Graph Theory

Fall 2025, Math 350 Course Notes

McGill University

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Despite all efforts, there may still be some typos, unclear explanations, etc. If you find potential mistakes, or any suggestions regarding concepts or formats, etc., feel free to reach out to the author at *zhangjohnson729@gmail.com.*

Contents

Lecture 1: Introduction to Graphs

We may think of graphs as tools for encoding pairwise relationships between objects, for example:

Objects	Relationships
Cities	Direct Flight Between Them
Atoms	Chemical Bonds

Definition

Definition 1. *A graph G is a pair of sets $(V(G), E(G))$ where $V(G)$ is the vertex set, $v \in V(G)$ called vertices. $E(G)$ is the edge set, $e \in E(G)$ called edges. Each edge has 1 or 2 vertices at its ends.*

More terminologies can be defined once we defined $G(V, E)$. The picture below will help to illustrate:

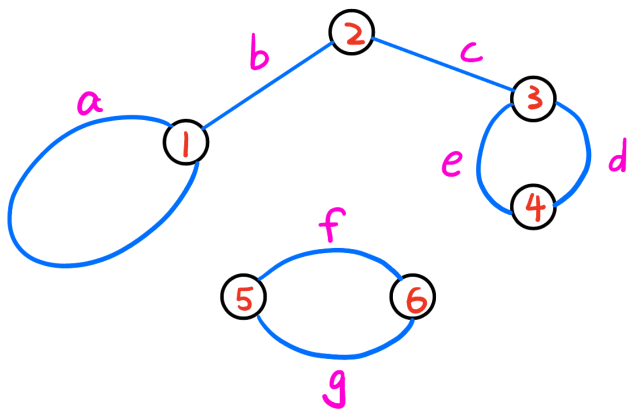


Figure 1: An example of the graph

(i) A loop is an edge with 1 end; like edge a above.

(ii) Two non-loop edges are parallel if they have the same ends; like edges e, d; f, g are parallel.

(iii) An edge joins its ends, which means an edge is connected by two or one vertices.

(iv) Two vertices are adjacent if they are joined by some edges; in above, vertices $1, 2$ are adjacent, but $1, 3$ are not adjacent.

Next we shall see two interesting examples of graph theory, both of which can be well understand by the terminologies we have introduced:

Example 1: Bridges of Königsberg (Euler 1735)

Euler studied this problem in 1734, where in the city of Königsberg, there are 7 bridges connecting 4 pieces of lands. The structure of the bridges is shown in the next figure. We are interested in the problem that, if a traveller walks through the bridges in his/her free time, is there a path such that all bridges can be visited exactly once?

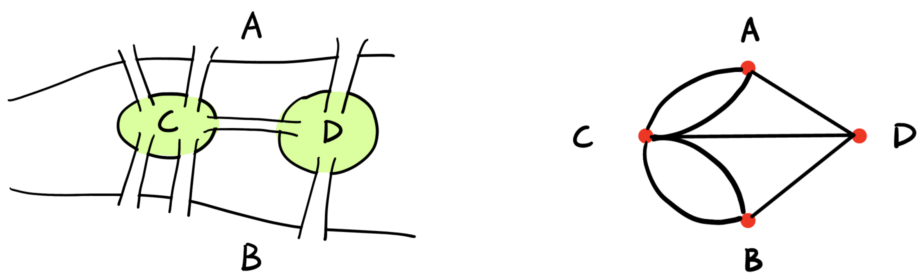


Figure 2: To the left: The bridges in Königsberg, where A, B, C, D are lands and there are 7 bridges in total connecting them; to the right: A simplified version of Königsberg's bridges using vertices and edges, while we connect two vertices with an edge if there is a bridge jointing them

The above question can be turned into a simple graph theory problem (see figure 2 right). Our goal now is: Is there a way to travel through all edges exactly once?

The answer is it doesn't exist.

Proof. Now, WLOG, suppose we start at point A and finally end at point B, then we will have a path $A \to$ some vertices $\to B$, then we know throughout the entire path, except for starting point and ending point, the points in between must have even number of vertices (or paths), since we need to complete an "in-and-out" action. That is, if there are n vertices, then we need at least $n-2$ of those vertices to have an even number of edges. But we see that in Königsberg's situation, there are only 1 vertex with even number of edges, so such a path is not possible. ∎

Not only we solved Königsberg's problem, we also generalized the problem to an arbitrary graph, and we can use this idea to see if we can travel all the bridges jointing Manhattan, Newark, Brooklyn and Queens! Next time you can show off your graph-theory knowledge to a traveler who wants to walk through all NYC bridges at once.

