

Integration of rational functions by partial fractions

In this section, we will learn how to solve integrals like $\int \frac{P(x)}{Q(x)} dx$ where $P(x), Q(x)$ are polynomials, and $Q(x) \neq 0$. The method we use is called **Partial Fraction Decomposition (PFD)**, which is basically decompose a fraction into the sums of several fractions, such that for each single term we can integrate by using basic methods. A simple example is to integrate $\int \frac{1}{x^2-1} dx$, in this case we may use $x^2 - 1 = (x+1)(x-1)$ and decompose the integral as

$$\int \frac{1}{(x+1)(x-1)} dx = \frac{1}{2} \left(\int \frac{1}{x-1} dx + \int \frac{1}{x+1} dx \right)$$

Thus for each part we may integrate quickly, that is

$$\frac{1}{2} \left(\int \frac{1}{x-1} dx + \int \frac{1}{x+1} dx \right) = \frac{1}{2} (\ln|x-1| + \ln|x+1|) + C$$

We will learn all the methods to do such integral. In the whole section, we will assume that $f(x) = \frac{P(x)}{Q(x)}$

Step 1 : Check if $f(x)$ is proper or not

Suppose $P(x)$ has a maximum degree of n and $Q(x)$ has a maximum degree of m , then:

- $f(x)$ is proper, if $m > n$, such as $f(x) = \frac{x^3+1}{x^4+2x}$ is proper
- $f(x)$ is improper, if $m \leq n$, such as $f(x) = \frac{x^3+1}{x^2-6}$ is improper

Note that partial fraction decomposition only works for proper fractions. For improper fractions, we need to do **long division** to simplify the polynomial. If $f(x)$ is improper, we may do:

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

by long division, and the remainder $\frac{R(x)}{Q(x)}$ would always be proper. Here is an example for doing long-division: Suppose we have $f(x) = \frac{x^3+x}{x-1} dx$, then we observe that $f(x)$ is improper, so the long division is like:

$$\begin{array}{r}
 x^2 + x + 2 \\
 \hline
 x-1 \overline{) x^3 + x} \\
 \underline{-x^3 + x^2} \\
 x^2 + x \\
 \underline{-x^2 + x} \\
 2x \\
 \underline{-2x + 2} \\
 2
 \end{array}$$

The steps are quite similar to integer divisions. We try to vanish the highest degree term in each step. On the first step we want to vanish x^3 term since it is the highest degree term, so we write x^2 to our result, and we have $x^2 \times (x-1) = x^3 + x^2$, and we subtract the original $x^3 + x$ with what we get, we then have $x^2 + x$ remaining. Then we will try to vanish x^2 term, so we put $+x$ to the result, and do the same thing, until we cannot subtract anymore. In the structure above, the final result should be like $f(x) = S(x) + \frac{R(x)}{Q(x)}$, and $S(x)$ is the result we get, which is $x^2 + x + 2$, and $Q(x)$ is the original denominator, $Q(x) = x-1$ and $R(x)$ is the reminder 2, so the result should look like

$$f(x) = x^2 + x + 2 + \frac{2}{x-1}$$

Thus, it is much more easier to integrate this new function.

• Exercises on long division •

$$(1) f(x) = \frac{x^4 - 2x^3 + 4x + 1}{x^3 - x^2 - x + 1}$$

$$(2) g(x) = \frac{x^4 + 1}{x^2 + 1}$$

Step 2 : When $f(x)$ is proper, apply PFD

Theorem 1 (PFD). Any polynomial $Q(x)$ can be factored into a product of factors.

In this course, the denominator will always be easy to factor, so you shouldn't worry much about it. For example, to factor $Q(x) = x^4 - 16$, $M(x) = 2x^3 + 3x^2 - 2x$.

Step 3 : Identify the PFD

• *PFD Type 1* : We have **distinct, linear factors** for the denominator.

