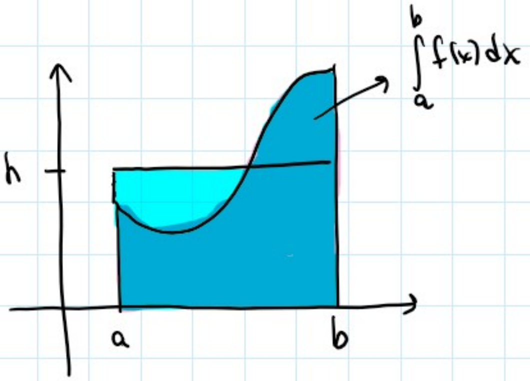
Announcements: We will replace your midterm grade with your final exam grade if the latter is greater than the former.

§6.5 Average value of a function:



Goal: Find a rectangle that has the same area as the area under the curve of $f(x)$, and with the same base. We need to find the height h.

Area of the rectangle = base $\cdot$ h

$$\int_a^b f(x)\,dx = (b-a)h$$

Solving for h, we get $h = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

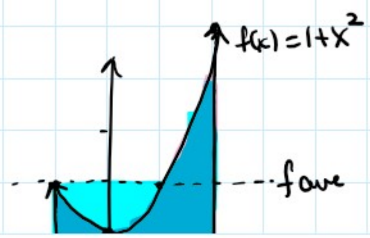
It turns out that this h is the average value of f over $[a,b]$.

Def: The average value of f on $[a,b]$ is:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex Find the average value of $f(x) = 1+x^2$ over $[-1,2]$.

Sol: $$f_{ave} = \frac{1}{(2)-(-1)}\int_{-1}^{2}(1+x^2)\,dx = \frac{1}{3}\left[x+\frac{x^3}{3}\right]_{-1}^{2} = \cdots = 2$$

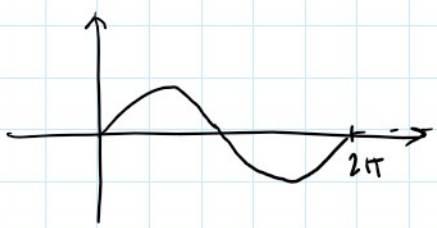


Minute Math:

a) guess the average value of $f(x) = \sin x$ over $[0, 2\pi]$

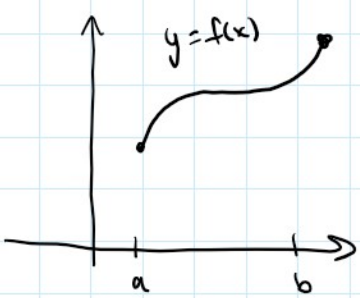
b) Use the formula $\frac{1}{b-a}\int_a^b f(x)dx$ to verify your guess.

b) $$\frac{1}{2\pi-(0)}\int_0^{2\pi} \sin x\,dx = \frac{1}{2\pi}\Big[-\cos(x)\Big]_0^{2\pi} = \frac{1}{2\pi}\left(\cancel{-\cos(2\pi)+\cos(0)}^{\,0}\right) = 0$$



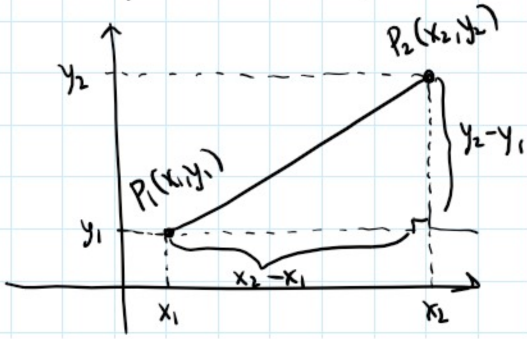
$\dashrightarrow f_{ave} = 0.$

§ 8.1 Arc length of a curve:



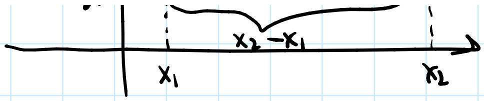
How do we measure the length of a curve?
(we say "arc length" to emphasise that the curve is curvy. ie "arc length of a curve" = "length of a curve")

