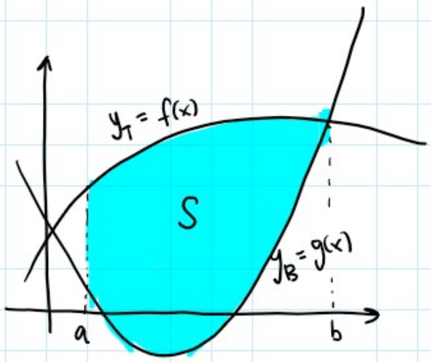
February 13, 2023 9:31 AM

Last time:

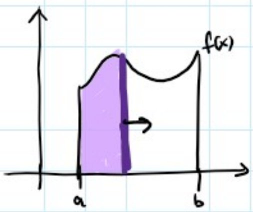


Area A of S:

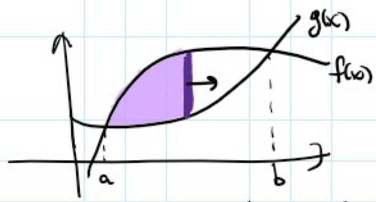
$$A = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_a^b [y_T - y_B]\,dx$$

A line of various heights, moved through space traces out an area:



$\int_a^b f(x)dx$

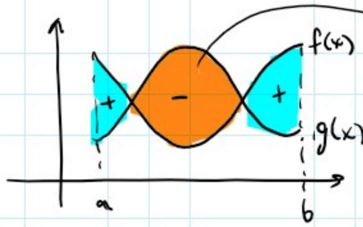


$\int_a^b [f(x) - g(x)]dx$

in this case the height of the line is $f(x) - g(x)$

Also last time:

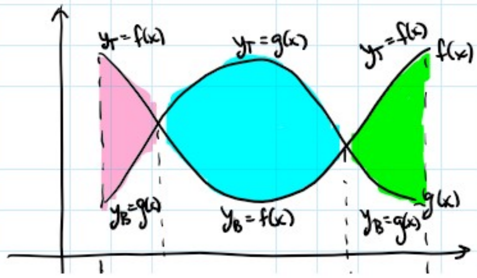
When we do area between curves we want positive areas. So, if $f(x)$ and $g(x)$ cross, we have issues of negative areas:

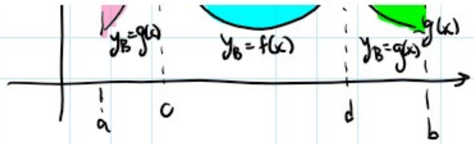


$f(x) - g(x)$ is negative.

To remedy this fact we really want $A = \int_a^b |f(x) - g(x)|\,dx$.

But, absolute values make for hard-to-evaluate integrands, so in practice we split the regions

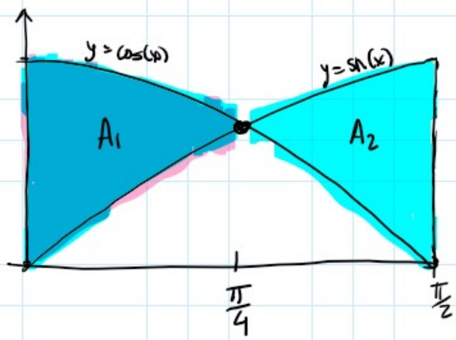


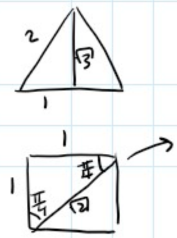


$$A = \int_a^b |f(x) - g(x)|\,dx = \int_a^c (f(x) - g(x))\,dx + \int_c^d (g(x) - f(x))\,dx + \int_d^b (f(x) - g(x))\,dx$$

<u>Minute Math/Ex 6</u>: Find the area of the region bounded by $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi/2$.

<u>Sol</u>: Point of intersection $\sin(x) = \cos(x)$ at $\pi/4$ (for the range x in $[0, \pi/2]$)





$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\sin(\frac{\pi}{4}) = 1/\sqrt{2}$

$$\text{Total area} = \int_0^{\pi/2} |\cos(x) - \sin(x)|\,dx = A_1 + A_2$$

$$= \int_0^{\pi/4} [\cos(x) - \sin(x)]\,dx + \int_{\pi/4}^{\pi/2} [\sin(x) - \cos(x)]\,dx$$

$$= \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4} + \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{\pi/2}$$

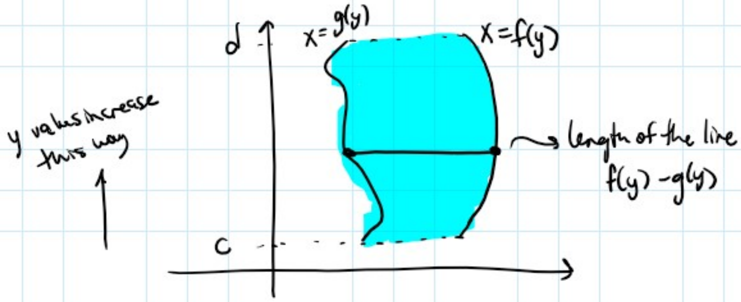
$$= \cdots$$

$$= 2\sqrt{2} - 2$$

We also could have noticed that by symmetry $A_1 = A_2$.

