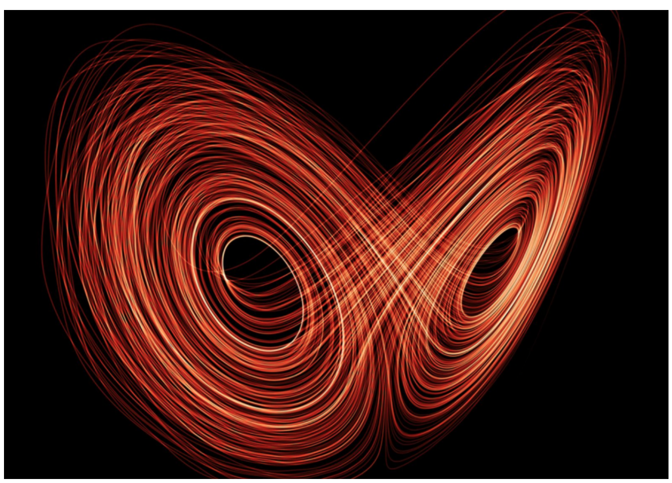
Ordinary Differential Equations

Winter 2025, Math 325 Course Notes

McGill University



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Despite all efforts, there may still be some typos, unclear explanations, etc. If you find potential mistakes, or any suggestions regarding concepts or formats, etc., feel free to reach out to the author at *zhangjohnson729@gmail.com*.

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Chapter 1

Classification and Existence & Uniqueness of ODEs

Definition and Examples of ODEs

Definition

Definition 1. *A differential equation is a relation involving an unknown function and some of its derivatives.*

Classification of Differential Equations

Definition

Definition 2. *An Ordinary Differential Equation (ODE) is a differential equation whose unknown function depends on one variable only. On the other hand, a Partial Differential Equation (PDE) has an unknown function that depends on more than one variable.*

Example: Consider the following differential equations:

• $my''(t) + \gamma y'(t) + mg = 0$ where γ, m, g are constants is an ODE, since the unknown function y depends on t only;

• $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial^2 x}$ (Heat Equation) is a PDE, since the unknown function u depends on two variables t and x.

Definition

Definition 3. *The order of an ODE is the order of the highest derivative appearing in the equation.*

Example: Consider the following differential equations:

• $y''(t) + \sin(t)y'''(t) = \cos(t)$ is a 3rd order ODE, since the order of the highest derivative is 3.

In general, an nth order ODE can be written as

$$F(t, y, y', y'', \cdots, y^{(n)}) = 0 \qquad (*)$$

where $F : \mathbb{R}^{n+2} \to \mathbb{R}$ is a function.

Now we would like to introduce systems of ODEs. Note that if $(*)$ takes the form

$$y^{(n)} = G(t, y, y', y'', \cdots, y^{(n-1)}),$$

then we can define intermediate variables as follows:

$$y_1 = y, y_2 = y', y_3 = y'', \cdots, y_n = y^{(n-1)}.$$

Then we have

$$
\begin{aligned}
y_1' &= y' = y_2 \\
y_2' &= y'' = y_3 \\
&\vdots \\
y_{n-1}' &= y_n \\
y_n' &= y^{(n)} = G(t, y, y', y'', \cdots, y^{(n-1)}).
\end{aligned}
$$

Define $\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \in \mathbb{R}^n$, then we have

$$
\boldsymbol{Y}'(t) = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ g \end{pmatrix}
$$

This leads to a system of ODEs of the form

$$
\boldsymbol{Y}'(t) = F(y(t), t) \qquad F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n
$$

Here F could be general.

Example: If we consider the ODE $y'' + 3y' + 4y = 0$, then let $y_1 = y, y_2 = y'$, we have

$$
\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -4y_1 - 3y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
$$

which is a system of 1st order linear ODEs.

Definition

Definition 4. *We say that the ODE $F(t, y, y', y'', \cdots, y^{(n)})$ is linear if the map*

$$
F : \mathbb{R}^{n+2} \to \mathbb{R}
$$

is a linear polynomial in the variables $y, y', \cdots, y^{(n)}$. In this case the ODE can be written as

$$
a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t) \qquad (**)
$$

where $a_i(t)$ are functions of t. And we say that the ODE is non-linear if it is not of the form $(**)$.

Examples : Consider the following ODEs:

- $y''(t) + 7ty(t) = \cos(t)$ is a linear ODE, and this is a 2nd order scalar linear ODE.

- $(y'(t))^2 + \cos(t)y = 3$ is not a linear ODE because of the term $(y'(t))^2$.

Definition

Definition 5. *If a first order ODE can be written as*

$$y'(t) = F(y(t)),$$

we say that the ODE is autonomous, otherwise it is non-autonomous.

Example : Consider the following ODEs:

- $y'(t) + \cos(t)y(t) = \sin(t)$ is non-autonomous.

- $my'' + \gamma y' + mg = 0$ is autonomous, since we have a linear system

$$\begin{pmatrix} y_1' \\ \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ \\ -\dfrac{\gamma}{m} y_2 - g \end{pmatrix}$$

which has the form

$$F(y(t), t) = \boldsymbol{A}(t)y(t) + \boldsymbol{b}(t).$$

- $\dfrac{dN}{dt} = RN$ where R is a constant is also autonomous.

Definition

Definition 6. *A solution to the ODE $y'(t) = F(y(t), t)$ on an interval $J \subseteq \mathbb{R}$ is a differentiable function $y : J \to \mathbb{R}^n$ that satisfies the ODE $y'(t) = F(y(t), t)$ for all $t \in J$, here t is the independent variable and y is the dependent variable.*

Example : $y(t) = e^{-t} + 1$ is a solution to the ODE $y' + y = 1$, in this case $y : \mathbb{R} \to \mathbb{R}$ and $J = \mathbb{R}$ is the maximum interval of existence.

However, if $y(t) = \dfrac{1}{t-1}$ is a solution to some ODE, then the maximum interval of existence is not $\mathbb{R}$ since at $t = 1$ the function is not differentiable. But we can pick $J = [2, +\infty)$ (is not a maximum interval of existence).

Existence and Uniqueness Theorem

Definition

Definition 7. *Suppose we have an ODE $y'(t) = F(y(t), t)$ and $y(t_0) = y_0 \in \mathbb{R}^n$ is given, we call the system*

$$\begin{cases} y'(t) = F(y(t), t) \\ y(t_0) = y_0 \end{cases}$$

an initial value problem (IVP).

Before introducing the existence and uniqueness theorem, we need some analysis background.

Definition

Definition 8. *Let $D \subseteq \mathbb{R}^n$ under any norm $\|\cdot\|$ on $\mathbb{R}^n$, a function $f : D \to \mathbb{R}^n$ is Lipschitz continuous (LC) if $\exists L > 0$, such that $\forall x, y \in D$,*

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad (*)$$

The smallest L which satisfies condition $(*)$ is called the Lipschitz constant.

Example : Consider the following functions:

- $f(x) = 10x - 3$ is LC, since

$$\|f(x) - f(y)\| = 10\|x - y\|,$$

and the Lipschitz constant is 10.

- $f(y) = \dfrac{1}{y-1}$ is not LC, but it is LLC on $(1, +\infty)$. See the definition below for LLC functions.

Definition

Definition 9. *Consider $D \subseteq \mathbb{R}^n$ to be an open set, a function $f : D \to \mathbb{R}^n$ is locally Lipschitz continuous (LLC) if for every compact set $K \subseteq D$, the function $f : K \to \mathbb{R}^n$ is Lipschitz continuous.*

Proposition

Proposition 1. *If a function $f : D \to \mathbb{R}^n$ is continuous differentiable and D is open, then f is LLC.*

Proof. By Mean-Value Theorem (MVT). ■

Now we would like to introduce the existence and uniqueness theorem:

Theorem

Theorem 1. *(Existence and Uniqueness Theorem)*

Consider $D \subseteq \mathbb{R}^n$ open and an open interval $I = (a, b) \subseteq \mathbb{R}$. Assume as well that $f : D \times (a, b) \to \mathbb{R}^n$ is continuous and that for all compact $K \subseteq D \times (a, b) \subseteq \mathbb{R}^{d+1}$, $\exists L(K)$ such that $\forall x, y \in K$

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\| \qquad (**)$$

Then $\exists$ open interval $J \subseteq \mathbb{R}$, $t_0 \in J$, over which the solution to the IVP

$$\begin{cases} y'(t) = f(y(t), t) \\ y(t_0) = y_0 \end{cases}$$

is defined. Furthermore any two solutions to that IVP agrees on the domain of their intersection.

To prove the theorem, we need to establish some lemmas and intermediate steps to help us.

Lemma

Lemma 1. *y solves the IVP*

$$\begin{cases} y' = f(y, t) \\ y(t_0) = y_0 \end{cases}$$

if and only if

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds \qquad (***)$$

Proof. $(\Longrightarrow)$: Assume y solves the IVP, then

$$\int_{t_0}^{t} y'(t)dt = \int_{t_0}^{t} f(y(s), s)ds,$$

which implies $y(t) - y(t_0) = y(t) - y_0$.

($\Longleftarrow$) Assume that the equation $(***)$ holds, then note that

$$y(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s), s)ds = y_0$$

which solves the second equation of the IVP. Also

$$\begin{aligned} y'(t) &= \frac{d}{dt} y_0 + \frac{d}{dt} \int_{t_0}^{t} f(y(s), s)ds \\ &= 0 + f(y(t), t)). \end{aligned}$$

Which solves the IVP.

■

So now our goal is to show that $(***)$ has a solution. Since $(y_0, t_0) \in D \times (a, b)$ open, then $\exists \alpha > 0, \delta > 0$, such that the compact cylinder

$$D_{\alpha,\delta} := \{(y, t) \in \mathbb{R}^{n+1} \Big| \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subseteq D \times (a, b).$$

Let

$$M_{\alpha,\delta} := \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y, t)\| < +\infty,$$

we then define the Picard operator:

Lemma

Lemma 2. *Let $\varepsilon > 0$ be defined by*

$$\varepsilon := \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right),$$

then $\varepsilon \leq \delta, D_{\alpha,\varepsilon} \subseteq D_{\alpha,\delta}$, and let $J = (t_0 - \varepsilon, t_0 + \varepsilon)$, then for any function $y(t)$ which satisfies $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha,\varepsilon}, \forall t \in J$, the function $T(y) : T \to \mathbb{R}^n$ defined by

$$T(y)(t) \equiv y_0 + \int_{t_0}^{t} f(y(s), s)ds$$

is called the Picard operator, and it satisfies the followings:

$$(1) \quad T(y_0)(t_0) = y_0;$$

$$(2) \quad (T(y)(t), t) \in D_{\alpha,\varepsilon}, \forall t \in J.$$

Before proving this lemma, below is a graph to illustrate $D_{\alpha,\delta}$ and $D_{\alpha,\varepsilon}$.

