Lecture

January 6, 2023 4:25 PM

Announcements:

Update: Mondays/Wednesdays/Fridays 10:35am - 11:30am
Same location

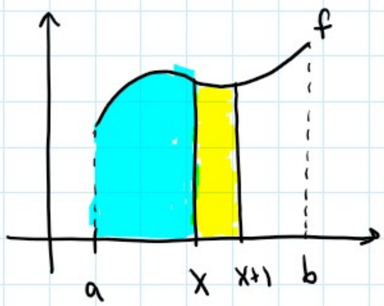
- There will be no lecture notes for Lecture 4 (last class) because Sid used the chalkboard.
- My office hours are ~~Tuesdays/Thursdays 10am - 11:30am~~ in Burnside 1031 or the nearby hallway.
  ↳ I will hold office hours today 10:30am - 11:45am at the same location

§5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) deals with functions of the form

$$A(x) = \int_a^x f(t)\,dt$$

Note that if x is fixed $\int_a^x f(t)dt$ is just a number. But, if x varies, then $\int_a^x f(t)dt$.



we think of $A(x)$ as the "area so far" function.

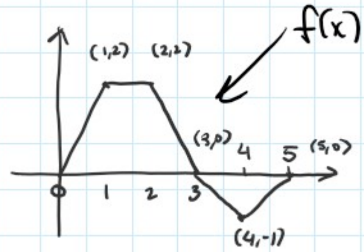
$A(x)$ = blue area

$A(x+1)$ = blue area + yellow area

Ex: Let f be the function given by this graph

Then let $A(x) = \int_0^x f(t)dt$, find

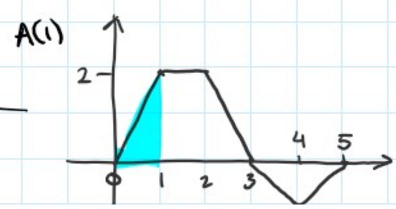
$A(0), A(1), \ldots, A(5)$ and sketch $A(x)$ from 0 to 5.



Sol:

$$A(0) = \int_0^0 f(t)dt = 0$$

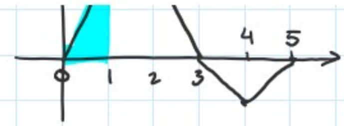
$$A(1) = \int_0^1 f(t)dt$$



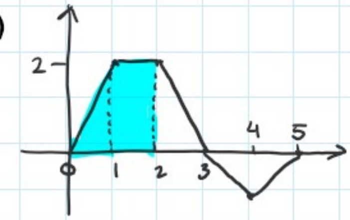
$A(1) = \int_0^1 f(t)\,dt$

$= \frac{1}{2}(1)(2)$

$= 1$



A(2)



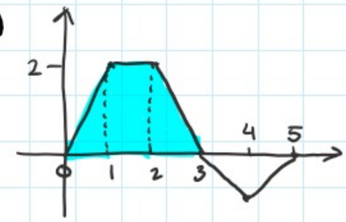
$A(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

$= A(1) + \int_1^2 f(t)\,dt$

$= 1 + (1)(2)$

$= 3$

A(3)

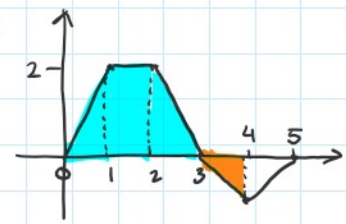


$A(3) = A(2) + \int_2^3 f(t)\,dt$

$= A(2) + 1$

$= 4$

A(4)

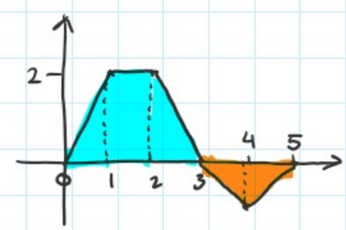


$A(4) = A(3) + \int_3^4 f(t)\,dt$

$= 4 - \frac{1}{2}(1)(1)$

$= 3.5$

A(5)

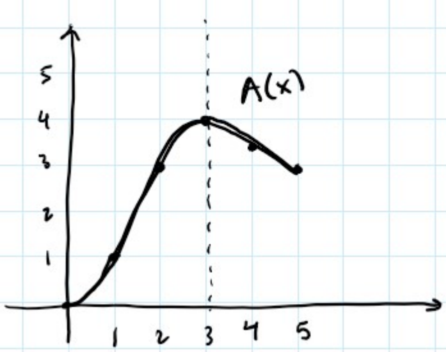


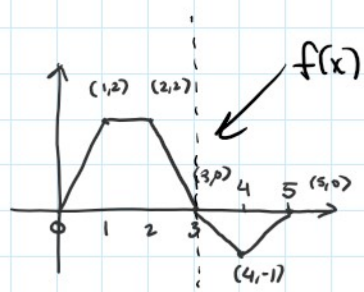
$A(5) = A(4) + \int_4^5 f(t)\,dt$

$= 3.5 - 0.5$

$= 3$

Sketch A:





at $x = 3$

A has a maximum

$f(3) = 0$.

at $x=3$
A has a maximum

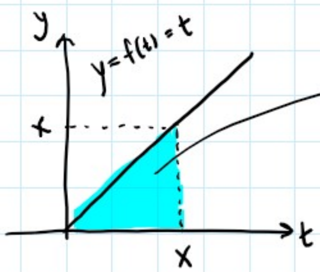
$f(3)=0$.
$f=0$ at $x=3$

Note that $A(x)$ is increasing when $f(x)$ is positive
$A(x)$ is decreasing when $f(x)$ is negative
$A(x)$ has its maximum when $f(x)=0$

It looks like $A'=f$, or in other words, A is an antiderivative of f.

