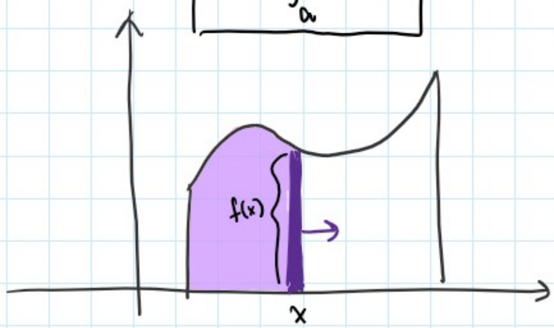
Last time:

$$\boxed{A = \int_a^b f(x)\,dx}$$

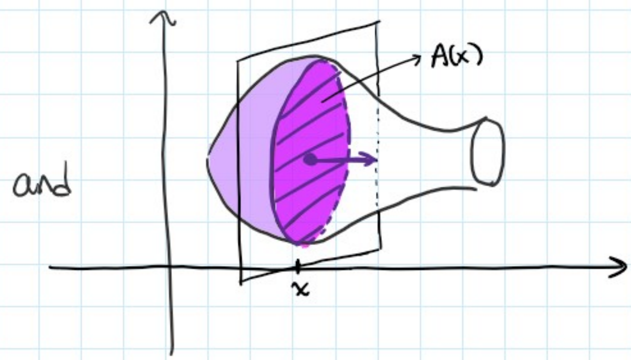


Moving a line with height $f(x)$ traces out an area

(lines: 1D, areas 2D)

and

$$\boxed{V = \int_a^b A(x)\,dx}$$



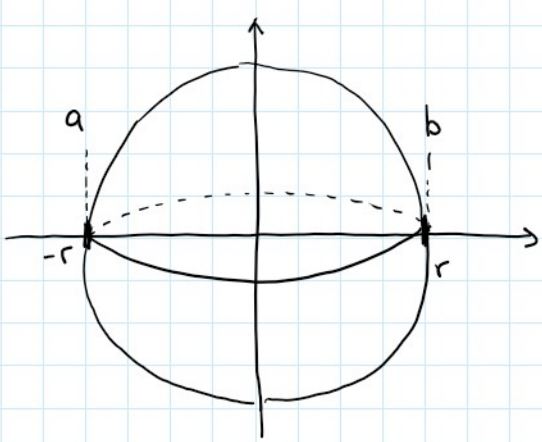
Moving a cross sectional area with area $A(x)$ traces out a volume

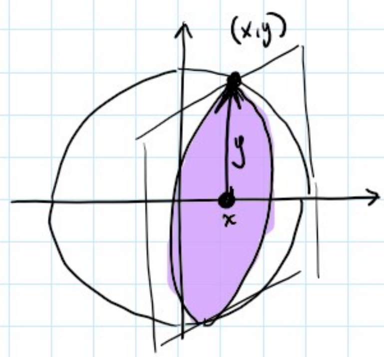
(areas: 2D, Volumes: 3D)

$$\boxed{\underline{\text{Volume}}: \quad V = \int_a^b A(x)\,dx}$$

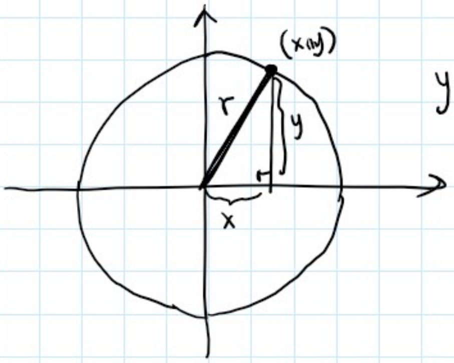
Ex1: show that the volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$

Sol: place our sphere's center at the origin:





the cross sectional area $A(x)$ is a circle.
the radius of the circle will by y, ie $A(x) = \pi y^2$



$y = \sqrt{r^2 - x^2}$ by the pythagorean theorem, ie $A(x) = \pi y^2 = \pi(r^2 - x^2)$

A(x) is even

thus $$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\pi \int_0^r r^2 - x^2\,dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

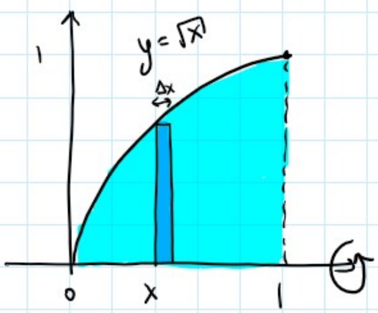
$$= 2\pi \left(r^3 - \frac{r^3}{3} \right)$$

