MATH 574 Final Project

Introduction to CR3BP: Stability, Orbits

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Contents

1 CR3BP : Circular Restricted Three-Body Problem

1.1 What we need to know from Two-Body problem?

Unlike Three-Body problem, two-body problem is well-studied and we are able to find a closed form solution for a general two-body problem. However my interests will focus on three-body problem hence I will not cover all the details in two-body problem. I will leave the deviation in the appendix, and I will state some key results from the two-body problem.

First of all, two-body problem is a planar motion, with circular orbits and constant angular velocity under regularity conditions. It means that two bodies problem is $2D$, which greatly simplified the situation. We introduce two bodies with mass m_1 and m_2, and we introduce the notion of *centre of mass*, which means the net force acting at this point is 0, and two bodies will orbit around this point in circular orbits, with the same constant angular velocity. Also if we use r_1, r_2 to denote the distance to the centre of mass from two bodies, Newton's law will tell us $\frac{m_1}{m_2} = \frac{r_2}{r_1}$. Usually we introduce the unit mass as well as unit distance for simplicity, where we make $m_1 + m_2 = 1$, then of course $r_1 + r_2 = 1$.

To better investigate the motion, we will use rotating coordinate system. Since we know both bodies orbits around the centre of mass with the same angular velocity, it is nice for us to use rotating coordinate system. Since m_1, m_2 and the centre of mass lie on a straight line at any time of the motion, we make this line as our new x axis, denote by x' and y' is perpendicular to x'. We may think of the transformation $(x, y) \to (x', y')$ as a transformation, of rotating the axis by angle $\theta = \omega t$, ω is the angular velocity and t is time. Just assume the rotation is counter-clockwise, as the figure shown below, we can easily find the transformation matrix by

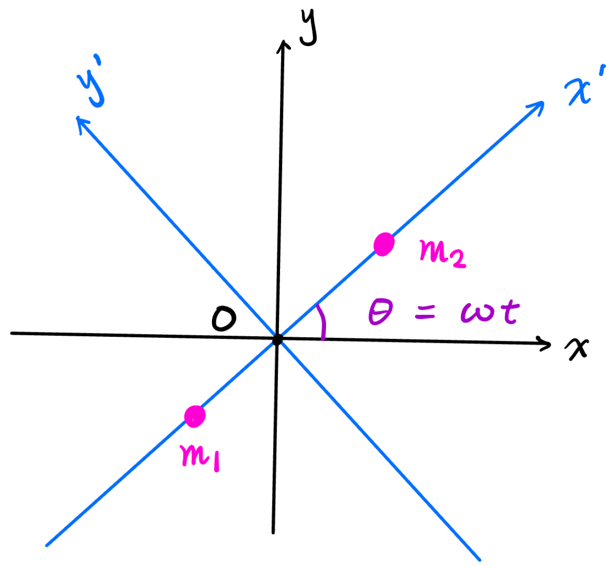

Figure 1: Rotating coordinate system $x'y'$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

If we were to add z axis which is perpendicular to x, y plane, its transformation matrix can also be easily computed. So from now on, for clarity, we shall use (x, y, z) as the coordinate in the rotating coordinate system!

1.2 Motion of the Third Body

Now we will move on to three body problem. In general there is no closed form for an arbitrary three-body problem, and any perturbation could lead to chaos due to the 18 degrees of freedom we have in the system. However, we could study the dynamics of restricted three-body problem, where we apply some restrictions to make the system less chaotic and more manageable. For example, in Euler's restricted three-body problem, all three bodies lie on the same line at any time of the motion; in, Lagrange's three body problem, all three bodies form the three vertices of an equilateral triangle at any time of the motion, and there are many famous three bodies problem.

For us, we would study the dynamics based on the well-developed two-bodies problem. Suppose in a circular two body problem, we now introduce a third body with negligible mass compared to the other two. In this case the gravitational force of the third body will not have any impact on the rest two and hence the rest two will still form a circular two bodies problem. This is known as CR3BP, namely, **Circular Restricted Three-Body Problem**. It has a lot of real life applications, for example the earth, the moon and the satellite could be a perfect example of CR3BP.

We initialize this problem by defining three bodies, and their masses, such that $m_1, m_2 >> m_3$, we call m_1, m_2 the primary masses. Also, two primary masses will orbit around the center of mass in circular planar orbits, with the same angular velocity ω. We may wonder how "good" are these constraints? How well do those assumptions match the reality? For our solar system, the eccentricity (*recall that the circular orbit has eccentricity* 0) of some two-bodies orbits are as follows:

Bodies	Eccentricity of the orbit
Earth around Sun	0.0549
Moon around Earth	0.0162
Charon around Pluto	0.00005
Europa around Jupiter	0.009

So we should have no problem assuming the orbits are perfect circle!

For simplicity, we define $\mu = m_2$ and $1 - \mu = m_1$, that is, we use the unit mass 1 as the sum of m_1, m_2, and use d, r as the relative distance of the third mass to m_1, m_2. Let (x, y, z) as the position of the third mass under the rotating coordinate system, we can get

$$d = \sqrt{((x+\mu)^2 + y^2 + z^2)}; r = \sqrt{((x-1+\mu)^2 + y^2 + z^2)} \tag{2}$$

In the system, we use (x, y, z) to denote the position of the third body. By letting the centre of mass of m_1, m_2 to be the origin, xy plane lie on the orbits of m_1, m_2 and z axis to be perpendicular to xy plane. So the motion of the third body can be described by the following system of ODEs:

$$\ddot{x} - 2\dot{y} - x = -\frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3} \tag{3}$$

$$\ddot{y} + 2\dot{x} - y = -\frac{(1-\mu)y}{d^3} - \frac{\mu y}{r^3} \tag{4}$$

$$\ddot{z} = -\frac{(1-\mu)z}{d^3} - \frac{\mu z}{r^3} \tag{5}$$

Also, the gravitational potential in this system is

$$\Omega = \frac{1-\mu}{d} + \frac{\mu}{r} + \frac{x^2+y^2}{2} \tag{6}$$

Then by finding $\Omega_x, \Omega_y, \Omega_z$, the original system of ODEs can be simplified to

$$\begin{cases} \ddot{x} = 2\dot{y} + \Omega_x \\ \ddot{y} = -2\dot{x} + \Omega_y \\ \ddot{z} = \Omega_z \end{cases} \tag{7}$$

1.3 Lagrange Points and their Stability

Imagine you had a marble and you wanted to put it at the point between the Earth and the Moon where the force of gravity was equal from both. The two forces would cancel out, and the marble would remain in that location forever. If the sun and the earth were not moving, there would only be one of these points, but because the Earth and the Moon are both orbiting around a common barycenter there are 5 of these points that always remain in constant relative positions to the main bodies. They are called Lagrange points, after the man who discovered them, Joseph-Louis Lagrange, and are some of the most interesting points in the 3-body problem.