Example 2: Ramsey Number

Lemma

Lemma 1. *For any group of* 6 *people, either:*

(i) 3 *of them know each other (pairwise).*

(ii) 3 *of them are pairwise strangers.*

Our set up is that, consider a graph with 6 vertices (representing 6 people), and an edge connects two people if and only if they know each other (here we do not talk about directed graphs, that is

A knows B will also imply B knows A).

Proof. First, we do case analysis. In fact most proofs in graph theory are based on case analysis.

1. Assume an individual v knows more than 3 of the others.

(i) If at least one pair of $w, x; w, y; x, y$ knows each other, then as we can see in the next figure (left) where we chose w, x knows each other, v, w, x forms a closed path and hence those 3 knows each other and that satisfies case (i)

(ii) No pair of w, x, y know each other. Which means w, x, y do not know each other and that satisfies case (ii).

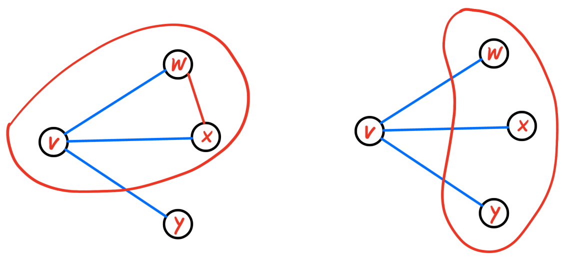


Figure 3: To the left: The case when one pair of $w, x; w, y; x, y$ know each other; to the right: The case when none of w, x, y knows each other. Set of vertices satisfying the conditions is circled in red.

2. Assume v knows at most 2 of the others

(i) If at least one of $w, x; w, y; x, y$ don't know each other, then as we can see (where we choose w, x don't know each other), then we have a set v, w, x where they don't know each other and it satisfies case (ii).

(ii) If else, then all of w, x, y knows each other and hence they satisfy case (i).

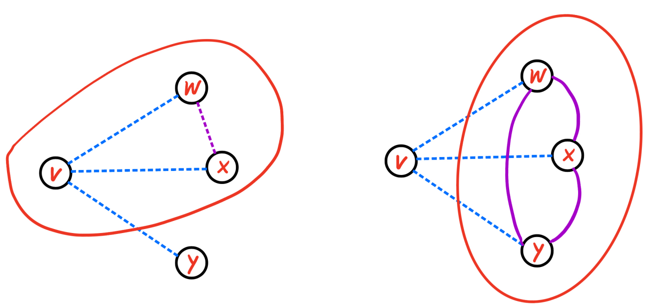


Figure 4: To the left: The case when one pair of $w, x; w, y; x, y$ don't know each other (marked in dashed line); to the right: The case when all of w, x, y knows each other. Set of vertices satisfying the conditions is circled in red.

■

The above corollary is a special case of Ramsey numbers. We give the formal definition:

Definition

Definition 2. *For $k \in \mathbb{N}, k \geq 2$, the Ramsey number $R(k)$ is the smallest integer n such that in a graph $G(V, E)$ of n distinct vertices, there are either:*

(i) k vertices that are pairwise adjacent;

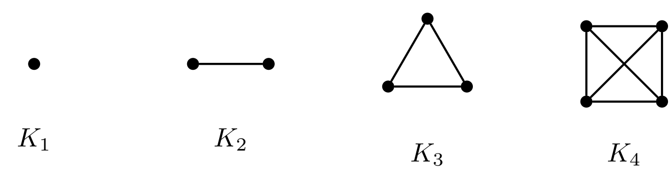
(ii) k vertices that are pairwise non-adjacent.

In our setting we just change the graph theory setting to a real world example. It can be shown that $R(2) = 2, R(3) = 6, R(4) = 18$, and $R(5) \in [43, 48]$.

Definition

Definition 3. *A graph $G(V, E)$ is simple if it has no loops or parallel edges, a simple graph with n vertices $\binom{n}{2}$ is called a clique (or complete graph) of size n, denote by K_n.*

We may think of, in a clique (complete graph), each two vertices are jointed by a different edge. See the figure below:



Then we may generalize Ramsey's number. Say in a clique, we use different colors to label edges (red or blue, for example), then $R(r, s) = n$ means the smallest integer n such that in any K_n with two colorings of edges, there is either a K_r with all blue edge or a K_s with all red edges. Same idea can be generalized for $R(r_1, r_2, \cdots, r_m) = n$. The above corollary can be viewed as $R(3, 3) = 6$.

