February 7, 2023 11:45 AM

Continue §7.5

Minute Math Find $\int x^8 \sqrt{x^3+1}\,dx$

$$\int x^8\sqrt{x^3+1}\,dx = \int \underbrace{x^6}_{(u-1)^2}\underbrace{\sqrt{x^3+1}}_{\sqrt{u}}\underbrace{x^2dx}_{\frac{du}{3}} = \frac{1}{3}\int (u-1)^2\sqrt{u}\,du = \cdots$$ From here it's easy.

$u = x^3+1 \longrightarrow u-1 = x^3$

$du = 3x^2dx \qquad (u-1)^2 = x^6$

$\frac{du}{3} = x^2dx$

Minute Math: Find $\int \frac{\cos(1/x)}{x^3}dx$

$$\int \frac{\cos(1/x)}{x^3}dx = \int \underbrace{\left(\frac{1}{x}\right)}_{u}\underbrace{\cos\left(\frac{1}{x}\right)}_{\cos(u)}\underbrace{\left(\frac{1}{x^2}\right)dx}_{-du} = -\int u\cos(u)\,du$$ From here use IBP (use a different variable than u!)

$u = 1/x$

$du = -\frac{1}{x^2}dx$

$-du = \frac{1}{x^2}dx$

Ex Find $\int\sqrt{1+e^x}\,dx$

Let $u = \sqrt{1+e^x}$ (this is a rationalizing substitution, reference: end of §7.4)

$du = \frac{1}{2\sqrt{1+e^x}}\cdot e^x dx$ $\qquad u^2 = 1+e^x$

$= \frac{1}{2u}e^x dx$ $\qquad$ ie $u^2-1 = e^x$

$2u\,du = e^x dx$

$2u\,du = (u^2-1)dx$

$2\frac{u}{u^2-1}du = dx$

$$\boxed{\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}$$

$$\int \underbrace{\sqrt{1+e^x}}_{u}\,\underbrace{dx}_{2\frac{u}{u^2-1}du} = 2\int\frac{u^2}{u^2-1}du = 2\left(\int 1\,du + \int\frac{1}{u^2-1}du\right) = 2\left(u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right) + C$$

$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$$

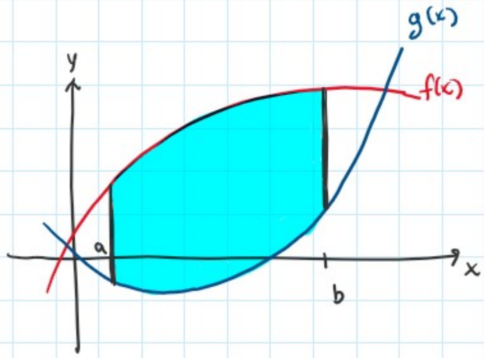
$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$$

Long division $\frac{u^2}{u^2-1} = 1 + \frac{1}{u^2-1}$

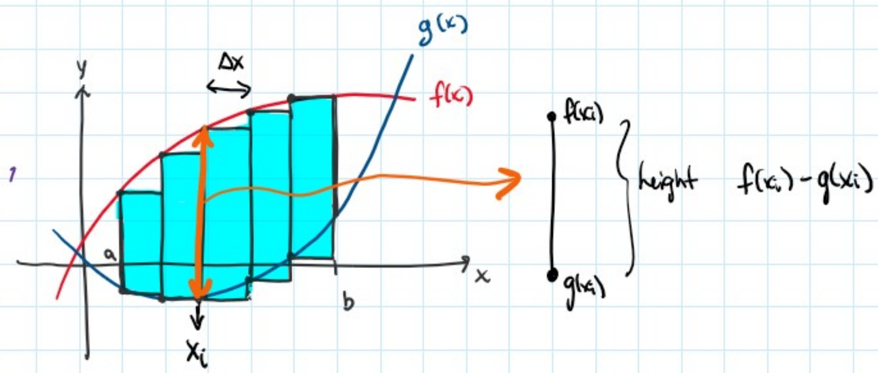
or you could see: $\frac{u^2}{u^2-1} = \frac{u^2-1+1}{u^2-1} = \frac{u^2-1}{u^2-1} + \frac{1}{u^2-1} = 1 + \frac{1}{u^2-1}$