A simple example is like $f(x) = \frac{P(x)}{(a_1x_1 + b_1)(a_2x_2 + b_2) \cdots (a_nx_n + b_n)}$, where $f(x)$ is proper and each term is distinct. In this case, we have the following property:

$$f(x) = \frac{P(x)}{(a_1x_1 + b_1)(a_2x_2 + b_2) \cdots (a_nx_n + b_n)} = \frac{C_1}{a_1x_1 + b_1} + \frac{C_2}{a_2x_2 + b_2} + \cdots + \frac{C_n}{a_nx_n + b_n}$$

Where C_1, C_2, \dots, C_n are all constants. Thus, we can solve C_1 to C_n . Then, we want to make the denominator to be the same for each term, so for each factor, we multiply the remaining terms on both the numerator and denominator, and sum the numerators up, we then have

$$\sum_{i=1}^n C_i \left(\prod_{j \neq i}^n (a_jx_j + b_j) \right) = P(x)$$

Now, depending on the degree of $P(x)$, we may have different numbers of equations. But the point is that if two polynomials are equal, then the coefficients for each degree must be the same, so we only need to compare the coefficients and then we have a systems of linear equations with unknowns C_1, \dots, C_n . And in this course, we can always assume that there is a unique solution, otherwise we can't integrate.

Example 1 : Evaluate $\int \frac{1}{x^2 - 1} dx$

First, we need to check if it is proper or not, and it is obvious that $f(x)$ is proper. So we may do the second step, and factor the term $x^2 - 1 = (x + 1)(x - 1)$, then notice that each factor is unique, so it belongs to *type 1*, thus we write

$$\int \frac{1}{x^2 - 1} dx = \int \left(\frac{C_1}{x + 1} + \frac{C_2}{x - 1} \right) dx$$

And we try to make the denominator the same for each factor, we do:

$$\int \left(\frac{C_1(x-1)}{(x+1)(x-1)} + \frac{C_2(x+1)}{(x+1)(x-1)} \right) dx$$

So, the sum of the numerators are $C_1x + C_2x + C_2 - C_1$, which is $(C_1 + C_2)x + (C_2 - C_1)$, this should be the same as $P(x)$, which is 1, so we have $(C_1 + C_2)x + (C_2 - C_1) = 1$, and as we have mentioned above, the coefficients for each degree must be the same, hence we have $C_1 + C_2 = 0$; $C_2 - C_1 = 1$, we get $C_1 = -\frac{1}{2}$, $C_2 = \frac{1}{2}$, and that's why we get

$$\frac{1}{2} \left(\int \frac{1}{x-1} dx + \int \frac{1}{x+1} dx \right) = \frac{1}{2} (\ln|x-1| + \ln|x+1|) + C$$

As we desired.

Exercise 1 : Evaluate $\int \frac{x+5}{x^2+x-2} dx$

Exercise 2 : Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$

Two techniques for faster integration : (Optional Material)

- *Factoring quadratic terms like $x^2 - 5x + 6$*

I would like to call this method *cross product*, though it is different from the real cross product in vector calculus, the process is to decompose the coefficients for x^2 and constant terms to be the product of two integers. Say we have $ax^2 + bx + c$, if $a = p_1p_2$ and $c = q_1q_2$, where each number is an integer. Then we take the sum of the cross of a and c , that is we calculate $p_1q_2 + p_2q_1$, if we have $p_1q_2 + p_2q_1 = b$, then we can factor the quadratic term into the product of two linear terms, namely $(p_1x + q_1)(p_2x + q_2)$. In the example $x^2 - 5x + 6$, we have $x = 1 \times 1$, thus $p_1 = p_2 = 1$, and we have $6 = -2 \times (-3)$, thus $q_1 = -2, q_2 = -3$, and we have $p_1q_2 + p_2q_1 = -3 + (-2) = -5$, the coefficient of x term. Thus we can write $x^2 - 5x + 6 = (x - 2)(x - 3)$. But notice that this method only applies for some quadratic polynomials, and we might need to try many ways of factorization to find the solution, or we might not be able to factor at all.

