Partial Differential Equations

Fall 2025, Math 475 Course Notes

McGill University

By Jiajun Zhang

September 2, 2025

Acknowledgments

I would like to extend my deepest thanks and appreciation to the following people, without whose support this note would not have been possible:

[Adam Oberman], *Professor, McGill University*
I would like to express my deepest gratitude to Professor Chen, the instructor for this course, whose guidance and expertise were invaluable throughout the development of my notes.

Despite all efforts, there may still be some typos, unclear explanations, etc. If you find potential mistakes, or any suggestions regarding concepts or formats, etc., feel free to reach out to the author at *zhangjohnson729@gmail.com*.

Contents

First-Order Linear PDEs

In this section, we will dive into some basic and elementary structures of partial differential equations. Throughout the section, consider $u(x_1, \cdots, x_n)$ as a multivariate function, $u_{x_1}, \cdots, u_{x_n}$ as its first order partial derivatives, and $u_{x_i x_j} \cdots$ as second order partial derivatives and so on.

1.1 Constant Coefficient Linear PDEs

In this part, let $u(x,y) : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a, b \in \mathbb{R}$ not both zero, and we are interested in the PDE of the form

$$\boxed{au_x + bu_y = 0} \tag{1.1}$$

A very intuitive example is $u(x,y) = bx - ay$, where $u_x = b, u_y = -a$ and $au_x + bu_y = ab - ab = 0$. But in general how can we solve it? We will give two methods.

1.1.1 Geometric Method Via Directional Derivative

When we have a multivariate function (in our example it suffices to illustrate with 2 variables) $z = u(x,y)$, the two partial derivatives are defined by

Definition

Definition 1. *Let $z = u(x,y)$ be a multivariate function, then the partial derivatives u_x, u_y at (x_0, y_0) are defined as*

$$u_x := \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \tag{1.2}$$

$$u_y := \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}. \tag{1.3}$$

Just like normal derivative with one variable, the partial derivatives u_x, u_y are the rate of change along the x, y axis respectively. We may further extend this idea to any direction, say the "partial derivative" or the "rate of change" along the line $y = x$, $y = 2x - 3$ (a linear combination of x, y coordinates) etc. That's where we introduce directional derivatives. So we may then define the directional derivative along a vector $\mathbf{v} = (a,b)$:

Definition

Definition 2. *The directional derivative of $u(x,y)$ at (x_0, y_0) along $=(a,b)$ is given by*

$$D_{\mathbf{v}} u(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0 + ha, y_0 + hb) - u(x_0, y_0)}{h} \tag{1.4}$$

and we can easily see that u_x, u_y is just the special case when $\mathbf{v} = (1,0), (0,1)$ respectively. Below is a figure to illustrate directional derivative geometrically:

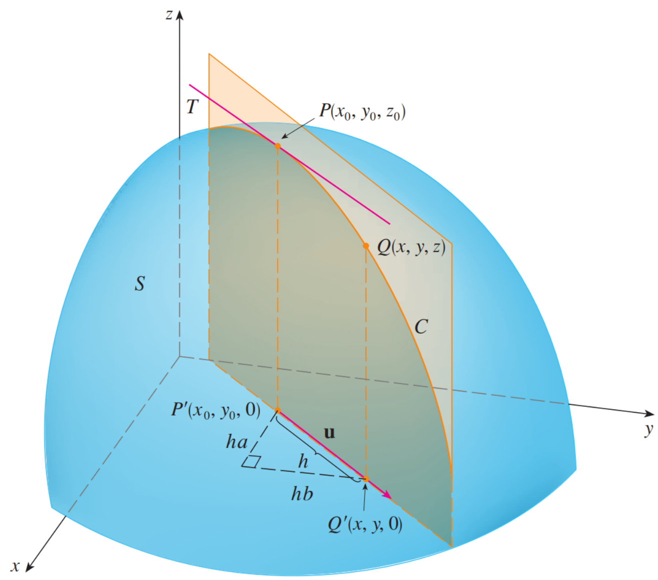

Figure 1: The directional derivative at (x_0, y_0, z_0) of the function $z = u(x, y)$. It can be viewed as the slope of tangent line of the curve obtained by slicing the function with a vertical plane that passes through the direction of the directional derivative. (*Credit: James-Stewart Calculus, early transcendentals, Eighth Edition, Page 946*)

Theorem

Theorem 1. *The directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ can be computed by*

$$D_{\mathbf{v}}u(x,y) = u_x(x,y) \cdot a + u_y(x,y) \cdot b \tag{1.5}$$

The proof of this theorem can be done by using Chain rule and hence the readers are encouraged to try it themselves.