At these Lagrange points, all the forces acting on the particles will be equal and cancel each-other out, which is were they get the name equilibrium points from. A more rigorous definition of a Lagrange point is one where all time derivatives of position are zero in the rotating reference frame. That means velocity and acceleration are both zero. So in equation (44) we let all accelerations and velocities to be zero, we will get

$$0 = \Omega_x = x - \frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3} \tag{8}$$

$$0 = \Omega_y = y - \frac{(1-\mu)y}{d^3} - \frac{\mu y}{r^3} \tag{9}$$

$$0 = \Omega_z = -\frac{\mu z}{d^3} - \frac{\mu z}{r^3} \tag{10}$$

Then, from equation (22) we see that $z = 0$, it means that the Lagrange points all lie on the xy plane! In fact there are 5 of them.

Collinear Lagrange Points: We set $y = 0$, now we have

$$0 = x - \frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3} = \Omega_x \tag{11}$$

Also since $y = 0$ we can further make $d^3 = (x+\mu)^3$ and $r^3 = (x-1+\mu)^3$ and hence we have

$$0 = x - \frac{(1-\mu)(x+\mu)}{(x+\mu)^3} - \frac{\mu(x-1+\mu)}{(x-1+\mu)^3} \tag{12}$$

From the fact that the above equation is third order, it should have 3 roots, and those three roots will be the positions of 3 Lagrange points. Because we set $y = 0$ to find these points, we call these Lagrange points the **collinear Lagrange points**. Now for some bad news. We don't have simple analytic expressions for the positions of the collinear Lagrange points, but we can find them out through numerical methods. The most popular method for finding them is *Newton-Raphson Method*. Below is a picture of the positions of collinear Lagrange points in the Earth-Moon system.

Newton-Raphson Method: This is one of the methods used in numerical analysis to approximate the root of a function $f(x)$. The algorithm is as follows:

1. Give an initial guess about the root, say x_0, then define

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{13}$$

2. Repeat this step, for a general n we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{14}$$

3. Repeat, until some source of convergence is met, i.e $|x_{n+1} - x_n| < \epsilon$ for some pre-fixed ϵ.

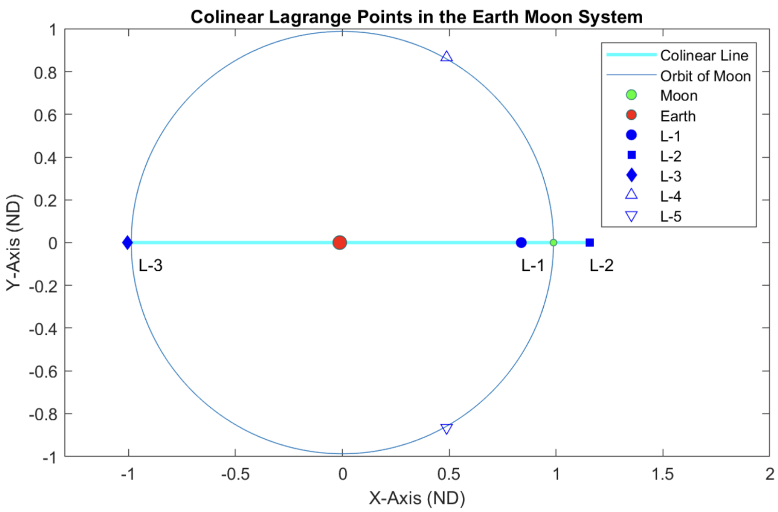

Figure 2: Position of Collinear Lagrange Points in Earth-Moon System. In this system we define the distance from the earth to moon as unit 1 and the earth to be the origin, and we use numerical methods to find that the x coordinates for collinear Lagrange points to be $L_1 = 0.609, L_2 = 1.260, L_3 = -1.042$

We use L_1, L_2, L_3 to indicate collinear Lagrange points.

Equilateral Lagrange Points: Now we assume $y \neq 0$, so we better have

$$1 - \frac{1-\mu}{d^3} - \frac{\mu}{r^3} = 0 \tag{15}$$

and hence we have $d = r$, also use the condition on x, we have another two Lagrange points: $x_{L_{4,5}} = 0.5 - \mu, y_{L_{4,5}} = \pm\sqrt{3}/2$. These are called equilateral Lagrange Points because they lie on the points of an equilateral triangle. The primary and secondary lie on the other points of the equilateral triangles. Below is a picture showing the other two Lagrange points in the Earth-Moon system.

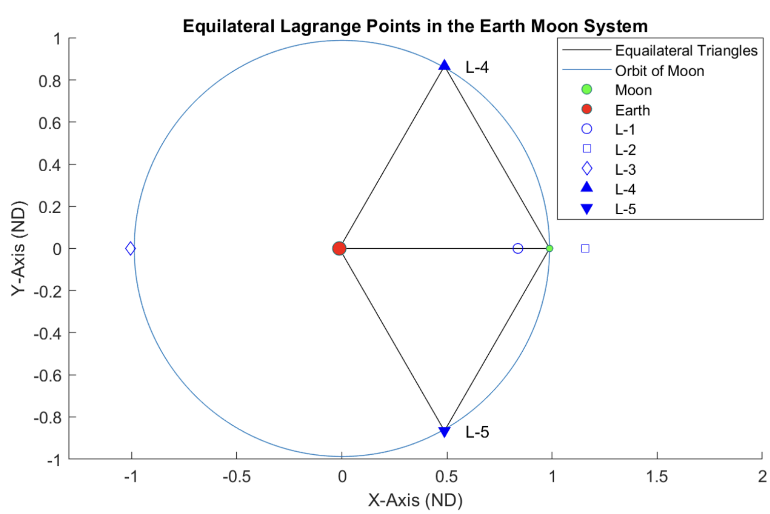

Figure 3: The other two Lagrange Points of the Earth-Moon system, denoted by L_4 and L_5.