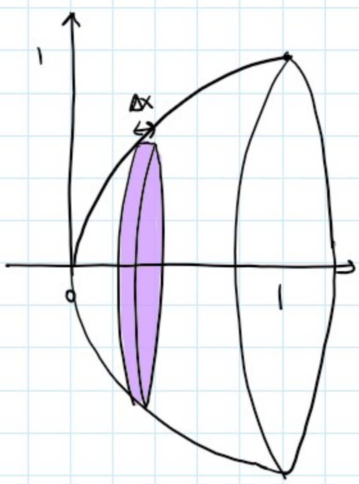
$$= 2\pi \left(\frac{2}{3} \right) r^3$$

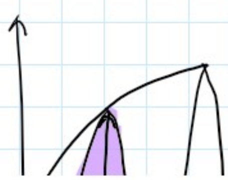
$$= \frac{4}{3}\pi r^3$$

Ex2: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

Sol: First we sketch the region under $y = \sqrt{x}$ from 0 to 1

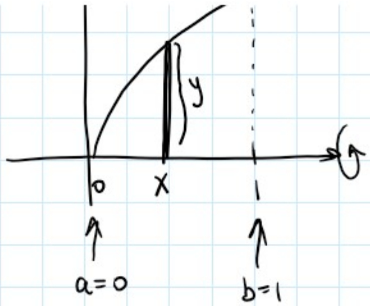


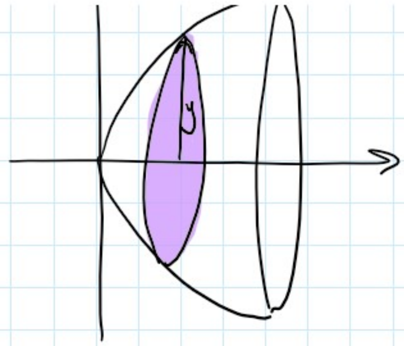




A(x) is a circle with radius y, $A(x) = \pi y^2$

$y = \sqrt{x}$





$A(x)$ is a circle with radius y, $A(x) = \pi y^2$

$= \pi(\sqrt{x})^2$

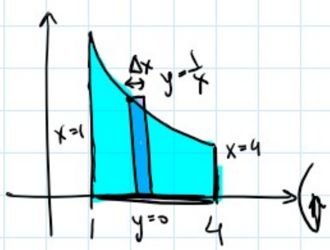
$= \pi x$

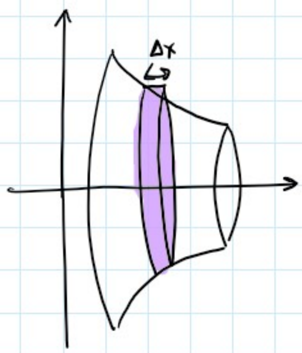
$y = \sqrt{x}$

$$V = \int_0^1 A(x)dx = \pi \int_0^1 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}$$

Minute Math: Find the Volume of the Solid obtained by rotating the region bounded by $y = 1/x$, $y=0$, $x=1$, $x=4$ around the x axis. Sketch the region, the solid, and an approximating cylinder. ✓

Sol:





$$V = \int_1^4 A(x)dx = \int_1^4 \pi y^2 dx$$

$$= \pi \int_1^4 \left(\frac{1}{x^2}\right) dx$$

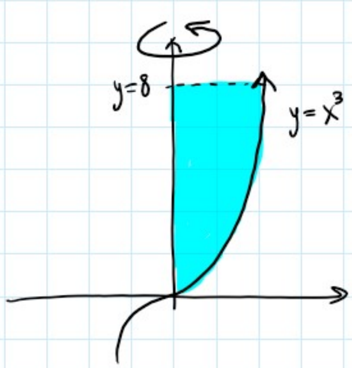
$$= \pi \left[-\frac{1}{x}\right]_1^4$$

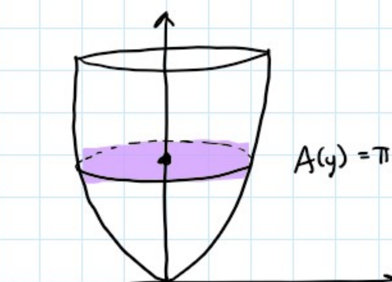
$$= \pi\left[-\frac{1}{4} + 1\right]$$

$$= \frac{3}{4}\pi$$

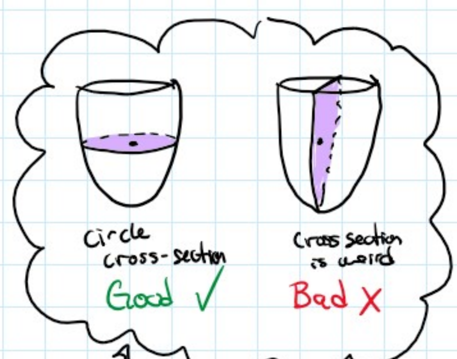
Ex3 Find the volume of the Solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ around the y-axis.

Sol:





$A(y) = \pi x^2$



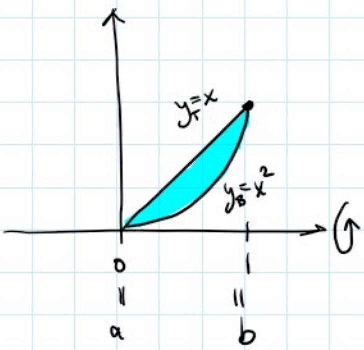
Let's integrate in y since we are rotating around the y axis (aka horizontal cross sections will be easier to calculate)

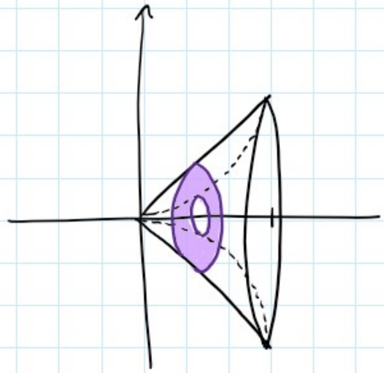
$$y = x^3 \qquad \text{get} \qquad x = \sqrt[3]{y}$$

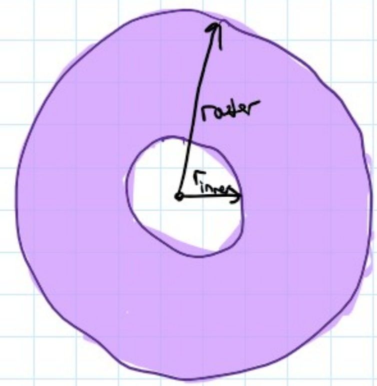
$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi x^2\,dy = \int_0^8 \pi y^{2/3}\,dy = \cdots = \frac{96\pi}{3}$$

<u>Ex 4</u>: The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated around the x-axis. Find the volume of the resulting solid.

<u>Sol</u>:







$$A(x) = \pi r_{outer}^2 - \pi r_{inner}^2$$

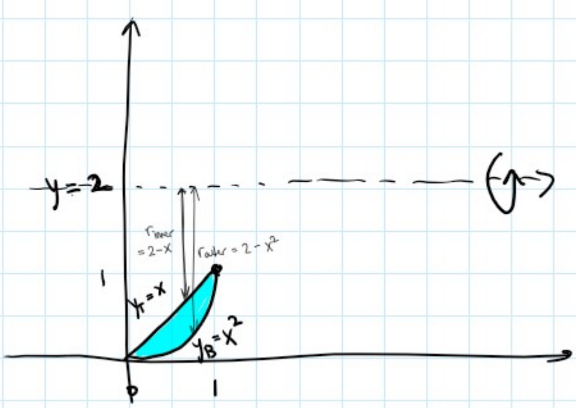
$r_{inner} = y_B = x^2$

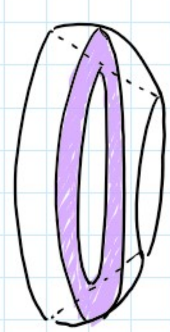
$r_{outer} = y_T = x$

$$A(x) = \pi x^2 - \pi (x^2)^2$$

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 x^2 - x^4\,dx = \cdots = \frac{2\pi}{15}$$

<u>Ex 5</u>: Find the volume of the same region R in ex 4 rotated around the line $y = 2$.





$$A(x) = \pi r_{outer}^2 - \pi r_{inner}^2$$

$$A(x) = \pi r_{outer}^2 - \pi r_{inner}^2$$
$$= \pi(2-x^2)^2 - \pi(2-x)^2$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left(\pi(2-x^2)^2 - \pi(2-x)^2\right) dx$$
$$= \cdots$$
$$= \frac{8\pi}{15}$$