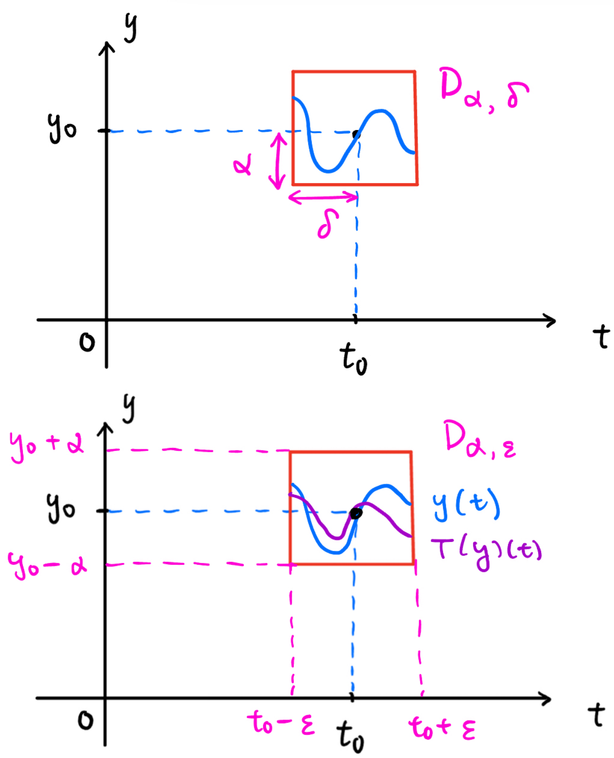


Figure 1.1: $D_{\alpha,\delta}$ and $D_{\alpha,\varepsilon}$

Proof. First we show that $T(y)(t_0) = y_0$. By definition,

$$T(y)(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s), s)dx = y_0,$$

which satisfies the equation.

Then we will show that

$$\|T(y)(t) - y_0\| \leq \alpha, |t - t_0| \leq \delta,$$

where the latter one is trivial since $\varepsilon \leq \delta$. Now

$$\begin{aligned}
\|T(y)(t) - y_0\| &= \left\| y_0 + \int_{t_0}^{t} f(y(s), s)ds - y_0 \right\| \\
&= \left\| \int_{t_0}^{t} f(y(s), s)ds \right\| \\
&\leq \left| \int_{t_0}^{t} \|f(y(s), s)\| ds \right| \\
&\leq M_{\alpha,\delta} \cdot \left| \int_{t_0}^{t} ds \right| \\
&\leq M_{\alpha,\delta} \cdot \varepsilon \\
&\leq \alpha.
\end{aligned}$$

■

The proof of existence relies on the process of Picard operators. We start by defining a sequence of functions $\{y_k\}$ with $y_0(t) = y_0 \in \mathbb{R}^n$ to be a constant function, and we define

$$y_k(t) \equiv T(y_{k-1})(t) = y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)dx, k \geq 1.$$

Lemma

Lemma 3. $\forall t \in J = (t_0 - \varepsilon, t_0 + \varepsilon)$, $\{y_k(t)\}$ *is Cauchy, and furthermore* $y(t) = \lim_{k\to\infty} y_k(t)$ *exists, and solves the IVP.*

Proof. We first have

$$\begin{aligned} \|y_1(t) - y_0(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(y_0(s), s)ds - y_0 \right\| \\ &= \left\| \int_{t_0}^{t} f(y_0(s), s)ds \right\| \end{aligned}$$

W.L.O.G, we assume that $t \in [0, t_0 + \varepsilon)$, hence

$$\begin{aligned} &\leq \int_{t_0}^{t} \|f(y_0(s), s)\| ds \\ &\leq M_{\alpha,\delta}(t - t_0) \end{aligned}$$

We further claim that

$$\|y_m(t) - y_{m-1}(t)\| \leq M_{\alpha,\delta} \cdot L^{m-1} \frac{(t-t_0)^m}{m!}.$$

We will prove it using induction. Assume it holds for all $N \leq m$, and we prove it on $m+1$. We have

$$\begin{aligned} \|y_{m+1}(t) - y_m(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(y_m(s), s)dx - y_0 - \int_{t_0}^{t} f(y_{m-1}(s), s)ds \right\| \\ &\leq \int_{t_0}^{t} \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds \\ (\textit{By LLC }) &\leq \int_{t_0}^{t} L\|y_m(s) - y_{m-1}(s)\| ds \\ (\textit{By induction hypothesis }) &\leq \int_{t_0}^{t} L \cdot M_{\alpha,\delta} \cdot L^{m-1} \frac{(s-t_0)^m}{m!} ds \\ &= \frac{L^m \cdot M_{\alpha,\delta}}{(m+1)!}(t-t_0)^{m+1}, \end{aligned}$$

which finishes the induction. Then $\forall l > 1$, we have

$$\begin{aligned} \|y_l(t) - y_{l-1}(t)\| &\leq M_{\alpha,\delta} \cdot L^{t-1} \frac{(t-t_0)^l}{l!} \\ &\leq \frac{M_{\alpha,\delta}}{L} \frac{(L\varepsilon)^l}{l!}. \end{aligned}$$

Now let $p, m \geq 1$, we rewrite

$$\|y_{m+p}(t) - y_{m+1}(t)\| = \|y_{m+p}(t) - y_{m+p-1}(t) + y_{m+p-1}(t) + \cdots + y_{m+2}(t) - y_{m+1}(t)\|$$

Hence

$$\begin{aligned}
L.H.S &\leq \sum_{k=1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\| \\
&\leq \sum_{k=1}^{p-1} \frac{M_{\alpha,\delta}}{L} \cdot \frac{(L\varepsilon)^{m+k+1}}{(m+k+1)!} \\
&= \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{p+m} \frac{(L\varepsilon)^j}{j!} \overset{m,p\to\infty}{\to} 0.
\end{aligned}$$

Thus $\{y_k(t)\}$ is Cauchy. Since $\mathbb{R}^n$ is complete, i.e every Cauchy sequence converges, so $\lim_{k\to\infty} y_k(t)$ exists, and we denote $y(t) = \lim_{k\to\infty} y_k(t)$. We will show that $y(t)$ solves the IVP.

Take $p \to \infty$, we know that

$$\sup_{t\in J} \|y_{m+p}(t) - y_{m+1}(t)\| \overset{p\to\infty}{\to} \|y(t) - y_{m+1}(t)\| \leq \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{\infty} \frac{(L\varepsilon)^j}{j!} \overset{m\to\infty}{\to} 0.$$

Which implies that $y_k(t) \to y(t)$ uniformly, and since $y_k(t)$ is continuous, so $y(t)$ is continuous. Then

$$\lim_{k\to\infty} y_k(t) = \lim_{k\to\infty} y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)ds$$

and by uniform convergence,

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds,$$

which solves the IVP. ■

Now we will prove the solution is unique, under all constructions above.

Proof. Suppose $y(t), z(t)$ both solves the IVP, and $W.L.O.G$, let $t \in (t_0, t_0 + \varepsilon)$. We know that

$$\begin{aligned}
\|y(t) - z(t)\| &\leq \int_{t_0}^{t} \|f(y(s), s)ds - f(z(s), s)ds\| \\
&\leq L \int_{t_0}^{t} \|y(s) - z(s)\| ds.
\end{aligned}$$

Let

$$g(t) \equiv \int_{t_0}^{t} \|y(s) - z(s)\| ds,$$

where g is non-negative and $g'(t) \leq Lg(t)$. We multiply both sizes of the inequality by an integrating factor $e^{-L(t-t_0)}$, then

$$L.H.S = \frac{d}{dt}\left(e^{-L(t-t_0)} g(t)\right) = e^{-L(t-t_0)}(g(t) - Lg(t)) \leq 0.$$

The function $t \mapsto e^{-L(t-t_0)}$ is decreasing on the chosen interval $(t_0, t_0 + \varepsilon)$, so

$$0 \leq e^{-L(t-t_0)} g(t) \leq g(t_0) = 0,$$

and g is non-negative, thus $g \equiv 0$, i.e $g'(t) = \|y(t) - z(t)\| = 0$. So we conclude that

$$y(t) = z(t),$$

and thus the solution is unique. Hence we have proved the existence and uniqueness theorem. ■

Example: Consider the IVP given by

$$\begin{cases} y' = y + 1 \\ \\ y(0) = 1 \end{cases},$$

this ODE is continuous, and f does not depend on $t, n = 1$, where $y' = y + 1 = f(y)$ which is LLC, then the theorem of existence and uniqueness applies. We take

$$\varepsilon = \min\left\{\delta, \frac{\alpha}{M_{\alpha,\delta}}\right\}, D_{\alpha,\delta} = \left\{(y,t) \middle| \|y - 1\| \leq \alpha, \|t - 0\| \leq \delta\right\} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta].$$

Since $f : \mathbb{R} \to \mathbb{R}$, so for this example α, β can be whatever as we want, and thus

$$M = \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y,t)\| = 2 + \alpha,$$

so

$$\varepsilon = \min\left\{\delta, \frac{\alpha}{2 + \alpha}\right\}.$$

For instance, if we take $\delta = 1, \alpha = 7$, we will get

$$J = \left(-\frac{7}{9}, \frac{7}{9}\right).$$

Hence by the theorem of existence and uniqueness, there is a solution that solves the IVP which is defined on J. Of course the choice of α, β varies in this example.

Definition

Definition 10. *Let $\varphi : I \to D, \psi : J \to D$ be solutions to $y' = f(y,t)$ where $f : D \times [a,b] \to \mathbb{R}^n$ is satisfying the hypothesis of the theorem of existence and uniqueness. We say that ψ is an extension of φ, if $I \subseteq J$ and $\varphi(t) = J(t), \forall t \in I$. If $I \subsetneq J$, then we say that ψ is a proper extension of φ. A solution is called a maximal solution if it has no proper extension. In this case J is called the maximal interval of existence and it's denoted by $J_{\max}$.*

Chapter 2

First Order Scalar Equations

First Order Linear Equations

In this section we consider the ODE of the form

$$a_0(t)y' + a_1(t)y = g(t)$$

where a_0, a_1, g are all continuous functions in $\mathbb{R}$. If $a_0(t) \neq 0$ then we have an ODE of a better form:

$$y'(t) + p(t)y = q(t)$$

and this is the type of equation we will be solving.

Theorem

Theorem 2. *If $p, q : (a, b) \to \mathbb{R}$ are continuous, $t_0 \in (a, b)$, then the unique solution to the IVP*

$$\begin{cases} y' = f(y, t) \\ \\ y(t_0) = y_0 \end{cases}$$

is such that $J_{\max} = (a, b)$.

To solve this ODE, we will use the notation of integrating factors. We multiply the ODE by a non-zero function $\mu(t)$, chosen in such a way that we can solve the ODE given by

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)q(t), \mu(t) \neq 0.$$

Using the product rule, we chose $\mu(t)$ such that

$$\mu(t)y' + \mu(t)p(t)y = (\mu(t) \cdot y)'$$

thus
$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t),$$
then integrating both sides,
$$\mu(t)y(t) = \int \mu(t)q(t)dt + C,$$
Hence we have a solution
$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right].$$
To find $\mu(t)$, we have
$$\frac{\mu'(t)}{\mu(t)} = p(t),$$
by chain rule, we have
$$\frac{d}{dt}\ln|\mu(t)| = \frac{\mu'(t)}{\mu(t)} = p(t),$$
thus
$$\ln|\mu(t)| = \int p(t)dt$$
$$\mu(t) = e^{\int p(t)dt}.$$

Note that without loss of generosity, we may omit the constant $+C$ for $\int p(t)dt$, the reason is that the constant on the exponential will result in a e^C term, and since we finally divide the whole equation by $\mu(t)$, so the two constants will cancel out.

