May 22, 2023 9:22 PM

Info regarding the final exam:

Ⓐ 15 Multiple choice questions (3 mrks each) = 45 marks
Ⓑ 4 long answer questions = 45 marks
↳ 1. Taylor Series
  2. Vector functions
  3. Differentiation of multivariate functions
  4. Integration of multivariate functions

No Calculators, No crib sheets
Exam is cumulative, covers all topics (not just after the midterm)

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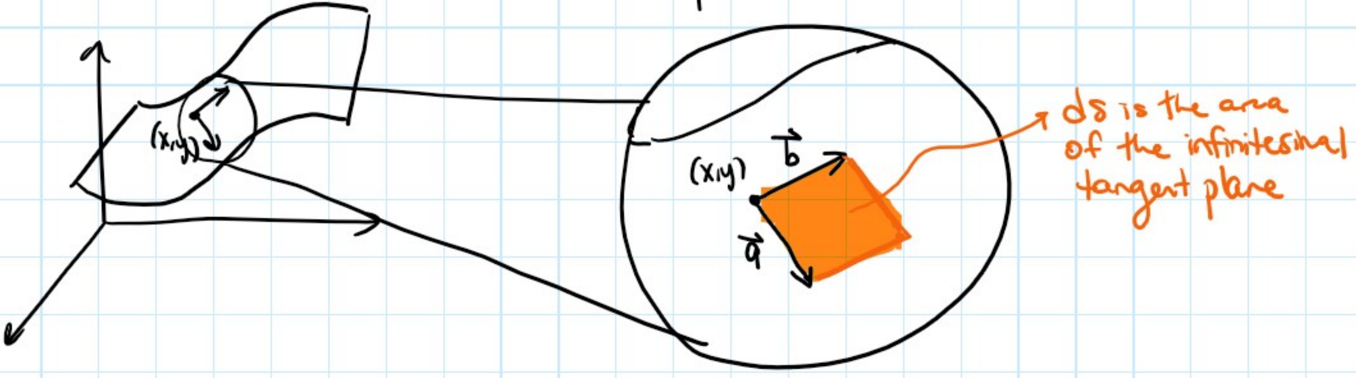
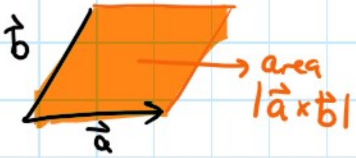
§15.5 Surface Area:

take a small patch of surface area dS, at (x,y)

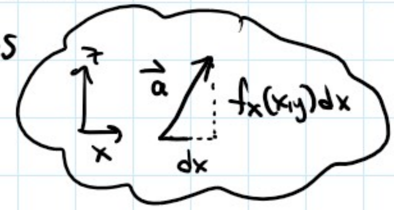


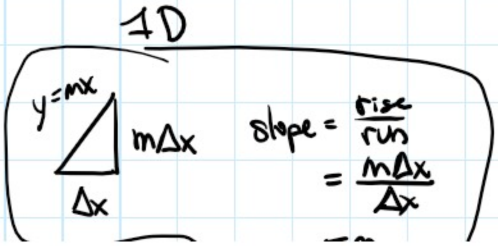
then the area dS is given by the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$,
so $dS = |\vec{a} \times \vec{b}|$



$\vec{a}$ is our tangent vector in the x direction.
it has slope $f_x(x,y)$. Thus

$\vec{a} = \hat{\imath}\,dx + \hat{k} f_x(x,y)\,dx$





Δx

$= \frac{m\Delta x}{\Delta x}$

$= m$

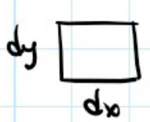
likewise

$\vec{b} = \hat{j}dy + \hat{k} f_y(x,y)dy$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = -f_x dx dy\,\hat{i} - f_y dx dy\,\hat{j} + dx\,dy\,\hat{k}$$

