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LAST TIME

Theorem 4: For any given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three posibilities:

i) The series converges only at $x=a$.

ii) The series conerges for all x

iii) There is some positive number R, such that, the series converges for $|x-a|<R$ and diverges when $|x-a|>R$

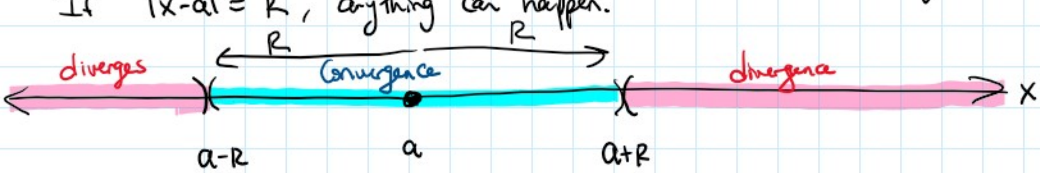
The number R in case iii) is called the radius of convergence (ROC)
By convention, we say $R=0$ in case i) and $R=\infty$ for case ii)

the interval of convergence (IOC) is the values of x for which the series converges

i) The IOC is just the single point a

ii) The IOC is $(-\infty, \infty)$

iii) In case iii) we know the series converges for $|x-a|<R$ and diverges $|x-a|>R$
If $|x-a|=R$, anything can happen.



$|x-a|=R$ then $x=a-R$ or $x=a+R$.

So the IOC is one of:

$(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$

Ex 5: Find the radius of convergence (ROC) and the Interval of Convergence (IOC) of

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Sol: Always always always use ratio test for these problems. $a_n = \frac{n(x+2)^n}{3^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left|\frac{n+1}{n} \cdot \frac{(x+2)^{\cancel{n+1}}}{\cancel{(x+2)^n}} \cdot \frac{\cancel{3^{n+1}}}{3^{\cancel{n+2}}}\right|$$

$$= \left| \frac{n+1}{n} \cdot \frac{(x+2)^{\cancel{n+1}}}{(x+2)^{\cancel{n}}} \cdot \frac{3^{\cancel{n+1}}}{3^{\cancel{n+2}}} \right|$$

$$= \frac{n+1}{n} |x+2| \cdot \frac{1}{3}$$

Therefore $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} |x+2| \cdot \frac{1}{3} = |x+2| \cdot \frac{1}{3}$

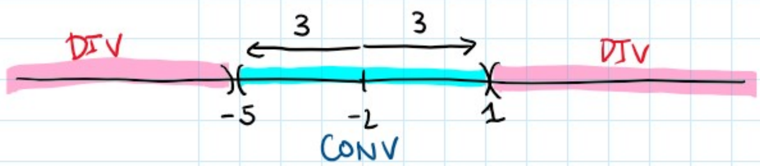
The ratio test says that if $L < 1$ ie if $|x+2| \cdot \frac{1}{3} < 1$ the series converges
and that if $L > 1$ ie if $|x+2| \cdot \frac{1}{3} > 1$ the series DIV.

ie if $|x+2| < 3$ series CONV
if $|x+2| > 3$ series DIV

Thus our ROC is $R = 3$.

Note that $|x+2| = |x-(-2)|$

$$\boxed{|x-a| < R}$$



Now we check the endpoints to determine the IOC.

when $x = -5$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n \left(\frac{-3}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} n(-1)^n$ DIV

By the test for divergence $n(-1)^n \not\to 0$.

when $x = 1$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{n(\cancel{3})^{\cancel{n}}}{\cancel{3}^{\cancel{n}}} = \frac{1}{3} \sum_{n=0}^{\infty} n$ DIV by the test for divergence
$n \to \infty \neq 0$

Thus the interval of convergence is $(-5, 1)$

<u>Ex</u> #20 §11.8 (8^{th} ed) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$ find ROC and IOC

<u>Sol</u> Always always always, ratio test.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2x-1)^{n+1}}{5^{n+1} \sqrt{n+1}} \cdot \frac{5^n \sqrt{n}}{(2x-1)^n} \right|$$

$$= \left| \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(2x-1)^{n+1}}{(2x-1)^n} \right|$$

$$= \left| \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(2x-1)^{n+1}}{(2x-1)^n} \right|$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{5} \cdot |2x-1|$$

thus $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{5} \cdot |2x-1| = \frac{1}{5}|2x-1|$