So for the ODE $y' + p(t)y = q(t)$, we have a general formula:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right]$$
where
$$\mu(t) = e^{\int p(t)dt}.$$

Example : Solve the ODE $y' - 2y = 3e^t$.

Here we have $p(t) = -2, q(t) = 3e^t$, so the integrating factor $\mu(t)$ is given by
$$\mu(t) = e^{\int -2dt} = e^{-2t},$$
and thus
$$\begin{aligned} y(t) &= \frac{1}{e^{-2t}}\left[\int e^{-2t}3e^t dt + C\right] \\ &= e^{2t}\left[3\int e^{-t}dt + C\right] \\ &= e^{2t}[-3e^{-t} + C] \\ &= -3e^t + Ce^{2t}, \end{aligned}$$

where $C \in \mathbb{R}$ is a constant.

Remark: In general, if the initial condition is given, we can then solve for C and get an exact solution.

Let's consider some applications of this type of ODE, and how do we solve them.

Example : (Tank Mixing Problem) A tank with capacity $120L$ contains originally $90L$ of brine water in which $90g$ of salt is dissolved. A brine of concentration $2g$ of salt per liter enters the tank at a constant rate of $4L$ per-minute. Assume that the salt water is well-mixed, and it also exists the tank at a rate of $3L$ per-minute. Then what is the quantity of salt in the tank when the tank is full?

Solution : We denote $y(t)$ to be the quantity of salt at time t (minute), then $y' = \dfrac{dy}{dt}$ would be the rate of change of the quantity of salt at time t, measured in minute. The rate of change can be described by

$$y' = R_{in} - R_{out},$$

where R_{in} is the rate of the quantity of salt that enters the tank, which is given by

$$R_{in} = 4L/min \times 2g/L = 8g/min.$$

For R_{out}, we first wish to find the quantity of salt at time t, which is given by $y(t)$ (measured in gram) as the assumption gives. Also at time t there will be $90 + (4L/min - 3L/min)t = 90 + t$ liters of water, thus we can also get the concentration of the salt at time t, given by $\dfrac{y(t)}{90 + (4-3)t}$. Hence

$$R_{out} = 3L/min \times \frac{y(t)}{90+t} g/L.$$

So finally we get an ODE with IVP of the form

$$\begin{cases} y'(t) + \dfrac{1}{90+t} y(t) = 8 \\ \\ y(0) = 90 \end{cases}.$$

So by solving this equation we get

$$y(t) = 180 + 2t - \frac{90^4}{(90+t)^3},$$

where $t \in [0, 30]$, hence $y(30) \approx 202$ grams of salt.

Example: (Newton's Law of Cooling) The temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100°C to 50°C in 160s. the same cooling occurs in 140s in another bath. Determine the initial temperature of the second bath. (Assume the water bath is large enough that its temperature is stationary.)

Solution: We assume $T(t)$ is the temperature of the object at time t, A be the temperature of the water bath, and K to be the cooling constant. Thus in the first bath we have $A_1 = 25$, and the IVP

$$\begin{cases} \dfrac{dT}{dt} = K(T - 25) \\ T(0) = 100 \\ T(160) = 50 \end{cases}$$

where the ODE is first order linear, and we obtain a general solution

$$T(t) = 25 + Ce^{Kt}.$$

Also we have $T(0) = 100$, thus $C = 75$, $T(160) = 50$, so the cooling constant $K = \dfrac{-\ln 3}{160}$.

Now, in the second bath, we assume the temperature of the water bath is A_2, and we have the following IVP:

$$\begin{cases} \dfrac{dT}{dt} = K(T - A_2) \\ T(0) = 100 \\ T(140) = 50 \end{cases}$$

Where the general solution is

$$T(t) = A_2 + Ce^{Kt}$$

where $K = \dfrac{-\ln 3}{160}$, and $T(0) = 100$, thus $C = 100 - A_2$, also $T(140) = 50$, so we get

$$50 = A_2 + (100 - A_2)e^{-140\ln(3)/160},$$

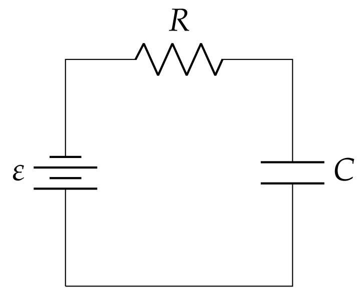
and we can solve for A_2, given by

$$A_2 = \frac{100 \times 3^{-7/8} - 50}{3^{-7/8} - 1}.$$

Example: (Circuits with a charging capacitor)

Consider the following circuit which has one battery with constant voltage ε and its resistance can be negligible; one resistor with resistance R, as well as a capacitor. At time $t = 0$ the circuit is turned on, and let $Q(t)$ denote the amount of charge on the capacitor, find $Q(t)$ at $t = t_0$ using the ODE

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{\varepsilon}{R}$$



This ODE is a first order scalar linear ODE, and the integrating factor is given by

$$\mu(t) = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}.$$

Separable Equations

Definition

Definition 11. *A first order scalar ODE of the form*

$$\frac{dy}{dx} = f(x, y)$$

is called separable, if we can write $f(x, y) = f_1(x) \cdot f_2(y)$.

Here are some examples:

① $\frac{dy}{dx} = e^{xy}$ is not separable;

② $\frac{dy}{dx} = e^{x+y}$ is separable;

③ $\frac{dy}{dx} = \frac{x^4}{\sin(y)}$ is not separable;

④ $\frac{dy}{dx} = x^2 + y^2$ is not separable.

We assume $f_2(y) \neq 0$, then we have the form

$$-f_1(x) + \frac{1}{f_2(y)} \cdot \frac{dy}{dx} = 0,$$

by making substitutions, we have

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (*)$$

We assume that $H_1(x), H_2(y)$ are anti-derivatives of M, N, then $H_1'(x) = M(x), H_2'(x) = N(x)$, so equation $(*)$ becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0 \qquad (**)$$

Think of y as a function of x, then by chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y(x)),$$

now equation $(**)$ becomes

$$\frac{d}{dx}\left[H_1(x) + H_2(y(x))\right] = 0,$$

meaning that $H_1(x) + H_2(y) = C$ where $C \in \mathbb{R}$.

So now we let $H_1(x) = \int M(x)dx, H_2(y) = \int N(y)dy = \int \frac{1}{f_2(y)}dy$, we now have a general solution of the form

$$\int \frac{dy}{f_2(y)}dy = \int f_1(x)dx + C$$

where $c \in \mathbb{R}$ is a constant.

Example 1: Solve the ODE

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}, \quad y \neq \pm 1.$$

We know that this ODE is separable, so its general solution is given by

$$\int (1-y^2)dy = \int x^2 dx + C,$$

thus

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C.$$

We may have noticed that this is not a "strict" function, y is implicitly given. But we can see that the graph is the level curve of the multi-variable function $f(x,y) = \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$, as the figure below shows.

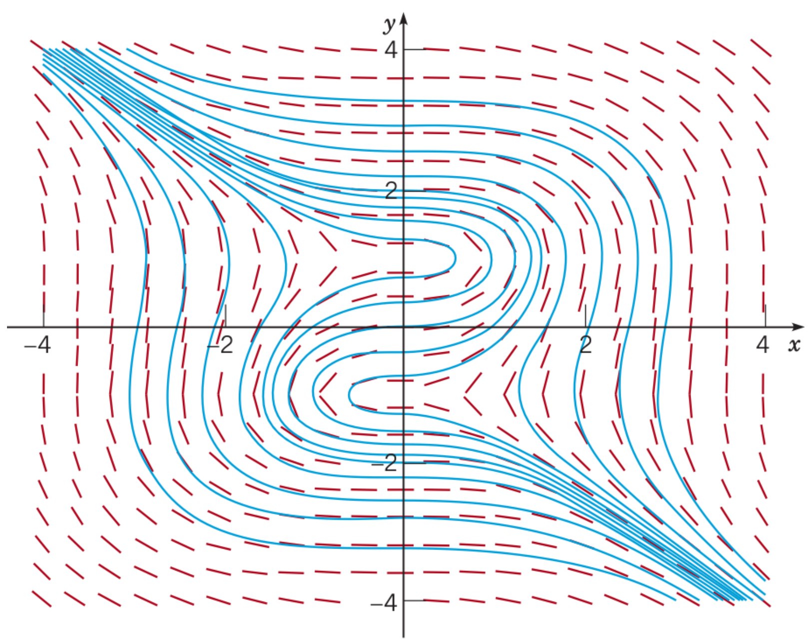


Figure 2.1: The graph of $y = \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$, each C represents a level curve of the function $f(x,y) = y - \frac{1}{3}y^3 - \frac{1}{3}x^3$ where $f(x,y) = C$.

The level curve of the can be defined by $\psi(x,y) = C$, and we denote

$$\Gamma_{x_0,y_0} = \{(x,y) \Big| \psi(x,y) = \psi(x_0,y_0) = C\}$$

to be the level curve of height $C = \psi(x_0, y_0)$ containing the point (x_0, y_0).

Example 2: Solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)} \\ \\ y(0) = -1 \end{cases}$$

This ODE is separable, so we have a general solution of the form

$$\int 2(y-1)dy = \int (3x^2+4x+2)dx + C,$$

where the general solution is given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

where C is a constant. To determine C, we substitute the condition that $x = 0, y = -1$ into the equation, and we get $C = 3$. Now in order to solve the equation explicitly, we need to solve y in terms of x, which in this case we have

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Since the initial condition states that $y(0) = -1$, so this will correspond to the equation with minus sign above, i.e

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

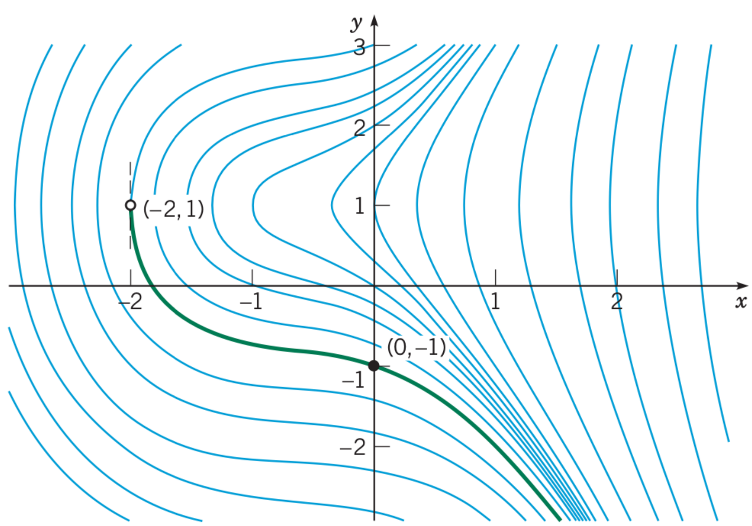


Figure 2.2: The solution satisfying $y(0) = -1$ is shown in green.

