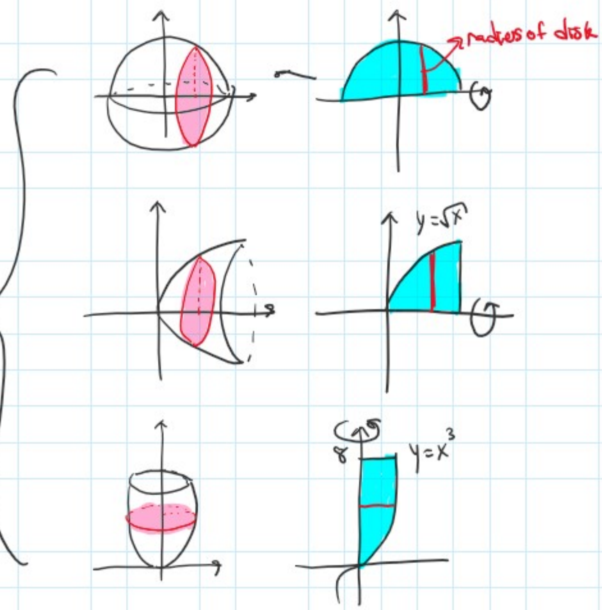
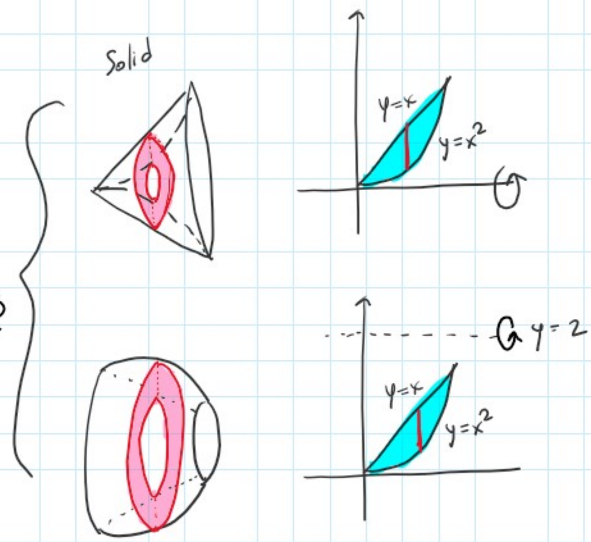
A review of §6.2:

The solids we've seen so far

DISK
Ex1 - Ex3



WASHER
Ex4 - Ex5



These are called "Solids of Revolution" (SoR), because they are obtained by revolving a region around a line. In general, we calculate their volume by using:

$$V=\int_a^b A(x)\,dx \qquad \text{or} \qquad V=\int_c^d A(y)\,dy$$

We find the cross sectional area in one of the following ways:

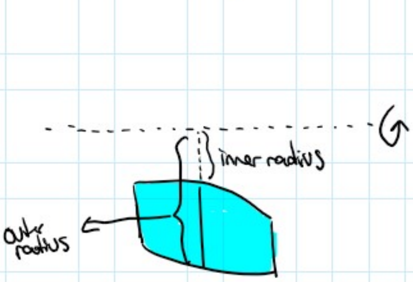
1) If the cross section is a <u>disk</u> (as in Ex1-3)

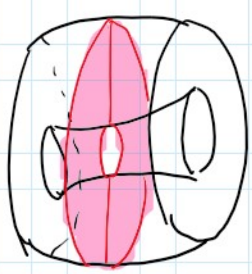
$$A=\pi(\text{radius})^2$$

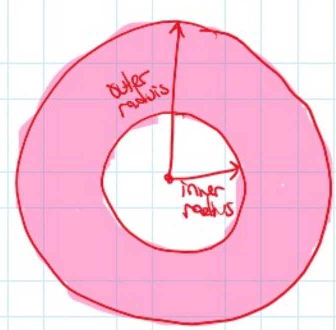
2) If the cross section is a <u>washer</u> (as in Ex4-5)

$$A=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$

⚠ ($A \neq \pi(\text{outer radius} - \text{inner radius})^2$) WRONG!

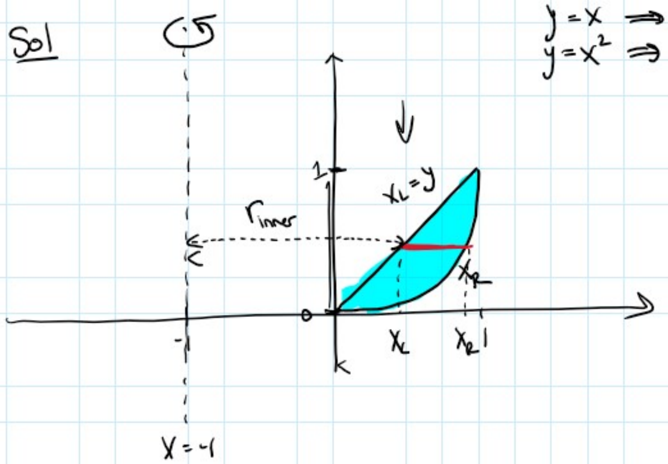




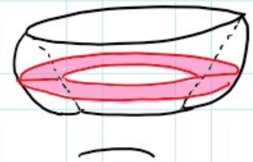


<u>Ex 6:</u> Find the volume of the SoR obtained by rotating the region enclosed by the curves $y=x$, $y=x^2$, around the line $x=-1$.