Now, we shall study the stability of the Lagrange points. Recall the systems of ODEs in equation (7), and also around a Lagrange Point (x, y, z), we apply a small perturbation $(x+\xi, y+\eta, z+\zeta)$, and see if this point will get closer to the Lagrange point or not. We plug these relations into the equations of motion, and we do a second order Taylor expansion to linearize the equations, we should have

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta + \Omega_{xz}\zeta \tag{16}$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta + \Omega_{yz}\zeta \tag{17}$$

$$\ddot{\zeta} = \Omega_{xz}\xi + \Omega_{zy}\eta + \Omega_{zz}\zeta \tag{18}$$

But note that at a Lagrange Point, we have $z = 0$, so we can ignore the equation given by $\ddot{\zeta}$, then the four partials are given by

$$\Omega_{xx} = 1 - \frac{1-\mu}{d^3} - \frac{\mu}{r^3} + \frac{3(1-\mu)(x+\mu)^2}{d^5} + \frac{3\mu(x-1+\mu)^2}{r^5} \tag{19}$$

$$\Omega_{yy} = 1 - \frac{1-\mu}{d^3} - \frac{\mu}{r^3} + \frac{3(1-\mu)y^2}{d^5} + \frac{3\mu y^2}{r^5} \tag{20}$$

$$\Omega_{xy} = \Omega_{yx} = \frac{3(1-\mu)(x+\mu)y}{d^5} + \frac{3\mu(x-1+\mu)y}{r^5} \tag{21}$$

Then we can rewrite the original equation into a system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \ddot{\xi} \\ \ddot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{xx} & \Omega_{xy} & 0 & 2 \\ \Omega_{yx} & \Omega_{yy} & -2 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} \tag{22}$$

The eigenvalues of A are given by the root of the characteristic polynomial:

$$\lambda^4 + (4 - \Omega_{xx} - \Omega_{yy})\lambda^2 + (\Omega_{xx}\Omega_{yy} - \Omega_{xy}\Omega_{yx}) = 0 \tag{23}$$

Now, we can convert this equation into a quadratic equation, let $\Lambda = \lambda^2$. If any of our eigenvalues have a positive real component than that Lagrange point is unstable. Because our eigenvalues come in positive and negative pairs, this means that any eigenvalue with a real component ensures the system is unstable. This leave purely imaginary eigenvalues as our only way to have a stable system.

1.3.1 Stability of Collinear Lagrange Points:

Here since we have $y = 0$, so $\Omega_{xy} = \Omega_{yx} = 0$, now equation (70) can be reduced to

$$\Lambda^2 + (4 - \Omega_{xx} - \Omega_{yy})\Lambda - \Omega_{xx}\Omega_{yy} = 0 \tag{24}$$

which has solutions given by

$$\Lambda = \frac{\Omega_{xx} + \Omega_{yy} - 4 \pm \sqrt{(4 - \Omega_{xx} - \Omega_{yy})^2 + 4\Omega_{xx}\Omega_{yy}}}{2} \tag{25}$$

Now, if we let $a = \frac{1-\mu}{d^3} + \frac{\mu}{r^3}$ for simplicity, then we have $\Omega_{xy} = \Omega_{yx} = 0$, also $\Omega_{yy} = 1 - a$. Use the relation that $d = x + \mu, r = x - 1 + \mu$, we can further make $\Omega_{xx} = 2a + 1$. Hence

$$\lambda^2 = \Lambda = \frac{a + 2 \pm \sqrt{8 - 7a^2}}{2} \tag{26}$$

In fact, it turns out that there will be 4 eigenvalues: One positive (λ_1), one negative (λ_2), and two purely imaginary that occurs in conjugate pairs (λ_3, λ_4). The positive eigenvalue already ensures that the Lagrange point is unstable.

1.3.2 Stability of Equilateral Lagrange Points

In this case, we have the geometric relation that $x = \mu - \frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$, we can simplify the characteristic polynomial to

$$\Lambda^2 + \Lambda + \frac{27}{4}(1 - \mu) = 0 \tag{27}$$

Because we know they appear in pairs, so we don't really care what the eigenvalues are, just whether they are purely real or imaginary. We investigate the discriminant:

$$d = 1 - 27\mu(1 - \mu) \tag{28}$$

and the original eigenvalues λ are given by

$$\lambda^2 = \frac{-1 \pm \sqrt{1 - 27\mu(1 - \mu)}}{2} \tag{29}$$

hence we need $1 > 27\mu(1 - \mu)$ to achieve purely imaginary roots, which is, $\mu \leq 0.0385208965$, purely imaginary roots means the motion around the Lagrange points L_4, L_5 are closed, and of course stable in this case. Recall that $\mu = \frac{m_2}{m_1 + m_2}$, in this case it means that if

$$m_2 \leq 0.04 m_1 \tag{30}$$

then we would achieve a steady state around Lagrange points L_4, L_5. Here $m_1 > m_2$, m_1 is the primary mass, m_2 is the secondary mass. This type of behavior has been observed in the Solar System. For instance, there is a sub-class of asteroids, known as the Trojan asteroids, which are trapped in the vicinity of the L_4 and L_5 points of the Sun-Jupiter system (which easily satisfies the above stability criterion), and consequently share Jupiter's orbit around the Sun, staying approximately 60° ahead of, and 60° behind, Jupiter, respectively. See the picture below.