After knowing some directional derivatives, we go back to the equation $a \cdot u_x + b \cdot u_y = 0$, we realized this means the directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ is equal to zero, meaning $u(x, y)$ is constant along $\mathbf{v}$. The line equation of $\mathbf{v}$ can be expressed by $bx - ay = 0$, and the set of lines parallel to $\mathbf{v}$ has the general form of $bx - ay = C$, where $C \in \mathbb{R}$. They are called the *characteristic lines*, and on each of those lines $D_{\mathbf{v}}u(x, y)$ is a constant and the entire plane $\mathbb{R}^2$ is generated by the set of all characteristic lines. So we see that in this case $u(x, y)$ only depends on $bx - ay$, i.e which characteristic line it belongs to, and hence the general solution to $au_x + bu_y = 0$ is

$$\boxed{u(x,y) = f(bx - ay)} \tag{1.6}$$

where f is a function of one variable, and the exact f may be obtained once extra conditions are given. To verify the correctness of (1.6), we two partial derivatives:

$$u_x = bf'(bx - ay), u_y = -af'(bx - ay) \tag{1.7}$$

and hence $au_x + bu_y = abf'(bx - ay) - abf'(bx - ay) = 0$.

Example: Solve the PDE $4u_x - 3u_y = 0$ with auxiliary condition $u(0,y) = y^3$.

Solution: It is first easy to see that $u(x,y) = f(-3x-4y)$ is the general solution, where $-3x-4y = C$ is the characteristic line. Then plugging in the auxiliary condition, $y^3 = u(0,y) = f(-4y)$ we get $f(-4y) = y^3$ and by a change of variable we see that $f(t) = -t^3/64$. So we have $u(x,y) = f(-3x-4y) = (-3x-4y)^3/64$.

1.1.2 Change of Variable Method

Now we present a different method using change of variables. Imagine using the direction of $\mathbf{v} = (a,b)$ as our new coordinates x' and $\mathbf{v}^\perp$ as y'. Then we can verify that

$$x' = ax + by \qquad y' = bx - ay \tag{1.8}$$

and hence we have the relation that

$$x = \frac{ax' + by'}{a^2+b^2} \qquad y = \frac{bx' - ay'}{a^2+b^2} \tag{1.9}$$

and by Chain rule, we get

$$u_x = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \tag{1.10}$$

$$u_y = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \tag{1.11}$$

so we have

$$au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2+b^2)u_{x'} \tag{1.12}$$

and we obtain a new equation $(a^2+b^2)u_{x'} = 0$, which indicates the solution $u(x,y)$ is independent of x' and hence we have $u(x',y') = f(y') = f(bx-ay)$. Note that this is the exact same answer we get using the directional derivative method.

Then, we are interested in a more general form

$$\boxed{au_x + bu_y = f(x,y)} \tag{1.13}$$

for a given function $f(x,y)$. If f reduces to zero then it is the case in section 1.1.1.

Theorem

Theorem 2. *The solution to (1.13) is given by*

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f ds + g(bx-ay) \tag{1.14}$$

where L is the characteristic line segment from y axis to (x,y).

Proof. We will again use the change of variable method, using the change of variables discussed in (1.8), and the relations stated in (1.9);(1.10);(1.11), we have

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = f\left(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}\right) \tag{1.15}$$

which simplifies to

$$u_{x'} = \frac{1}{a^2+b^2} \cdot f\left(\frac{ax'+by'}{a^2+b^2}, \frac{bx'-ay'}{a^2+b^2}\right) \tag{1.16}$$

hence integrate 1.16 with respect to x' gives

$$u(x',y') = \int_{x'} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar+by'}{a^2+b^2}, \frac{br-ay'}{a^2+b^2}\right) dr + g(y') \tag{1.17}$$

and we now use the original coordinates and we have

$$u(x,y) = \int_{ax+by} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar+b(bx-ay)}{a^2+b^2}, \frac{br-a(bx-ay)}{a^2+b^2}\right) dr + g(bx-ay) \tag{1.18}$$

by a simple change of variable $s = \dfrac{r}{\sqrt{a^2+b^2}}$, we have

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_{\frac{ax+by}{\sqrt{a^2+b^2}}} f\left(\frac{b^2x+a(\sqrt{a^2+b^2}s-by)}{a^2+b^2}, \frac{a^2y+b(\sqrt{a^2+b^2}s-ax)}{a^2+b^2}\right) ds + g(bx-ay) \tag{1.19}$$

and denoted by

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f ds + g(bx-ay) \tag{1.20}$$

and as we can see L is the line segment from y axis to the point (x,y).

■

Example: Solve the PDE $u_x + u_y = 1$.