Example 3: Solve the ODE $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

This ODE is clearly separable, its general solution has the form

$$\int y dy = -\int x dx + C,$$

thus we have $y^2 = -x^2 + C'$.

The level curves of the potential $\psi(x,y)$, which are sometimes called the integral curve, are in this case given by circles centered at $(0,0)$, and $\Gamma(x_0,y_0) := \{(x,y) \big| \psi(x,y) = \psi(x_0,y_0)\}$ is the integral curve that contains the point (x_0,y_0). If the initial condition is given by $y(x_0) = y_0$, then the unique solution to the IVP is contained in $\Gamma(x_0,y_0)$.

Example 4: (Logistic Equation)

Suppose we have a population model with its population N given by

$$\frac{dN}{dt} = r\left(1 - \frac{N}{k}\right) \cdot N$$

with initial condition $N(0) = N_0$, where r is the growth rate initially for small population, K is the carrying capacity, N_0 is the initial population. This ODE is separable, so we have

$$\int \frac{1}{N(\frac{N}{k} - 1)} dN = -\int r dt + C,$$

so we have

$$\begin{aligned}
\int \frac{k}{N(N-k)} dn &= -rt + C \\
\implies \int \left(\frac{1}{N-k} - \frac{1}{N}\right) dN &= -rt + C \\
\implies \ln\left|\frac{N-k}{N}\right| &= -rt + C \\
\implies \frac{N-k}{N} &= C_0 e^{-rt}.
\end{aligned}$$

With initial condition $N(0) = N_0$, we may solve the ODE, its solution is given by

$$N(t) = \frac{KN_0}{N_0 + (k - N_0)e^{-rt}}$$

Now, for different initial condition N_0, the graph of $N(t)$ would also behave differently.

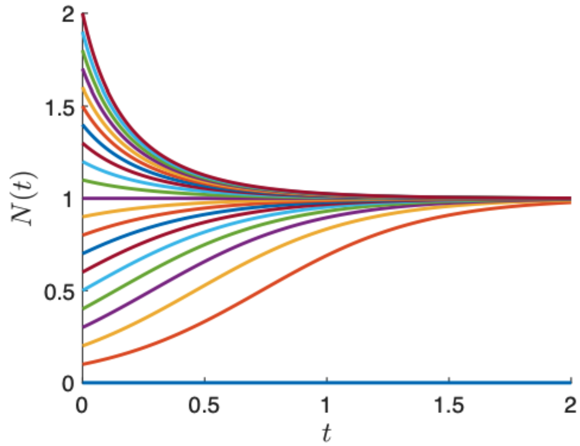


Figure 2.3: We fix $r = 3, K = 1$, and we have solutions of the ODE with initial value N_0.

Exact Equations

Recall that separable equations can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

In general, a first order scalar ODE can be written as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Definition

Definition 12. *Consider $D \subseteq \mathbb{R}^2$ open, we say that the ODE*

$$M(x,y) + N(x,y)\frac{dx}{dy} = 0$$

is exact in the domain D if there exists a potential function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y).$$

If the ODE is exact, then we have

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0,$$

which by Chain rule, we have

$$\frac{d}{dx}[\psi(x,y)] = 0.$$

Hence we have a general solution for an exact equation:

$$\boxed{\psi(x,y) = C}$$

Theorem

Theorem 3. *Let M and N, $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ be continuous functions of x, y on some simply connected open domain $D \subseteq \mathbb{R}^2$, then the ODE*

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact in D if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \forall (x,y) \in D.$$

Remark : Separable equations are exact.

Proof. We will prove for both directions:

$(\Longrightarrow)$ Assume $M(x,y)+N(x,y)\frac{dy}{dx}=0$ is exact, thus $\exists\psi(x,y)$ such that $M=\frac{\partial\psi}{\partial x}, N=\frac{\partial\psi}{\partial y}$, thus we have

$$\begin{cases}\dfrac{\partial M}{\partial y}=\dfrac{\partial}{\partial y}\cdot\dfrac{\partial\psi}{\partial x}=\dfrac{\partial^2\psi}{\partial y\partial x}\\[2ex] \dfrac{\partial N}{\partial x}=\dfrac{\partial}{\partial x}\cdot\dfrac{\partial\psi}{\partial y}=\dfrac{\partial^2\psi}{\partial x\partial y}\end{cases}\Longrightarrow\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

$(\Longleftarrow)$ Assume we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, and assume D is a rectangle $[x_0,x]\times[y_0,y]$ (general case requires a bit of knowledge from topology), what we aim at finding is a function $\psi(x,y)$ such that $M=\frac{\partial\psi}{\partial x}, N=\frac{\partial\psi}{\partial y}$. If we integrate M, we get

$$\psi(x,y)=\int_{x_0}^{x}M(s,y)ds+f(y) \qquad (*)$$

Now we differentiate $(*)$ with respect to y, we have

$$\begin{aligned}\frac{\partial\psi}{\partial y}&=\frac{\partial}{\partial y}\int_{x_)}^{x}M(s,y)dx+f'(y)\\ &=\int_{x_0}^{x}\frac{\partial M}{\partial y}(s,y)ds+f'(y) \qquad (**)\end{aligned}$$

our goal is to show that $(**)$ is equal to $N(x,y)$, i.e

$$f'(y)=N(x,y)-\int_{x_0}^{x}\frac{\partial M}{\partial y}(s,y)ds$$

Note that

$$\frac{\partial}{\partial x}\left[N(x,y)-\int_{x_0}^{x}\frac{\partial M}{\partial y}(s,y)ds\right]=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0,$$

which proved the theorem.

■

Example 1: Solve the ODE

$$y\cos(x)+2xe^y+(\sin(x)+x^2e^y-1)\frac{dy}{dx}=0$$

Solution: We first note that $M, N, M_y, N_x\in C^\infty(\mathbb{R})$, so we may proceed. Note that

$$M_y=\frac{\partial}{\partial y}[y\cos(x)+2xe^y]=\cos(x)+2xe^y; N_x=\frac{\partial}{\partial x}[\sin(x)+x^2e^y-1]=\cos(x)+2xe^y,$$

which means the ODE is exact. So there exists a potential function $\psi(x, y)$ such that $\psi_x = M, \psi_y = N$. Now we integrate M with respect to x, then we have

$$\begin{aligned} \psi(x, y) &= \int M(x, y)dx + h(y) \\ &= \int (y\cos(x) + 2xe^y)dx + h(y) \\ \implies \psi(x, y) &= y\sin(x) + x^2e^y + h(y). \end{aligned}$$

Now we differentiate $\psi(x, y)$ with respect to y, and hence

$$\frac{\partial}{\partial y}\psi(x, y) = N(x, y) = \sin(x) + x^2e^y + h'(y),$$

and compared to $N(x, y)$, we have

$$\sin(x) + x^2e^y - 1 = \sin(x) + x^2e^y + h'(y),$$

which means $h'(y) = 1 \implies h(y) = y$. The reason why we dropped the $+C$ constant is because it will finally be generalized in the level curve $\psi(x, y) = C$. Thus the potential function is given by $\psi(x, y) = y\sin(x) + x^2e^y - y$ and hence the general solution is given by the level curve $\psi(x, y) = C$, namely

$$y\sin(x) + x^2e^y - y = C, C \in \mathbb{R}.$$

In the ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0,$$

one question may arise: What if an ODE is not exact? That is, $M_y \neq N_x$. In this case are we still be able to find a general solution? The answer is somehow yes. We will use integrating factors, as we have seen in first order linear ODEs. We would like to construct a function $\mu(x, y)$ such that now the new ODE

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)\frac{dy}{dx} = 0$$

is exact. Then we may proceed with the similar technique, that is to find a potential function $\psi(x, y)$ with $\psi_x = \mu M, \psi_y = \mu N$. Let's investigate this case further. If our new PDE is exact, it means that

$$(\mu M)_y = (\mu N)_x.$$

By chain rule, we have

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x},$$

which is, we have a equation like this:

$$\boxed{\frac{\partial \mu}{\partial y}M - \frac{\partial \mu}{\partial x}N + \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0}$$

But there is one problem: This equation is a PDE, since the unknown μ depends on its two different partial derivatives, so we currently cannot develop a technique to solve them.

We may want to re-investigate our μ. Why not being picky? Let assume that μ only depends on one variable, then the equation above becomes an ODE!

• **We assume μ is a function of x only, i.e $\mu_y = 0$:**

Then in this case we have

$$\frac{\partial \mu}{\partial x} N = \mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

since $\frac{\partial \mu}{\partial y} = 0$. We may re-arrange the equation above to get

$$\frac{\frac{d\mu}{dx}}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Now, it looks like a separable equation, but not quite. In order for the ODE to become separable, we need to further assume that $\frac{M_y - N_x}{N}$ is a function of x only. Then we can solve this ODE, and we have a general solution of $\mu(x)$ given by

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

Note that again the constant $+C$ doesn't matter since we are multiplying the original ODE all by μ, so they will eventually cancel out with each other.

• **We assume that μ is a function of y only, i.e $\mu_x = 0$:**

Similarly, we have

$$\frac{\partial \mu}{\partial y} M = -\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu$$

which is

$$\frac{\frac{d\mu}{dy}}{\mu} = -\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

If $-\frac{M_y - N_x}{M}$ is a function of y only, we can solve this separable ODE, and we get

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M} dy}$$

Example 2: Solve the ODE

$$3xy + y^2 + (x^2 + xy)\frac{dy}{dx} = 0$$

First, it is easy to see that $M, N, M_y, N_x \in C^\infty(\mathbb{R})$, so we may proceed. Since $M_y = 3x + 2y \neq N_x = 2x + y$, so this ODE is not exact. We now aim to find an integrating factor. And fortunately we have

$$\frac{M_y - N_x}{N} = \frac{x+y}{x^2+xy} = \frac{1}{x}$$

is a function of x only, so such an integrating factor exists, and is given by the ODE

$$\frac{d\mu}{dx} = \mu\left(\frac{M_y - N_x}{N}\right) = \mu\frac{1}{x}$$

which is

$$\mu(x) = e^{\int \frac{1}{x}dx} = x.$$

So now we multiply the original ODE by the integrating factor $\mu(x) = \frac{1}{x}$, we get

$$(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0,$$

which is exact, since $M_y = 3x^2 + 2xy = N_x = 3x^2 + 2xy$. Let $\psi(x,y)$ to be the potential function, then $\psi_x = M, \psi_y = N$. Integrate ψ_x with respect to x, we have

$$\psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(y),$$

and differentiate with respect to y, we have

$$\frac{\partial}{\partial y}\psi(x,y) = x^3 + x^2y + h'(y),$$

and we have

$$x^3 + x^2y + h'(y) = x^3 + x^2y,$$

which means $h(y) \equiv 0$, and thus the potential function is given by

$$\psi(x,y) = x^3y + \frac{1}{2}x^2y^2,$$

and the general solution is of the form

$$x^3y + \frac{1}{2}x^2y^2 = C.$$

Chapter 3

Systems of Linear ODEs

Higher order Equations and Systems

We denote $M_n(\mathbb{R})$ to be the set of all $n \times n$ real matrix, and we would like to study the systems of linear ODEs like

$$y'(t) = A(t)y(t) + r(t)$$

Here,

$$y'(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, y : J \mapsto \mathbb{R}^n, n > 1$$

and for each $t \in J$, $a(t)$ is an $n \times n$ matrix, $r(t)$ is defined by

$$r(t) = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix} \in \mathbb{R}^n, n > 1$$

in the case when $r(t) = 0$, we say the system $y'(t) = A(t)y(t) + r(t)$ is homogeneous, if not we then say it is non-homogeneous.

