Trigonometric Integrals

• Techniques used in this part: *u-Substitution and Integration by Parts*

$$\int f(g(x))g'(x)dx = \int f(u)du \text{ where } u = g(u) \quad \textit{(u-Substitution)}$$
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \quad \textit{(Integration By Parts)}$$

• Trigonometric Properties:

Derivatives and Integrals:

$$(\sin(x))' = \cos(x); (\cos(x))' = -\sin(x); (\tan(x))' = \sec^2(x)$$

$$\int \sin(x)dx = -\cos(x) + C; \int \cos(x)dx = \sin(x) + C$$

Half-Angle / Double Angle / Square and other properties

$$\sin(2x) = 2\sin(x)\cos(x) \tag{1}$$
$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \tag{2}$$
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \tag{3}$$
$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)) \tag{4}$$
$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)) \tag{5}$$
$$\sin^2(x) + \cos^2(x) = 1 \tag{6}$$
$$\sec^2(x) = 1 + \tan^2(x) \tag{7}$$
$$\sin(2x) = \frac{2\tan(x)}{1+\tan^2(x)} \tag{8}$$
$$\cos(2x) = \frac{1-\tan^2(x)}{1+\tan^2(x)} \tag{9}$$
$$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right) \tag{10}$$
$$\sin(x)\sin(y) = -\frac{1}{2}(\cos(x+y) - \cos(x-y)) \tag{11}$$
$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y)) \tag{12}$$

Integral of type 1 : $\int \sin^m(x)\cos^n(x)dx$

Example 1 : Evaluate the integral $\int \cos^3(x)dx$.

We we may do $\cos^3(x) = \cos^2(x)\cos(x) = (1-\sin^2(x))\cos(x)$ by formula 6. Then the integral we are evaluating will become $\int (1-\sin^2(x))\cos(x)dx$. Now By *u-Substitution*, we assume that $\sin(x) = u$, then we also know that $du = \cos(x)dx$, thus our integral can be written as $\int (1-u^2)du$, which we can evaluate it since it's a polynomial. Hence we have $\int (1+u^2)du = u - \frac{1}{3}u^3 + C$, lastly we replace u with $\sin(x)$, and we get $\int (1-u^2)du = \sin(x) - \frac{1}{3}\sin^3(x) + C$

Exercise 1 : Evaluate the integral $\int \sin^5(x)\cos^2(x)dx$.

Hint : Rewrite as $\int ((1-\cos^2(x))^2\cos^2(x)\sin(x)dx)$

General strategy for integrals like $\int \sin^m(x)\cos^n(x)dx$:

(1) If we have $n = 2k+1, k \in \mathbb{Z}$, rewrite as $\int \sin^m(x)(1-\sin^2(x))^k\cos(x)dx$

(2) If we have $m = 2k+1, k \in \mathbb{Z}$, rewrite as $-\int (1-\cos^2(x))^k\cos^n(x)\sin(x)dx$

(3) Both m,n are odd, then we can do for both m,n, by the same strategy.

(4) Both m,n are even, then we use the formula (1)(2)(4)(5) to reduce the power

Exercise 2: Evaluate the integral $\int_0^\pi \sin^2(x)dx$ *(Hint: Use formula (4))*

Exercise 3: Evaluate the integral $\int \sin^4(x)dx$ *(Hint: Apply formula (4) twice.)*

Integral of type 2 : $\int \sec^m(x)\tan^n(x)dx$

We may do the same, since we have $\sec^2(x) = 1 + \tan^2(x)$, and keep in mind, that

$$\frac{d}{dx}\tan(x) = \sec^2(x); \frac{d}{dx}\sec(x) = \tan(x)\sec(x)$$

Example 2 : Evaluate the integral $\int \tan^6(x)\sec^4(x)dx$

In this problem, we rewrite the original integral as $\int \tan^6(x)(1+\tan^2(x))\sec^2(x)dx$, and we let $u = \tan(x)$, thus we know $du = \sec^2(x)dx$, then it becomes $\int u^6(1+u^2)du$, which can be calculated as

$$\int u^6(1+u^2)du = \frac{1}{7}\tan^7(x) + \frac{1}{9}\tan^9(x) + C$$

Exercise 4 : Evaluate the integral $\int \tan^5(x)\sec^7(x)dx.$

$$\textit{Hint : Rewrite as } \int (\sec^2(x)-1)^2\sec^6(x)\tan(x)\sec(x)dx$$

Exercise 5 : Evaluate the integral $\int \sec(x)dx.$

$$\textit{Hint : Rewrite as } \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx$$

Exercise 6 : Evaluate the integral $\int \sec^3(x)dx.$ *Hint :* $(\tan(x))' = \sec^2(x)$

MORE EXERCISES ON INTEGRATION

★★ *Evaluate the integral* $\int \frac{\sin(2x)}{1+\cos^2(x)}dx$

★★★ *Evaluate the integral* $\int \frac{1}{1-\cos(x)}dx$

★★★★★ *Evaluate the integral* $\int \sqrt{(\tan(x))}dx$ *Hint :* $x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

Trigonometric Substitution

Example 3 : Evaluate the integral $\int \frac{1}{x^2\sqrt{x^2+4}}dx$.

In forms like this, we will perform a trig-sub to simplify the terms inside the square root. Usually we do $\sec(x)$ sub if inside the square root there is ax^2+b, and we do $\sin(x)$ or $\cos(x)$ sub if inside the square root there is ax^2-b. In this problem, we will do $\sec(x)$ sub. If we let $x = 2\tan(y)$, then we have $x^2+4 = 4\tan^2(y)+4 = 4\sec^2(y)$, and we have $dx = 2\sec^2(y)dy$, so the integral now becomes $\int \frac{2\sec^2(y)}{4\tan^2(y)(2\sec(y))}dy$, which is $\frac{1}{4}\int \frac{\sec(y)}{\tan^2(y)}dy$, if we let $\tan(y) = \sin(y)/\cos(y)$ and $\sec(y) = 1/\cos(y)$, we will get $\frac{1}{4}\int \frac{\cos(y)}{\sin^2(y)}dy$, and we can achieve this by another u-sub, let $u = \sin(y)$ thus we get $\frac{1}{4}\int \frac{\cos(y)}{\sin^2(y)}dy = \frac{1}{4}\int \frac{1}{u^2}du$, which we get $\frac{1}{4}\int \frac{1}{u^2}du = -\frac{1}{4}\csc(y)+C$, then we perform the last step, by the equation $x = 2\tan(y)$, we solve $\csc(y)$ in terms of x, which we get $-\frac{\sqrt{x^2+4}}{4x}+C$

Exercise 7 : Evaluate the integral $\int \frac{x}{\sqrt{x^2+4}}dx$.

Hint : Use $\sec(x)$ *sub as example 3, or do normal u sub*

Exercise 8 : Evaluate the integral $\int_0^a \frac{1}{(a^2+x^2)^{3/2}}dx$.

Exercise 9 : Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}dx.$

Hint : We have $3-2x-x^2 = 4-(x+1)^2$