PDE HW 1

1.1

Q1 : (i) $Lu = u_x + x u_y$

Let u, v be functions , $c \in \mathbb{R}$

$$\begin{aligned} L(u+cv) &= (u+cv)_x + x(u+cv)_y \\ &= u_x + c(v_x) + x u_y + c x (v_y). \\ &= u_x + x u_y + c(v_x + x v_y) \\ &= L(u) + c L(v) \end{aligned}$$

thus (i) is linear.

(ii) $Lu = u_x + u \cdot u_y$

Let u, v be functions, $c \in \mathbb{R}$

$$\begin{aligned} L(u+cv) &= (u+cv)_x + (u+cv)\cdot(u+cv)_y \\ &= u + c(v)_x + (u+cv)(u_y + c(v_y)) \\ &= u + c v_x + u \cdot u_y + c u \cdot v_y + c v \cdot u_y + c^2 v \cdot v_y \end{aligned}$$

$L(u) + cL(v) = u_x + u \cdot u_y + c v_x + c v \cdot v_y$

which is not equal to $L(u+cv)$, so (ii) is not linear.

Q2

(i) $u_t - u_{xx} + 1 = 0$: Second order, linear, inhomogeneous.

(ii) $u_t - u_{xt} + u \cdot u_x = 0$: Second order, Nonlinear, homogeneous.

Proof.

Q3 Suppose $Lu_1 = g$ and $Lu_2 = g$, then by linearity $Lu_1 - Lu_2 = L(u_1 - u_2) = g - g = 0$

$$\Rightarrow \quad L(u_1 - u_2) = 0$$

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Q4 In $u_n(x,y) = \sin(nx) \cdot \sinh(ny)$

By direct computation,

$$\cancel{(u_n)_x = n\cos(nx)\cdot\sinh(ny) + n\sin(nx)\cdot\cosh(ny)}$$

$$(u_n)_x = n\sinh(ny)\cdot\cos(nx)$$

$$\boxed{(u_n)_{xx} = -n^2\sinh(ny)\cdot\sin(nx)} \qquad (*)$$

$$(u_n)_y = n\sin(nx)\cdot\cosh(ny)$$

$$\boxed{(u_n)_{yy} = n^2\sin(nx)\cdot\sinh(ny)} \qquad (**)$$

By $(*)$, $(**)$, we see that $(u_n)_{xx} + (u_n)_{yy} = 0$. and hence is a solution.

Q5 In $u(x,y) = x^2 + g(y)$

$$u_x = 2x \qquad u_{xx} = 2$$

$$u_y = g'(y) \qquad u_{yy} = g''(y).$$

By $u_{xx} + u_{yy} = 0 \quad \Rightarrow \quad \boxed{g''(y) + 2 = 0} \qquad (***)$

turns into an ODE $\dfrac{d^2y}{dx^2} = -2$

$$\rightarrow \quad \frac{dy}{dx} = -2x + C_1 \qquad y = -x^2 + C_1 x + C_2$$

$$\Rightarrow \quad g(y) = -x^2 + C_1 x + C_2, \qquad C_1, C_2 \in \mathbb{R}.$$

PDE HW1.

1.2

Q1. (a) $y \cdot u_x + x \cdot u_y = 0$

Along $\vec{v} = (y, x)$ the directional derivative is 0, and u is constant. The slope is

$$\frac{dy}{dx} = \frac{x}{y}$$

→ which is separable and we have $\int y\, dy = \int x\, dx + C$

⇒ $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, and $\frac{1}{2}y^2 - \frac{1}{2}x^2 = C$ is the set of characteristic curves, $C \in \mathbb{R}$, and hence

$$u(x,y) = g\left(\tfrac{1}{2}y^2 - \tfrac{1}{2}x^2\right).$$

(b) $u_x + u_y + u_z = 0$

Along $\vec{v} = (1,1,1)$ the directional derivative is 0 and u is constant. So we have ~~deriva~~ slope of directional derivative

$$\begin{cases} \dot{x} = \frac{dx}{dz} = 1 \\ \dot{y} = \frac{dy}{dx} = 1 \\ \dot{z} = \frac{dz}{dy} = 1 \end{cases} \Rightarrow \begin{cases} y - x = C_1 \\ z - y = C_2 \\ x - z = C_3 \end{cases}$$

which can be compressed into 2 eqns: $\begin{cases} y - x = C_1 \\ z - y = C_2 \end{cases}$

So the function along those 2 characteristic lines are constant, and we have the general solution

$$u(x,y,z) = g(y-x,\ z-y).$$

(c). $\cancel{CE}(\sqrt{1-x^2})u_x + u_y = 0 \qquad u(0,y) = y$

Along ~~On~~ $\vec{v} = \cancel{\xi}(\sqrt{1-x^2}, 1)$, the function u is constant, the slope of the directional derivative is

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

and this ODE is separable where we have

$$\int 1\, dy = \int \frac{1}{\sqrt{1-x^2}}\, dx + C$$

$$\Rightarrow \quad y = \arcsin(x) + C.$$

$\Rightarrow \quad y - \arcsin(x) = C$ is the characteristic curves for $C \in \mathbb{R}$ and u has a general solution

$$u(x,y) = g(y - \arcsin x).$$

with auxiliary condition

$$y = g(y)$$

and we hence have

$$u(x,y) = y - \arcsin x.$$

as the solution to the PDE.