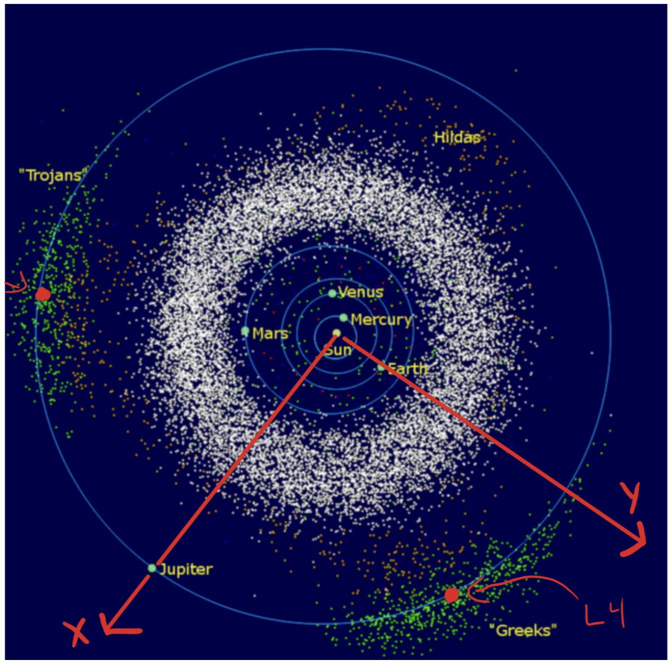

Figure 4: The green dots in the picture are the Trojan asteroids, they behave almost exactly as predicted in the CR3BP, and in Sun-Jupiter system, we have $\mu = 0.00095$, and $e = 0.0487$. The group of asteroid near L_4 are called Greek Camp and the group of asteroid near L_5 is called Trojan camp. Isn't it nice?

Below is a table showing the number of asteroids near L_4, L_5 Lagrange point of each planet in the solar system:

Planet	Number in L4	Number in L5	List (L4)	List (L5)
Mercury	0	0	—	—
Venus	1	0	2013 ND_{15}	—
Earth	2	0	(706765) 2010 TK_7, (614689) 2020 XL_5	—
Mars	2	13	(121514) 1999 UJ_7, 2023 FW_{14}	many
Jupiter	7508	4044	Greek camp	Trojan camp
Saturn	1	0	2019 UO_{14}	—
Uranus	2	0	(687170) 2011 QF_{99}, (636872) 2014 YX_{49}	—
Neptune	24	4	many	many

Figure 5: Number of asteroids near the L_4, L_5 Lagrange point of each planet in sloar system.

1.4 Periodic Orbits Near Collinear Lagrange Points

Let's recall the first three-body problem I mentioned: Earth-moon-satellite system. When launching a satellite to the space, we would wish to launch the satellite to the place near the Lagrange points to make it stable. Since we know that the collinear Lagrange points are unstable, but since we have showed there are two imagery eigenvalues, so it gives us the possibility to have periodic orbits near the Lagrange point.

So in a neighborhood of a collinear Lagrange point, by linearizing, the solution to the differential equation takes the form (*recall that there are* 4 *eigenvalues,* 1 *positive,* 1 *negative and two purely imaginary complex conjugate pairs. Denote the imaginary part to be* k.)

$$\xi = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 \sin(kt) + A_4 \cos(kt) \tag{31}$$

$$\eta = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 \sin(kt) + B_4 \cos(kt) \tag{32}$$

Where A_i, B_i shall be determined by the initial conditions. So is it possible that $A_1 = A_2 = B_1 = B_2 = 0$? Then we will have purely imaginary eigenvalues, and we have a periodic orbit! We call this type of orbits as Lyapunov Orbits. In fact, as we grow the radius of the Lyapunov orbit, it will become more kidney shaped. The figure below shows the periodic orbits near L_1, L_2 of the earth-moon system:

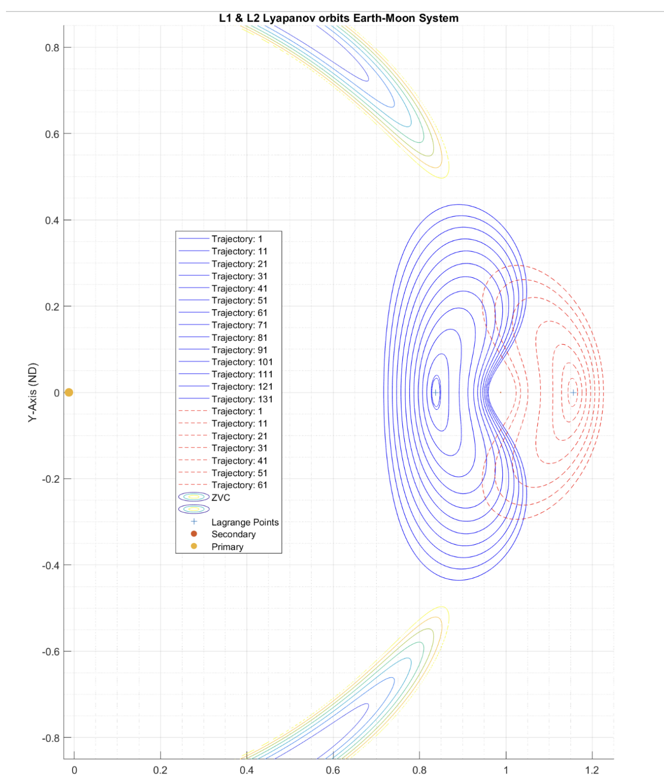

Figure 6: The Lyapunov orbits near L_1, L_2 of the earth-moon system

A generalization of Lyapunov orbits, called Halo orbits, are also periodic orbits but in a $3D$ environment near the Lagrange points. Recall the system of ODEs used to describe the motion of the third body:

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \\ \ddot{z} &= \Omega_z\end{aligned}$$

Halo orbits are obtained by adjusting values of $x_0, z_0, \dot{y}_0$ at a perpendicular crossing of the xz plane (i.e., $y_0 = \dot{x} = \dot{z}_0 = 0$), so that the next crossing of this plane, at $y_1 = 0, t = t_1, \operatorname{sgn}(\dot{y}_1) = -\operatorname{sgn}(y_0)$, is also perpendicular, so that $y_1 = \dot{x}_1 = \dot{z}_1 = 0$. Furthermore, because of the invariance of Equations under the reflection $y \to -y, t \to -t$, it follows that such an orbit will necessarily repeat with period $T = 2t_1$. Hence we will have a set of different Halo orbits near the Lagrange points. The figure below shows the Halo orbit around L_2 in earth-moon system.