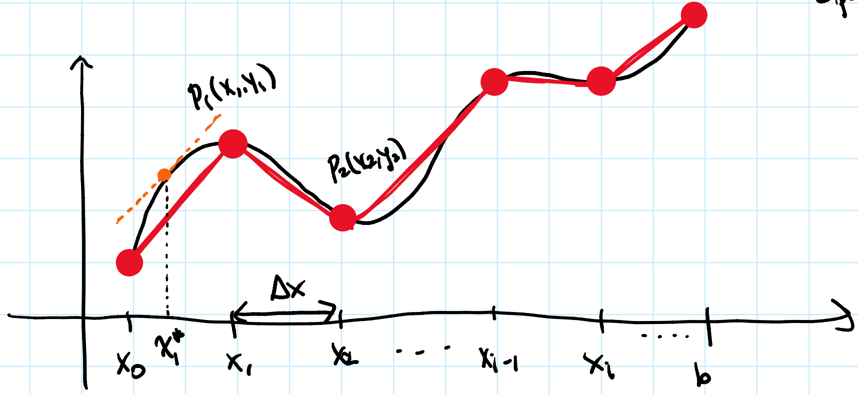
The length of a straight line is easy to calculate.



$|P_1P_2| \rightarrow$ length of the line from P_1 to P_2

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$





Space x_i evenly from a to b, with distance Δx.

Set $y_i = f(x_i)$,

Call P_i the point $P_i(x_i, y_i)$

thus the arc length is approximately

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

What is $|P_{i-1}P_i|$?

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Define

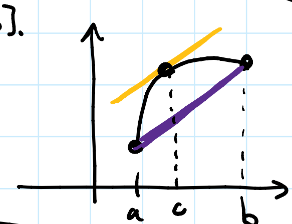
$$\Delta y_i = y_i - y_{i-1}$$

Recall: MVT (Mean Value theorem)

Let f be differentiable on (a,b) and continuous on $[a,b]$.

then there exists some $c \in (a,b)$ such that

$$\underbrace{f'(c)}_{\text{slope of tangent at } c} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{slope of secant over } [a,b]}$$



We use the MVT applied to f on the interval $[\underset{\uparrow\, a}{x_{i-1}}, \underset{\uparrow\, b \text{ in MVT}}{x_i}]$ to find the existence of some point $\underset{\text{our } c \text{ in MVT}}{x_i^*} \in [x_{i-1}, x_i]$

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}$$

ie, by rearranging $\boxed{\Delta y_i = f'(x_i^*) \Delta x}$

So, returning to $|P_{i-1}P_i|$:

So, returning to $|P_{i-1}P_i|$:

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + (f'(x_i^*))^2 \Delta x^2}$$

$$= \Delta x \sqrt{1 + [f'(x_i^*)]^2}$$

Then, we see the length L of our curve is:

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

Thus, $L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

Let $g(x) = \sqrt{1 + [f'(x)]^2}$
then we know that
$\lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x = \int_a^b g(x)\,dx$

<u>To summarize</u>: If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

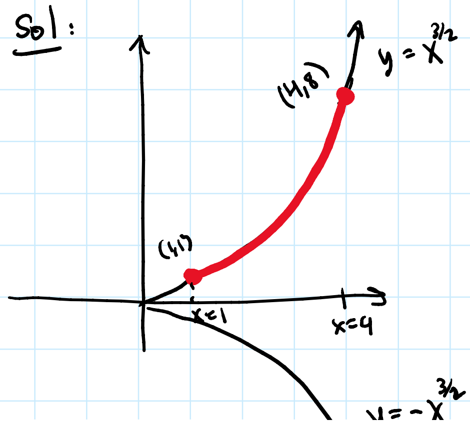
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

and in Leibniz notation:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$\boxed{\text{Ex 1}}$ Find the length of the curve $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$

<u>Sol</u>:



For the top half of the curve, $y = x^{3/2}$

So $\frac{dy}{dx} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$

So: $L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

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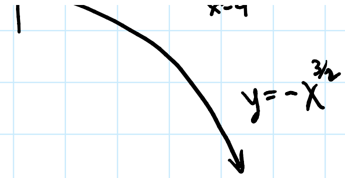

$$= \int_1^4 \sqrt{1+\left(\tfrac{3}{2}\sqrt{x}\right)^2}\,dx$$

$$= \int_1^4 \sqrt{1+\tfrac{9}{4}x}\,dx$$

$$= \int_{\frac{13}{4}}^{10} \sqrt{u}\left(\tfrac{4}{9}\,du\right)$$

$u = 1+\frac{9}{4}x$

$du = \frac{9}{4}dx$

$\frac{4}{9}du = dx$

$$= \frac{4}{9}\left[\frac{2}{3}\cdot u^{3/2}\right]_{13/4}^{10}$$

$$= \cdots$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$