b 2\pi y \, ds = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_c^d 2\pi y \, ds = \int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$



circumference $2\pi x$

$$S = \int 2\pi x \, ds$$