- *Solving systems of linear equations using matrices*

Notice that in this part, we will only discuss the form where $Ax = y$ has **one unique solution**, and we will not discuss the cases when there are no solutions or multiple solutions (infinity many). This method is extremely helpful to solve for C_1, \dots, C_n as we had seen before. After we match the coefficients with $P(x)$ on each term, suppose we have the following equation:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

We call this **Systems of Linear Equations** of n unknowns $(x_1, x_2 \dots x_n)$, where a_{ij} are coefficients, and

b_i are the results. for example, a system of linear equations of 3 unknowns can be:

$$\begin{cases} 2x + y + 2z = 1 \\ 3x + 2y - z = 2 \\ -2x + 0y + 0z = -1 \end{cases}$$

We may write this equation into the matrix form, which is

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ -2 & 0 & 0 & -1 \end{array} \right]$$

In this form, we do not write the unknowns, as it is easier to perform operations, we only write the coefficients. On each row, it represents one equation, and the coefficients are ordered, column i represents the coefficients for x_i and row j represents the j th equation. On the left of the vertical bar are the coefficients, namely a_{ij} and on the right of the vertical bar are the results for each equation, b_i . Our goal is to perform different operations on each column and rows such that in the end we will get the identity matrices like:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$$

Only the diagonal has elements 1, and 0 for all other positions. In this case, we shall easily see that $x_1 = a, x_2 = b, x_3 = c$ and thus we have solved the equations. In order to achieve that, we will perform the following operations:

Theorem 2 (Gauss - Elimination). *In the matrix form of a systems of linear equations, we can perform the following operation, and the result will not be affected.*

OPERATION 1 : *Interchanging any two rows / columns*

OPERATION 2 : *Multiplying one row with a constant*

OPERATION 3 : *Adding the corresponding elements from one row to another row*

Now, let's solve the matrix we have seen before by using **OPERATION 1,2,3**. The common method is to start from column 1, and then move to column 2, and so on. So after the operation on the first column, we should get:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ -2 & 0 & 0 & -1 \end{array} \right] \xrightarrow{\text{After Some Operations}} \left[\begin{array}{ccc|c} 1 & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{array} \right]$$

So, to make $a_{21} = a_{31} = 0$, we see that on the third column, we have $a_{31} = -2 = -a_{11}$, so we may add row 1 to row 3 to make $a_{31} = 0$, thus this is what we do:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ -2 & 0 & 0 & -1 \end{array} \right] \xrightarrow{R_3 + R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

Then, for row 2, we can do $2 \times R_2 - 3 \times R_1$, since $2a_{21} = 6, 3a_{11} = 6$, thus:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 3 & 2 & -1 & 2 \\ -2 & 0 & 0 & -1 \end{array} \right] \xrightarrow{2R_2 - 3R_1} \left[\begin{array}{ccc|c} 6 & 3 & 6 & 3 \\ 0 & 1 & -8 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

Thus, we are finished on column 1, now we shall move on to column 2. This times, we want to make a_{22} non-zero and the rest to be all zero. Since row 1 is a multiple of 3, we may divide by 3. We get

$$\left[\begin{array}{ccc|c} 6 & 3 & 6 & 3 \\ 0 & 1 & -8 & 2 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_1/3} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 1 & -8 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right]$$

Then, we can do $R_1 - R_2$ and $R_3 - R_2$ to cancel out the 1 on the position a_{12}, a_{32} , and we will get:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 1 & -8 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_1 - R_2; R_3 - R_2} \left[\begin{array}{ccc|c} 2 & 0 & 10 & 0 \\ 0 & 1 & -8 & 1 \\ 0 & 0 & 10 & -1 \end{array} \right]$$

You see the advantage of starting from column 1? In this case we will not create any extra non-zero values on the terms we had done before, so only in this way we can make our results easier. Now we come to the third column, and we want to keep a_{33} non zero while cancelling out a_{13}, a_{23} . Thus we do :

$$\left[\begin{array}{ccc|c} 2 & 0 & 10 & 0 \\ 0 & 1 & -8 & 1 \\ 0 & 0 & 10 & -1 \end{array} \right] \xrightarrow{R_1 - R_3; 10R_2 + 8R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 10 & 0 & 2 \\ 0 & 0 & 10 & -1 \end{array} \right]$$