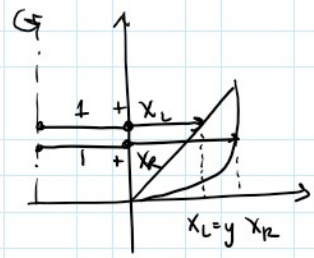
<u>Sol</u>



$y = x \Rightarrow x_L = y$ left boundary of region
$y = x^2 \Rightarrow x_R = \sqrt{y}$ right boundary of region



$b-a = (x_L)-(-1) = 1+x_L$



$$r_{inner} = 1 + x_L = 1 + y$$
$$r_{outer} = 1 + x_R = 1 + \sqrt{y}$$

$$A(y) = \pi r_{outer}^2 - \pi r_{inner}^2$$
$$= \pi(1+\sqrt{y})^2 - \pi(1+y)^2$$

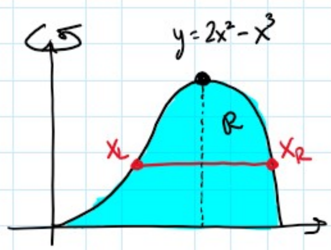
$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 \left[(1+\sqrt{y})^2 - (1+y)^2\right] dy = \cdots = \frac{\pi}{2}$$

We won't do Ex7 - Ex9 because they concern solids which aren't SoR.

<u>§6.3 Volumes by Cylindrical Shells:</u>

<u>Motivating Example:</u>

Consider the region R given by:



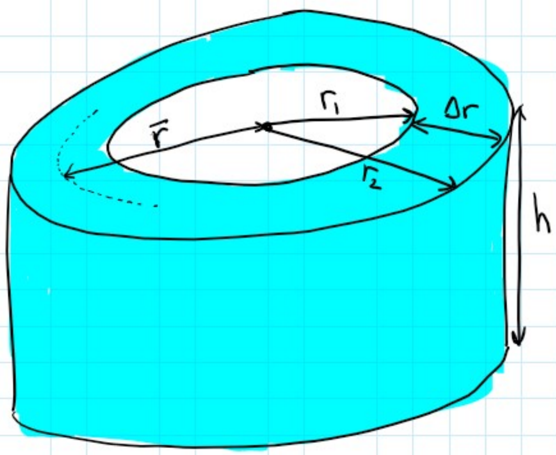
The volume of the SoR obtained by revolving R around the y axis is not easy to do via cross sectional area (ie washers in this case)

In particular solving $y = 2x^2 - x^3$ for x is not easy to do.

But also, we need x_L and x_R so we would need to find the x-coordinate of the maximum

This seems like a ton of work. Fortunately there is another method, called the method of cylindrical shells (or just "shells" for short) that is easier in a case like this.

<u>What is a cylindrical shell?</u>



Volume of a cylindrical shell:

$$V = \overbrace{V_2}^{\text{outer cylinder}} - \overbrace{V_1}^{\text{inner cylinder}}$$

$$= \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi h\left(r_2^2 - r_1^2\right) \qquad \Delta r = r_2 - r_1$$

$$= \pi h(r_2 + r_1)\underbrace{(r_2 - r_1)}_{\Delta r}$$

$$= \pi h(r_2 + r_1)\Delta r$$

$$= 2\pi h\left(\frac{r_2 + r_1}{2}\right)\Delta r$$

$$= 2\pi h \bar{r} \Delta r$$

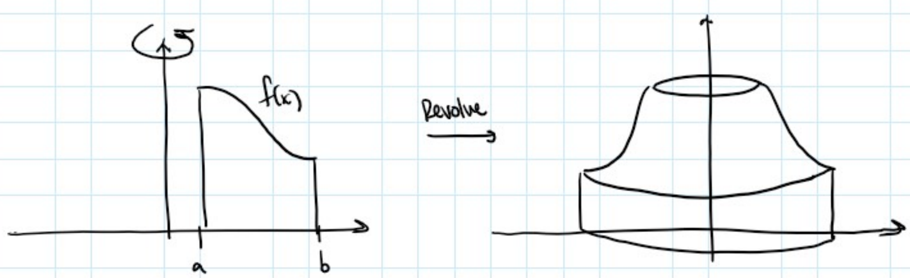
where $\bar{r} = \frac{r_2 + r_1}{2}$ the midpoint of the radii.