Theory of Linear Systems

The existence and uniqueness theorem also applies here:

Theorem

Theorem 4. *Assume that each entry of the $n \times n$ matrix $A(t)$ and the n-dimensional vector $r(t)$ are continuous on some open interval $I \subseteq \mathbb{R}$, then if $t_0 \in I$, then the IVP*

$$\begin{cases} y'(t) = A(t)y(t) + r(t) \\ \\ y(t_0) = y_0 \in \mathbb{R}^n \end{cases}$$

has a unique solution which is defined on I. In order words, $J_{\max} = I$.

We begin the theory by studying the first order homogeneous equations, that is systems of the form

$$y'(t) = A(t)y(t)$$

Denote the solution set of $y'(t) = A(t)y(t)$ by

$$\mathcal{S} := \{y \in C^1(I) : \ y \text{ is a solution}\}$$

Lemma

Lemma 4. *$\mathcal{S}$ is an n-dimensional vector subspace of $C^1(I)$.*

Proof. Let $y_1, y_2 \in S$, take $\alpha_1, \alpha_2 \in \mathbb{R}$. We will show that $\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{S}$. Indeed, let $y := \alpha_1 y_1 + \alpha_2 y_2$, so

$$\begin{aligned} y' = \alpha_1 y_1' + \alpha_2 y_2' &= \alpha_1 A(t) y_1 + \alpha_2 A(t0 y_2 \\ &= A(t)[\alpha_1 y_1 + \alpha_2 y_2] \\ &= A(t)y \end{aligned}$$

Thus $y = \alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{S}$, and $\mathcal{S}$ is a vector subspace. This is also known as the principle of superposition.

Now let's construct n linearly independent solutions which span $\mathcal{S}$. Fix $t_0 \in I$, for each $i = 1, 2, \cdots, n$. Denote $\boldsymbol{e}_i$ to be the ith unit vector, and denoted by $y_i(t)$, the unique solution of the IVP $\begin{cases} y' = A(t)y \\ y(t_0) = \boldsymbol{e}_i \end{cases}$. Denote $\mathcal{B} := \{y_1(t), y_2(t), \cdots, y_n(t)\}$, it remains to show that $\mathcal{S} = span(\mathcal{B})$, i.e: (i) $\boldsymbol{s} = \alpha_1 y_1 + \cdots + \alpha_n y_n, \forall \boldsymbol{s} \in S, \alpha_i \in \mathbb{R}$; (ii) y_i are linearly independent.

For (i), first it is trivial that $span(\mathcal{B}) \subseteq \mathcal{S}$, this follows from the fact of the principle of superposition. Now, consider $\forall z \in \mathcal{S}$, denote

$$z(t_0) = \begin{pmatrix} z_0^1 \\ z_0^2 \\ \vdots \\ z_0^n \end{pmatrix} = \sum_{i=1}^{n} z_0^i \boldsymbol{e}_i$$

Define the function $w(t) = \sum_{i=1}^{n} z_0^i y_i(t) \in \mathcal{S}$. Since $w(t_0) = \sum_{i=1}^{n} z_0^i y_i(t_0) = \sum_{i=1}^{n} z_0^i \boldsymbol{e}_i = z(t_0)$, then by the theorem of existence and uniqueness, we have

$$w(t) = z(t), \forall t \in I, \sum_{i=1}^{n} z_0^i y_i(t) = z(t) \in span(\mathcal{B}).$$

For (ii), assume that $c_1, \cdots, c_n \in \mathbb{R}$ and assume that

$$c_i y_1(t) + \cdots + c_n y_n(t) = 0, \forall t \in I.$$

I particular, at $t = t_0$,

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = 0 = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

thus $c_1 = c_2 = \cdots = c_n = 0$. ■

Definition

Definition 13. *A matrix valued function $Y : I \mapsto M_n(\mathbb{R})$ is a matrix solution of $y'(t) = A(t)y(t)$ if each of its column is a solution of $y'(t) = A(t)y(t)$. In particular, a matrix solution whose columns are linearly independent is called a fundamental matrix of solution.*

Proposition

Proposition 2. *Let $Y(t)$ be a matrix solution to $y'(t) = A(t)y(t)$, then either $\det(Y(t)) = 0, \forall t \in I$ or $\det(Y(t)) \neq 0, \forall t \in I$.*

Proof. First assume that $t_0 \in i$ such that $\det(Y(t_0)) = 0$, meaning that $Y(t_0)$ is not invertible, and the columns of $Y(t_0)$ are linearly independent, so we have a non-trivial linear combination $c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = 0$ where

$$Y(t) = \begin{pmatrix} | & | & & | \\ y_1(t) & y_2(t) & \cdots & y_n(t) \\ | & | & & | \end{pmatrix}.$$

Let $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$, then $Y(t_0)\boldsymbol{c} = 0$, and we define $y_1(t) \equiv Y(t)\boldsymbol{c} \in \mathcal{S}$. Note that $y_1(t_0) = Y(t_0)\boldsymbol{c} = 0$. Also $y_2(t) \equiv 0$ for all $t \in I$ also solves the IVP, $y' = Ay$, $y(t_0) = 0$. Then by the theorem of existence and uniqueness, $y_1(t) = Y(t)\boldsymbol{c} = y_2(t) = 0, \forall t$. Thus $\forall t \in I$, the columns of $y(t)$ are linearly dependent, thus $\det(Y(t)) = 0$.

■

Definition

Definition 14. *Given n vector functions $y_i : I \mapsto \mathbb{R}^n$, we define their Wronskain by*

$$W(t) = W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} | & | & & | \\ y_1(t) & y_2(t) & \cdots & y_n(t) \\ | & | & & | \end{pmatrix}.$$

From what we just saw, if $y_1, \cdots, y_n \in S$, then either $W(t) \equiv 0, \forall t \in I$ or $W(t) \neq 0, \forall t \in I$.

Theorem

Theorem 5. *(Liouville's Formula)*

If $Y(t)$ is a matrix solution to $y' = Ay$ satisfying $Y(t_0) = Y_0 \in M_n(\mathbb{R})$, then $\forall t \in I$,

$$\det Y(t) = \det(Y_0) \cdot e^{\int_{t_0}^{t} tr(A(s))ds}$$

Proof. Omitted for now. ■

Constant Coefficient Homogeneous Linear Systems

In this section, suppose we have $y' = Ay$ where $A \in M_n(\mathbb{R})$ does not depend on t. In the case $n = 1$, $y' = ay$ is separable, and has a solution of the general form $y(t) = Ce^{at}$. In general, for $n > a$, it is natural to look for solutions of $y' = Ay$ of the form $y(t) = e^{\lambda t}u$ where u is a vector, so we have

$$\begin{aligned} y'(t) = \frac{d}{dt}e^{\lambda t}u &= \lambda e^{\lambda t}u \\ &= e^{\lambda t}(\lambda u) \\ &= Ay(t) = e^{\lambda t}(Au) \end{aligned}$$

Meaning we have $Au = \lambda u$, and λ is an eigenvalue with associated eigenvector u.

Definition

Definition 15. *For $A \in M_n(\mathbb{R})$, its kernel is defined by*

$$\ker(A) = \{u \in \mathbb{C}^n \Big| Au = 0\}$$

here $\mathbb{C}$ is the set of complex numbers.

Definition

Definition 16. *The spectrum of a matrix $A \in M_n(\mathbb{R})$ is*

$$\sigma(A) := \{\lambda \in \mathbb{C} \Big| \dim\ker(A - \lambda I) > 0\}$$

i.e, the set of all eigenvalues of A.

Example: Consider the ODE given by $y' = Ay$ where $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Find the general solution $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

We first find the eigenvalues of A, we have

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (\lambda - 4)(\lambda + 2)$$

Thus we have 2 eigenvalues, $\lambda_1 = -2$, $\lambda_2 = 4$. Then, the corresponding eigenvectors for u_1 is given by $u_1 \in \ker(A - \lambda_1 I)$, thus

$$\ker(A - \lambda_1 I) = Span\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}, u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Similarly,

$$\ker(A - \lambda_2 I) = Span\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So we have two solutions, namely

$$y_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}; y_2(t) = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

And they are linearly independent, thus by the principle of super position, the general solution is given by

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

where $a_1, a_2 \in \mathbb{R}$ are constants.

Definition

Definition 17. *$A \in M_n(\mathbb{R})$ is diagonalizable if their exists an invertible matrix P such that $P^{-1}AP = Q$ where Q is diagonal.*

Lemma

Lemma 5. *If $A \in M_n(\mathbb{R})$ has n distinct real eigenvalues, then A is diagonalizable.*

Proof. ■

Theorem

Theorem 6. *$A \in M_n(\mathbb{R})$ is diagonalizable iff it has n linearly independent eigenvectors.*

Proof. ■

Proposition

Proposition 3. *Assume A is a $n \times n$ matrix, and $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$ are distinct, real eigenvalues and $u_1, \cdots, u_n$ to be the corresponding eigenvectors, then the general solution to the ODE $y' = Ay$ is given by*

$$y(t) = c_1 e^{\lambda_1 t} u_1 + \cdots + c_n e^{\lambda_n t} u_n$$

where $c_i \in \mathbb{R}$.

This leaves two cases open for us: What if we have complex eigenvalues? What if the matrix is not diagonalizable?

We first investigate complex eigenvalues:

Theorem

Theorem 7. *If $A \in M_n(\mathbb{R})$ is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$ (possibly complex) and corresponding eigenvectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$, then the set*

$$\{e^{\lambda_i t}\boldsymbol{u}_i | i = 1, \cdots, n\}$$

is a basis for the solution set $y' = Ay$.

If we have a complex eigenvalue. i.e a complex root fo the characteristic polynomial, then according to the fundamental theorem of algebra, complex solutions appear in conjugate pairs, thus if $\lambda = \alpha + i\beta$ is a eigenvalue of A, then $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of A. Similarly, if $\boldsymbol{u} + i\boldsymbol{v}$ is the corresponding eigenvector for λ, then $\boldsymbol{u} - i\boldsymbol{v}$ is the corresponding eigenvector for $\overline{\lambda}$. By the previous theorem, we may assume that

$$y(t) = e^{(\alpha+i\beta)t}(\boldsymbol{u} + i\boldsymbol{v})$$

is a solution of $y' = Ay$. Our goal now is to find a real polynomial to represent this complex solution. To achieve this, we first introduce Euler's formula:

Lemma

Lemma 6. *(Euler's Formula)*

$$e^{ix} = \cos(x) + i\sin(x) \quad \forall x \in \mathbb{R}.$$

Proof. Via Taylor Expansion.