Theorem

Theorem 1. ***Ramsey's Theorem:***

$R(r, s)$ exists for every $r, s \geq 2$, $r, s \in \mathbb{N}$.

We will get back to this later.

Lecture 2: Paths, Cycles, and Other Terminologies

Definition

Definition 4. *A path on n vertices, denoted P_n, is a graph with vertices $\{v_0, \cdots, v_{n-1}\}$, edges $\{e_1, \cdots, e_{n-1}\}$ such that e_i has ends v_{i-1} and v_i, for each $i \in \{1, \cdots, n-1\}$.*

Definition

Definition 5. *A cycle on n vertices, denoted C_n, is a graph with vertices $\{v_0, \cdots, v_{n-1}\}$, edges $\{e_1, \cdots, e_n\}$ such that e_i has ends v_{i-1} and v_i for all $i \in \{1, \cdots, n-1\}$ and additionally e_n has ends v_{n-1}, v_0.*

We say the *length* of a path/cycle is the number of edges it has. In comparison to a path, we also define a *walk* on a graph $G(V, E)$, which is a sequence of the form $(v_0, e_1, v_1, e_2, \cdots, e_k, v_k)$, where e_i is an edge with ends v_{i-1} and v_i for $i \in \{1, \cdots, k\}$. But the point is the edges and vertices can be repeated, and that differs us from a "path". We say an edge e is incident to vertex v if v is an end of e.

Definition

Definition 6. *The degree of a vertex v in $G(V, E)$, denoted as $\deg_G(v)$ is the number of edges incident to v, with loops counted twice.*

Theorem

Theorem 2. *(Handshaking Lemma)*

For any graph $G(V, E)$, we have

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|.$$

Proof. We define a function

$$i(e, v) = \begin{cases} 2 & \text{if } e \text{ is a loop incident to } v \\ 1 & \text{if } e \text{ is a non-loop edge incident to } v \\ 0 & \text{otherwise} \end{cases},$$

then

$$
\begin{aligned}
\sum_{v\in V(G)} \deg_G(v) &= \sum_{v\in V(G)} \sum_{e\in E(G)} i(e,v) \\
&= \sum_{e\in E(G)} \sum_{v\in V(G)} i(e,v) \\
&= \sum_{e\in E(G)} \begin{cases} 1+1 & \text{if } e \text{ is a non-looping edge} \\ 2 & \text{if } e \text{ is a looping edge} \end{cases} \\
&= \sum_{e\in E(G)} 2 \\
&= 2|E(G)|
\end{aligned}
$$

■

Definition

Definition 7. *For graphs G, H say H is a subgraph of G if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and all edges of H have the same ends in H as in G. Denote by $H \subseteq G$*

If $H_1, H_2 \subseteq G$, then $H_1 \cup H_2 \subseteq G$, where $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$. Same for the intersection, $H_1 \cap H_2 \subseteq G$, where $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$.

Definition

Definition 8. *Graphs G, H are isomorphic if they are identical up to relabeling. That is, there exists a bijection $\varphi : V(G) \cup E(G) \to V(H) \cup E(H)$ such that*

(i) For all $v \in V(G)$, $\varphi(v) \in V(H)$;

(ii) For all $e \in E(G)$, if it has ends u, v, then $\varphi(e) \in E(H)$ and $\varphi(e)$ has ends $\varphi(u), \varphi(v)$.

Lecture 3: Graph Connectivity

Below, denote $G = (V(G), E(G))$ be a graph

Definition

Definition 9. *Vertices u, v of a graph G are connected in G written as $u \xleftrightarrow{G} v$ if there exists a walk in G with ends u, v.*

It is natural to think that, if $u \xleftrightarrow{G} v$ then there exists a path in G with ends u, v. Why we still use the terminology "walk"? That's because in a walk, we can easily concatenate two walks and

still get a walk, say two walks: $u = u_0e_1u_1 \cdots e_ku_k = v$ and $v = v_0f_1v_1 \cdots f_lv_l = w$, then combining them will still give us a walk: $u = u_0e_1u_1 \cdots e_ku_kf_1v_1 \cdots f_lv_l = w$. However concatenating two paths will not necessary give us another path, see the figure below:
However, a walk joints u, v will imply a path jointing u, v:

