March 17, 2023 2:58 PM

Today: A review of some topics.

Substitution Find $\int x^8(x^3+1)^{10}dx$

$$= \int \underbrace{x^6}\underbrace{(x^3+1)^{10}}\underbrace{x^2dx}$$

$u = x^3+1 \rightarrow u-1 = x^3$

$du = 3x^2dx \quad (u-1)^2 = x^6$

$\frac{du}{3} = x^2dx$

$$= \int (u-1)^2 u^{10} \left(\frac{du}{3}\right)$$

$$= \frac{1}{3}\int (u^2-2u+1)u^{10}du$$

$$= \frac{1}{3}\int u^{12}-2u^{11}+u^{10}du$$

$= \cdots$

Rationalizing Substitutions: Find $\int \frac{\sqrt{1+\sqrt{x}}}{x}dx$

$u = \sqrt{1+\sqrt{x}} \longrightarrow u^2 = 1+\sqrt{x}$

$u^2-1 = \sqrt{x}$

$(u^2-1)^2 = x$

$du = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}dx$

$= \frac{1}{2u}\cdot\frac{1}{2(u^2-1)}dx$

$dx = 4u(u^2-1)du$

$$= \int \frac{u}{(u^2-1)^2}4u(u^2-1)du$$

$$= 4\int \frac{u^2\cancel{(u^2-1)}}{(u^2-1)^{\cancel{2}}}du$$

$$= 4\int \frac{u^2}{u^2-1}du$$

$$= 4\int \frac{u^2-1+1}{u^2-1}du$$

$$= 4\int \frac{(u^2-1)}{(u^2-1)} + \frac{1}{(u^2-1)}du$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

$$= 4\int 1 + \frac{1}{u^2-1}du$$

$$= 4\left(u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right)+C$$

$$= 4\left(\sqrt{1+\sqrt{x}} + \frac{1}{2}\ln\left|\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right|\right)+C$$

Length of a curve: Find the arc length of $y = 1 - e^{-x}$ from $0 \le x \le 2$.

Arc length formula is $\int ds$ where $(ds)^2 = (dx)^2 + (dy)^2$

we have $y = f(x)$ and x boundary points so let's use

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$y = 1 - e^{-x}$

$\frac{dy}{dx} = e^{-x}$

$$\int ds = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + e^{-2x}}\, dx$$

$u = \sqrt{1 + e^{-2x}} \longrightarrow u^2 = 1 + e^{-2x}$ $\quad u^2 - 1 = e^{-2x}$

$du = \frac{1}{2\sqrt{1+e^{-2x}}} \cdot e^{-2x} \cdot (-2)\, dx$

$= \frac{1}{\cancel{2}u} \cdot (u^2 - 1) \cdot (-\cancel{2})\, dx$

$= -\frac{u^2-1}{u}\, dx$

so $dx = -\frac{u}{u^2-1}\, du$

Bounds: when $x = 0$ $\quad u = \sqrt{1+1} = \sqrt{2}$

when $x = 2$ $\quad u = \sqrt{1 + e^{-4}} = \beta$

$$= -\int_{\sqrt{2}}^{\beta} \frac{u^2}{u^2-1}\, du$$

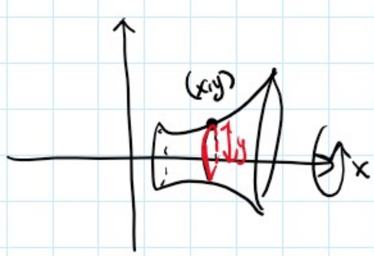
$$= \int_{\beta}^{\sqrt{2}} \frac{u^2}{u^2-1}\, du$$

$$= \left[u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| \right]_{\beta}^{\sqrt{2}}$$

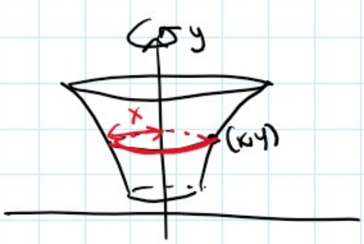
$= \dots$

Surface Area: Formula for surface area:

Surface area $= \int \left(\begin{array}{c}\text{Circumference of the circle} \\ \text{traced out by a point } (x,y)\end{array}\right) ds$



$\longrightarrow \int 2\pi y\, ds$



$\longrightarrow \int 2\pi x\, ds$

you can pick whether you want

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

Ex: Find SA of curve $y = \sqrt{1 + e^x}$, $0 \le x \le 1$ revolved around x-axis.