■

Thus by Euler's formula, if $\alpha + i\beta$ is an eigenvalue of A, we have

$$\begin{aligned} y(t) &= e^{\alpha t}e^{i\beta t}(\boldsymbol{u} + i\boldsymbol{v}) \\ &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\boldsymbol{u} + i\boldsymbol{v}) \\ &= e^{\alpha t}((\boldsymbol{u}\cos(\beta t) - \boldsymbol{v}\sin(\beta t)) + ie^{\alpha t}(\boldsymbol{u}\sin(\beta t) + \boldsymbol{v}\cos(\beta t)) \end{aligned}$$

If we let

$$y_1(t) = Re(y(t)) = e^{\alpha t}(\boldsymbol{u}\cos(\beta t) - \boldsymbol{v}\sin(\beta t)) \quad y_2(t) = Im(y(t)) = e^{\alpha t}(\boldsymbol{u}\sin(\beta t) + \boldsymbol{v}\cos(\beta t))$$

Then, $y(t) = y_1(t) + iy_2(t)$ and also $y' = Ay$, which means

$$y_1'(t) + iy_2'(t) = Ay(t) = Ay_1(t) + iAy_2(t)$$

This tells us $y_1'(t) = Ay_1(t)$ and $y_2'(t) = Ay_2(t)$, so $y_1(t), y_2(t)$ are two real valued solutions of $y' = Ay$ associated to a complex eigenvalue.

Proposition

Proposition 4. *If $\beta \neq 0$, then the two solutions $y_1(t), y_2(t)$ are linearly independent.*

Proof. ■

Example: Consider $\alpha, \beta, \gamma \in \mathbb{R}$ and the matrix given by $A = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$, find the general solutions to $y' = Ay$.

We first find all the eigenvalues and eigenvectors. Its characteristic polynomial is given by

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} \alpha - \lambda & -\beta & 0 \\ \beta & \alpha - \lambda & 0 \\ 0 & 0 & \gamma - \lambda \end{pmatrix} = (r - \lambda)[(\alpha - \lambda)^2 + \beta^2]$$

By solving $p(\lambda) = 0$, we obtain three roots:

$$\lambda_1 = \alpha + i\beta, \;\; \lambda_2 = \alpha - i\beta, \;\; \lambda_3 = \gamma$$

Then clearly A is diagonalizable, the corresponding eigenvectors are given by:

$$\boldsymbol{u}_1 \in \ker(A - \lambda_1 I) = \ker \begin{pmatrix} -i\beta & -\beta & 0 \\ \beta & -i\beta & 0 \\ 0 & 0 & \gamma - \alpha - i\beta \end{pmatrix} \in span \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \implies \text{choose } \boldsymbol{u}_1 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

For $\boldsymbol{u}_2$, since we know that complex solutions appear in conjugate pairs, so we automatically have a choice for $\boldsymbol{u}_2$, namely $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$. Also

$$\boldsymbol{u}_3 \in \ker(A - \lambda_3 I) = span \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies \text{choose } \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus for the complex conjugate pairs, since we have $\alpha + i\beta$ as the complex eigenvalue, then we have the real valued solution given by

$$\begin{aligned} y(t) &= e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) \cdot \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &= e^{\alpha t}\left[\cos(\beta t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin(\beta t) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] + ie^{\alpha t}\left[\sin(\beta t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \cos(\beta t) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] \end{aligned}$$

Thus we have two real-valued solution gien by

$$y_1(t) = e^{\alpha t}\begin{pmatrix}\cos(\beta t)\\ \sin(\beta t)\\ 0\end{pmatrix} \quad y_2(t) = e^{\alpha t}\begin{pmatrix}\sin(\beta t)\\ -\cos(\beta t)\\ 0\end{pmatrix}$$

Also for a real eigenvalue $\lambda_3 = \gamma$, we have $y_3 = e^{\gamma t}\boldsymbol{u}_3$. So our general solution to this ODE is given by

$$y(t) = c_1 e^{\alpha t}\begin{pmatrix}\cos(\beta t)\\ \sin(\beta t)\\ 0\end{pmatrix} + c_2 y_2(t) + e^{\alpha t}\begin{pmatrix}\sin(\beta t)\\ -\cos(\beta t)\\ 0\end{pmatrix} + c_3 e^{\gamma t}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Example: Find the general solution to the ODE given by $\begin{pmatrix}y_1'(t)\\ y_2'(t)\end{pmatrix} = \begin{pmatrix}6 & -13\\ 1 & 0\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}$

Solution: The characteristic polynomial of the matrix is given by

$$p(\lambda) = \det\begin{pmatrix}6-\lambda & -13\\ 1 & -\lambda\end{pmatrix} = -\lambda(6-\lambda) + 13 = 0$$

which means we have 2 complex solutions, given by $\lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$, then the eigenvector for λ_1, denoted by $\boldsymbol{u}_1$ is given by

$$\boldsymbol{u}_1 = \ker\begin{pmatrix}3-2i & -13\\ 1 & -3-2i\end{pmatrix} = \ker\begin{pmatrix}1 & -3-2i\\ 1 & -3-2i\end{pmatrix} = Span\begin{pmatrix}3+2i\\ 1\end{pmatrix} =$$

Let $\boldsymbol{u}_1 = \boldsymbol{v} + i\boldsymbol{w}$, where $\boldsymbol{v} = \begin{pmatrix}3\\ 1\end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix}-2\\ 0\end{pmatrix}$. Since complex solutions appear in conjugate pairs, so we already know that $\boldsymbol{u}_2 = \boldsymbol{v} - i\boldsymbol{w}$. Then, recall the eigenvalues, where $Re(\lambda_1) = 3, Im(\lambda_1) = 2$, so the general solution takes the form

$$\boldsymbol{y}(t) = c_1 e^{3t}(\boldsymbol{u}\cos(2t) - \boldsymbol{v}\sin(2t)) + c_2 e^{3t}(\boldsymbol{u}\sin(2t) + \boldsymbol{v}\cos(2t))$$

which is,

$$\boldsymbol{y}(t) = c_1 e^{3t}\begin{pmatrix}3\cos(2t) - 2\sin(2t)\\ \cos(2t)\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}2\cos(2t) + 3\sin(2t)\\ \sin(2t)\end{pmatrix}.$$

Non-Diagonalizable Matrices

In the case of non-diagonalizable matrices, we do not have enough eigenvalues, in this chapter we will introduce the concepts of generalized eigenvectors as well as matrix exponentials to solve this kind of problem.

Definition

Definition 18. *Let $A \in M_n(\mathbb{R})$ and $p(\lambda)$ be its characteristic polynomial. Also we let $p(\lambda_0) = 0$, i.e λ_0 is an eigenvalue. The algebraic multiplicity of λ_0 is defined to be the largest integer m such that $(\lambda - \lambda_0)^m$ divides $p(\lambda)$. The geometric multiplicity of λ_0 is $\dim(\ker(A - \lambda_0 I))$.*

Based on this definition, one can also justify if a matrix is diagonalizable or not:

Theorem

Theorem 8. *Let $A \in M_n(\mathbb{R})$, then A is diagonalizable if and only if for all eigenvalues λ, the algebraic multiplicity is equal to the geometric multiplicity.*

Let $A \in M_n(\mathbb{R})$, if the sum of the geometric multiplicity of all eigenvalues is less than n, then it's easy to check that A is not diagonalizable.

Definition

Definition 19. *Let $A \in M_n(\mathbb{R})$, an eigenvalue λ with algebraic multiplicity m is defected if $\dim(\ker(A - \lambda I)) < m$. We define the defect of the defected eigenvalue λ by*

$$d_\lambda \equiv m - \dim(\ker(A - \lambda I)).$$

If we have an defected eigenvalue, we may still define a generalized version:

Definition

Definition 20. *If λ is an eigenvalue of $A \in M_n(\mathbb{R})$ with algebraic multiplicity m, we say $\boldsymbol{u}$ is a generalized eigenvector associated to λ if $(A - \lambda I)^m \boldsymbol{u} = \mathbf{0}$.*

Now, with generalized eigenvectors introduced, we have a very powerful theorem:

Theorem

Theorem 9. *Let λ be an eigenvalue of $A \in M_n(\mathbb{R})$ with algebraic multiplicity $m \geq 1$, then*

$$\dim(\ker(A - \lambda I)^m) = m$$

Moreover, denoted by $\mathcal{G}_{\lambda_i} = \ker(A - \lambda_i I)^{m_i}$, where $\lambda_1, \cdots, \lambda_d$ are eigenvalues with algebraic multiplicity $m_1, \cdots, m_d$, and $m_1 + \cdots + m_d = n$. Moreover, we have

$$\mathbb{C}^n = \mathcal{G}_{\lambda_1} \oplus \cdots \oplus \mathcal{G}_{\lambda_d}$$

In other words, for any $A \in M_n(\mathbb{R})$, it is always possible to find a basis for generalized eigenvectors. (A consequence of spectrum theorem).

To continue our study, we would like to investigate a bit more about the kernel operator.

Definition

Definition 21. *Let $A \in M_n(\mathbb{R})$, then the kernel of A is defined as*

$$\ker(A) := \{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} = \mathbf{0}\}.$$

Thus it is easy to check that the kernel operator is a linear operator, i.e closed under vector addition and scalar multiplication. Moreover, the kernel operator has the property that

$$\{\mathbf{0}\} \subseteq \ker A \subseteq \ker A^2 \cdots \subseteq \ker A^n$$

To see this, let $\boldsymbol{x} \in \ker A^m$, then $A^m \boldsymbol{x} = \mathbf{0}$, thus $A^{m+1}\boldsymbol{x} = A(A^m \boldsymbol{x}) = A\mathbf{0} = \mathbf{0}$. Now we assume that λ is a defected eigenvalue of A with

$$\dim(\ker(A - \lambda I)^m) = m$$

Then for the integers $j = 1, 2, \cdots, m-1, m$, we may denote

$$K_j := \ker(A - \lambda I)^j,$$

by our previous construction, we already know that

$$\{\mathbf{0}\} \subseteq K_1 \subseteq K_2 \cdots \subseteq K_{m-1} \subseteq K_m$$

and there exists $\boldsymbol{\omega} \in K_i / K_{i-1}$, so in general we have the relation

$$(A - \lambda i)\boldsymbol{u} = \boldsymbol{\omega}$$

for $\boldsymbol{u} \in K_{i+1}/K_i$ and $\boldsymbol{\omega} \in K_i / K_{i-1}$, because $(A - \lambda I) : K_j \to K_{j-1}$.

Example: Find a basis of generalized eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -2 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Solution: We first find all eigenvalues, that is we solve

$$p(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 & 1 \\ 0 & -2-\lambda & -1 & -1 \\ 1 & 1 & -\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{pmatrix} = \lambda^3(\lambda+1) = 0$$

where we get 2 eigenvalues, $\lambda_1 = 0$ with algebraic multiplicity 3 and $\lambda_2 = -1$ with algebraic multiplicity 1. Let $\boldsymbol{u}_2$, $\boldsymbol{u}_1$ to be the eigenvector correspond to λ_1, λ_2, we then have

$$\boldsymbol{u}_2 \in \ker \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \mathbf{0}$$

also

$$\boldsymbol{u}_1 \in \ker \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -2 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = Span \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Exponential of Matrices

In this chapter, we will define the exponential of a matrix, and we will see how can we solve an ODE using matrix exponential.