$$= \langle -f_x, -f_y, 1\rangle\, dxdy$$

$$= \langle -f_x, -f_y, 1\rangle\, dA$$



$dA = dxdy$

So $dS = |\langle -f_x, -f_y, 1\rangle| dA$

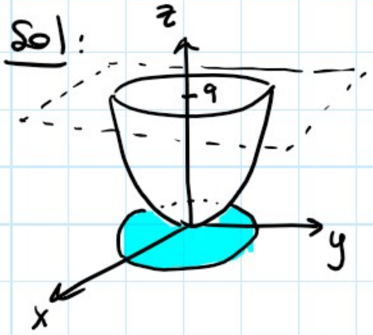
$= \sqrt{f_x^2 + f_y^2 + 1}\; dA$

so sufrace area, for $z = f(x,y)$

$$A(S) = \iint_D dS = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\; dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA$$

Ex2: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Sol:



Because $z = x^2 + y^2$ the plane $z = 9$ intersects the parabola in the circle $x^2 + y^2 = 9$.

so $D = \{(x,y) : x^2 + y^2 \leq 9\}$

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA$$

$\frac{\partial z}{\partial x} = 2x$ $\quad z = x^2 + y^2$

$\frac{\partial z}{\partial y} = 2y$

$$A = \iint_D \sqrt{1 + 4(x^2 + y^2)}\; dA$$

$$A = \int_0^{2\pi}\int_0^{3} \sqrt{1 + 4r^2}\; r\, dr\, d\theta$$

$$\int_0 \int_0 \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{3} \sqrt{1+4r^2}\; r\right)$$

$$= \cdots$$

$$= \frac{\pi}{6}\left[37\sqrt{37}-1\right]$$

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§15.6 : Triple integrals

Integrals of a function $f(x,y,z)$ have a similar Riemann sum definition as double integrals did. But, at the end of the day, we'll just use Fubini's theorem.

<u>Fubini's Theorem for triple integrals</u>: If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s] = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy$$

$$= \cdots$$

all 6 possibilities are valid

$dx\,dy\,dz$

$dy\,dz\,dx$

$dx\,dz\,dy$

⋮

etc.

<u>Ex 1</u> Evaluate $\iiint_B xyz^2\,dV$ where $B = [0,1]\times[-1,2]\times[0,3]$

<u>Sol</u>: we can choose any of the six orders of integration. Let's do $dx\,dy\,dz$

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\,dz$$

3 2

$$\int_0 \int_{-1} \Big[\quad 2 \quad \Big]_{x=0}$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz$$

$$= \int_0^3 \left[\frac{y^2 z^2}{4} \right]_{y=-1}^{y=2} dz$$

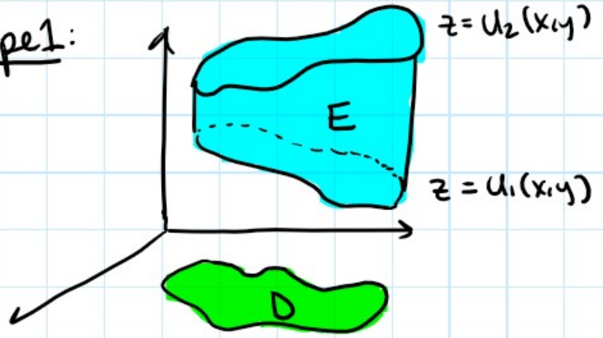
$$= \int_0^3 \frac{3z^2}{4}\, dz$$

$$= \left[\frac{z^3}{4} \right]_0^3$$

$$= \frac{27}{4}$$

<u>Triple integrals over general Regions</u>: we now consider integrating over a general solid E.

<u>Type 1</u>:



E is type 1 if it lies between the graphs of two surfaces $z = u_1(x,y)$, $z = u_2(x,y)$.

$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ where D is the projection of E onto the xy-plane.

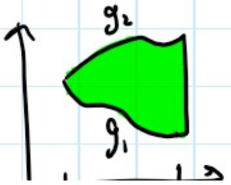
In this case: $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

↳ this means we "integrate out" z and then get a double integral.