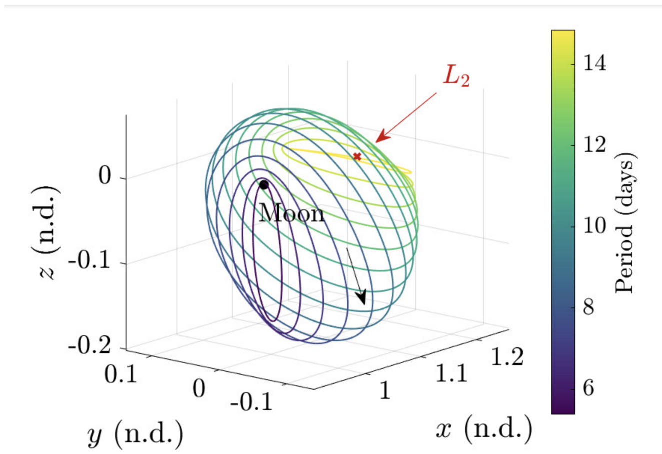

Figure 7: The Halo Orbits around L_2 of the earth-moon system. Source: *Leveraging Intermediate Dynamical Models for Transitioning from the Circular Restricted Three-Body Problem to an Ephemeris Model*

In general, for simplicity we define

$$c = \frac{1-\mu}{d^3} + \frac{\mu}{r^3}, c > 0 \tag{33}$$

then the equation of motion is now

$$\ddot{x} - 2\dot{y} - (1 + c_2)x = 0 \tag{34}$$
$$\ddot{y} + 2\dot{x} + (c_2 - 1)y = 0 \tag{35}$$
$$\ddot{z} + c_2 z = 0 \tag{36}$$

We can see that the motion on the z axis is harmonic and does not depend on x or y, and we can easily see that the eigenvalues are given by $\pm i\sqrt{\omega_v}, \omega_v = c$. In general on xy plane we would have 4 eigenvalues, with one positive, one negative and one purely imaginary complex conjugate pairs. We denote those eigenvalues to be $\pm\lambda, \pm i\sqrt{\omega_p}$. The solution to the linearized equations are

$$x = -A_x \cos(\omega_p t + \phi) \tag{37}$$
$$y = \kappa A_x \sin(\omega_p t + \phi) \tag{38}$$
$$z = A_z \sin(\omega_v t + \psi) \tag{39}$$

A_x, A_z are amplitudes, and they will decide the size of the orbit, κ is a constant related to ω_v and ω_p. Those orbits are called *Lissajous Orbit*. In general it is not a periodic orbit, and the figure below shows the trajectory of the sun-earth L_1 Lissajous orbit:

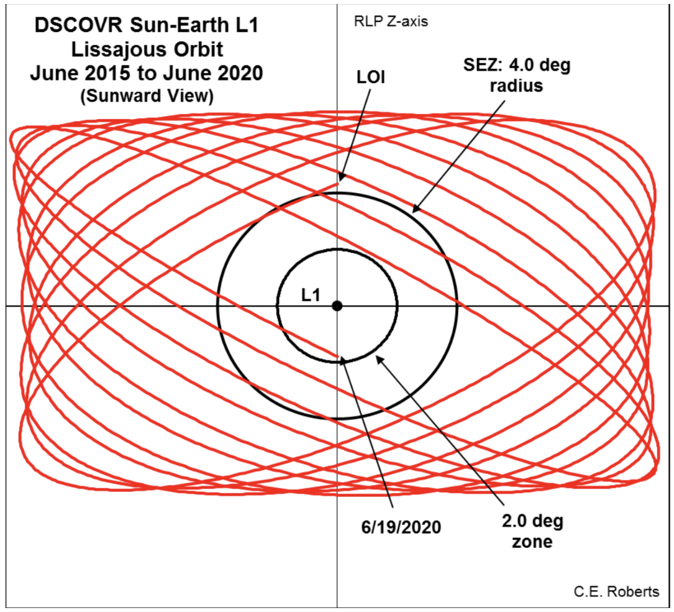

Figure 8: L_1 Lissajous orbit of sun-earth system

Finally, let's talk about the real-world applications: Launching satellite. Consider our nearest neighbor: moon. Due to *tidal locking effect*, we are only able to see one side of the moon from earth and the other side remains unseen for us. Thus lunching satellite to the other side of the moon can be challenging: Since the back of the moon can't be seen and all the communication signals will be blocked. Hence, we first launch a satellite to the L_2 Lagrange point, which forms a stable Halo orbit. Then we use this satellite as a relay satellite to capture the signal, and then we could land rovers on the back of the moon!

This ambitious mission was launched by China. The relay satellite was called "*Que-Qiao*", which translates in English, as "*Magpie Bridge*". The rover landed on the back of the moon was called "*Chang'e* 4". *Maybe I can talk about the interesting story of Chang'e and Que-Qiao in ancient Chinese myth to end my presentation.*