Definition

Definition 22. *For a matrix $A \in M_n(\mathbb{R})$, we define the exponential of A by*

$$e^A \equiv \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

If $AB = BA$, then $e^{A+B} = e^A \cdot e^B$. Also we define a map $t \mapsto e^{At}$ by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k, t \in \mathbb{R}.$$

Chapter 4

Stability Analysis and Phase Portraits

Chapter 5

Laplace Transform

Definition and Examples

Definition

Definition 23. *We say that a function* $f : [0, \infty) \to \mathbb{R}$ *is piecewise continuous, if there exists partitions of* $\mathbb{R}$: $I_1, \cdots, I_n$ *such that* $f : I_i^o \to \mathbb{R}$ *is continuous and the limit of L.H.S, R.H.S both exist for all* $i = 1, \cdots, n$.

Another definition follows:

Definition

Definition 24. *We say that a function* $f : [0, \infty) \to \mathbb{R}$ *is of exponential order, if there exists constants* a, K, M *such that*

$$|f(t)| \leq Ke^{at}$$

for all $t \geq M$

We denote the smallest possible a to be the order of $f(t)$, and we say that f is PCEO if it is piecewise continuous and of exponential order.

For PCEO functions, we define the notion of Laplace transform:

Definition

Definition 25. *Let a PCEO function, then the Laplace transform of f, denoted by $\mathcal{L}[f]$, is a map such that*

$$f : [0, \infty) \to \mathbb{R} \xrightarrow{\mathcal{L}} F : (a, \infty) \to \mathbb{R}$$

defined by

$$\mathcal{L}[f](s) = F(s) = \int_a^\infty e^{-st} f(t) dt$$

where a is the order of $f(t)$.

Examples :

Consider $f(t) = K$, a constant function, then

$$\begin{aligned}\mathcal{L}[f](s) &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^\infty e^{-st} K dt \\ &= \frac{K}{s}, s > 0.\end{aligned}$$

Also, consider $f(t) = t^n$, then

$$\mathcal{L}[t^n](s) = \int_0^\infty e^{-st} t^n dt = \frac{n!}{s^{n+1}}, s > 0$$

by Euler-Gamma function.

Consider $f(t) = e^{at}$, then

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{s-a}, s > a$$

Consider $f(t) = \begin{cases} t & t \in [0,1) \\ K & t = 1 \\ 0 & t > 1 \end{cases}$, then

$$\mathcal{L}[f(t)](s) = \int_0^1 e^{-st} \cdot 1 dt = \frac{1}{s}(1 - e^{-s}), s > 0.$$

Proposition

Proposition 5. *Let f be PCEO with order a, then the function $F : (a, \infty) \to \mathbb{R}$ defined by*

$$F(s) = \mathcal{L}[f](s)$$

is smooth, and furthermore

$$\frac{d^k}{ds^k} F(s) = (-1)^k \mathcal{L}[t^k f(t)](s), s > a.$$

We introduce another proposition:

Proposition

Proposition 6. *The Laplace transform $\mathcal{L}$ is a linear operator, i.e*

$$\mathcal{L}[af+bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$$

Proof. Let f, g be PCEO with order m_1, m_2, then denote $s := \max\{m_1, m_2\}$, we have

$$\begin{aligned}\mathcal{L}[af+bg](s) &= \int_0^\infty e^{-st}(af(t)+bg(t))dt \\ &= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt \\ &= a\mathcal{L}[f](s) + b\mathcal{L}[g](s).\end{aligned}$$

■

We can use this proposition to find the Laplace transform for more complicated functions.

Example : Consider $f(t) = \cosh(\alpha t)$, then we have

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}\left[\frac{e^{\alpha t}+e^{-\alpha t}}{2}\right] \\ &= \frac{1}{2}\mathcal{L}[e^{\alpha t}](s) + \frac{1}{2}\mathcal{L}[e^{-\alpha t}](s) \\ &= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}, s > a\end{aligned}$$

Laplace transform is also defined for complex valued PCEO functions, so we can combine with Euler's formula to show the Laplace transform of trigonometric functions:

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, s > 0.$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}, s > 0.$$

Proposition

Proposition 7. *(Uniqueness of Laplace Transform)*

Assume $f, g : [0,\infty) \to \mathbb{R}$ are PCEO, with $\mathcal{L}[f](s) = \mathcal{L}[g](s)$ for all $s > a$, then $f \equiv g$. In other words, the Laplace transform is injective on the space of PCEO functions.

Proof. The proof is not easy. We will use Stone-Weierstrass Theorem and Dominated Convergence Theorem.

■

Because of the uniqueness, we introduce the inverse Laplace transform, for example,

$$\mathcal{L}\left[\frac{s}{s^2+\omega^2}\right](t) = \cos(\omega t)$$

Using Laplace Transform to Solve ODEs

Theorem

Theorem 10. *Assume $f : [0, \infty) \to \mathbb{R}$ is PCEO, and $f' : [0, \infty) \to \mathbb{R}$ is piecewise continuous, then $\mathcal{L}[f']$ exists, and*

$$\mathcal{L}[f'](s) = s\mathcal{L}[f] - f(0)$$

The proof can be done integration by parts. We can also further show that

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f] - sf(0) - f'(0).$$

Proposition

Proposition 8. *Let $f : [0, \infty) \to \mathbb{R}$ be C^n and is PCEO, with order a. Then for every $k = 1, \cdots, n$, we have*

$$\mathcal{L}[f^{(k)}](s) = s^k\mathcal{L}[f] - \sum_{j=0}^{k-1} f^{(j)}(0) \cdot s^{k-1-j}, s > a$$

Now, given a general ODE

$$ay'' + by' + cy = f(t), y(0) = y_0, y'(0) = y_0'$$

Then applying Laplace transform on both sides,

$$\mathcal{L}[ay'' + by' + cy] = \mathcal{L}[f(t)],$$

apply linearity, we have

$$a\mathcal{L}[y''] + b\mathcal{L}[y'] + c\mathcal{L}[y] = \mathcal{L}[f(t)],$$

using the proposition stated before, we have

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0), \mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0),$$

then let $Y(s) = \mathcal{L}[y](s)$, then we have

$$a[s^2Y(s) - sy_0 - y_0'] + b[sY(s) - y_0] + cY(s) = F(s).$$

Also we let $y(t) = \mathcal{L}^{-1}[Y(s)]$, then we have

$$(as^2 + bs + c)Y(s) = F(s) + asy_0 + ay_0' + by_0,$$

and hence

$$Y(s) = \frac{1}{as^2 + bs + c}(F(s) + asy_0 + ay_0' + by_0),$$

and hence

$$\boxed{y(t) = \mathcal{L}^{-1}\left[\frac{F(s) + asy_0 + ay_0' + by_0}{as^2 + bs + c}\right]}.$$

and we have a general formula for the IVP. Then, it suffies to find the inverse Laplace transform, and we will develop some techniques to solve the inverse Laplace transform.

Proposition

Proposition 9. *Let $f : [0, \infty) \to \mathbb{R}$ be PCEO, let $F(s) = \mathcal{L}[f](s)$, then we have*

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}F(s),$$

or equivalently,

$$\mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)du$$

Proof. Let $g(t) = \int_0^t f(u)du$, one can show that $g(t)$ is PCEO, also since $g'(t) = f(t)$, then by the previous result

$$\mathcal{L}[g'(t)] = s\mathcal{L}[g] - g(0) = s\mathcal{L}\left[\int_0^t f(u)du\right] - 0 = F(s).$$

∎

Example : Consider $G(s) = \frac{1}{s(s^2+1)}$. To find the inverse Laplace transform, we first write

$$G(s) = \frac{1}{s}F(s), F(s) = \frac{1}{s^2+1},$$

then by the above proposition, we have

$$\mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)du$$

where

$$F(s) = \frac{1}{s^2+1} = \mathcal{L}[f](s) \implies f(t) = \sin(t),$$

so we would have

$$\mathcal{L}^{-1}[G(s)] = \int_0^t \sin(u)du = 1 - \cos(t).$$

Also, we can use another approach, by partial fractions:

$$G(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

and now we apply linearity of the Laplace transform:

$$\mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = 1 - \cos(t).$$

So as we see, knowing Laplace transform table is very important! We will give the table here, which contains the Laplace transform for elementary functions:

Table of Laplace Transforms and Some Standard Integrals.

function $f(t)$	Laplace transform $F(s)$
1	$1/s \quad (s>0)$
t^n	$n!/s^{n+1} \quad (s>0)$
e^{at}	$1/(s-a) \quad (s>a)$
$\sin at$	$a/(s^2+a^2) \quad (s>0)$
$\cos at$	$s/(s^2+a^2) \quad (s>0)$
$\sinh at$	$a/(s^2-a^2) \quad (s>\lvert a\rvert)$
$\cosh at$	$s/(s^2-a^2) \quad (s>\lvert a\rvert)$
$e^{-at}f(t)$	$F(s+a)$
$\mathcal{U}(t-a)$ or $\mathcal{U}_a(t)$ $(a \geqslant 0)$	$e^{-as}/s \quad (s>0)$
$\delta(t-a) \quad (a>0)$	e^{-as}
$\mathcal{U}(t-a)f(t-a)$ or $\mathcal{U}_a(t)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-f^{(n-1)}(0)$
$(-t)^nf(t)$	$F^{(n)}(s)$
$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

Figure 5.1: The Laplace Transform for Elementary Functions

Now, we see that Laplace transform satisfies linearity, but it is absolutely **not** true that $\mathcal{L}[f_1f_2]=\mathcal{L}[f_1]\mathcal{L}[f_2]$!, but there is a similar form, which is the convolution of functions:

Theorem

Theorem 11. *The convolution operator of functions f_1, f_2 is defined by $f_1 * f_2$, given by*

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\tau)\cdot f_2(\tau)d\tau$$

furthermore, we have

$$\mathcal{L}[f_1 * f_2] = \mathcal{L}[f_1]\cdot\mathcal{L}[f_2]$$

Proof. We assume f_1, f_2 are PCEO, furthermore it can be shown that $f_1 * f_2$ is PCEO. Now

$$\begin{aligned}\mathcal{L}[f_1]\cdot\mathcal{L}[f_2] &= F_1(s)\cdot F_2(s)\\ &= \left(\int_0^\infty f_1(\xi)e^{-s\xi}d\xi\right)\cdot\left(\int_0^\infty f_2(\tau)e^{-s\tau}d\tau\right)\\ &= \int_0^\infty f_2(\tau)\int_0^\infty f_1(\xi)e^{-s(\xi+\tau)}d\xi d\tau\end{aligned}$$

Now we let $t = \xi + \tau$, we now have

$$\begin{aligned}
\mathcal{L}[f_1]\mathcal{L}[f_2] &= \int_0^\infty f_2(\tau) \int_\tau^\infty f_1(t-\tau)e^{-st}dtd\tau \\
&= \int_0^\infty \int_0^t f_2(\tau) f_1(t-\tau)e^{-st}d\tau dt \\
&= \int_0^\infty e^{-st} \int_0^t f_1(t-\tau) f_2(\tau)d\tau dt \\
&= \int_0^\infty e^{-st}(f_1 * f_2)dt \\
&= \mathcal{L}[f_1 * f_2](s).
\end{aligned}$$

■

Also, we have

$$\mathcal{L}^{-1}[F_1(s) \cdot F_2(s)] = f_1 * f_2$$

Example: Consider $H(s) = \dfrac{a}{s^2(s^2+a^2)}$, $a \in \mathbb{R}$, then we let $F_1(s) = \dfrac{1}{s^2}$ and $F_2(s) = \dfrac{a}{s^2+a^2}$, and from the Laplace transform table we would have

$$f_1(t) \equiv \mathcal{L}^{-1}[F_1(s)] = t, f_2(t) \equiv \mathcal{L}^{-1}[F_2(s)] = \sin(at).$$

Hence

$$\begin{aligned}
\mathcal{L}^{-1}[H(s)] &= \mathcal{L}^{-1}[F_1(s) \cdot F_2(s)] \\
&= f_1(t) * f_2(t) \\
&= \int_0^t (t-\tau)\sin(a\tau)d\tau \\
&= \frac{at - \sin(at)}{a^2}.
\end{aligned}$$

Proposition

Proposition 10. *(Properties of Convolutions)*

(i) (Commutative) $f * g = g * f$

(ii) (Associativity) $(f * g) * h = f * (g * h)$

(iii) (Distributivity) $f * (g + h) = f * g + f * h$

(iv) $f * 0 \equiv 0$.

