Lecture
January 6, 2023 4:25 PM

Announcements:

- I changed my office hours to be Mon/Wed/Fri 10:35am-11:30am. Same location (Burnside 1031 or the nearby hallway)

Last time: The Fundamental theorem of Calculus (FTC)

FTC1:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

FTC2:

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

for any F such that $F' = f$.

Let's do some examples of FTC1:

Minute Math:

FTC1:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

a) Find $\frac{d}{dx}\left[\int_5^x \frac{\sin(t)}{t} dt\right] = \frac{\sin(x)}{x}$

b) Find $\frac{d}{dx}\left[\int_{-1}^{\sin(x)} t^2 dt\right]$

b) $A(u) = \int_{-1}^{u} t^2 dt$ $\quad A'(u) = u^2$ $\quad A(\sin(x)) = \int_{-1}^{\sin(x)} t^2 dt$

$$\frac{d}{dx}\left[\int_{-1}^{\sin(x)} t^2 dt\right] = \frac{d}{dx}\left[A(\sin(x))\right]$$

$$= A'(\sin(x)) \cdot \cos(x)$$

$$= \sin^2(x)\cos(x)$$

c) Find $\frac{d}{dx}\left[\int_x^5 \arctan(t)\, dt\right]$

c) $\frac{d}{dx}\left[\int_x^5 \arctan(t)\, dt\right]$

$$= \frac{d}{dx}\left[-\int_5^x \arctan(t)dt\right]$$

$$= -\arctan(x)$$

d) Find $\frac{d}{dx}\left[\int_{-x}^{x} f(t)\, dt\right]$

(where f is a generic continuous function)

d) $\frac{d}{dx}\left[\int_{-x}^{x} f(t)\, dt\right]$

$$= \frac{d}{dx}\left[\int_{-x}^{0} f(t)dt + \int_0^x f(t)dt\right]$$

$$= \frac{d}{dx}\left[-\int_0^{-x} f(t)dt + \int_0^x f(t)dt\right]$$

$$= \frac{d}{dx}\left[-\int_0^{-x} f(t)\,dt + \int_0^{x} f(t)\,dt\right]$$

$$= -f(-x)\cdot(-1) + f(x)$$

$$= f(-x) + f(x)$$

Part b inspires a general formula:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x))\cdot g'(x)$$

Now for FTC2:

> **FTC2:**
> $$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$
> for any F such that $F' = f$.

Ex Evaluate $\int_3^6 \frac{dx}{x}$ (this notation is shorthand for $\int_3^6 \left(\frac{1}{x}\right)dx$)

Sol: an antiderivative for $1/x$ on $(0,\infty)$ is $\ln(x)$

$$\int_3^6 \frac{dx}{x} = \left[\ln(x)\right]_3^6 = \ln(6) - \ln(3) = \ln\left(\frac{6}{3}\right) = \ln(2)$$

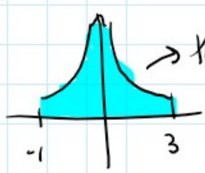
Ex What's wrong with the following calculation?

> x^n has antiderivative $\frac{x^{n+1}}{n+1} + C$ when $n \neq -1$.

WRONG

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{3} = -\left[\frac{1}{x}\right]_{-1}^{3} = -\frac{4}{3}$$

$\frac{1}{x^2} \geq 0$ for $x \neq 0$, so



→ the area should be positive!

What's wrong? $\frac{1}{x^2}$ is not continuous on $[-1, 3]$, so FTC2 does not apply.

§5.4 Indefinite integrals and the net change theorem

Notation: "$\int f(x)\,dx = F(x)$" means "$F'(x) = f(x)$" ← we call this the indefinite integral

ie, that F is an antiderivative of f.

Ex: $\int x^2 \ldots x^3$

ie, that F is an antiderivative of f. integral

Ex: $\int x^2\,dx = \frac{x^3}{3} + C$

WARNING: ⚠

Indefinite integral	Definite Integral
$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
this is an antiderivative ie it's a function	this is the area under the curve ie it's a number (for fixed a and b)

Here are some (not all) entries of the antiderivative table in your textbook

$$\int c f(x)\,dx = c\int f(x)\,dx$$

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

n can be non-integer ie $n = 1/2$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

(remember the absolute value!)

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

... etc.

Ex: Find $\int 10x^4 - 2\sec^2 x\,dx$

$\frac{d}{dx}[\tan(x)] = \sec^2 x$

$\int \sec^2 x\,dx = \tan x + C$

Sol:
$$\int 10x^4 - 2\sec^2 x\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$
$$= 10\left(\frac{x^5}{5}\right) - 2\tan(x) + C$$
$$= 2x^5 - 2\tan(x) + C$$

Ex Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

$n = 1/2$

$n+1 = 3/2$

Sol: First find $\int \frac{2\cancel{t^2} + \cancel{t^2}\sqrt{t} - 1}{t^2}\,dt = \int (2 + t^{1/2} - t^{-2})\,dt$

$n+1 = 3/2$

Sol: First find $\displaystyle\int \frac{2\cancel{t^2}}{\cancel{t^2}} + \frac{\cancel{t^2}\sqrt{t}}{\cancel{t^2}} - \frac{1}{t^2}\,dt = \int \left(2 + t^{1/2} - t^{-2}\right) dt$

$$= \int 2\,dt + \int t^{1/2}\,dt - \int t^{-2}\,dt$$

$$= 2t + \frac{t^{3/2}}{(3/2)} - \frac{t^{-1}}{-1} + C$$

$$= 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} + C$$

So $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \left[2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right]_1^9$

$$= \left(2(9) + \frac{2}{3}(9)^{3/2} + \frac{1}{9}\right) - \left(2(1) + \frac{2}{3}(1)^{3/2} + \frac{1}{1}\right)$$

$$= \dots$$

$$= 32 + \frac{4}{9}$$

Ex: $\displaystyle\int_0^1 5\,dx = \left[5x\right]_0^1 = 5(1) - 5(0) = 5$

$\displaystyle\int_0^1 5\,dx = \left[5x + 1\right]_0^1 = (5(1)+1) - (5(0)+1) = 5 \cancel{+1} - (0 \cancel{+1}) = 5$

Why do we not care about the "a" in FTC1?:

Let $\displaystyle\int f(x)\,dx = F(x)$

then $\displaystyle\int_a^x f(t)\,dt \overset{FTC2}{=} \left[F(t)\right]_a^x = F(x) - F(a)$

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \frac{d}{dx}\left[F(x) - F(a)\right] = \frac{d}{dx}\left[F(x)\right] - \underbrace{\cancel{\frac{d}{dx}\left[F(a)\right]}}_{\text{constant}\;\; 0}$$

$$= F'(x)$$

$$= f(x)$$

$\displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

The Net change theorem:

plug this in

FTC2 says $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ where $F' = f$

Another way to write this $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

Thm (Net change theorem): The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

<u>Ex</u>: If an object moves along a straight line with position function $s(t)$, then its velocity $v(t) = s'(t)$, so:

$$\int_{t_1}^{t_2} v(t)\,dt = \underbrace{s(t_2) - s(t_1)}_{\text{net change in position}}$$

(ie the displacement of the particle during the time period t_1 to t_2)