Laplace's Equations

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Content of this report

- Fundamental Solutions to Laplace's Equation and Poisson's Equations.

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- Fundamental Solutions to Laplace's Equation and Poisson's Equations.
- Properties of harmonic functions.

Laplace's Equations

Laplace's Equations

Definition

Let $u(x_1, \cdots, x_n) : U \subset \mathbb{R}^n \to \mathbb{R}$ be a unknown function, the Laplacian of the function is defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{0.1}$$

Laplace equation is of the form

$$\Delta u = 0 \tag{0.2}$$

for all $x \in U$, any function satisfying the above is also called harmonic.

Laplace's Equations

Laplace's Equations

Lemma

Laplace's equation is invariant under rotation, i.e if P is an orthogonal matrix with entries p_{ij}, then $\Delta u(Px) = 0$ for all $x \in U$.

Proof.

By Chain rule, we have

$$D_{x_i}(u(Px)) = \sum_{k=1}^{n} D_{x_k}(u(Px)p_{ik}) \tag{0.3}$$

and similarly

$$D_{x_i x_j}(u(Px)) = \sum_{l=1}^{n}\sum_{k=1}^{n} D_{x_k x_l}(u(Px)p_{ik}p_{il}). \tag{0.4}$$

Since P is orthogonal, thus $p_{ik}p_{il} = 1$ iff $k = l$ and hence

$$\Delta u(Px) = \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n} D_{x_k x_l}(u(Px)p_{ik}p_{il}) = \Delta u = 0. \tag{0.5}$$

□

Fundamental Solutions

Given this, we seek for radial solutions which takes the form $u(x) = v(r)$ where $r = |x|$. Denote $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, let $x \neq 0$, then we see that

$$\frac{\partial r}{\partial x_i} = (x_1^2 + \cdots + x_n^2)^{-1/2} x_i = \frac{x_i}{r} \tag{0.6}$$

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and by Chain rule we have

$$u_{x_i} = v'(r)\frac{x_i}{r} \tag{0.7}$$

$$u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\frac{r - x_i\frac{x_i}{r}}{r^2}, v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \tag{0.8}$$

Fundamental Solutions

using the fact that $x_1^2 + \cdots + x_n^2 = r^2$, we would have

$$\Delta u = \sum_{i=1}^{n} \left(v''(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \right) = v''(r) + \frac{n-1}{r} v'(r). \quad (0.9)$$

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$$v''(r) + \frac{n-1}{r}v'(r) = 0. \quad (0.10)$$

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$$v''(r) + \frac{n-1}{r}v'(r) = 0. \quad (0.10)$$

Now if $v'(r) \neq 0$, we see that

$$\log(v'(r))' = \frac{v''(r)}{v'(r)} = \frac{1-n}{r} \quad (0.11)$$

and we integrate with respect to r, we have

$$\log(v'(r)) = (1-n)\log r + C \tag{0.12}$$

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which simplifies to $v'(r) = C_0/r^{n-1}$ for some constant C_0, hence the general solution $v(r)$ takes the form

$$v(r) = \begin{cases} A\log r + B & n = 2 \\ \frac{C}{r^{n-2}} + D & n \geq 3 \end{cases} \tag{0.13}$$

where A, B, C, D are all constants. This motivates the fundamental solution to Laplace's equation.

Fundamental Solutions

Definition

The function defined by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases} \tag{0.14}$$

is the fundamental solution to Laplace's equation $\Delta u = 0$ for all $x \in \mathbb{R}^n / \{0\}$ where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Poisson's Equations

Laplace's equation is a special class of Poisson's equations, we define Poisson's equation as

Definition

For a given $f : \mathbb{R}^n \to \mathbb{R}$, and the unknown $u(x_1, \cdots, x_n) : U \subset \mathbb{R}^n \to \mathbb{R}$, the Poisson's equation takes the form

$$-\Delta u = f. \tag{0.15}$$

Fundamental Solutions

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Theorem

Define $u(x) = \displaystyle\int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$ *and assume that* $f \in C_C^2(\mathbb{R}^n)$*, then:*
(i) $u \in C^2(\mathbb{R}^n)$*;*
(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$*.*