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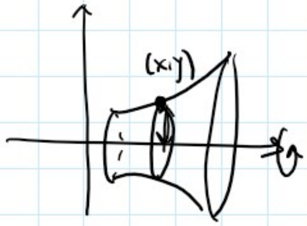

$$SA = \int 2\pi y \, ds$$

<u>Method 1</u> $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+e^x}} \cdot e^x$$

so

$$SA = \int_0^1 2\pi y \sqrt{1+\left(\frac{e^x}{2\sqrt{1+e^x}}\right)^2}\,dx$$

$$= \int_0^1 2\pi \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx$$

$$= \int_0^1 2\pi \sqrt{(1+e^x)\left(1+\frac{e^{2x}}{4(1+e^x)}\right)}\,dx$$

$$(1+e^x)\left(1+\frac{e^{2x}}{4(1+e^x)}\right)$$

$$(1+e^x) + \cancel{(1+e^x)}\left(\frac{e^{2x}}{4\cancel{(1+e^x)}}\right)$$

$$= \int_0^1 2\pi \sqrt{1+e^x+\tfrac{1}{4}e^{2x}}\,dx$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$a = \tfrac{1}{2}e^x \quad b=1$$

$$(\tfrac{1}{2}e^x+1) = \tfrac{1}{4}e^{2x}+e^x+1$$

$$= \int_0^1 2\pi \sqrt{(\tfrac{1}{2}e^x+1)^2}\,dx$$

$$= \int_0^1 2\pi (\tfrac{1}{2}e^x+1)\,dx$$

$$= \cdots$$

<u>Method 2</u>: $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$$y=\sqrt{1+e^x}$$
$$y^2 = 1+e^x$$
$$y^2-1 = e^x \;\rightarrow\; x = \ln(y^2-1)$$

$$\frac{dx}{dy} = \frac{1}{y^2-1}\cdot 2y = \frac{2y}{y^2-1}$$

$$SA = \int 2\pi y \, ds = \int_{\sqrt{2}}^{\sqrt{1+e}} 2\pi y \cdot \sqrt{1+\left(\frac{2y}{y^2-1}\right)^2}\,dy$$

Bounds: $x=0 \quad y=\sqrt{1+e^x}=\sqrt{2}$

$x=1 \quad y=\sqrt{1+e}$

$$= 2\pi \int^{\sqrt{1+e}} y\sqrt{1+\frac{4y^2}{(y^2-1)^2}}\,dy$$

$$= 2\pi \int_{\sqrt{2}}^{\sqrt{1+e}} y \sqrt{1 + \frac{4y^2}{(y^2-1)^2}}\, dy$$

$$= 2\pi \int_{\sqrt{2}}^{\sqrt{1+e}} y \sqrt{\frac{(y^2+1)^2}{(y^2-1)^2}}\, dy$$

$$= 2\pi \int_{\sqrt{2}}^{\sqrt{1+e}} y \left(\frac{y^2+1}{y^2-1}\right) dy$$

$$= \cdots \quad \text{use partial Fractions.}$$

$x = 1 \qquad y = \sqrt{1+e^1}$

$$1 + \frac{4y^2}{(y^2-1)^2} = \frac{(y^2-1)^2 + 4y^2}{(y^2-1)^2} = \frac{y^4 - 2y^2 + 1 + 4y^2}{(y^2-1)^2}$$

$$= \frac{y^4 + 2y^2 + 1}{(y^2-1)^2}$$

$$= \frac{(y^2+1)^2}{(y^2-1)^2}$$

§11.7 Exercises:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1} \qquad \text{(Test for Divergence)}$$

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$$

thus $\lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1}$ Does not exist.

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} \qquad \text{(Root test)}$$

$$\sum_{n=1}^{\infty} n^2 e^{-n^3} \qquad \text{(Integral test)} \qquad \int x^2 e^{-x^3}\, dx$$

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}} \qquad \left(\text{Comparison} \quad \frac{1}{k\sqrt{k^2+1}} \le \frac{1}{k\sqrt{k^2}} = \frac{1}{k^2} \right)$$

$$\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n} \qquad \left(\text{Show absolute convergence} \quad \left|\frac{\sin(2n)}{1+2^n}\right| \le \frac{1}{1+2^n} \le \frac{1}{2^n} \right)$$