Lemma

Lemma 2. *If there is a walk in G with ends u, v, then there exists a path in G with ends u, v.*

Proof. If $u = v$, then the one vertex walk is indeed a path. Otherwise, fix a walk $u = u_0e_1u_1 \cdots e_ku_k = v$ with ends u, v with minimum length, if $u_0, \cdots, u_k$ are distinct vertices it is then already a path. Otherwise assume there are repetitions, that is we can find $0 \le i < j \le k$ such that $u_i = u_j$, then another walk can be defined: $u = u_0e_1u_1 \cdots e_iu_ie_{j+1} \cdots e_ku_k$, however now the length is shorter than the original walk and that is not possible. ■

Definition

Definition 10. *A graph G is connected if for all $u, v \in V(G)$ we have $u \overset{G}{\longleftrightarrow} v$.*

It is equivalent to say that there is a path in G with ends u, v, or a walk with ends u, v.

Lemma

Lemma 3. *A graph G is not connected if $\exists$ a partition (X, Y) of $V(G)$ such that $X, Y \neq \varnothing$ and such that there is no $e \in E(G)$ with one end in X and one end in Y.*

Proof. ($\Longrightarrow$) Let (X, Y) be such a partition, fix $u \in X, v \in y$. Suppose $\exists$ a walk $u = u_0e_1u_1 \cdots e_ku_k = v$ with ends u, v. Let u_i be the first vertex on the walk with $u_i \in Y$, then e_i has ends $u_{i-1} \in X$ but $u_i \in Y$ and this yields a contradiction. ($\Longleftarrow$) If G is not connected so $\exists u, v \in V(G)$ with $u \neq v$ such that there is no walk connecting u and v and let $X := \{x \in V(G) : u \overset{G}{\longleftrightarrow} x\}$ and $Y = V(G) \setminus X$. Note that $u \in X, v \in Y$, if $e \in E(G)$ has ends $x \in X$ and $y \in Y$ then we can fix a path $u = u_0e_1u_1 \cdots e_ku_k = x$, then $u_0e \cdots e_ku_key$ is a path with ends u, y so $y \in X$ which is a contradiction. ■

Lemma

Lemma 4. *If $H_1, H_2 \subseteq G$ are connected subgraphs, and $V(H_1) \cap V(H_2) \neq \varnothing$ then $H_1 \cup H_2$ is also connected.*

Proof. Let $v \in V(H_1) \cup V(H_2)$ and let $x, y \in V(H_1 \cup H_2)$ then either $v, x \in V(H_1)$ or $v, x \in V(H_2)$. Either case there is a walk $v = v_0e_1 \cdots e_kv_k = x$ with ends v, x in $H_1 \cup H_2$. Likewise we have either $v, y \in V(H_1)$ or $v, y \in V(H_2)$ and either case there is also a walk $y = y_0f_1 \cdots f_ly_l = v$ with ends v, y in $H_1 \cup H_2$. Then their concatenation is, by definition also a walk with ends x, y and $H_1 \cup H_2$ is connected. ■

Definition

Definition 11. *A connected component of a graph G is a maximal connected subgraph of G. i.e if $H \subseteq G$ is a connected component if $H \subseteq G$ connected and for any other connected H' with $H \subseteq H' \subseteq G$, then $H = H'$.*

Lemma

Lemma 5. *For any graph G, every vertex $v \in V(G)$ belongs to a unique connected component.*

Proof. For every vertex $v \in V(G)$, the subgraph $(\{v\}, \varnothing)$ is a connected subgraph containing v so there exists a maximal connected subgraph containing v. Suppose H_1, H_2 are distinct maximal connected subgraphs containing v then we know $H_1 \cup h_2$ is a connected subgraph of G containing v, but $H_1, H_2 \neq H_1 \cup h_2$ and $H_1, H_2 \subseteq H_1 \cup H_2$ and we arrive at a contradiction. ■

Exercise: A subgraph $H \subseteq G$ is a connected component of G iff H is connected and if $e \in E(G)$ has an end in $V(H)$ then $e \in E(H)$.

Definition

Definition 12. *For $e \in E(G)$, $v \in V(G)$, define $G \setminus e$ is the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) - \{e\}$.*

We use Comp(G) as the number of connected components of G.

