March 17, 2023 2:57 PM

Last time: we saw that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Today: Continuing §11.2:

$\boxed{\text{Theorem 6: If } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} a_n = 0.}$

Proof: Let $S_N = a_1 + a_2 + \cdots + a_N$

Then $a_N = (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{N-1}} + a_N) - (\cancel{a_1} + \cancel{a_2} + \cdots + \cancel{a_{N-1}})$

$= S_N - S_{N-1}$

then because the series converges, $\lim_{N\to\infty} S_N = S$ $\left(S = \sum_{n=1}^{\infty} a_n\right)$

also $\lim_{N\to\infty} S_{N-1} = S$

So $\lim_{N\to\infty} a_N = \lim_{N\to\infty} (S_N - S_{N-1}) = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = S - S = 0.$

⚠ The Reverse Statement is not true!

The "reverse statement" would be "If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges"

This is not true, an example is the harmonic series $(a_n = 1/n)$

$\lim_{n\to\infty} \frac{1}{n} = 0$ However we saw $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

Test for divergence: If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ doesn't exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex 10: Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges

Sol: We note that $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$

So by the test for divergence, the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Minute Math: Find the sum (if it converges):

$\sum_{n=1}^{\infty} \left(\frac{1}{\cdots} - \frac{1}{\cdots} \right)$

Minute Math: Find the sum (if it converges):

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$$

Sol: This is a telescoping series:

$$S_N = \sum_{n=1}^{N}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}} = \left(\frac{1}{\sqrt{1}}-\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{\frac{1}{\sqrt{2}}}-\cancel{\frac{1}{\sqrt{3}}}\right)+\left(\cancel{\frac{1}{\sqrt{3}}}-\cancel{\frac{1}{\sqrt{4}}}\right)+\cdots+\left(\cancel{\frac{1}{\sqrt{N}}}-\frac{1}{\sqrt{N+1}}\right)$$

$$= 1-\frac{1}{\sqrt{N+1}}$$

Thus $$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1-\overset{0}{\cancel{\frac{1}{\sqrt{N+1}}}}\right) = 1$$

Remark: A finite number of terms doesn't affect the convergence or divergence of a series.

Ex: If $\sum_{n=M}^{\infty} a_n$ converges, then

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \cdots + a_{M-1}}_{\text{finite \#}} + \underbrace{\sum_{n=M}^{\infty} a_n}_{\text{also a number}} \longrightarrow \text{thus } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Likewise if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=M}^{\infty} a_n$ converges as well.

Sometimes in Math we call $\sum_{n=M}^{\infty} a_n$ the "tail"

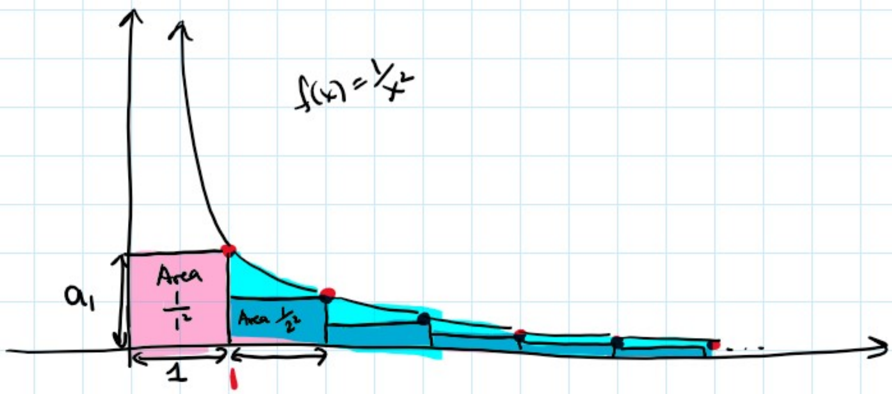
$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_{M-1}}_{\text{the "head"}} + \underbrace{\sum_{n=M}^{\infty} a_n}_{\text{the "tail"}}$$

§11.3 the integral test.

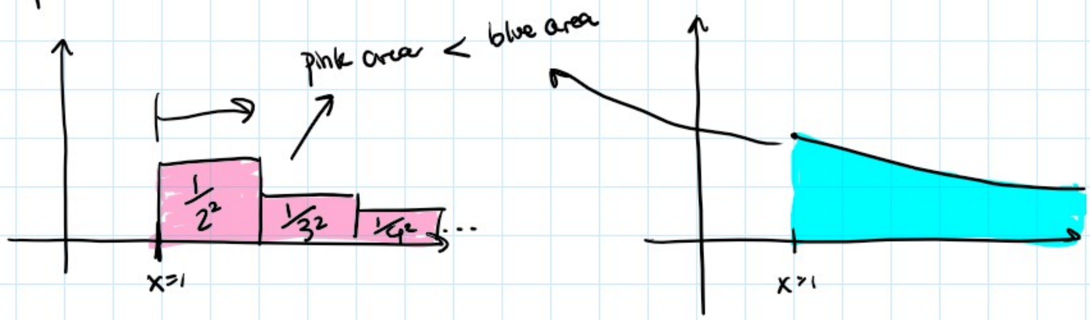
Motivating Ex: Consider $\sum_{n=1}^{\infty}\frac{1}{n^2}$

There is no simple formula for the partial sum S_N. However a computer approximation to S_N makes it look like the series converges.

To confirm this suspicion, we use a geometric argument:



We compare:



That means $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2} < \int_1^{\infty}\frac{1}{x^2}\,dx$

So $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \sum_{n=2}^{\infty}\frac{1}{n^2} = 1 + \sum_{n=2}^{\infty}\frac{1}{n^2} < 1 + \underbrace{\int_1^{\infty}\frac{1}{x^2}\,dx}_{=1}$$

$$= 1 + 1$$

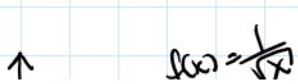
$$= 2$$

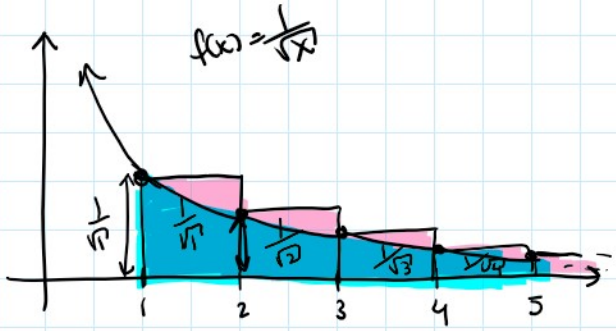
Thus $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} < 2$

So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} \neq \infty$, Hence $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ must converge.

Let's see if this technique can prove divergence!

Consider: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$





by §7.8 the p-test $p = \frac{1}{2} < 1$

thus $\quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \infty$

this tells us $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$, so it diverges.

The integral test Suppose f is a continuous, positive, decreasing function on $[1,\infty)$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)\,dx$ converges.

ie a) If $\int_1^{\infty} f(x)\,dx$ is CONV then $\sum_{n=1}^{\infty} a_n$ is CONV

b) If $\int_1^{\infty} f(x)\,dx$ is DIV then $\sum_{n=1}^{\infty} a_n$ is DIV.