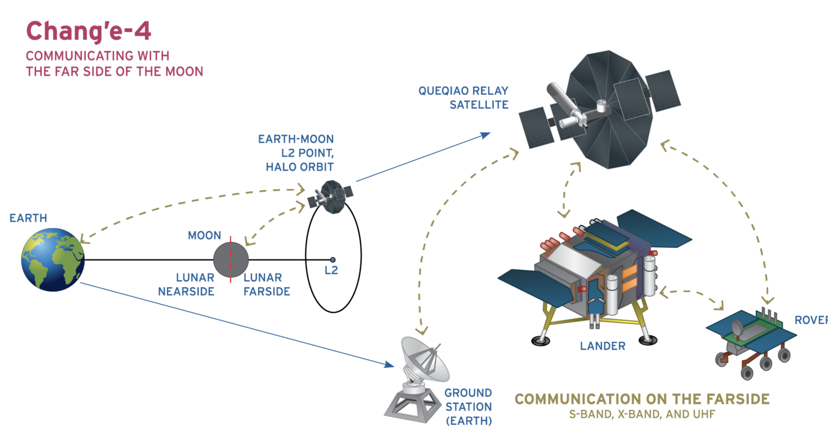

Figure 9: Queqiao and Chang'e

2 Appendix

A Preliminary Result: Euler-Lagrange Equation

In Newtonian mechanics, the analysis of a system is largely based on the analysis of net force acting on it, and it relies on Newton's second law, given by $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$. In some cases, like a system of double pendulum, studying the net force on each object would become extremely difficult, and Newtonian mechanics will no longer be efficient. However, Lagrangian mechanics can easily solve this problem. instead of studying the net force, Lagrangian mechanics focus on the study of energy. We define the Lagrangian to be

$$\mathcal{L} = T - U \tag{40}$$

where T is the kinetic energy and U is the potential energy, where in a dynamical system, the Lagrangian can be represented as $\mathcal{L} = \mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$, where $\boldsymbol{q}(t)$ represents the position of the object at time t, and $\dot{\boldsymbol{q}}(t)$ is the velocity of the object. If we are only interested in the starting point $t = a$ and ending point $t = b$, then there are infinitely many smooth paths from a to b, along a path $\mathcal{L}$, we know that at time t the state of the object can be described by $\mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$, then we may integrate $\mathcal{L}$ along this path with respect to t to get a "quantity" of $\mathcal{L}$, It turns out that in reality, the object will always obey the $\mathcal{L}$ such that the path integral is minimized.

Theorem 1. *Let $(X, \mathcal{L})$ be a dynamical system with n degrees of freedom. Where X is a smooth manifold, and $\mathcal{L} = L(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ is the Lagrangian, $\boldsymbol{q}(t) \in X$ also be smooth. Let $\mathcal{P}(a, b, \boldsymbol{x}_a, \boldsymbol{x}_b)$ be the set of all smooth paths $\boldsymbol{q} : [a, b] \to X$ for which $\boldsymbol{q}(a) = \boldsymbol{x}_a$ and $\boldsymbol{q}(b) = \boldsymbol{x}_b$, then the action functional $S : \mathcal{P}(a, b, \boldsymbol{x}_a, \boldsymbol{x}_b) \to \mathbb{R}$ is defined via*

$$S[\boldsymbol{q}] = \int_a^b \mathcal{L} = \mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))dt. \tag{41}$$

Then, a path $\boldsymbol{q} \in \mathcal{P}(a, b, \boldsymbol{x}_a, \boldsymbol{x}_b)$ is a stationary point of S if and only if

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_i} = 0, \quad i = 1, \cdots, n. \tag{42}$$

(42) is also known as the Euler-Lagrange Equation. We will show the proof only for the one-dimensional case.

Proof. We wish to find a function f which satisfies the boundary conditions $f(a) = A, f(b) = B$ and which extermizes the functional

$$S[f] = \int_a^b \mathcal{L}(x, f(x), f'(x))dx \tag{43}$$

Also we will assume that $\mathcal{L} \in C^2$. If f extremizes the functional subject to the boundary conditions, then any slight perturbation of f that preserves the boundary values must either increase S or decrease S. Let $f + \epsilon\eta$ be the result of such a perturbation, where ϵ is small and η is smooth and compactly supported with $\eta(a) = \eta(b) = 0$, then we define

$$\Phi(\epsilon) = S[f + \epsilon\eta] = \int_a^b \mathcal{L}(x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x))dx \tag{44}$$

Now we compute the total derivative of Φ w.r.t ϵ:

$$\frac{d\Phi}{d\epsilon} = \frac{d}{d\epsilon}\int_a^b \mathcal{L}(x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x))dx \tag{45}$$

$$= \int_a^b \frac{d}{d\epsilon}\mathcal{L}(x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x))dx \tag{46}$$

$$= \int_a^b \left[\eta(x)\frac{\partial\mathcal{L}}{\partial f}(x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x)) + \eta'(x)\frac{\partial\mathcal{L}}{\partial f'}(x, f(x) + \epsilon\eta(x), f'(x) + \epsilon\eta'(x))\right] dx \tag{47}$$

where the third line follows from the fact that x does not depend on ϵ.
When $\epsilon = 0$, Φ has an extremum, so that

$$\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left[\eta(x)\frac{\partial\mathcal{L}}{\partial f}(x, f(x), f'(x)) + \eta'(x)\frac{\partial\mathcal{L}}{\partial f'}(x, f(x), f'(x))\right] dx = 0 \tag{48}$$

Using integration by parts on the second term of the intergrad, we have

$$\int_a^b \left[\frac{\partial\mathcal{L}}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'}(x, f(x), f'(x))\right]\eta(x)dx + \left[\eta(x)\frac{\partial\mathcal{L}}{\partial f'}(x, f(x), f'(x))\right]_a^b = 0 \tag{49}$$

Using the boundary conditions $\eta(a) = \eta(b) = 0$, then

$$\int_a^b \left[\frac{\partial\mathcal{L}}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'}(x, f(x), f'(x))\right]\eta(x)dx = 0 \tag{50}$$

Applying the fundamental lemma of calculus of variations, which is, if a continuous function f on (a, b) satisfies satisfies

$$\int_a^b f(x)h(x)dx = 0 \tag{51}$$

for all compactly supported smooth functions h on (a, b), then $f \equiv 0$, meaning that in our case we have

$$\frac{\partial\mathcal{L}}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'}(x, f(x), f'(x)) = 0 \tag{52}$$

and hence we have the Euler-Lagrange equation:

$$\boxed{\frac{\partial\mathcal{L}}{\partial f} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial f'} = 0} \tag{53}$$

$\square$

Particularly, Lagrange's approach was to set up independent generalized coordinates for the position and speed of every object, which allows the writing down of a general form of Lagrangian (total kinetic energy minus potential energy of the system) and summing this over all possible paths of motion of the particles yielded a formula for the "action", which he minimized to give a generalized set of equations. This summed quantity is minimized along the path that the particle actually takes, and it is indeed the stationary point satisfying the Euler-Lagrange equation. This choice eliminates the need for the constraint force to enter into the resultant generalized system of equations. There are fewer equations since one is not directly calculating the influence of the constraint on the particle at a given moment.