Example: Consider $\mathcal{L}^{-1}\left[\dfrac{1}{(s-a)^2}\right]$ for some $a \in \mathbb{R}$, then we would have

$$\mathcal{L}^{-1} = \mathcal{L}^{-1}\left[\frac{1}{s-a} \cdot \frac{1}{s-a}\right] = f_1 * f_2$$

where we have

$$f_1(t) = f_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

and thus

$$f_1 * f_2 = \int_0^t e^{a(t-\tau)} e^{a\tau} d\tau = te^{at}.$$

Example: Solve the integral-differential equations given by

$$y' + 2y + \int_0^t y(\tau)d\tau = \sin(t), y(0) = 1$$

We first define $Y(s) = \mathcal{L}[y]$, then we know that

$$\mathcal{L}\left[\int_0^t y(\tau)d\tau\right] = \frac{1}{s}Y(s)$$

then we apply Laplace transform on both sides:

$$\begin{aligned}
&\implies \mathcal{L}\left[y' + 2y + \int_0^t y(\tau)d\tau\right] = \mathcal{L}[\sin(t)] \\
&\implies s\mathcal{L}[y] - y(0) + 2\mathcal{L}[y] + \mathcal{L}\left[\int_0^t y(\tau)d\tau\right] = \frac{1}{s^2+1} \\
&\implies sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \\
&\implies \left(s + 2 + \frac{1}{s}\right)Y(s) = 1 + \frac{1}{s^2+1} \\
&\implies Y(s) = \frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}
\end{aligned}$$

So finally we apply the inverse Laplace transform, and we get

$$\begin{aligned}
\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{1}{1+s}\right] - \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{(1+s)^2}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
&= e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin(t).
\end{aligned}$$

which is the solution to the IVP!

Step Functions and Translation Theorems

In this section, we will deal with piecewise continuous functions, where there will be discontinuity. We start by defining step functions:

Definition

Definition 26. *We define the tep function $u_c : [0, \infty) \to \mathbb{R}$ by*

$$u_c(t) = \begin{cases} 0 & t \in [0, c) \\ 1 & t \geq c \end{cases}$$

When $c = 0$ we denote $u_0(t) = H_0(t)$ as the Heaviside function.

Example: Suppose we have

$$f(t) = \begin{cases} 2 & t \in [0, 4) \\ 5 & t \in [4, 7) \\ -1 & t \in [7, 9) \\ 1 & t \geq 9 \end{cases}$$

then, we may actually rewrite f in terms of a linear combination of step functions. Namely,

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

Proposition

Proposition 11. *If we have a function $f : [0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} K & x \in [a, b) \\ 0 & \textit{otherwise} \end{cases}$$

where K is a constant, then we may write $f(x) = u_a(t) - u_b(t)$.

Now we introduce the first translation theorem:

Theorem

Theorem 12. *(Translation Theorem (1))*

$$\mathcal{L}[u_c](s) = \frac{1}{s} e^{-cs}$$

Proof. The proof is very straight-forward, we just apply the definition of Laplace transform, we have

$$\mathcal{L}[u_c](s) = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{1}{s} e^{-cs}.$$

■

The first translation theorem gives us the Laplace transform of a step function, while the second translation theorem will deal with the step function acting on a function. We first define a translation operator:

Definition

Definition 27. *Given a function $f : [0, \infty) \to \mathbb{R}$ whose graph is in the half-right plane, and $c > 0$, we define the function $g(t) = u_c(t)f(t-c)$ by translating on the right of the graph f at distance c, denote by*

$$t_c(f) \equiv u_c(t)f(t-c).$$

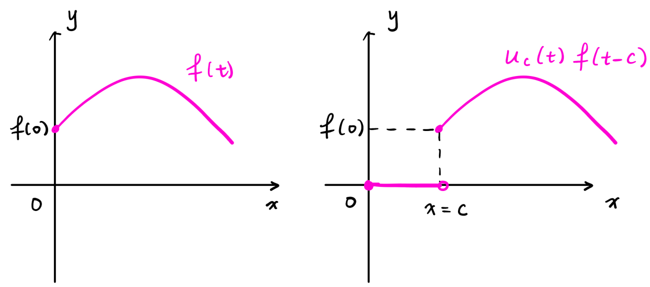


Figure 5.2: A graphical illustration of $t_c(f)$

Theorem

Theorem 13. *(Translation Theorem (2))*

Let $f : [0, \infty) \to \mathbb{R}$ be PCEO of order $a \in \mathbb{R}$, also let $c > 0$, then

$$\mathcal{L}[t_c(f)](s) = e^{-cs}\mathcal{L}[f](s), s > a$$

Proof. Now, we have

$$\begin{aligned}
\mathcal{L}[t_c(f)](s) &= \int_0^\infty u_c(t)f(t-c)e^{-st}dt \\
&= \int_0^\infty f(t-c)e^{-st}dt \\
&= \int_0^\infty f(u)e^{-s(u+c)}du \\
&= e^{-sc}\int_0^\infty e^{-su}du = e^{-sc}\mathcal{L}[f](s), s > a.
\end{aligned}$$

■

Example: Find the Laplace transform of the function defined by

$$f(t) = \begin{cases} 2 & t \in [0,1) \\ 1+t & t \in [1,4) \\ t^2 & t \geq 4 \end{cases}.$$

Solution: First we express f in terms of step functions, namely

$$f(t) = 2(u_0 - u_1) + (1+t)(u_1 - u_4) + t^2u_4 = 2u_0 + (t-1)u_1 + (t^2 - t + 1)u_4$$

so now by linearity of Laplace transform,

$$\mathcal{L}[f(t)](s) = 2\mathcal{L}[u_0] + \mathcal{L}[(t-1)u_1] + \mathcal{L}[(t^2 - t + 1)u_4]$$

Note that by translation theorem (1), $2\mathcal{L}[u_0] = 2\mathcal{L}[1] \equiv \frac{2}{s}$; also using translation theorem (2), we may consider $(t-1)u_1$ as a right shift of $g(t) = t$ by one unit, hence

$$\mathcal{L}[(t-1)u_1] = e^{-s}\mathcal{L}[t](s) = e^{-s} \cdot \frac{1}{s^2}.$$

Finally, we will modify the last part so that it looks like a "right shift by 4" function. We do

$$\mathcal{L}[(t^2 - t + 1)u_4] = \mathcal{L}[((t-4)^2 + 7(t-4) + 11)u_4](s) = e^{-4s}\mathcal{L}[t^2 + 7t + 11]$$

Hence

$$\mathcal{L}[f(t)](s) = \frac{2}{s} + e^{-s}\frac{1}{s^2} + e^{-4s}\left[\frac{2}{s^3} + \frac{7}{s^2} + \frac{11}{s}\right].$$

Example: Find $\mathcal{L}^{-1}\left[\frac{1-e^{-2s}}{s^2}\right]$.

Solution: By linearity, we can easily see that $\mathcal{L}^{-1}\left[\frac{1-e^{-2s}}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right]$, then we apply translation theorem (2), and we will have

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - \mathcal{L}^{-1}(e^{-2s}F(s)) = t - t_2(f) = t - u_2(t)(t-2).$$

Theorem

Theorem 14. *(Translation Theorem (3))*

Let f be PCEO, $f : [0, \infty) \to \mathbb{R}$ with order a, let $c \in \mathbb{R}$, then

$$\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}f(t)](s), s > a + c,$$

or equivalently,

$$\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$$

Proof.

$$\begin{aligned}
\mathcal{L}[e^{ct}f(t)](s) &= \int_0^\infty e^{-st}e^{ct}f(t)dt \\
&= \int_0^\infty e^{t(c-s)}f(t)dt \\
&= \mathcal{L}[f](s-c), s > a + c.
\end{aligned}$$

■

Example: Solve the ODE with IVP

$$y'' + 4y' + 3y = e^{-3t}\cos(t), y(0) = y'(0) = 0.$$

Of course, we can solve it using the standard technique from second order ODEs, but now let's try it by using Laplace transform:

$$\mathcal{L}[y'' + 4y' + 3y] = \mathcal{L}[e^{-3t}\cos(t)]$$

The L.H.S is trivially $(s^2 + 4s + 3)\mathcal{L}[y]$, while the Laplace transform of the R.H.S can be viewed as a shift of 3 unit. After the shift the function is $\cos(t)$, so before the shift we would have $\cos(3 + t)$, and hence by translation theorem (3) we have

$$\mathcal{L}[e^{-3t}\cos(t)] = \mathcal{L}[\cos(t+3)](s) = \frac{s+3}{(s+3)^2+1}$$

Hence, we have

$$y = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+3)} \cdot \frac{s+3}{(s+3)^2+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{[(s+3)-2][(s+3)^2+1]}\right] \overset{\text{Translation 2}}{=} e^{-3t}\mathcal{L}[G(s)]$$

where $G(s) = \dfrac{1}{(s-2)(s^2+1)}$, then

$$\begin{aligned} y &= e^{-3t}\mathcal{L}^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right] \\ &= e^{-3t}\mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}\right] \\ &= e^{-3t}\left(\frac{1}{5}e^{2t} - \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t)\right). \end{aligned}$$

Impulse Functions

Let $\tau > 0$, and we consider the function

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, t \in [-\tau, \tau] \\ \text{otherwise} \end{cases}$$

such that we have $\int_{\mathbb{R}} d_\tau(t)dt = 1$. As $\tau \to 0+$, we call this the impulse function, denote as $\delta(t)$, and we have

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}$$

Chapter 6

Power Series and Numerical Solutions