Lastly, we make the elements on the diagonal to be 1, so our last step is:

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 10 & 0 & 2 \\ 0 & 0 & 10 & -1 \end{array} \right] \xrightarrow{R_1/2; R_2/10; R_3/10} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{1}{10} \end{array} \right]$$

So, the result we get is $a = \frac{1}{2}, b = \frac{1}{5}, c = -\frac{1}{10}$. Again, I want to state that we cannot always find a solution to such a linear system, but in this course, in this technique, one can always assume that there will be exactly one unique solution. The discussion on cases where there won't be solutions are beyond the scope of this course. If the rows and columns don't match (not the same number), we will discuss later. Please see my notes on Linear Algebra for if the reader would like to know more about it.

• Practice Exercises

1. Try to factor the quadratic terms below into the product of two linear terms.

$$(1) x^2 + 11x - 10 \quad (2) 3x^2 + 4x + 1 \quad (3) -6x^2 - 3x + 3$$

2. Try to solve the following systems of linear equations, using *Gauss - Elimination*.

$$x + y + z = 2$$

$$x + y + 2z = 1$$

$$2x + y - z = 4$$

and

$$x + y - 2z = -3$$

$$2x + y + 2z = 0$$

$$-x - y + 4z = 4$$

• *PDF Type 2 : Linear Terms with Repetitions*

Compared to type 1, the only difference is that we have some multiples of a linear term after we factored the denominator. Say we have :

$$f(x) = \frac{P(x)}{(a_1x + b_1)^{r_1}(a_2x + b_2)^{r_2} \cdots (a_kx + b_k)^{r_k}}$$

Where we will restrict r_1, \dots, r_n to be positive integers. Then, we can decompose it into $r_1 + r_2 + \cdots + r_n$ terms, and for each $(a_ix + b_i)^{r_i}$, we have:

$$\frac{Q_i(x)}{(a_ix + b_i)^{r_i}} = \frac{A_{i1}}{(a_ix + b_i)} + \frac{A_{i2}}{(a_ix + b_i)^2} + \cdots + \frac{A_{ir_i-1}}{(a_ix + b_i)^{r_i-1}} + \frac{A_{ir_i}}{(a_ix + b_i)^{r_i}}$$

A simple example would be :

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(x-1)} + \frac{A_4}{(x-1)^2} + \frac{A_5}{(x-1)^3}$$

Where we may use the same strategy as we did in type 1, We try to make the denominator the same for each term, then we add the numerator together and refactor them into different terms, compare with the coefficients, and solve the systems of linear equations.

• *PFD Type 3 : Factors with Irreducible Terms*

In this case, we will deal with some terms that will not be factored into linear terms. Like $\frac{x}{(x^2 + 1)(x^2 + 4)}$. A quadratic term $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$. In this case, we may decompose the whole term, but somehow we need to add a linear term to the numerator, like

$$\frac{x}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{(x^2 + 4)}$$

Then, by using the same method we can solve for unknowns A, B, C, D and thus solve the integral. In cases where there are repetitions like we had seen in type 2, we can also decompose multiple terms from it with different degrees, like

$$\frac{x^3 + x + 1}{(x^2 + x + 1)(x^2 + 1)^3} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + 1)} + \frac{Dx + E}{(x^2 + 1)^2} + \frac{Fx + G}{(x^2 + 1)^3}$$

And we can use the same strategy like we did before.

• Rationalizing Substitution

In terms of $f(x) = \sqrt[n]{g(x)}$, we will do rationalizing substitution, that is, by making $u = \sqrt[n]{g(x)}$, it will be more effective.