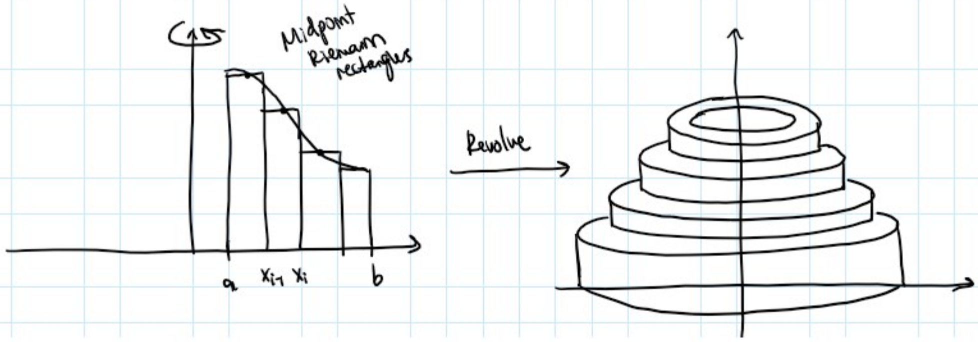
Thus the volume of a cylindrical shell can be remembered by:

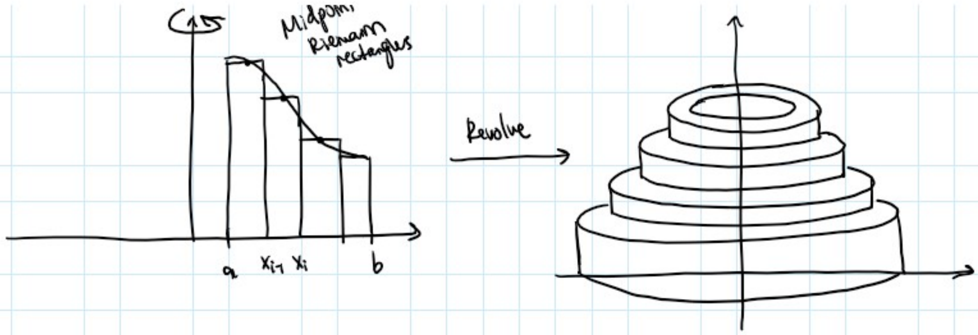
$$V = (2\pi \bar{r})\, h\, \Delta r$$

$$V = [\text{circumference}]\,[\text{height}]\,[\text{thickness}]$$

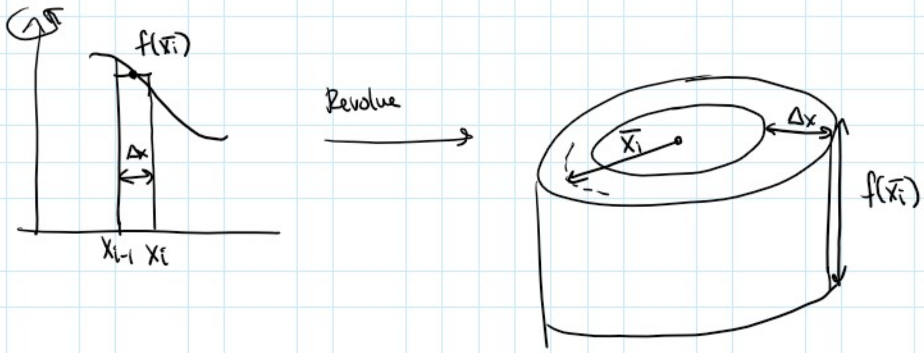
Let's use this to calculate volumes of SoRs:







Let's find the volume of one of these approximating cylindrical shells:



Volume of the i^{th} shell (obtained from the interval $[x_{i-1}, x_i]$) is

$$V_i = [\text{circumference}][\text{height}][\text{thickness}]$$

$$= (2\pi \bar{x}_i)\, f(\bar{x}_i)\, \Delta x$$

$$= 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$$

So $\quad V \approx \displaystyle\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\, \bar{x}_i\, f(\bar{x}_i) \Delta x$

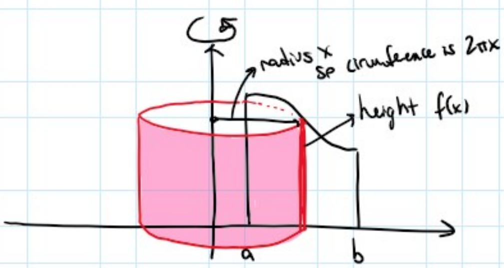
The approximation gets better as $n \to \infty$, therefore:

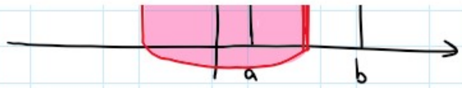
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x = \int_a^b 2\pi x f(x)\, dx$$

} Volume of the SoR obtained by revolving around the y axis the region under f(x) from a to b.

The way to remember this formula is:

$$V = \int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \overbrace{f(x)}^{\text{height}} \underbrace{dx}_{\text{thickness}}$$




a
b