Series Tests

1. Limit Test

Theorem 1. *Let* (a_n) *be an infinite series, if* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

It is important to know that the reverse is not true.

Theorem 2. *Let* (a_n) *be an infinite series, if* $\lim_{n\to\infty} a_n \neq 0$ *or doesn't exist, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Exercises :

Justify if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$
2. $\sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$
3. $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

2. Integral Test

Theorem 3. *Let (a_n) be an infinite series, and $f(x)$ is a continuous function defined on $(0,\infty)$. If for all positive integer n, we have $a_n = f(n)$, then by integral test, we say $\sum_{n=1}^{\infty} a_n$ converges if the followings hold:*

1. $f(x) \geq 0$ **for all** $n \geq 1$.

2. $f(x)$ **is decreasing on** (a,∞) **for some** $a \geq 1$**, and** $\lim_{n\to\infty} f(x) = 0$**.**

3. $\int_1^{\infty} f(x)dx$ **converges.**

Exercises : Justify if the following series converge or diverge. 1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ 2. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ 3. $\sum_{n=1}^{\infty} ne^{-n^2}$

4. Find the values of p, such that the series $\sum_{n=1}^{\infty} n(1+n^2)^p$ is convergent.

3. Comparison Test

Theorem 4. *Let $(a_n), (b_n)$ be* ***positive****, infinite series with the property that $a_n \leq b_n$, then*

$$\text{If} \sum_{n=1}^{\infty} b_n \text{ is convergent, then} \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\text{If} \sum_{n=1}^{\infty} a_n \text{ is divergent, then} \sum_{n=1}^{\infty} b_n \text{ is divergent.}$$

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Theorem 5. *Let $(a_n), (b_n)$ be* ***positive****, infinite series, if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then both $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges or diverges.*

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$
2. $\displaystyle\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$

4. Alternating Series Test

Theorem 6. *Let (b_n) be an infinite series where $b_n \geq 0$ for all n, if $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then $\displaystyle\sum_{i=1}^{\infty}(-1)^n b_n$ and $\displaystyle\sum_{i=1}^{\infty}(-1)^{n+1} b_n$ converge.*

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+1}$

Definition :

Let (a_n) be an infinite series, the sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. Series that converge but not absolutely converge are called **conditionally convergent**.

Exercise : Test the sum $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$.

5. Ratio Test and Root Test

Theorem 7. *Let (a_n) be an infinite series, define $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then:*

(1) *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is* ***absolutely convergent****.*

(2) *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is* ***divergent****.*

(3) *If $L = 1$, then the ratio test is* ***inconclusive***

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Theorem 8. *Let (a_n) be an infinite series, define $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, then:*

(1) *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is* ***absolutely convergent****.*

(2) *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is* ***divergent****.*

(3) *If $L = 1$, then the root test is* ***inconclusive***

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{n+1}}{3^{2n+1}}$

More series tests like Dirichlet's test, Abel's test, Weierstrass M-test, are beyond the slope of this course and hence will not be introduced. You can look it up if you are interested.

6. Strategy for Testing Series

(1) If (a_n) is of the form $\dfrac{1}{n^p}$, then, we should know that $\sum a_n$ converges for $p > 1$ and diverges for $p \leq 1$, also if (a_n) is a geometric series, then we can use the formula to find its sum.

(2) If (a_n) has a form which is similar to p-series or geometric series, then we can try to use comparison test.

(3) If $\lim_{n\to\infty} a_n \neq 0$, then (a_n) is divergent.

(4) If $(a_n) = (-1)^n b_n$, then we shall apply alternating series test.

(5) If inside (a_n) there are terms like $n!$, or some products related to n, then we shall consider ratio test.

(6) If $a_n = (b_n)^n$. then we shall consider root test.

(7) If $a_n = f(n)$ for some positive elementary function $f(x)$, if it is easy to integrate $f(x)$, we shall consider integral test.

Exercises : Justify if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

2. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln(n)}{(1+n)^3}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^n(n)}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\tan(n)}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n^2+1}}$

7. Power Series, Radius of Convergence and Interval of Convergence

Theorem 9. *For any given power series* $a_n = \sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

(1) *The series converges only when* $x = a$*;*

(2) *The series converges for all* x*;*

(3) *There exists a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*. In this case,* R *is called the* ***radius of convergence****. The interval of convergence is one of* $(a-R, a+R); (a-R, a+R]; [a-R, a+R); [a-R, a+R]$*, depending on the convergence of boundary points.*

Exercises :

1. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

2. For what values of x is the series $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

3. Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt[3]{n}}$