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We will prove both results separately:

For (i) :Use the property of a convolution, we have

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x - y) dy \tag{0.16}$$

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then denote e_i as the unit vector in ith position of $x = (x_1, \cdots, x_n)$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x_i}(x) &= \lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} \\
&= \int_{\mathbb{R}^n} \Phi(y) \cdot \lim_{h \to 0} \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right] dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f}{\partial x_i}(x-y) dy.
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Similarly one can show

$$\frac{\partial u}{\partial x_i \, \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial^2 f}{\partial x_i \, \partial x_j}(x - y) dy. \tag{0.17}$$

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$$u(x) = \int_{B(0,\epsilon)} \Phi(y) f(x-y) dy + \int_{\mathbb{R}^n / B(0,\epsilon)} \Phi(y) f(x-y) dy \tag{0.18}$$

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hence

$$\Delta u(x) = \int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^n / B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy. \tag{0.19}$$

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Denote the first term as I_ϵ, second term as J_ϵ. Our first goal is to bound I_ϵ, proceed as follows:

When $n = 2$, we have

$$
\begin{aligned}
|I_\epsilon| &= \left| - \int_{B(0,\epsilon)} \frac{1}{2\pi} \log|y| \cdot \Delta_x f(x-y) dy \right| \\
&\leq C ||Df||_{L^\infty(\mathbb{R}^n)} \cdot \left| \int_{B(0,\epsilon)} \log|y| dy \right|
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for some constant C, and we further have

$$\int_{B(0,\epsilon)} \log|y| dy = \int_0^\epsilon \int_0^{2\pi} \log|r| r d\theta dr = \epsilon^2 \log|\epsilon| \tag{0.20}$$

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so we have $|I_\epsilon| < C\epsilon^2 |\log \epsilon|$ for some constant C. When $n \geq 3$, similarly we can show that $|I_\epsilon| < C\epsilon$, hence the term I_ϵ is bounded in terms of arbitrary ϵ.

Now for J_ϵ, first note that $\Delta_x f(x-y) = \Delta_y f(x-y)$, we have

$$J_\epsilon = \int_{\mathbb{R}^n / B(0,\epsilon)} \Phi(y) \Delta_y f(x-y) dy. \tag{0.21}$$

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To compute this integral, we need some technical lemmas:

Lemma

(Gauss-Green Theorem) Let $u(x_1, \cdots, x_n) \in C^1(\overline{U})$, let $\nu = (\nu_1, \cdots, \nu_n)$ be the outward pointing unit normal vector of U defined on ∂U then

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu_i dS. \tag{0.22}$$

We can use this lemma to further derive an integration by parts formula:

Lemma

(Integration by Parts) Let $u, v \in C^1(\overline{U})$, *then*

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS. \tag{0.23}$$

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Then we may rewrite J_ϵ as

$$J_\epsilon = -\int_{\mathbb{R}^n/B(0,\epsilon)} D\Phi(y) \cdot D_y f(x-y) dy + \int_{\partial B(0,\epsilon)} \Phi(y) \nu D f(x-y) dS(y) \tag{0.24}$$

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where ν is the inward pointing unit normal vector. We now denote the first term by K_ϵ and the second term by L_ϵ, then using the same technique as we did for I_ϵ, we can check that

$$|L_\epsilon| \leq C||Df||_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)| dS(y) \leq \begin{cases} C\epsilon|\log\epsilon| & n = 2 \\ C\epsilon & n \geq 3 \end{cases} \tag{0.25}$$

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hence L_ϵ is also bounded in terms of ϵ. We again perform integration by parts in K_ϵ, and we have