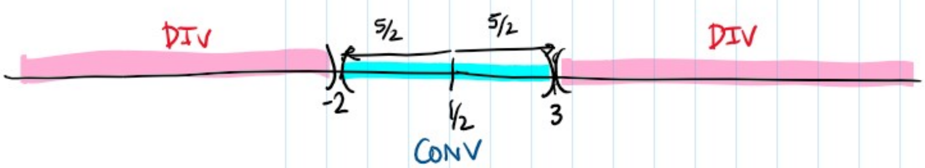
when $L<1$ ie $\frac{1}{5}|2x-1|<1$ the series CONV

when $L>1$ ie $\frac{1}{5}|2x-1|>1$ Series DIV

$\frac{1}{5}|2x-1|<1$ is the same as $|2x-1|<5$ is the same as $2|x-\frac{1}{2}|<5$
is the same as $|x-1/2|<5/2$

thus the series CONV for $|x-1/2|<5/2$ $\quad |x-a|<R \quad a=1/2$
DIV for $|x-1/2|>5/2$ $\quad |x-a|>R \quad R=5/2$

thus our ROC is $R=5/2$



$1/2 + 5/2 = 3$

we check the endpoints:

when $x=-2$ $\quad \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Converges by the alternating series test since $b_n = \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} \to 0$.
and b_n is decreasing in n.

when $x=3$ $\quad \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Diverges by the p-test
$(p=1/2<1)$

So our IOC is $[-2, 3)$

Remark, Note that the ROC is always half of the length of the IOC.

$$R = \frac{1}{2}(3-(-2)) = \frac{1}{2}(5) = \frac{5}{2}$$

A review of seq. & series:

An important theorem:

$$-|a_n| \le a_n \le |a_n|$$

Theorem: If $|a_n| \to 0$ then $a_n \to 0$

Ex: #44 §11.1 (8th ed) let $a_n = 2^{-n}\cos(n\pi)$, find $\lim_{n\to\infty} a_n$ if it exists.

Sol: Note that $|a_n| = |2^{-n}\cos(n\pi)| = 2^{-n}\underbrace{|\cos(n\pi)|}_{|\cos(x)|\le 1} \le 2^{-n}$

So $0 \le |a_n| \le 2^{-n}$ and $\lim_{n\to\infty} 0 = 0$

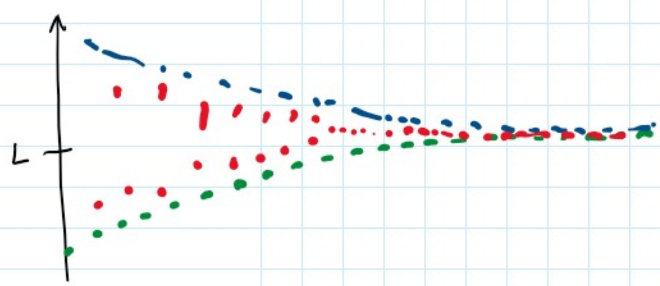
$\lim_{n\to\infty} 2^{-n} = 0$

therefore by the squeeze theorem $\lim_{n\to\infty} |a_n| = 0$

and thus $\lim_{n\to\infty} a_n = 0$.

Squeeze theorem:

If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$



Ex #42 §11.1 (8th ed) Calculate $\lim_{n\to\infty} \ln(n+1) - \ln(n)$

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$

Sol $\lim_{n\to\infty} \ln(n+1) - \ln(n)$

$= \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$

ln is a continuous function

$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$

for continuous

$$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$$ $\ln$ is a continuous function

$$= \ln(1)$$

$$= 0$$

for continuous

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

if this exists

§11.7 Strategies for testing series

How would you test...?

$$\frac{e}{n^2} \geq \frac{e^{1/n}}{n^2}$$

#28 (8th ed) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ Compare to $\sum_{n=1}^{\infty} \frac{e}{n^2}$

#24 (8th ed) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ Test for divergence since $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$