Question 1 (a)

Proof. To show that $\lim \left(\frac{n^2+n}{2n^2-3}\right) = \frac{1}{2}$, we need to show that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t $\forall n \geq N$: $|a_n - L| < \epsilon$.

In this case, $(a_n) = \frac{n^2+2n}{2n^2-3}$, $L = \frac{1}{2}$.

$|a_n - L| = \left|\frac{n^2+n}{2n^2-3} - \frac{1}{2}\right| = \left|\frac{2n+3}{4n^2-6}\right|$. When n is enfficently large, especially, say $n \geq 3$, then we have $\left|\frac{2n+3}{4n^2-6}\right| < \left|\frac{2n+3}{4n^2-9}\right| = \frac{2n+3}{4n^2-9}$, because $2n+3 > 0$ and $4n^2 - 9 > 0$ in this case.

We may observe that $\frac{2n+3}{4n^2-9} = \frac{2n+3}{(2n+3)(2n-3)} = \frac{1}{2n-3}$. This operation is totally doable because $2n-3 > 0$ and $2n+3 > 0$.

Then, if we let $\frac{1}{2n-3} < \epsilon$, solve for n we get $n > \frac{1}{\epsilon} + 3$. Then, by our previous steps, we say that

$$|a_n - L| < \frac{1}{2n-3} < \epsilon \quad \text{for all} \quad n > \frac{1}{\epsilon} + 3$$

If we let $N > \frac{1}{\epsilon} + 3$, then we conclude that $\forall n \geq N$, we have $|a_n - L| < \epsilon$, which means

$$\lim \left(\frac{n^2+n}{2n^2-3}\right) = \frac{1}{2}$$

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Question 2

Proof. By definition, if $\lim(x_n) = x$, it means $\exists N \in \mathbb{N}$, $\forall x \geq N$, $|x_n - x| < \epsilon$, for all $\epsilon > 0$.

$$\implies |x_n - x| < \epsilon$$

$$\implies -\epsilon < x_n - x < \epsilon$$

$$\implies -\epsilon + x < x_n < \epsilon + x$$

If we choose $\epsilon = \frac{1}{2}x$, then we have $\frac{1}{2}x < x_n < \frac{3}{2}x$, for all $x > N$ for some N. In this case, it is the same as saying $\frac{1}{2}x < x_n < 2x$, because $x > 0$. So we have proved the inequility.

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