March 10, 2023 10:52 AM

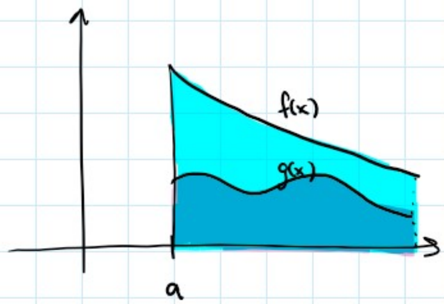
Continuing Improper Integrals:

In cases where we only need to know if an improper integral converges (and not its value) we can appeal to the following theorem.

Comparison Theorem: Suppose f and g are cont. with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a) If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.

b) If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges



Rmk: If $\int_a^\infty f(x)dx$ diverges then the theorem says nothing about $\int_a^\infty g(x)dx$

Ex: $f(x)=1$
$g(x)=1/x$
$a=1$

$1 \geq \frac{1}{x} \geq \frac{1}{x^2}$

$\int_1^\infty 1dx = \infty$

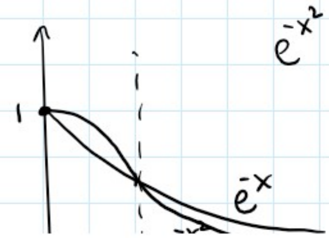
$\int_1^\infty \frac{1}{x}dx = \infty$ diverges $\qquad \int_1^\infty \frac{1}{x^2}dx$ converges

If $\int_a^\infty g(x)dx$ converges then the theorem says nothing about $\int_a^\infty f(x)dx$

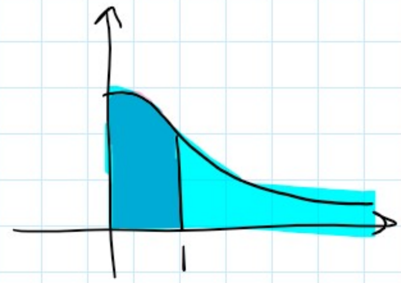
Ex 9: Show that $\int_0^\infty e^{-x^2}dx$ is convergent.

Sol Note that e^{-x^2} has no elementary antiderivative.
We want to compare it to a simpler function, the most obvious choice to try is e^{-x}

Let's graph the two:



We write $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \underbrace{\int_0^{1} e^{-x^2}\,dx}_{\substack{\text{a finite \#} \\ \text{a definite,} \\ \text{not improper,} \\ \text{integral!}}} + \int_1^{\infty} e^{-x^2}\,dx$



So $\displaystyle\int_0^{\infty} e^{-x^2}\,dx$ converges if $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ converges.

We use the Comparison theorem, with $f(x) = e^{-x}$, $g(x) = e^{-x^2}$, $a = 1$

We check:

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx = \lim_{t\to\infty} \left[-e^{-x}\right]_1^{t} = \lim_{t\to\infty} e^{-1} - \cancel{e^{-t}}^{\,0} = e^{-1}$$

thus $\displaystyle\int_1^{\infty} f(x)\,dx$ converges, and so $\displaystyle\int_1^{\infty} g(x)\,dx$ converges, ie $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ converges.

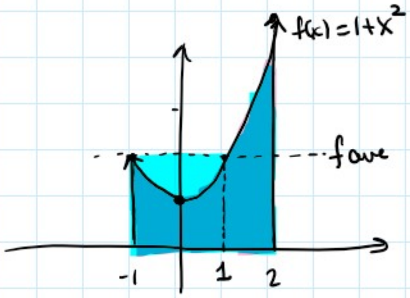
Therefore $\displaystyle\int_0^{\infty} e^{-x^2}\,dx$ converges.

One quick addition to §6.5: Averge value of a function:

The Mean Value Theorem for integrals: If f is cont. on $[a,b]$, then there exists a number c, such that $a < c < b$ so that:

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex $f(x) = 1+x^2$, over $[-1, 2]$.



we calculated that

$$f_{ave} = \frac{1}{(2)-(-1)} \int_{-1}^{2} (1+x^2)\,dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = \cdots = 2$$

Then we see that $f(1) = 2$

So $c = 1$, gives that $f(c) = f_{ave}$

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§11.1: Sequences:

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, \ a_2, \ a_3, \ \ldots, \ a_n, \ \ldots$$

a_1 is our first term

a_2 is our second term

a_n is our n^{th} term

You could also think of a_n as being a function $f(n)$ with domain in the whole numbers, but generally we will stick to subscript notation.

Notation: Sometimes we write $\{a_1, a_2, a_3, \ldots\}$ as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Ex 1:

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$

b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$, $a_n = \frac{(-1)^n(n+1)}{3^n}$, $\left\{-\frac{2}{3}, +\frac{3}{9}, -\frac{4}{27}, +\frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

n	$(-1)^n$
1	-1
2	1
3	-1
4	1
5	-1

c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$, $a_n = \sqrt{n-3}$ where $n \geq 3$, $\left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

this is a_3

$5 \mid -1$

Ex2/Minute Math: Find a formula for the general term a_n of the sequence:

$$\left\{ \underset{\uparrow \atop a_1}{\frac{3}{5}},\ \underset{\uparrow \atop a_2}{-\frac{4}{25}},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125},\ \dots \right\}$$

Sol: Let's do sign, numerator, and denominator separately:

sign:

n	1	2	3	4	5	6
sign	1	-1	1	-1	1	-1

$(-1)^{n+1}$ or $(-1)^{n-1}$

numerators:

n	1	2	3	4	5
num.	3	4	5	6	7

num. $= n+2$

denominators:

n	1	2	3	4
den	5	25	125	625

den $= 5^n$

$$a_n = (-1)^{n+1} \frac{n+2}{5^n}$$

$$a_n = \frac{(-2-n)}{(-5)^n} = \frac{(-1)(n+2)}{(-1)^n 5^n} = (-1)^{n+1} \frac{(n+2)}{5^n}$$