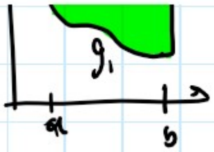
if $g(x,y) = \displaystyle\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$ then $\displaystyle\iiint_E f(x,y,z)\,dV = \iint_D g(x,y)\,dA$

If D is a type 1 plane region $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



$$\iiint_E f(x,y,z)\,dV = \int_a \int_{g_1(x)} \int_{u_1(x,y)} f(x,y,z)\,dz\,dy\,dx$$



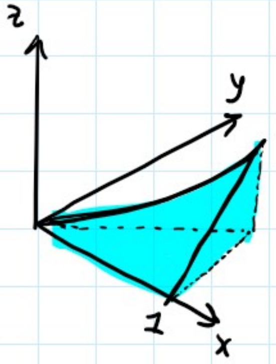
If D is a type 2 plane region $D = \{(x,y):\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

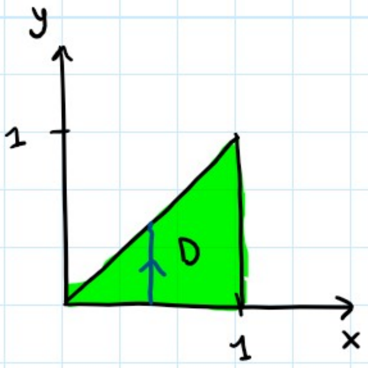
$x \ge 0$
$y \ge 0$
$z \ge 0$

Ex2: Evaluate $\iiint_E z\,dV$ where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$, $x = 1$

Sol Handy tip, it's usually good to draw the solid E and the domain D.



$x = 1$
$z = 12xy$
$= 12y$



$$E = \{(x,y,z):\ 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 12xy\}$$

thus

$$\iiint_E z\,dV = \int_0^1 \int_0^x \int_0^{12xy} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_0^{12xy} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^x (12xy)^2\,dy\,dx$$

$$= 72 \int^1 \int^x x^2y^2\,dy\,dx$$

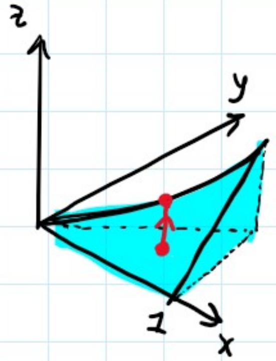
$$= 72\int_0^1\int_0^x x^2y^2\,dy\,dx$$

$$= 72\int_0^1\left[x^2\frac{y^3}{3}\right]_0^x dx$$

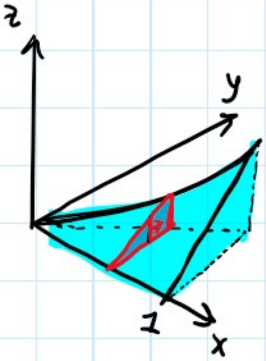
$$= 24\int_0^1 x^5\,dx$$

$$= 24\left(\frac{1}{6}\right) = 4$$

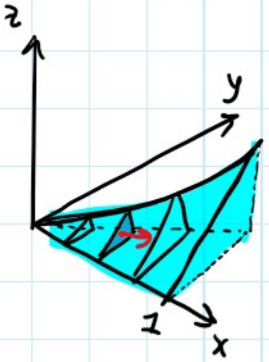
It can be useful to imagine our integration "sweeping out" the volume.



z varies from 0 to 12xy



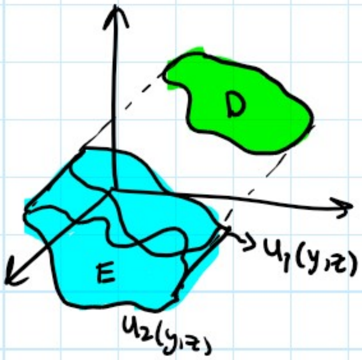
y varies from 0 to x while x is constant



x varies from 0 to 1

Type 2 E is type 2 if it is of the form $E=\{(x,y,z):\ (y,z)\in D,\ u_1(y,z)\le x\le u_2(y,z)\}$

D is the projection of E onto the yz-plane.