Deviation of the Two-Body Problem

We will consider a general isolated 2-body system: Two bodies with mass M_1, M_2, and they will orbit under the gravitational force. With the origin chosen, we use $\boldsymbol{r}_1, \boldsymbol{r}_2$ to denote the position vector of the two bodies, and we use $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ to denote the relative position between the two, then we know that $\boldsymbol{r}_1 = \boldsymbol{r}_2 - \boldsymbol{r}, \boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{r}$. if we investigate the line segment $\overline{\boldsymbol{r}_1\boldsymbol{r}_2}$, then at any point it will receive gravitational force from opposite directions, so we are able to find a point, such that the net force is zero. This point is also known as the Lagrange point, or center of mass, denoted by position vector $\boldsymbol{R}$, and then we have

$$\boldsymbol{R} = \frac{M_1\boldsymbol{r}_1 + M_2\boldsymbol{r}_2}{M_1 + M_2} \tag{54}$$

for simplicity, we will also denote $M = M_1 + M_2$, then using $\boldsymbol{R}, \boldsymbol{r}$, we can rewrite the position vector $\boldsymbol{r}_1, \boldsymbol{r}_2$ as

$$\boldsymbol{r}_1 = \boldsymbol{R} - \frac{M_2}{M}\boldsymbol{r}, \boldsymbol{r}_2 = \boldsymbol{R} + \frac{M_1}{M}\boldsymbol{r} \tag{55}$$

by taking the derivative w.r.t t, we will get the velocity of the two bodies:

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{R}} - \frac{M_2}{M}\dot{\boldsymbol{r}}, \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{R}} + \frac{M_1}{M}\dot{\boldsymbol{r}} \tag{56}$$

then, using the formula for kinetic energy $T = \frac{1}{2}M\boldsymbol{v}^2$, the total kinetic energy in this system is given by

$$T = \frac{1}{2}M_1\left[\dot{\boldsymbol{R}} - \frac{M_2}{M}\dot{\boldsymbol{r}}\right]^2 + \frac{1}{2}\left[\dot{\boldsymbol{R}} + \frac{M_1}{M}\dot{\boldsymbol{r}}\right]^2 \tag{57}$$

which can be simplified to

$$T = \frac{1}{2}(M_1 + M_2)\dot{\boldsymbol{R}}^2 + \frac{1}{2}\frac{(M_1 + M_2)M_1M_2}{M^2}\dot{\boldsymbol{r}}^2 \tag{58}$$

which is $T = \dfrac{1}{2}M\dot{\boldsymbol{R}}^2 + \dfrac{1}{2}\mu\dot{\boldsymbol{r}}^2$, where we denote $\mu = \frac{M_1M_2}{M}$ as the "reduced mass".

For the potential energy, it is a function that only depends on $G, M_1, M_2, r = |\boldsymbol{r}|$ *(we later will sees that in fact $U(r) = -\frac{GM_1M_2}{r}$)*, we will use $U(r)$ to denote the potential energy. Then we may write down the Lagrangian:

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(r) \tag{59}$$

then, since it is a $3D$ model, we will apply Euler-Lagrange equation on x, y, z directions. For x direction, we have

$$\frac{\partial L}{\partial \boldsymbol{R}_x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{R}}_x} = 0 \tag{60}$$

note that in L, we do not have dependency on $\boldsymbol{R}$! so $\frac{\partial L}{\partial \boldsymbol{R}_x} = 0$, and hence $\frac{d}{dt}M\dot{\boldsymbol{R}}_x = 0$. We can do the same for y, z direction, and we will get $\frac{d}{dt}M\dot{\boldsymbol{R}} = 0$, meaning the center of mass is moving at a constant speed, by changing the reference frame, we may just let $\boldsymbol{R}$ to be stationary for simplicity, hence we can now let $\boldsymbol{R} = 0$, the center of mass to be the origin, as well as $\dot{\boldsymbol{R}} = 0$.

To make further simplifications, we will use the conservation of angular momentum. *This part requires a lot of physics and vector calculus, namely torque, cross product and so on, so I want to skip the derivation and jump right into the result.* It can be shown that the net torque is 0, which means the system will stay in a $2D$ plane. So now we may only work with xy plane with center of mass to be the origin, which greatly simplified the problem. Then by converting into polar coordinates, we have

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases} \tag{61}$$

thus the Lagrangian for the reduced mass is now

$$L = \frac{1}{2}\mu\left[(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2\right] - U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{62}$$

Now, we use Euler-Lagrange equation for r, where $U(r) = -\frac{GM_1M_2}{r}$:

$$\frac{\partial L}{\partial r} = -\frac{GM_1M_2}{r^2} + \mu r\dot{\theta}^2, \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \mu\ddot{r} \tag{63}$$

which means

$$-\frac{GM_1M_2}{r^2} + \mu r\dot{\theta}^2 = \mu\ddot{r} \tag{64}$$

Since the angular momentum is given by $l = \mu r^2\dot{\theta}$ where l is a fixed constant because of conservation, so we would have $\dot{\theta} = \frac{l}{\mu r^2}$, and thus, we will get an equation with only one variable r:

$$-\frac{GM_1M_2}{r^2} + \mu r\frac{l^2}{\mu^2 r^4} = \mu\ddot{r} \tag{65}$$

which is, we have a second-order non-linear scalar ODE with variale r:

$$\boxed{\frac{d^2r}{dt^2} = \frac{l^2}{\mu^2}\cdot\frac{1}{r^3} - \frac{GM_1M_2}{\mu}\cdot\frac{1}{r^2}} \tag{66}$$

which means, now the system will only depend on one variable r, and hence it is easy to study the dynamics of this system. In fact we can try to solve this ODE!