Lemma

Lemma 6. *G is connected if and only if $Comp(G) = 1$.*

Proof. ($\Longrightarrow$) if G is connected, then $\forall u, v \in V(G)$, there is a walk with ends u and v, and thus u, v belongs to the same connected components, and indeed all vertices $v \in V(G)$ are in the same connected components and so Comp$(G) = 1$. ($\Longleftarrow$) If Comp$(G) = 1$, then by definition there is only one connected component, so for any vertices $u, v \in V(G)$ there is a walk jointing them. Hence G is connected. ■

Definition

Definition 13. *e is a cut edge of G if $e \in E(G)$ and e is not an edge of any cycle in G.*

Exercise: If e is a cut edge, then e has distinct ends u, v and there is no walk from u to v in $G \setminus e$.

Lemma

Lemma 7. *Let $e \in E(G)$ with ends u, v then exactly two of the following holds: (i) e is a cut edge, u, v are in different connected components of $G \setminus e$ and $Comp(G \setminus e) = Comp(G) + 1$; (ii) e is not a cut edge, u, v are in the same connected components of $G \setminus e$ and $Comp(G \setminus e) = Comp(G)$.*

Definition

Definition 14. *Let G be a graph, the compliment of G, denote as G^c, is the graph defined by:*

$$G^c := \Big(V(G), \Big\{\{e, f\} \subseteq V(G), ef \notin E(G)\Big\}\Big).$$

(we use e, f for vertices and ef as an edge with ends e and f).

It can be viewed as, if vertices u, v is jointed by an edge in G (the edge has ends u, v), then u, v is not joint by any edge in G^c; and likewise if u, v is not joint by any edge in G, then they are joint by some edge $e' \in E(G^c)$ with ends u, v.

Lemma

Lemma 8. *For any graph G either G or G^c is connected.*

Proof. By definition of connectedness it suffices to consider simple graphs G. Let $|V(G)| = n$, and when $n = 1$, it is the graph with one vertex and by definition it is connected, and when $n = 2$, the edges (if exist) will only joint those two vertices and the case is trivial. Consider $n \geq 3$: For all $u, v \in V(G)$: (i) Suppose u, v are not jointed by any edge in $E(G)$, then by definition u, v will be jointed by an edge $e' \in E(G^c)$ and hence a walk from u to v is possible and u, v are connected; (ii) If u, v is jointed by some edge (say e_1) with ends u, v, then u, v is connected in G and hence they belong to the same connected component, say $u, v \in C_1$. Then, for all $w \in V(G) \setminus C_1$, there is no walk from w to u and no walk from w to v, hence they are not jointed by any edge in $E(G)$, so in G^c there will be an edge (say e_2) with ends u, w and another edge (say e_3) with ends w, v, then in G^c we see that there are two walks: $u = u_0 e_1 u_1 \cdots e_k u_k := w$ and $w = v_0 f_1 v_1 \cdots f_l v_l := v$, and concatenating them will gives us a walk with ends u, v and hence u, v is connected in G^c. ■

Lecture 4: Trees and Forests

Just for notation, we say the null graph G is the graph such that it has no edges or vertices, that is $V(G) = \varnothing, E(G) = \varnothing$.

Definition

Definition 15. *Let G be a graph, G is a forest if G contains no cycles; G is a tree if it is a (non-null) connected forest.*

Lemma

Lemma 9. *Let F be a non-null forest, then $Comp(F) = |V(F)| - |E(F)|$.*

Proof. We perform induction $|E(F)|$: If $|E(F)| = 0$, each vertex is in its own connected component. So $\text{Comp}(F) = |V(F)|$, if $|E(F)| \geq 1$, fix $e \in E(F)$ and by definition e must be a cut edge otherwise a cycle arises. Then using induction hypothesis,

$$\text{Comp}(F \setminus e) = |V(F \setminus e)| - |E(F \setminus e)| = |V(F)| - (E(F) - 1)$$

Now $\text{Comp}(F \setminus e) = \text{Comp}(F) + 1$, and hence we have

$$\text{Comp}(F) + 1 = |V(F)| - (|E(f)| - 1)$$

which implies $\text{Comp}(F) = |V(F)| - |E(F)|$. ■

An corollary follows trivially from the above lemma, that is, if T is a tree then $|V(T)| = |E(T)|+1$ since $\text{Comp}(T) = 1$.