§6.1: Area between curves

In this section we want to find the area between two curves $f(x)$ and $g(x)$



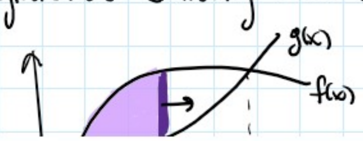
To find a formula for this area, we will appeal to Riemann Sums.
A Riemann Sum approximation to this area looks like:



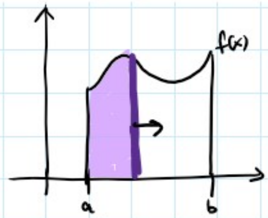
So $A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i)-g(x_i)]\Delta x = \int_a^b [f(x)-g(x)]\,dx$

$$\boxed{A = \int_a^b [f(x)-g(x)]\,dx}$$

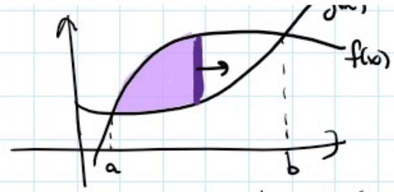
Intuitively we remember this formula by thinking of integration as extruding a line of varying heights through space.



$\int^b [f(x)-g(x)]\,dx$



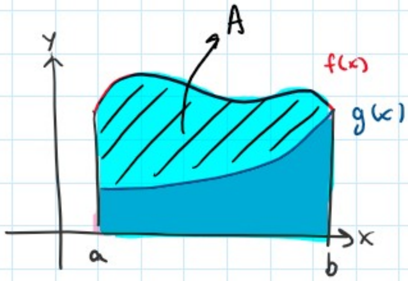
$\int_a^b f(x)dx$



$\int_a^b [f(x)-g(x)]dx$

in this case the height of the line is $f(x)-g(x)$

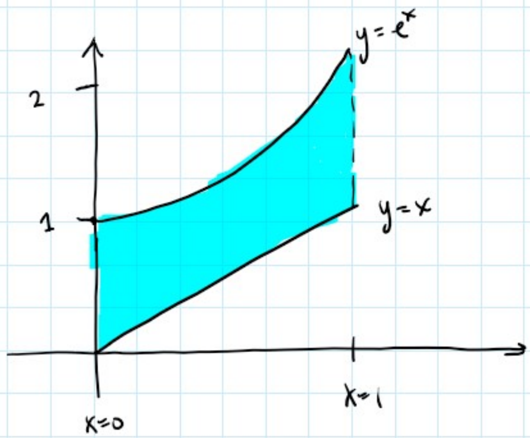
→ If both f and g are positive, this makes sense geometrically



$$A = [\text{area under } f] - [\text{area under } g]$$
$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$
$$= \int_a^b [f(x)-g(x)]dx.$$

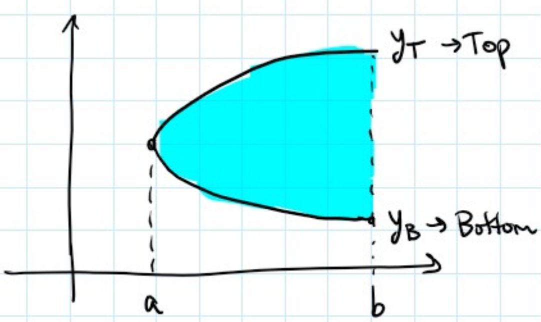
Ex1: Find the area of the region bounded above by $y=e^x$, bounded below by $y=x$ and bounded on the sides by $x=0$, and $x=1$.