Solution: We follow the result of theorem 2, we have $f(x,y) = 1, a = b = 1$, then we plug these values into (1.20), we have

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} ds + g(x-y) \\ &= \frac{x+y}{2} + g(x-y) \end{aligned}$$

and we get

$$u_x = \frac{1}{2} + g'(x-y) \qquad u_y = \frac{1}{2} - g'(x-y) \tag{1.21}$$

where $g(x-y)$ is an arbitrary single variable function that depends through $x-y$, and notice that $u_x + u_y = 1$ which means the solution is valid.

Example: Solve the PDE $u_x + u_y = xy$.

Solution: In this example, we have $f(x,y) = xy, a = b = 1$ and (1.20) gives us

$$u(x,y) = \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \left(\frac{x+\sqrt{2}s-y}{2} \right) \cdot \left(\frac{y+\sqrt{2}s-x}{2} \right) ds + g(x-y) \tag{1.22}$$

$$= \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \frac{2s^2-(x-y)^2}{4} ds + g(x,y) \tag{1.23}$$

$$= \left[\frac{2s^3-3s(x-y)^2}{12} \right] \Bigg|_{s=\frac{x+y}{\sqrt{2}}} + g(x-y) \tag{1.24}$$

$$= -\frac{(x+y)(x^2+y^2-4xy)}{12} + g(x-y) \tag{1.25}$$

hence we get the general solution to the original PDE:

$$u(x,y) = -\frac{(x+y)(x^2+y^2-4xy)}{12} + g(x-y) \tag{1.26}$$

where $g(x-y)$ is an arbitrary single valued function which depends through $x-y$. To verify, note that we have

$$\begin{aligned} u_x &= -\frac{1}{12}\Big(x^2+y^2-4xy+(x+y)(2x-4y)\Big) + g'(x-y) \\ &= \frac{1}{4}\Big(-x^2+y^2+2xy\Big) + g'(x-y) \end{aligned} \tag{1.27}$$

and similarly

$$\begin{aligned} u_y &= -\frac{1}{12}\Big(x^2+y^2-4xy+(x+y)(2y-4x)\Big) - g'(x-y) \\ &= \frac{1}{4}\Big(x^2-y^2+2xy\Big) - g'(x-y) \end{aligned} \tag{1.28}$$

where

$$\begin{aligned} u_x+u_y &= \frac{1}{4}\Big(-x^2+y^2+2xy+x^2-y^2+2xy\Big) + g'(x-y) - g'(x-y) \\ &= xy. \end{aligned} \tag{1.29}$$

1.2 Solving General Linear PDEs Using Characteristic Curves

In this part, we will consider a more general first order linear PDE which takes the form

$$\boxed{a(x,y)u_x + b(x,y)u_y = f(x,y)u + f_1(x,y)} \tag{1.30}$$

where a, b, f, f_1 are all given functions of (x,y). We will again use the idea of characteristic lines introduced in the previous section, but we will generalize them into "characteristic curves". To see this, first consider $f = f_1 = 0$, and we aim to solve

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{1.31}$$

which is, the directional derivative along $\mathbf{v} = (b(x,y), a(x,y))$ is zero, meaning the function is constant along $\mathbf{v}$. We notice that the slope of $\mathbf{v}$ satisfies

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \tag{1.32}$$

if (1.32) is separable, then we can solve the ODE to get the relation between x, y:

$$\int a'(y)dy = \int b'(x)dx + C \tag{1.33}$$

which we will write in the form of $y = h(x,C)$, where C is arbitrary constant, and the characteristic curve is given by $C = h'(x,y)$ hence we obtain the general solution

$$u(x,y) = f(h'(x,y)). \tag{1.34}$$

Example: Solve the PDE $u_x + 2xy^2 u_y = 0$.

Solution: First we see that along $\mathbf{v} = (1, 2xy^2)$ the directional derivative is zero, and the slope is given by

$$\frac{dy}{dx} = 2xy^2, \tag{1.35}$$

which is separable, and we have

$$\int \frac{1}{y^2} dy = \int 2x dx + C \tag{1.36}$$

which gives us

$$-\frac{1}{y} = x^2 + C \tag{1.37}$$

and the characteristic curves are defined by $\left\{ (x,y) \middle| -x^2 - \frac{1}{y} = C, C \in \mathbb{R}, y \neq 0 \right\}$, and hence the general solution takes the form

$$u(x,y) = f\left(-x^2 - \frac{1}{y}\right) \tag{1.38}$$

for any function f defined through $-x^2 - \dfrac{1}{y}$.

References

[1] CHOKSI Rustum. *Partial Differential Equations, a first course*. McGill University.

[2] STEWART James. *James Stewart Calculus, early transcendentals. Eighth Edition*. McMaster University and University of Toronto.

[3] STRAUSS R. Walter. *Partial Differential Equations, an introduction*. Brown University.