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n/B(0,\epsilon)} \Delta\Phi(y) f(x-y) dy - \int_{\partial B(0,\epsilon)} \nu D\Phi(y) f(x-y) dS(y) \\ &= -\int_{\partial B(0,\epsilon)} \nu D\Phi(y) f(x-y) dS(y). \end{aligned}$$

Since $D\Phi(y) = \frac{-1}{n\alpha(n)} \frac{y}{|y|^n}$ when $y \neq 0$ and $\nu = -y/|y| = -y/\epsilon$ on $\partial B(0, \epsilon)$, so we will have

$$\nu D\Phi(y) = \frac{1}{n\alpha(n)\epsilon^{n-1}} \tag{0.26}$$

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also $n\alpha(n)\epsilon^{n-1}$ is the surface area of the ball $\partial B(0, \epsilon)$, so

$$K_\epsilon = -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} f(x-y)dS(y) = -⨍_{\partial B(0,\epsilon)} f(y)dS(y) \to -f(x) \tag{0.27}$$

as $\epsilon \to 0$.

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as $\epsilon \to 0$. Since we have shown that I_ϵ, L_ϵ are all bounded by terms of ϵ so by setting $\epsilon \to 0$ they will vanish as well. Hence we have

$$-\Delta u(x) = f(x). \tag{0.28}$$

Properties of Harmonic functions

Now we shall introduce some properties of harmonic functions:

- Mean- Value Property

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- Mean- Value Property
- Maximum Principle
- Smoothness
- Liouville's Theorem
- Harnack's Inequality
- Convergences

Mean-Value Property

Theorem

Let $U \in \mathbb{R}^n$ open, if $u(x_1, \cdots, x_n) \in C^2(U)$ is harmonic then for each ball $B(x, r) \in U$,

$$u(x) = \fint_{\partial B(x,r)} u(y) dS(y) = \fint_{B(x,r)} u(y) dy \tag{0.29}$$

where

$$\fint_{\partial B(x,r)} u(y) dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y), \tag{0.30}$$

$$\fint_{B(x,r)} u(y) dy = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u(y) dy \tag{0.31}$$

$\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and $n\alpha(n)$ is the surface area of the unit ball in $\mathbb{R}^n$.

Let $u(x_1, \cdots, x_n) \in C^2(U)$ be harmonic, and define

$$\phi(r) = \begin{cases} \fint_{\partial B(x,r)} u(y)dS(y) & r > 0 \\ u(x) & r = 0 \end{cases}. \tag{0.32}$$

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If u is a smooth function, then $\lim_{r\to 0} \phi(r) = u(x)$ and hence ϕ is continuous. Note that by change of variables, we have

$$\phi(r) = \fint_{\partial B(0,1)} u(x + rz)dS(z) \tag{0.33}$$

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and we take the derivative of $\phi(r)$:

$$
\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot z dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r} dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \nabla(\nabla u) dy \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y) dy \equiv 0.
\end{aligned}
$$

which means ϕ is a constant function, and hence we have

$$u(x) = \fint_{\partial B(x,r)} u(y)dS(y). \tag{0.34}$$

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Furthermore we have

$$\begin{aligned} \int_{B(x,r)} u(y) dy &= \int_0^r \left(\int_{\partial B(x,s)} u dS \right) ds \\ &= u(x) \int_0^r n\alpha(n) s^{n-1} ds \\ &= \alpha(n) r^n u(x). \end{aligned}$$

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\end{aligned}$$

and that finishes the proof.

Theorem

If $u \in C^2(U)$ satisfies

$$u(x) = \fint_{\partial B(x,r)} u dS \tag{0.35}$$

for each ball $B(x,r) \in U$, then u is harmonic.

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Proof.