Definition

Definition 16. *A leaf in a graph G is a vertex $v \in V(G)$ with $\deg_G(v) = 1$.*

Lemma

Lemma 10. *Let T be a tree with at least 2 vertices, let X be the set of leaves of T, Y be the set of vertices in T with degree at least 3, then $|X| \geq |Y| + 2$.*

Proof. By handshaking lemma we have $2|E(G)| = \sum \deg_G(v)$ so it follows that $2|E(G)|-2|V(G)| = \sum(\deg_G(v) - 2)$, with the lemma before, we have $|E(G)| - |V(G)| = -1$ and then

$$\begin{aligned} -2 &= \sum(\deg_G(v) - 2) && (1)\\ &= -|X| + \sum_{\deg_G(v)\geq 3} (\deg_G(v) - 2) && (2)\\ &\leq -|X| + |Y| && (3) \end{aligned}$$

where $|X| + |Y| \geq 2$ follows. ■

Lemma

Lemma 11. *If T is a tree with exactly 2 leaves u, v, then T is a path with ends u, v.*

Proof. First note that T must be connected. By the previous lemma, $|X| = 2 \geq |Y| + 2$, which means $|Y| = 0$, i.e all vertices has degree less than 3, and so all vertices apart from u, v has degree exactly 2, and that's the definition of a path. ■

Lemma

Lemma 12. *Let T be a tree and v be a leaf, then $T \setminus v$ is also a tree.*

Proof. Note $T \setminus v$ cannot have any cycles and hence is a forest. By definition, v has a unique neighbor $u \in V(T)$, and

$$\begin{aligned} \text{Comp}(T \setminus v) &= |V(T \setminus v)| - |E(T \setminus v)| && (4) \\ &= |V(T)| - 1 - |(E(T)| - 1) && (5) \\ &= 1 && (6) \end{aligned}$$

so $T \setminus v$ is also a tree. ■

Lemma

Lemma 13. *Let v be a leaf in G, if $G \setminus v$ is a tree, then G is a tree.*

Proof. We claim there is no cycle in G. If not, then v must not be in the cycle because $\deg_G(v) = 1$, then the cycle lies somewhere else and that will make $G \setminus v$ not a tree. Let u be the unique neighbor of v, and let $D \subseteq G$ be the subgraph containing only u, v and the unique edge jointing them. So D is connected, also $G \setminus v$ is also connected, hence their union is connected (because they share one vertex in common) and so $G := (G \setminus v) \cup D$ is a tree. ■

Lemma

Lemma 14. *Let T be a tree, let $u, v \in V(T)$, then there is a unique path in T with ends u, v.*

Proof. When $|V(T)| = 1$, it is trivial. Assume $|V(T) \geq 2$, fix $u, v \in V(T)$ and a path P in T from u to v. (i) If T has a leaf $w \neq u, v$, then for sure $w \notin V(P)$, then P is still a path from u to v in $T \setminus w$. Assume Q is a another path rather than P in T, then by induction it is also a path in $T \setminus w$, and P, Q must be identical by induction hypothesis. (ii) If T has no leaf $w \neq u, v$ ■

Lecture 5: Spanning Trees

Definition

Definition 17. *Let G be a graph, a spanning subgraph of G is a subgraph $H \subseteq G$ with $V(H) = V(G), E(H) \subseteq E(G)$. A spanning forest is a spanning subgraph that is a forest. A spanning tree is a spanning subgraph that is a tree.*

Note that if G is not connected then there is no spanning tree.

Lemma

Lemma 15. *Suppose G is a non-null connected graph, $H \subseteq G$ be a connected spanning subgraph of G chosen to be minimal subject to this, then H is a spanning tree of G.*

Proof. It suffices to show H has no cycles. Suppose there is a cycle $C \subseteq H$, then $\forall e \in E(C)$, it is not a cut edge, meaning $\text{Comp}(H \setminus e) = \text{Comp}(H)$ but then $H \setminus e$ is a smaller connected subgraph compare to H. ■

Lemma

Lemma 16. *Suppose G is a non-null connected graph, let H be a spanning forest of G chosen to be maximal subject to this, then H is a spanning tree of G.*

Observation: Let H be a graph, $u, v \in V(H)$ belonging to two different connected components. Let H' be the graph by adding a new edge e with ends u, v. Then e is a cut edge.