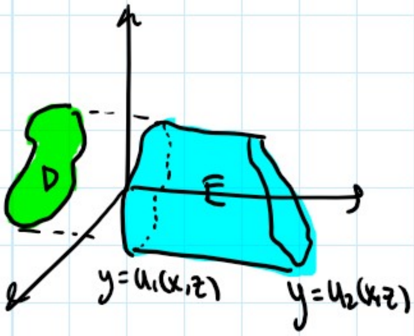


then

$$\iiint_E f(x,y,z)\,dV = \iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right]dA$$

Type 3: E is type 3 if it looks like $E=\{(x,y,z):(x,z)\in D,\ u_1(x,z)\le y\le u_2(x,z)\}$

D is the projection of E onto the xz-plane



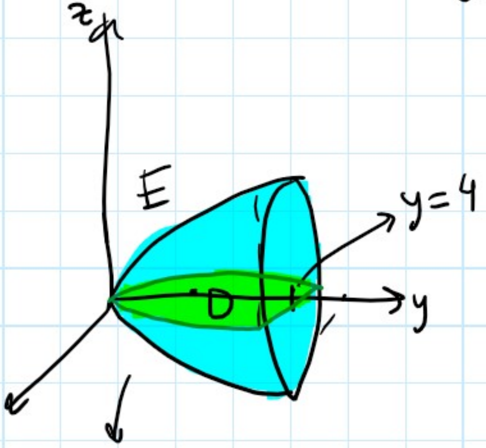
D is the projection of E onto the xz-plane

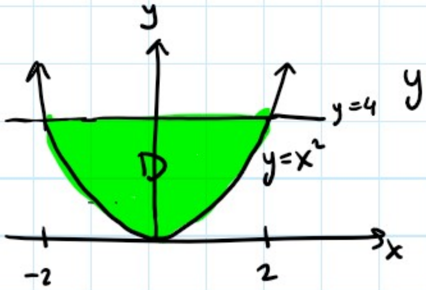
then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

Ex3: Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$.

Sol If we view E as a type1 solid:





$y = x^2+z^2 = x^2$
when $z=0$
ie on the xy plane

$y = x^2+z^2$ so $z^2 = y-x^2$ so $z = \pm\sqrt{y-x^2}$

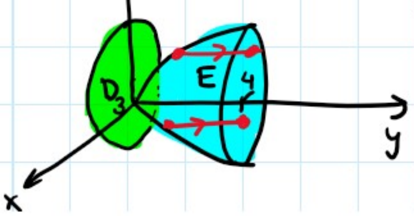
we got $E = \{(x,y,z) : -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y-x^2} \le z \le \sqrt{y-x^2}\}$

thus

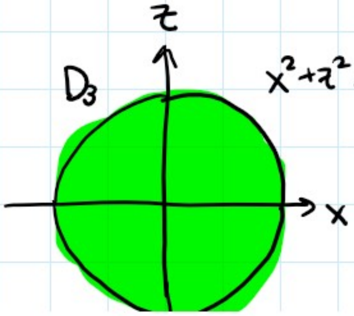
$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx \leftarrow$$

which looks <u>awful</u> to evaluate.

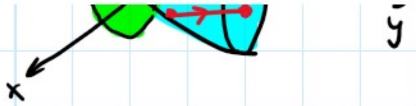
Let's try looking at E as a type 3 solid.

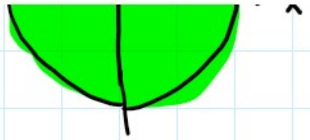


when $y=4$
$4 = x^2+z^2$



$x^2+z^2 \le 4$





thus $E = \{(x,y,z) : (x,z) \in D_3, \; x^2+z^2 \le y \le 4\}$

So

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}\left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right]dA$$

$$= \iint_{D_3}(4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

convert to polar
$x = r\cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 4r^2 - r^4\,dr\right)$$

$$= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2$$

$$= \frac{128\pi}{15}$$

Changing the order of integration: Fubini's theorem lets us change the order of integration. This is best seen in an example.