Suppose $\Delta u \neq 0$, then $\exists B(x,r) \in U$ such that $\Delta u > 0$ say, but then

$$\phi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y) dy > 0 \tag{0.36}$$

which contradicts the fact that ϕ is a constant function. $\square$

Maximum Principle

Theorem

(Strong Maximum Principle) Suppose $u \in C^2(U) \cap C(\overline{U})$ *is harmonic within* U*, then*
(i) $\max_{\overline{U}} u = \max_{\partial U} u$*;*
(ii) Furthermore, if U *is connected and there exists a point* $x_0 \in U$ *such that* $u(x_0) = \max_{\overline{U}} u$*, then* u *is constant within* U*.*

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(Strong Maximum Principle) Suppose $u \in C^2(U) \cap C(\overline{U})$ *is harmonic within* U*, then*
(i) $\max_{\overline{U}} u = \max_{\partial U} u$*;*
(ii) Furthermore, if U *is connected and there exists a point* $x_0 \in U$ *such that* $u(x_0) = \max_{\overline{U}} u$*, then* u *is constant within* U*.*

It suggests that if u is harmonic on a bounded domain U, then u attains its maximum value on the boundary of U

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$$M = u(x_0) = \fint_{B(x_0,r)} u(y)dy \tag{0.37}$$

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and such will hold only if $u \equiv M$ within $B(x_0, r)$, and $u(y) = M$. □

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Theorem

Let $g \in C(\partial U), f \in C(U)$, then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{0.39}$$

Proof.

Assume u_1, u_2 both solves the boundary value problem, then let $w = u_1 - u_2$, then we have

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{in } \partial U \end{cases} \tag{0.40}$$

then by maximum principle, $\max_{\partial U} w = \max_{\partial \bar{U}} w = 0$, i.e $u_1 = u_2$. □

Smoothness

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The proof of this theorem requires some knowledge on mollifiers, so before the proof we shall spend some time on mollifiers.

Definition

Define a smooth function $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{0.41}$$

for some constant $C > 0$ so that $\int_{\mathbb{R}^n} \eta dx = 1$. Then for each $\epsilon > 0$, set

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right). \tag{0.42}$$

We call η the standard mollifier, the functions η_ϵ are C^∞ as well.

Definition

Denote $U_\epsilon = \{x \in U |\ \text{dist}(x, \partial U) > \epsilon\}$, if a function $f : U \to \mathbb{R}$ is locally integrable, its mollification is defined by

$$f^\epsilon = \int_U \eta_\epsilon(x-y)f(y)dy \tag{0.43}$$

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Now we prove the theorem.

Let η be the standard mollifier, set $u^\epsilon = \eta_\epsilon * u$ in U_ϵ, we first show $u^\epsilon \in C^\infty$, which is left as an exercise to the reader. Then to show u is smooth, we will show in fact $u \equiv u^\epsilon$ on U_ϵ for each $\epsilon > 0$. Let $x \in U_\epsilon$, then we know the support of $\eta_\epsilon(x - y)$ as a function of y is given by $\{y : |x - y| < \epsilon\}$, and thus we have

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$$\begin{aligned}
u^\epsilon(x) &= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)dy \\
&= \frac{1}{\epsilon^n} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) \left(\int_{\partial B(x,r)} udS\right) dr \\
&= \frac{1}{\epsilon^n} u(x) \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) n\alpha(n) r^{n-1} dr \\
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so $u \in C^\infty(U_\epsilon)$ for all $\epsilon > 0$.

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Lemma

Assume u is harmonic in U, then

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x_0,r))}, C_0 = \frac{1}{\alpha(n)}, C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)} \tag{0.44}$$

for each ball $B(x_0, r) \in U$ and each multiindex α of order $|\alpha| = k$

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The proof to the lemma is left as an exercise.

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Fix $x_0 \in \mathbb{R}^n$ and $r > 0$, we have

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Liouville's theorem can also be used to conclude the uniqueness of Poisson's equation:

Theorem

Let $f \in C_C^2(\mathbb{R}^n)$ and $n \geq 3$, then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy + C \tag{0.46}$$

for some constant C.