<u>Ex2</u> Let $f(t)=t$, and $A(x)=\int_0^x t\,dt$. Find another formula for A that doesn't use integrals.

<u>Sol</u>:



$$A(x)=\int_0^x t\,dt=\frac{1}{2}(x)(x)=\frac{x^2}{2}$$

Notice that $\frac{d}{dx}[A(x)]=\frac{d}{dx}\left[\frac{x^2}{2}\right]=\frac{2x}{2}=x=f(x)$

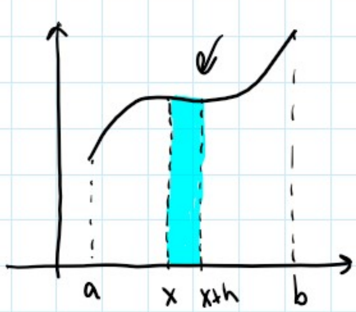
So A is an antiderivative of f

<u>Theorem</u>: (FTC1) If f is continuous on $[a,b]$, then the function defined by $A(x)=\int_a^x f(t)\,dt$ for $a\le x\le b$ is continuous on $[a,b]$, differentiable on (a,b) and A is an antiderivative of f, meaning $A'(x)=f(x)$ for x in (a,b).

<u>Proof Idea</u>: We use the definition of the derivative:

$$A'(x)=\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}$$

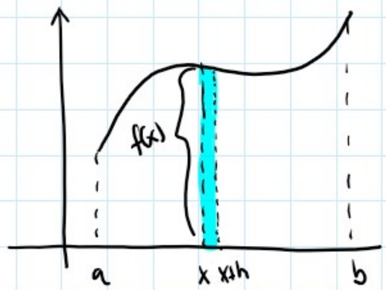
$]a,b[\ =(a,b)$



$$A(x+h)-A(x)$$
$$=\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt$$
$$=\left(\cancel{\int_a^x f(t)\,dt}+\int_x^{x+h} f(t)\,dt\right)-\cancel{\int_a^x f(t)\,dt}$$
$$=\int_x^{x+h} f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

So $\dfrac{A(x+h)-A(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$



because f is continuous, when h is small, this region is approximately a rectangle

$$\int_x^{x+h} f(t)dt \simeq f(x)\cdot h$$

thus $\dfrac{A(x+h)-A(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)dt \simeq \dfrac{1}{\cancel{h}}\cdot(f(x)\cdot \cancel{h}) = f(x)$

thus $A'(x) = \displaystyle\lim_{h\to 0} \dfrac{A(x+h)-A(x)}{h} = f(x)$.

In short FTC1 says: $\dfrac{d}{dx}\left[\displaystyle\int_a^x f(t)\,dt\right] = f(x)$

Ex Find the derivative of $A(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$

Sol: Since $f(t) = \sqrt{1+t^2}$ is continuous, FTC1 gives

$$A'(x) = f(x) = \sqrt{1+x^2}$$

Ex Find $\dfrac{d}{dx}\left[\displaystyle\int_1^{x^4} \sec(t)\,dt\right]$

Sol: Let $A(y) = \displaystyle\int_1^y \sec(t)dt$, then FTC1 says $A'(y) = \sec(y)$

However, we want $\dfrac{d}{dx}[A(x^4)]$ $\quad$ $A(x^4) = \displaystyle\int_1^{x^4} \sec(t)dt$

we use the chain rule:

$$\frac{d}{dx}[A(x^4)] = A'(x^4)\cdot \frac{d}{dx}[x^4] \longrightarrow \frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$$

$$= \sec(x^4)\,4x^3$$

Theorem (FTC2) If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f (ie $F' = f$)

Proof: Let $A(x) = \int_a^x f(t)dt$. Then FTC1 says A is an antiderivative of f.

And, by assumption F is also an antiderivative of f. Therefore A and F can differ only by a constant, ie $\boxed{F(x) = A(x) + C}^{*}$

Now, $F(a) \overset{*}{=} A(a) + C = \cancel{\int_a^a f(t)dt}^{0} + C = C$

thus $C = F(a)$

Thus $F(x) \overset{*}{=} A(x) + F(a)$

Plugging in $x=b$: $F(b) = A(b) + F(a)$

rearranging: $F(b) - F(a) = A(b) = \int_a^b f(x)dx$

So $\int_a^b f(x)dx = F(b) - F(a)$

Ex: Use FTC2 to evaluate $\int_1^3 e^x dx$

Sol: $f(x) = e^x$ is continuous, and $F(x) = e^x$ is an antiderivative

$$\int_1^3 e^x dx = F(3) - F(1) = e^3 - e.$$

So $[e^x]_1^3 = e^3 - e^1$

Notation: We use the notation $[F(x)]_a^b = F(b) - F(a)$

other notations: $F(x)\Big|_a^b$ or $F(x)\Big]_a^b$

Remark: $[cF(x)]_a^b = cF(b) - cF(a)$

$= c(F(b) - F(a))$

$= c[F(x)]_a^b$

$$= c\left(F(b) - F(a)\right)$$
$$= c\left[F(x)\right]_a^b$$

$$\left[F(x) + G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$

Therefore <u>FTC2</u> can be summarized as:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b \quad \text{where } F' = f$$