• *Example 2 : Evaluate $\int \frac{x^3 - 4x + 1}{x^2 - 3x + 2} dx$*

Note that this fraction is improper, thus we will do the long division first.

$$\begin{array}{r}
 \overline{) \begin{array}{r} x^3 \\ - x^3 + 3x^2 - 2x \\ \hline 3x^2 - 6x + 1 \\ - 3x^2 + 9x - 6 \\ \hline 3x - 5 \end{array} } \\
 x + 3
 \end{array}$$

Thus, we can rewrite the integral as

$$\int \frac{x^3 - 4x + 1}{x^2 - 3x + 2} dx = \int \left(x + 3 + \frac{3x - 5}{x^2 - 3x + 2} \right) dx$$

By the properties of the integral, we can do the following:

$$\int \left(x + 3 + \frac{3x - 5}{x^2 - 3x + 2} \right) dx = \int (x + 3) dx + \int \frac{3x - 5}{(x - 1)(x - 2)} dx$$

For the fraction part, we observe that this is of *Type I*, so we decompose the fraction as

$$\frac{3x - 5}{(x - 1)(x - 2)} = \frac{C_1}{(x - 1)} + \frac{C_2}{(x - 2)}$$

And we have the systems of linear equations $C_1(x-2) + C_2(x-1) = 3x-5$, which is $(C_1 + C_2)x - (2C_1 + C_2) = 3x-5$, by making the corresponding coefficients the same, we then have $C_1 + C_2 = 3$, $2C_1 + C_2 = 5$, hence $C_1 = 2$, $C_2 = 1$, so the integral we are evaluating is actually

$$\int (x+3)dx + \int \left(\frac{2}{x-1} + \frac{1}{x-2} \right) dx$$

Which is

$$\frac{1}{2}x^2 + 3x + 2\ln|x-1| + \ln|x-2| + C$$

• *Example 3 : Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$*

In this fraction, we see that it is proper, so there is no need to do long division. We may factor the denominator like

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{2x^2 - x + 4}{x(x^2 + 4)} dx$$

Notice that $(x^2 + 4)$ is irreducible, because $b^2 - 4ac = 0 - 16 = -16 < 0$, so we see that is it of *Type 3*, thus we decompose the fraction as

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Then, by making the denominator the same, we get the equation $A(x^2 + 4) + x(Bx + C)$ for the numerator. Then compare it with the original fraction, we have $2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$, i.e e have the following systems of linear equations.

$$A + B = 2$$

$$C = -1$$

$$4A = 4$$

We can solve it using *Gauss-Elimination* in the matrix form or by other methods, in the end we will get

$A = 1, B = 1, C = -1$, then we have the integral

$$\int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx = \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx + \int \frac{-1}{x^2+4} dx$$

We may perform a *u-Substitution* on the second term, let $u = x^2 + 4$, thus we have $du = 2x dx$, then we have

$$\int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{u} du + \int \frac{-1}{x^2+4} dx$$

Thus, the final result is given by

$$\int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{u} du + \int \frac{-1}{x^2+4} dx = \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$$

Notice that there is a very important formula used in this problem:

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

I also listed some useful tricks: **DO NOT USE THEM DIRECTLY IN EXAMS (LONG ANSWER).**

The standard method to prove them is by *Trig – Sub* using the equality $1 + \tan^2 x = \sec^2 x$.

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln \left| x + \sqrt{x^2-a^2} \right| + C$$

$$\int \frac{\sqrt{x^2-a^2}}{x} dx = \sqrt{x^2-a^2} - a \arccos\left(\frac{a}{x}\right) + C$$

$$\int \frac{1}{x^2\sqrt{x^2+a^2}} dx = -\frac{\sqrt{x^2+a^2}}{a^2 x} + C$$

EXERCISES ON PFD

• 1. Evaluate $\int \frac{4+x}{(1+2x)(3-x)} dx$

• 2. Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx$

• 3. Evaluate $\int \frac{x^3+4x^2+x-1}{x^3+x^2} dx$

• 4. Evaluate $\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3} dx$

• 5. Evaluate $\int \frac{\sin x}{\cos^2 x - 3 \cos x} dx$

• 6. Evaluate $\int \frac{e^{2x}}{e^{2x} + 3e^x + 2} dx$

• 6. Evaluate $\int \frac{1}{\sqrt{x} - \sqrt[3]{x}} dx$