Proof.

If u' is another solution, then $u - u'$ is a constant by Liouville's theorem. $\square$

Harnack's Inequality

We write $V \subset\subset U$ to denote $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact, we say V is compactly contained in U.

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Theorem

For each connected open set $V \subset\subset U$, there exists a positive constant C depending only on V such that

$$\sup_V u \leq C \inf_V u \tag{0.47}$$

for all non-negative harmonic functions u in U. In particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \tag{0.48}$$

for all $x, y \in V$.

Proof.

Consider a ball $B(a, r) \subset U$, and let $x, y \in B(a, \frac{1}{4}r)$, we first claim that $B(x, \frac{1}{4}r) \subset B(y, \frac{3}{4}r) \subset B(a, r)$.

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$$u(x) = \fint_{B(x,\frac{1}{4}r)} u(z)dz \leq \fint_{B(\frac{3}{4}r,y)} u(z)dz = 3^n u(y). \tag{0.49}$$

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Now since V is compact so there exists a finite cover of balls $\{B_i\}_{i=1}^M$ where $B_i \equiv B(a_i, r_i) \subset U$.

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$$u(x) \leq 3^n u(x_1) \leq \cdots \leq 3^{n\ell} u(y) \leq 3^{nM} u(y) \tag{0.50}$$

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for all $x, y \in V$, and hence we have

$$\sup_V u \leq C \inf_V u. \tag{0.51}$$

Convergence Theorems

Theorem

(Weierstrass Convergence Theorem) If a sequence of harmonic functions $\{h_n\}$ converges locally uniformly on U, then the limit $h = \lim_{n\to\infty} h_n$ is also harmonic on U, furthermore the derivatives converge locally uniformly as well, i.e $\lim_{n\to\infty} D_{x_i} h_n = D_{x_i} h$.

Proof.

Let $K \subset U$ be compact, we choose a larger compact set $\tilde{K}$ and $r := \mathrm{dist}(K, \mathbb{R}^n/\tilde{K}) > 0$, we use the derivative estimate to get

$$|D^\alpha h_l(a) - D^\alpha h_k(a)| \leq \frac{C}{r^{|\alpha|}} \sup_{\tilde{K}} |h_l - h_k| \tag{0.52}$$

and from the uniform convergence of $\{h_n\}$ on $\tilde{K}$, and we see that $\{D^\alpha h_n\}$ is a Cauchy sequence on K, thus uniformly convergent on K, and locally uniformly convergent on U. We can also show that $\lim_{n\to\infty} D^\alpha h_n = D^\alpha h$ locally uniformly on U for all index α. In particular we have $\Delta h = \lim_{n\to\infty} \Delta h_n$ and hence h is harmonic. □

Theorem

(Harnack Convergence Theorem) Consider an increasing sequence of harmonic functions $\{h_n\}_{n=1}^{\infty}$ *on* $U \subset \mathbb{R}^n$*, then either* $\lim_{n\to\infty} h_n = \infty$ *or it converge locally uniformaly to a harmonic function. In particular, if* $\exists x_0 \in U$ *such that* $\lim_{n\to\infty} h_n(x_0) \neq \infty$*, then we may conclude locally uniform convergence.*

Proof.

Assume $x_0 \in U$ such that $\lim_{n\to\infty} h(x_0) \neq \infty$. For all $k \leq l$, K compact, and $x_0 \in K \in U$, we have $h_l - h_k$ as a non-negative harmonic function, and hence by Harnack's inequality,

$$\max_K (h_l - h_k) \leq C \min_K (h_l - h_k) \leq C(h_l(x_0) - h_k(x_0)) \tag{0.53}$$

which implies the uniform Cauchy property of $\{h_n\}$ on K, and hence locally uniform convergence is achieved. We now let $\lim_{n\to\infty} h_n = h$ and it remains to show h is harmonic, which follows from Weierstrass convergence theorem.

□

Thanks