Sol:



$$\text{so } A = \int_0^1 [f(x)-g(x)]dx$$
$$= \int_0^1 [e^x - x]dx$$
$$= \left[e^x - \tfrac{1}{2}x^2\right]_0^1$$
$$= (e^1 - \tfrac{1}{2}) - (e^0 - 0)$$
$$= e - \tfrac{1}{2} - 1$$
$$= e - \tfrac{3}{2}.$$

Another notation for our formula:



$$A = \int_a^b [y_T - y_B]dx$$

Ex2 Find the area of the region enclosed by the parabolas

Ex 2 Find the area of the region enclosed by the parabolas

$$y = x^2 \quad \text{and} \quad y = 2x - x^2$$

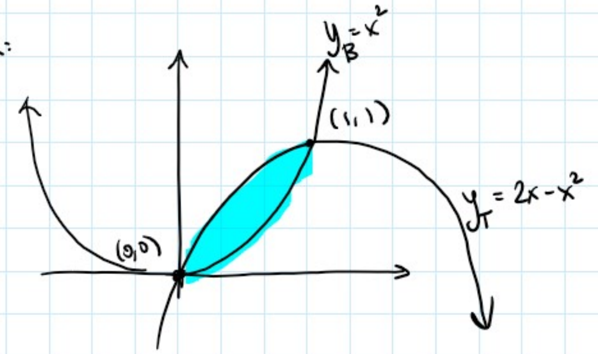
Sol: We need to sketch the region in order to find a and b.

First we find their points of intersection. $y = y$ gives:

$$x^2 = 2x - x^2$$
$$2x^2 - 2x = 0$$
$$2x(x-1) = 0$$

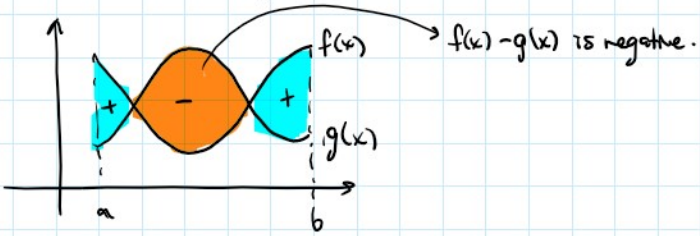
$$\left.\begin{array}{l} x = 0 \\ x = 1 \end{array}\right\} \xrightarrow{\text{use } y = x^2} \begin{array}{ll} x = 0 \text{ gives } y = 0 & (0,0) \\ x = 1 \text{ gives } y = 1 & (1,1) \end{array}$$

Sketch:



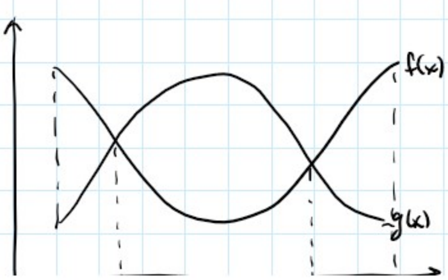
$$A = \int_0^1 [y_T - y_B]\,dx = \int_0^1 (2x - x^2 - x^2)\,dx = \cdots = \frac{1}{3}.$$

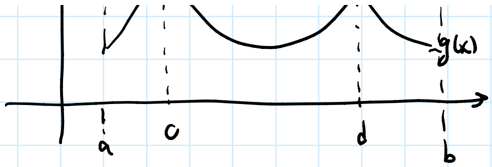
When we do area between curves we want positive areas. So, if $f(x)$ and $g(x)$ cross, we have issues of negative areas:



To remedy this fact we really want $A = \int_a^b |f(x) - g(x)|\,dx$.

But, absolute values make for hard-to-evaluate integrands, so in practice we split the regions





$$A = \int_a^b |f(x) - g(x)|\,dx = \int_a^c (f(x) - g(x))\,dx + \int_c^d (g(x) - f(x))\,dx + \int_d^b (f(x) - g(x))\,dx$$