From here, we will derive the orbits of the system. The term $U(r)$ is in fact the gravitational potential, and it is defined by $U(r) = -\frac{GM_1M_2}{r} = -\frac{\alpha}{r}$, we make $\alpha = GM_1M_2$ for simplicity. We now define the effective potential, given by $U_{\text{eff}} = \frac{l^2}{2\mu r^2} - \frac{\alpha}{r}$, then we have

$$\frac{dU_{\text{eff}}}{dr} = -\frac{l^2}{\mu r^3} + \frac{\alpha}{r^2} \tag{67}$$

where $l, \mu, \alpha > 0$, so we see that U_{eff} will attains its global minimum at $r = \frac{l^2}{\alpha\mu} \equiv p$, with minimum $U_{\text{eff}}(p) = -\frac{\alpha}{2p}$.

Then, we define the relative (total) energy in he system, given by $E_{\text{rel}} = \frac{\mu \dot{r}^2}{2} + U_{\text{eff}}(r)$, which is the sum of effective potential and its kinetic energy. Then since the kinetic energy is non-negative, so we would have

$$E_{\text{rel}} = \frac{\mu \dot{r}^2}{2} + U_{\text{eff}}(r) \geq U_{\text{eff}}(r) \tag{68}$$

Since $U_{\text{eff}} \geq -\frac{\alpha}{2p}$, then we will know that the motion is possible if we have $E_{rel} > -U_{\text{eff}}(p)$. We denote $|\min U_{\text{eff}}| = U_0$. Now under this condition, we will study the orbits. Here let's further consider a simpler two body problem, where $M_1 >> M_2$ so we may think M_1 is stationary and the center of mass $\boldsymbol{R}$ is just the center of M_1. So we would have M_1 being stationary at the origin, and M_2 will orbit around M_1, like earth orbits around the sun. We now define an important terminology called *eccentricity*, which is a positive constant defined by

$$e = \sqrt{1 + \frac{E_{\text{rel}}}{U_0}} \tag{69}$$

from here, we have $E_{\text{rel}} = (e^2 - 1)U_0$, so we have the following statements: If $0 < e < 1$, then $-U_0 \leq E_{\text{rel}} < 0$, if $e = 1$, then $E_{\text{rel}} = 0$, if $e > 1$ then $E_{\text{rel}} > 0$. Now, in equation (29), we can rewrite the equation as

$$\frac{1}{\dot{r}} = \sqrt{\frac{\mu}{2(E_{\text{rel}} - U_{\text{eff}})}} \tag{70}$$

since we have conservation of momentum, so $l = \mu r^2 \dot{\theta}$, hence $\dot{\theta} = \frac{l}{\mu r^2}$, then we have

$$\frac{d\theta}{dr} = \frac{d\theta/dt}{dr/dt} = \frac{\dot{\theta}}{\dot{r}} = \frac{l}{r^2\sqrt{2\mu(E_{\text{rel}} - U_{\text{eff}})}} \tag{71}$$

now we move $\dot{r}$ to the other side of the equation, and we integrate both sides, then we have

$$\theta = \int \frac{l dr}{r^2\sqrt{2\mu(E_{\text{rel}} - U_{\text{eff}})}} \tag{72}$$

now let $u = \frac{p}{r}$, hence we have $\frac{dr}{r^2} = -\frac{du}{p}$, also

$$E_{\text{rel}} - U_{\text{eff}} = (e^2 - 1)U_0 - \frac{l^2}{2\mu r} + \frac{\alpha}{r} \tag{73}$$

Now we replace $r = p/u = l^2/(p\alpha u)$ and $U_0 = \mu\alpha^2/(2l^2)$, so now we have

$$E_{\text{rel}} - U_{\text{eff}} = \frac{\mu\alpha^2}{2l^2}(e^2 - (u-1)^2) \tag{74}$$

back to the integral, we substitute everything we defined, tus

$$\theta = -l \int \frac{du}{p} \frac{1}{\sqrt{2\mu \frac{\mu\alpha^2}{2l^2}(e^2 - (u-1)^2)}} = -\int \frac{du}{\sqrt{e^2 - (u-1)^2}} \tag{75}$$

which is an elementary integral if we use trig-sub! *Note that the condition* $0 < (u-1)^2 < e^2$ *is always obeyed*, then if we let $u = 1 + \cos x$, then trivially the integral is

$$\phi = -\int \frac{-e \sin x dx}{e \sin x} \equiv x. \tag{76}$$

Finally, since $u = \frac{p}{r}, u = 1 + e\cos x$, the equation of the trajectory in polar coordinate is

$$\boxed{\frac{p}{r} = 1 + e\cos\theta} \tag{77}$$

which is known as the conic section. We claim that when $0 \leq e < 1$, the orbit is closed and hence has periodical orbits. When $e = 0$ the orbit is a perfect circle, when $0 < e < 1$ the orbit is an ellipse, when $e = 1$ the orbit is a parabola and finally when $e > 1$ the orbit is a hyperbola. Here, we will only consider circular orbits, $e = 0$. In this case the bodies move with a constant angular velocity, and hence we can replace θ with $\theta = \omega t$, and we have the motion given by

$$\frac{p}{r} = 1 + \cos(\omega t) \tag{78}$$

3 References

My presentation mainly takes references from the following papers and websites:

1. The 'Halo' Family of 3-Dimensional Periodic Orbits in the Earth-Moon Restricted 3-Body Problem, by *John V. Breakwell and John V. Brown.*

2. Series 3R3BP, by *Ari Rubinsztejn* , website can be found here: https://gereshes.com/category/math/astrodynamics/cr3bp/. I would like to thank the author to this website who provided many nice pictures.

3. The Circular Restricted Three-Body Problem, by *Richard Frnka.*

4. Lectures notes from MIT: Central Force Motion: Kepler's Laws, by *J. Peraire, S. Widnall*, Fall 2008.