Ex4: Express $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ as a triple integral over some solid E and then rewrite it as an iterated integral in the following orders:

a) $dx\,dz\,dy$

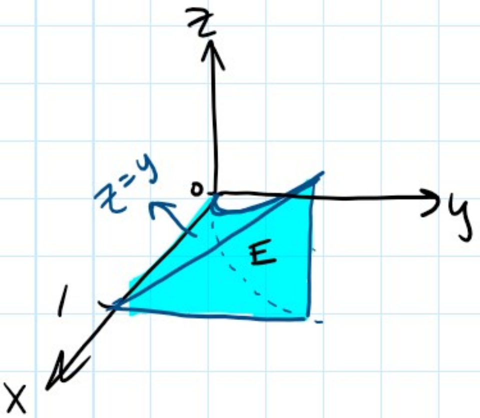
b) $dy\,dx\,dz$

Sol

$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$$

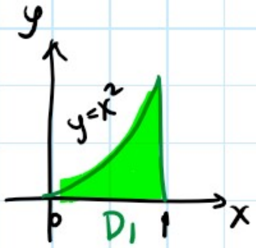
where $E = \{(x,y,z) : 0 \le x \le 1, \; 0 \le y \le x^2, \; 0 \le z \le y\}$

where $E = \{(x,y,z):\ 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$

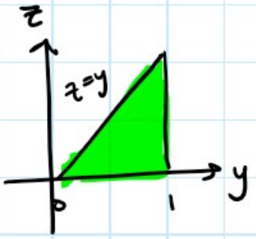


let's project E onto the coordinate planes:

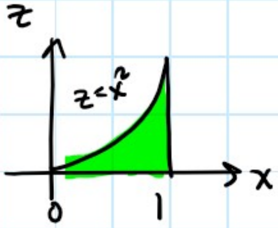
<u>xy-plane</u>: $D_1 = \{(x,y):\ 0 \le x \le 1,\ 0 \le y \le x^2\}$
$= \{(x,y):\ 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$



<u>yz-plane</u>: $D_2 = \{(y,z):\ 0 \le y \le 1,\ 0 \le z \le y\}$
$= \{(y,z):\ 0 \le z \le 1,\ z \le y \le 1\}$



<u>xz-plane</u>: $D_3 = \{(x,z):\ 0 \le x \le 1,\ 0 \le z \le x^2\}$
$= \{(x,z):\ 0 \le z \le 1,\ \sqrt{z} \le x \le 1\}$



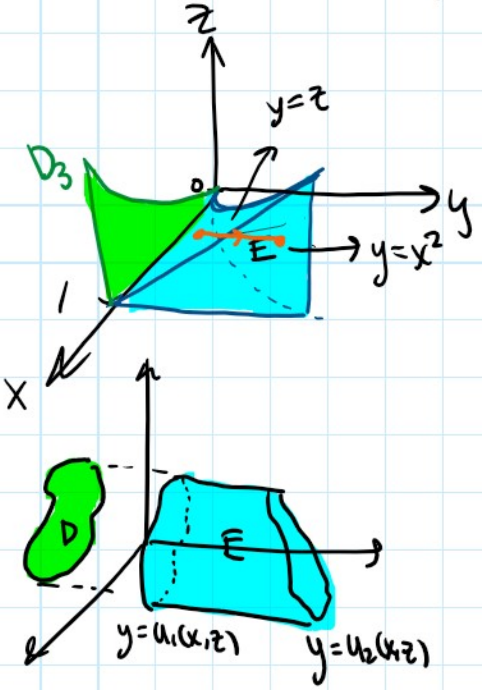
a) <u>dxdzdy</u> first we'll indegrate at x: Using $y = x^2$ we get $x = \sqrt{y}$

$$\iiint_E f(x,y,z) = \iint_{D_2} \left[\int_{\sqrt{y}}^{1} f(x,y,z)\,dx \right]$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy$$

b) <u>dydxdz</u> first integrate at y

b) <u>dy dx dz</u> first integrate at y



$$\iiint_E f(x,y)\,dV = \iint_{D_3}\left[\int_z^{x^2} f(x,y,z)\,dy\right]dA$$

$$= \int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$

Applications of triple integrals:

Volume: in 2D $A(D) = \iint_D dA$

so in 3D $V(E) = \iiint_E dV$

Center of Mass: If $\rho(x,y,z)$ gives the density of a point $(x,y,z)\in E$ then

the total mass: $m = \iiint_E \rho(x,y,z)\,dV$

and the center of mass $(\bar{x},\bar{y},\bar{z})$ is given by:

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,dV,\quad \bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV,\quad \bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV$$

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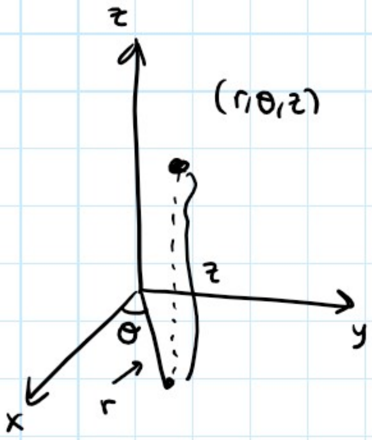
§15.7 Cylindrical Coordinates

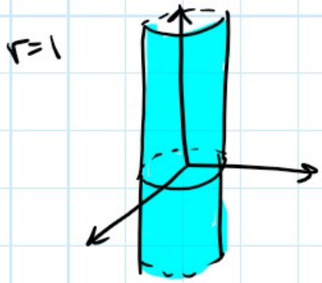
As we saw in Ex3 of last section, it can be useful to convert two of the coordinates into polar coordinates. This motivates <u>cylindrical coordinates</u>.



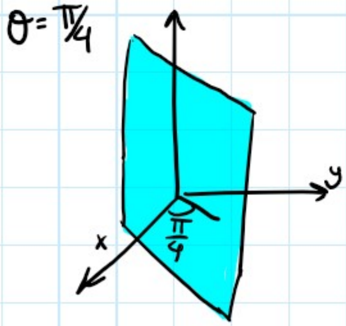
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

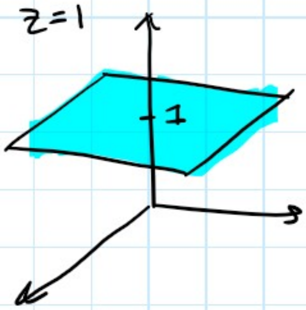
Cylindrical coordinates are 3D coordinates where we write the xy plane in polar coordinates.

Let's look at some surfaces in cylindrical coordinates:

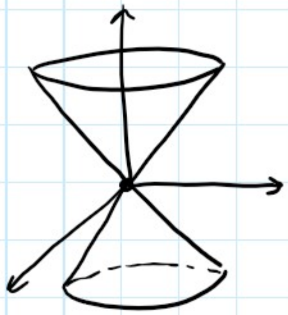


$x^2+y^2=1^2$



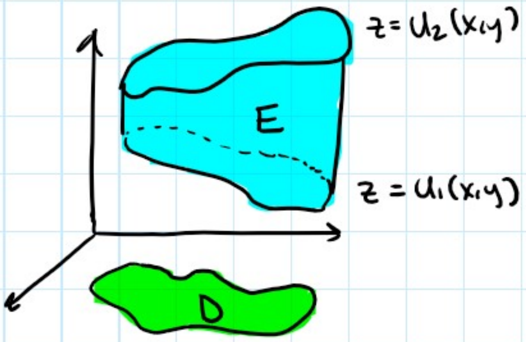


Minute Math what does $z = r$ look like?



2D version of

Triple integrals in cylindrical coordinates:



if E is type 1 we saw

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

Convert to polar

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

don't forget the r

if $D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$

general D in polar coordinates.