Proof. It suffices to show H is connected. Suppose not then we can find $u, v \in V(H)$ belonging to different connected components. Since G is connected so wlog say there is an edge e jointing u, v in G, this edge must be a cut edge since $\text{Comp}(H + e) = \text{Comp}(H) - 1$, but now $H + e$ is a larger connected subgraph compare to H. ■

Definition

Definition 18. *Let T be a spanning tree of a graph G, and fix $f \in E(G) \setminus E(T)$, a fundamental cycle of f w.r.t T is a cycle $C \subseteq G$ such that $C \setminus f \subseteq T$.*

Lemma

Lemma 17. *For any graph G, any spanning tree T of G, any edge $f \in E(G) \setminus E(T)$, there exists a unique fundamental cycle of f w.r.t T.*

Proof. Suppose f has ends $u, v \in V(C)$, then C is a fundamental cycle iff $C \setminus f$ is a path with ends u, v in T. Also such a path P is unique in T, so $C = P + f$ must also be unique. ■

Lemma

Lemma 18. *Let T be a spanning tree of G, let $f \in E(G) \setminus E(T)$ and let C be the fundamental cycle of f w.r.t T. Fix $e \in E(C)$ and $T' = (T + f) \setminus e$ is a spanning tree of G.*

Proof. We first show C is the unique cycle in $T + f$. Then $T' = T + f \setminus e$ has no longer a cycle and it is hence a forest. Then $\text{Comp}(T') = |V(T')| - |E(T')| = |V(T)| - |E(T) + 1 - 1| = 1$, and T' is a tree. ■

Any non-loop edge in a connected graph G is in some spanning tree of G.

Definition

Definition 19. *Fix a graph G, $\omega : E(G) \to \mathbb{R}_+ := [0, \infty)$, for $H \subseteq G$ define*

$$\omega(H) := \sum_{e \in E(H)} \omega(e).$$

A spanning tree T of G is a minimal spanning tree (MST) if it has minimum weight over all spanning trees of G.

Observations: If T is a minimum spanning tree (MST) of (G, ω) , $f \in E(G) \setminus E(T)$, C is the fundamental cycle of f w.r.t T, then $\forall e \in E(C)$, $\omega(f) \geq \omega(e)$.

Proof. We know that $T' = (T + f) \setminus e$ is another spanning tree, so

$$\omega(T') = \omega(T) + \omega(f) - \omega(e) \geq \omega(T) + \omega(e) - \omega(e).$$

■

Definition

Definition 20. *(Kruskal's Algorithm)*

INPUT: *G, a connected non-null graph, $\omega : E(G) \to \mathbb{R}_+$;*

FOR $i = 1, 2, \cdots, |V(G)| - 1$, DO:
Choose e_i with $\omega(e_i)$ minimum such that: (i) $e_i \notin \{e_1, \cdots, e_{i-1}\}$; (ii) $\{e_1, \cdots, e_{i-1}\} \cup \{e_i\}$ doesn't contain the edge set of a cycle.

OUTPUT: *A spanning tree T with $V(T) = V(G)$ and $E(T) := \{e_1, \cdots, e_{|V(G)|-1}\}$.*

When edge weights are all distinct, then Kruskal's algorithm outputs the unique MST of G.

Theorem

Theorem 3. *Let G be a connected graph, let $\omega : E(G) \to \mathbb{R}_+$ be injective and T be an MST for (G, ω), List $E(T)$ as $e_1, \cdots, e_k$ with $\omega(e_1) < \cdots < \omega(e_k)$ where $k = |V(G)| - 1$. Then for every $1 \leq i \leq k$, e_i is the minimum weight edge of G such that: (i) $e_i \notin \{e_1, \cdots, e_{i-1}\}$; (ii) $\{e_1, \cdots, e_i\}$ doesn't contain the edge set of a cycle.*

Proof. ■

Lecture 6: Counting Trees

Suppose we label the vertices of a tree T to be $1, \cdots, n$, then how many pairwise non-isomorphism labeled trees are there? Well Cayley's formula tells us the answer:

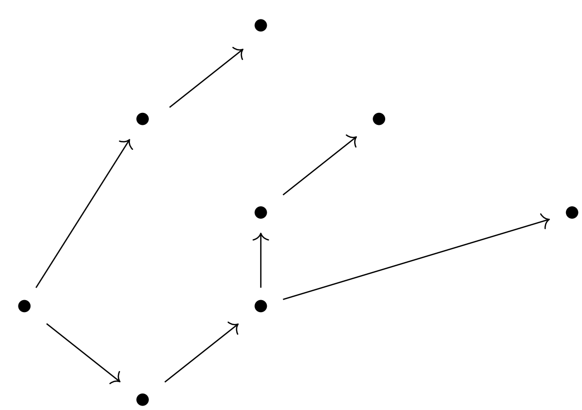
Theorem

Theorem 4. *Let $f(n)$ be the number of (non-isomorphism) labeled trees with vertex labels $1, \cdots, n$ then $f(n) = n^{n-2}$.*

A nice equivalent statement about Cayley's formula is that the complete graph K_n has n^{n-2} spanning trees. The idea of the proof is that, rather than counting trees, we count forest in order to be able to use induction on the number of edges, also we count rooted forests rather then non-rooted ones.

Definition

Definition 21. *A rooted forest is a forest together with a choice of a root vertex for each tree in the forest.*



Once the roots are chosen, there is only one way to direct the edges away from the root. If a tree has root r then edge with ends u, v is directed from u to v iff u lies on the unique path from r to v. Like in the figure above we have chosen the left-most vertex as the root and we directed the edges. But note that each vertex also has a label which I didn't list.

Definition

Definition 22. *Let $\mathcal{T}_{n,k}$ to be the set of rooted directed forest with vertex labeled $1, \cdots, n$ and has k connected components.*

Then by definition, elements in $\mathcal{T}_{n,1}$ is just a labeled tree but with a choice of root. Since we have n choices of root, $|\mathcal{T}_{n,1}|$ is n times larger than what we want to compute: The number of labeled trees (non-isomorphism). Also it is easy to see that $|\mathcal{T}_{n,n}| = 1$.

Lemma

Lemma 19. *For $1 \leq k \leq n$, we have*

$$n(k-1) \cdot |\mathcal{T}_{n,k}| = (n-k+1) \cdot |\mathcal{T}_{n,k-1}|.$$

Proof. Note that for sets A, B and a relation between A, B, if every element of A is related to α elements of B and every element of B is related to β elements of A, then $\alpha|A| = \beta|B|$. We shall use this fact to construct relations between them.

Say a forest $F \subseteq \mathcal{T}_{n,k}$ is a child of a forest $F' \subseteq \mathcal{T}_{n,k-1}$ if F is obtained from F' by deleting an edge and adding a root (while leaving edge direction unchanged, and this root is automatically obtained). We say a forest $F' \subseteq \mathcal{T}_{n,k-1}$ has $|E(F')| = |V(F')| - \mathrm{Comp}(F') = n-(k-1) = n-k+1$ edges. Now how many parents does a forest have? To construct a parent of F in $\mathcal{T}_{n,k-1}$, we can add an edge from any root to any vertex not in the same component. Equivalently from any vertex v of F to any root in a different component of F from v. So each $F \subseteq \mathcal{T}_{n,k}$ has $n(k-1)$ parents, hence we have

$$n(k-1)|\mathcal{T}_{n,k}| = (n-k+1)|\mathcal{T}_{n,k-1}|$$

■

Theorem

Theorem 5.

$$|\mathcal{T}_{n,k}| = \binom{n}{k} k n^{n-1-k}$$

for $1 \leq k \leq n$.

Proof. We perform induction on $n-k$, the base case is when $n-k=0$, then we just have $\mathcal{T}_{n,n}$ and it is obviously true that $|\mathcal{T}_{n,n}| = 1$. Assume that $\mathcal{T}_{n,k}| = \binom{n}{k} k n^{n-1-k}$, then we know by $n(k-1)|\mathcal{T}_{n,k}| = (n-k+1)|\mathcal{T}_{n,k-1}|$ we have

$$\begin{aligned}
|\mathcal{T}_{n,k-1}| &= \frac{n(k-1)}{(n-k+1)} \binom{n}{k} k n^{n-1-k} && (7)\\
&= \frac{n(k-1)}{n-k+1} \cdot \frac{n!}{k!(n-k)!} \cdot k n^{n-1-k} && (8)\\
&= (k-1)\binom{n}{k-1} n^{n-1-(k-1)}. && (9)
\end{aligned}$$

■

Then a corollary suggests that $|\mathcal{T}_{n,1}| = n^{n-1}$, also we said it is n times more than what we actually need to count, so Cayley's formula follows trivially, that

$$f(n) = n^{n-2}.$$