March 17, 2023 2:58 PM

Last Time:

Power Series centered about a: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

Ex1: $\sum_{n=0}^{\infty} n!\, x^n$ converges only when $x = 0$

Ex2: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges when $|x-3|<1$, diverges when $|x-3|>1$ (Ratio test)
converges when $x = 2$, diverges when $x=4$ (checked the endpoints)
thus it converges only when $2 \le x < 4$.

Today:

Ex3: What is the domain (ie x for which the series converges) of

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Sol: let $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

We calculate

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right|$$

$2(n+1) = 2n+2$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{x^{2n+2}}{x^{2n}} \cdot \frac{2^{2n}}{2^{2n+2}} \cdot \frac{(n!)^2}{((n+1)!)^2} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \cdot x^2 \cdot \frac{1}{4} \cdot \frac{1}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{4(n+1)^2}$$

$$= 0.$$

$$\frac{(n!)^2}{((n+1)!)^2} = \frac{(n!)^2}{(n!\cdot(n+1))^2} = \frac{\cancel{(n!)^2}}{\cancel{(n!)^2}(n+1)^2}$$

OR, ALTERNATIVELY:

$(n!)^2 = (1\cdot 2\cdot 3 \cdot \cdots \cdot (n-1)n)^2 = 1^2\cdot 2^2 \cdot 3^2 \cdots (n-1)^2 n^2$

$((n+1)!) = 1^2 \cdot 2^2 \cdot 3^2 \cdot \cdots \cdot n^2 (n+1)^2$

$$\frac{(n!)^2}{((n+1)!)^2} = \frac{\cancel{1^2}\cdot\cancel{2^2}\cdot\cancel{3^2}\cdots\cancel{n^2}}{\cdots} = \frac{1}{(n+1)^2}$$

$$= \frac{(n!)^2}{\cancel{(n!)^2}(n+1)^2}$$
$$= \frac{1}{(n+1)^2}$$

$$\frac{(n!)^2}{((n+1)!)^2} = \frac{\cancel{1^2}\cdot\cancel{2^2}\cdot\cancel{3^2}\cdot \cdots \cdot\cancel{n^2}}{\cancel{1^2}\cdot\cancel{2^2}\cdot\cancel{3^2}\cdot \cdots \cdot\cancel{n^2}(n+1)^2} = \frac{1}{(n+1)^2}$$

Thus for all x, $L = 0 < 1$, so the series converges for all x.

In other words, the domain of J is $(-\infty, \infty)$

<u>Theorem 4</u>: For any given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three posibilities:

i) The series converges only at $x = a$.

ii) the series converges for all x

iii) There is some positive number R, such that, the series converges for $|x-a| < R$ and diverges when $|x-a| > R$

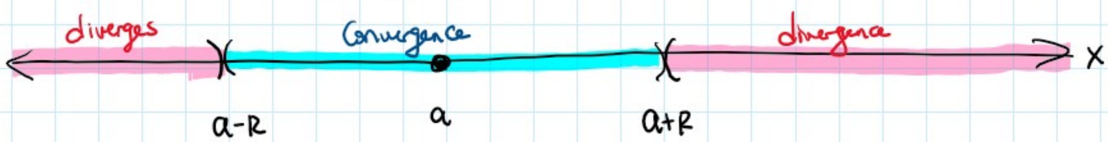
The number R in case iii) is called the <u>radius of convergence</u> (ROC)
By convention, we say $R = 0$ in case i) and $R = \infty$ for case ii)

The <u>interval of convergence</u> (IOC) is the values of x for which the series converges

i) The IOC is just the single point a

ii) The IOC is $(-\infty, \infty)$

iii) In case iii) we know the series converges for $|x-a| < R$ and diverges $|x-a| > R$
If $|x-a| = R$, anything can happen.



$|x-a| = R$ then $x = a - R$ or $x = a + R$.

So the IOC is one of:

$(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$

In our previous examples we saw:

Ex1: $ROC = 0$ and IOC is the single point 0

Ex2: $ROC = 1$ and IOC is $[2,4)$

Ex3: $ROC = \infty$ and IOC is $(-\infty,\infty)$

Ex4 Find the ROC and IOC for $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\qquad \left(\begin{array}{l} a = 0 \\ c_n = \frac{(-3)^n}{\sqrt{n+1}} \end{array} \right)$

Sol:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \left| \frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= \left| (-3) x \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{\sqrt{1+1/n}}{\sqrt{1+2/n}} = \frac{\sqrt{1}}{\sqrt{1}} = 1$$

$$\to 3|x|$$

Thus $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x|$

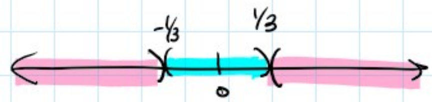
This series converges when $L<1$ and diverges when $L>1$ (Ratio test)

$\hookrightarrow |x| > 1/3$

ie converges when $3|x| < 1$

thus $|x| < \frac{1}{3} \rightarrow R.O.C = 1/3$

So the series converges for x in $(-1/3, 1/3)$



We now test the endpoints:

when $x = -1/3$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{\overbrace{(-3)^n\left(-\frac{1}{3}\right)^n}^{=1}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \rightarrow$ diverges by p-test.

set $k = n+1$

when $x = 1/3$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \leftarrow$ converges by alternating series test.

IOC $(-1/3, 1/3]$

$$(-3)^n(-\tfrac{1}{3})^n = ((-3)(-\tfrac{1}{3}))^n = 1^n = 1$$

$$(-3)(1/3)^n = ((-3)(\tfrac{1}{3}))^n = (-1)^n$$