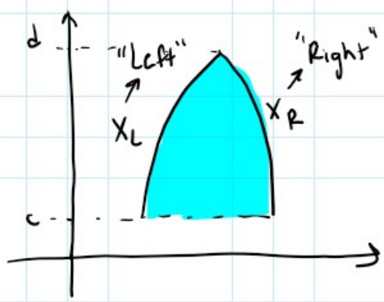
thus $A = 2A_1 = 2\int_0^{\pi/4} [\cos(x) - \sin(x)]\,dx$

Sometimes it is easier to integrate regions by regarding x as a function of y.



$$A = \int_c^d [f(y)-g(y)]\,dy$$

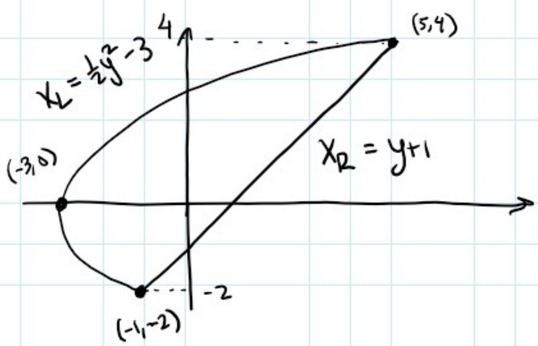
In the other notation:



$$A = \int_c^d (x_R - x_L)\,dy$$

<u>Ex7</u> Find the area enclosed by the line $y = x-1$ and the parabola $y^2 = 2x+6$

<u>Sol</u>: By solving we get points of intersection $(-1,-2)$ and $(5,4)$



Solving the equations for x:

$y = x-1 \rightarrow x = y+1$

$y^2 = 2x+6 \rightarrow x = \frac{1}{2}y^2 - 3$

when $y=0$ $x=-3$

So $(-3,0)$ is the vertex of the parabola.

$$\text{So} \quad A = \int_{-2}^{4} (x_R - x_L)\,dy$$

$$= \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy$$

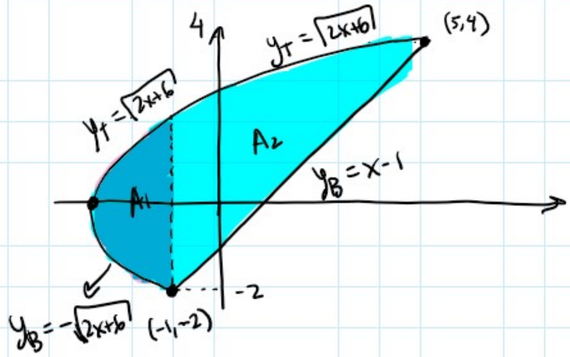
$$= \int^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

$$= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy$$

$$= \cdots$$

$$= 18$$

WARNING: If we had tried to integrate with respect to (w.r.t) x we would have had to split the region up and used gross formulas, namely $y = \pm\sqrt{2x+6}$



$$A_1 = \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx$$

$$A_2 = \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx$$

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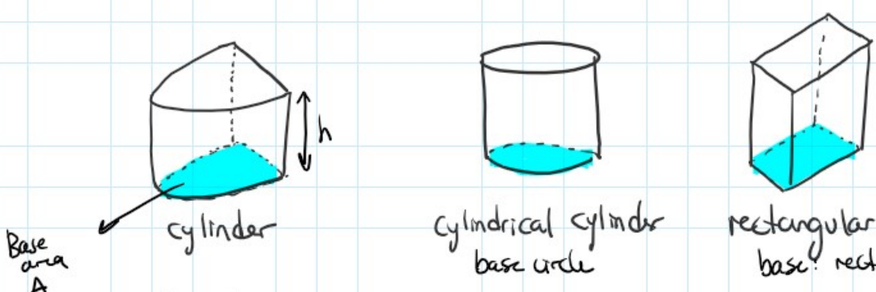
§ 6.2 Volumes In this section we will use integration to calculate volumes

To begin, we introduce the idea of a cylinder:



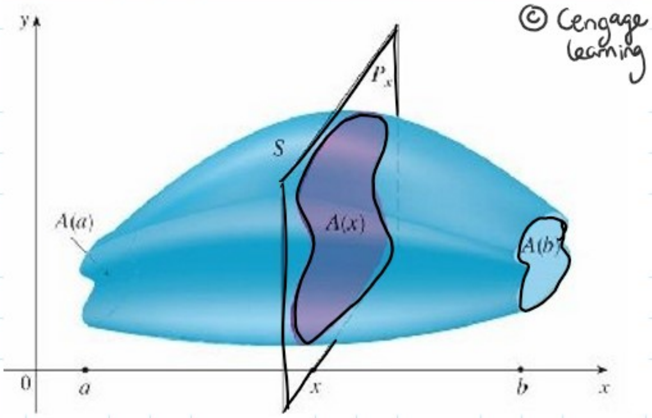
cylinder	cylindrical cylinder base circle	rectangular box base: rectangle
$V = Ah$	$V = \pi r^2 h$	$V = lwh$
	$(A = \pi r^2)$	$A = lw$

A cylinder is a prism with any base.

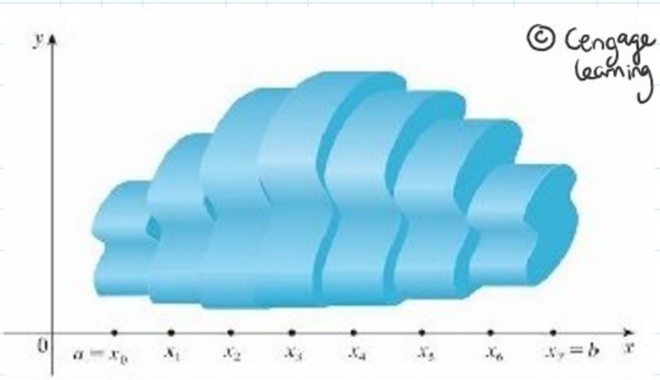
For a solid S that isn't a cylinder we will approximate its volume V by "cutting" S into pieces and approximating each piece by a cylinder.

To begin, say we have $A(x)$ which is the area of the cross-section of S in a plane perpendicular to the x axis passing through a point x (call the plane P_x).

The cross sectional area $A(x)$ will vary as x increases from a to b

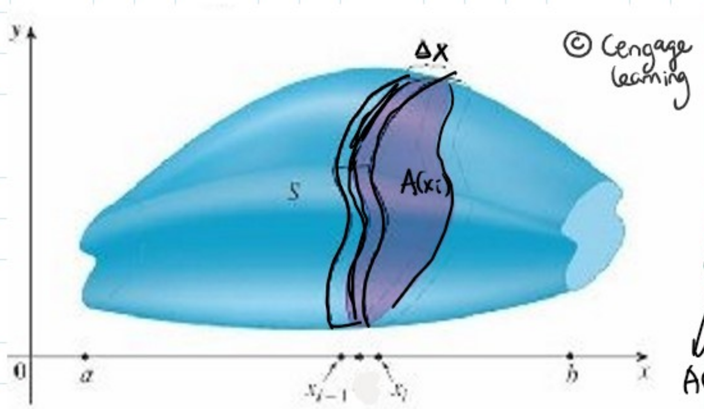


We now divide S into n "slabs" of equal width using planes $P_{x_1}, P_{x_2}, \dots$ to slice the solid.



$a = x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots \quad x_7 = b$

$n = 7.$



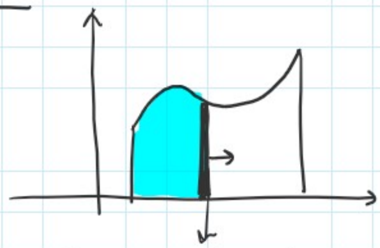
We approximate the volume of the i^{th} slab (call it S_i, the region between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i)$ and "height" Δx

ie $\quad V(S_i) \approx A(x_i)\Delta x$

so $\quad V = \sum_{i=1}^{n} A(x_i)\Delta x$

So we define the volume of S from a to b as $V = \int_a^b A(x)\,dx$.

<u>Note</u>:

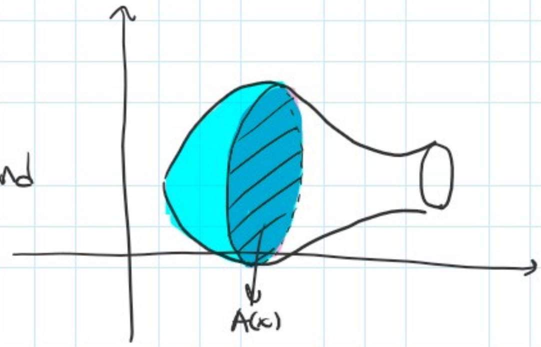


Moving a line with height $f(x)$ traces out an area

(lines: 1D, areas 2D)

$$A = \int_a^b f(x)\,dx$$

and



Moving a cross sectional area with area $A(x)$ traces out a volume

(areas: 2D, volumes: 3D)

$$V = \int_a^b A(x)\,dx$$