Problem 1

1.1 part (a)

Proof. In order to prove $\forall x, y \in \mathbf{R} : y^3 - x^3 = (y-x)(x^2+xy+y^2)$,

We can expand the $R.H.S$ of the equation,that is

$$\begin{aligned}(y-x)(x^2+xy+y^2) &= yx^2+xy^2+y^3-x^3-x^2y-xy^2\\ \bullet &= y^3-x^3+(yx^2-yx^2)+(xy^2-xy^2)\\ &= y^3-x^3\end{aligned}$$

Those two parts got cancelled in step $\bullet$

So for the right hand side, we get $R.H.S = y^3 - x^3$

Compared to the origional equation, we found that

$$(y-x)(x^2+xy+y^2) = y^3-x^3$$

That is, $L.H.S = R.H.S$

So, we have proved that $y^3 - x^3 = (y-x)(x^2+xy+y^2)$.

□

1.2 part (b)

Proof. In order to prove $\forall x, y \in \mathbf{R} : x^2+xy+y^2 \geq 0$,

Using direct proof, we firstly noticed that:

$$(x+y)^2 \geq 0 : \forall x, y \in \mathbf{R}$$

By expanding the $L.H.S$ of the above,we get :

$$x^2+2xy+y^2 \geq 0$$

Then, we substract both $L.H.S$ and $R.H.S$ by xy, and still holds for:

$$x^2+xy+y^2 \geq -xy \qquad \blacktriangle$$

Thus, there are three cases for this:

- Case 1: If x, y has the same sign, (i.e $xy > 0$) then

  From the $L.H.S$ of $\blacktriangle$, $x^2 > 0; y^2 > 0; xy > 0$

  So it is obvious that $x^2+xy+y^2 > 0$

- Case 2: If x, y has the different sign, (i.e $xy < 0$) then

  It is obvious that $-xy > 0$ instead

  Then from the $R.H.S$ of ▲, we see that $R.H.S > 0$

  Since the inequality says that $L.H.S > R.H.S$

  So it demonstrates that $x^2 + xy + y^2 > -xy > 0$

  Which is $x^2 + xy + y^2 > 0$

- Case 3: If $xy = 0$ then

  For cases like $x = 0$ **or** $y = 0$

  It is clear that from ▲, $L.H.S > 0$ and $R.H.S = 0$, the inequality holds

  For a special case, where $x = 0$ **and** $y = 0$

  In this case only, from ▲ we can apply that $L.H.S = R.H.S = 0$

In conclusion, we can say that $\forall x, y \in \mathbf{R}, x^2 + xy + y^2 \geq 0$

$x^2 + xy + y^2 = 0$ *if and only if* $x = 0, y = 0$

□

1.3 part (c)

Proof. In order to prove $f : \mathbf{R} \longrightarrow \mathbf{R}, x \mapsto x^3$ is strictly increasing, we need to show

$\forall x, y \in \mathbf{R}$, if $x > y$, then $f(x) > f(y)$

That is, prove $x^3 > y^3$ if $x > y$

By direct proof, if $x > y$, then we can conclude that $x - y > 0$

As shown in part (b), $\forall x, y \in \mathbf{R}, x^2 + xy + y^2 \geq 0$

Since $x \neq y$, which means x, y cannot be all zero

So the statement $x^2 + xy + y^2 > 0$ is true for this case

So $(x^2 + xy + y^2)(x - y) > 0$ is also true

By expanding the formula above, we get $x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 > 0$

That is, $x^3 - y^3 > 0$, which is exactly what we need to prove

So, $f : \mathbf{R} \longrightarrow \mathbf{R}, x \mapsto x^3$ is strictly increasing.

□

Problem 2 and Problem 3 are omitted

Problem 4

4.1 part (a)

Proof. In order to show that $\sqrt{6}$ is irrational,

Using proof by contradiction, assume that $\sqrt{6}$ is rational, so it can be written as

$$\sqrt{6} = \frac{a}{b}; \quad a, b \in \mathbb{N}; \quad b \neq 0; \quad \gcd(a, b) = 1$$

$$\Longrightarrow 6 = \frac{a^2}{b^2}$$
$$\Longrightarrow 6b^2 = a^2$$

Since $6b^2$ must be an even number, it applies that a^2 is also an even number

So, a is an even number, also a is divisible by 6, meaning that

$$\exists c \in \mathbb{N} : a = 6c$$
$$\Longrightarrow 6b^2 = 36c^2$$
$$\Longrightarrow b^2 = 6c^2$$

Since $6c^2$ must be an even number, it applies that b^2 is also an even number

So, b is an even number

If both a, b are even, meaning that $\gcd(a, b) \geq 2$, thus $\gcd(a, b) \neq 1$

Which leads to a contradiction. So $\sqrt{6}$ must be irrational.

□

4.2 part (b)

Proof. In order to prove that $\sqrt{2} + \sqrt{3}$ is irrational,

Using proof by contradiction, assume that both$\sqrt{2}, \sqrt{3}$ are rational

So $\sqrt{2} + \sqrt{3}$ is rational

By taking the square, $(\sqrt{2} + \sqrt{3})^2$ is also rational

By expanding the square, $2 + 3 + \sqrt{2 \times 3}$ is also rational

Which is, $5 + \sqrt{6}$ is rational

Since 5 is rational, and $\sqrt{6}$ is irrational (shown in part a)

Then $5 + \sqrt{6}$ can not be rational, that is $5 + \sqrt{6}$ is irrational.

That leads to a contradiction, so $\sqrt{2} + \sqrt{3}$ is irrational.

□

Problem 5

5.1 part (a)

Proof. In order to prove

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{\text{n nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

By induction, we know that the range of n is $n \in \mathbf{N}$

- **The Base Case :**

▲ When $n = 1$

From the $L.H.S$ of the equation ♣, we know $L.H.S = \sqrt{2}$

Also, $R.H.S = 2\cos\left(\frac{\pi}{4}\right) = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$

So, $L.H.S = R.H.S$ for base case, which is true.

- **The Inductive Case :**

▲ Suppose ♣ is true for one $n \in \mathbb{N}$, that is

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{\text{n nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

▲ Then we need to prove it also holds for $n + 1$, which is to prove

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{2}}}}}}_{\text{n+1 nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+2}}\right)$$

Assume that

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{\text{n nested sqyare roots}} = t$$

So, we only need to prove that

$$\sqrt{2+t} = 2\cos\left(\frac{\pi}{2^{n+2}}\right)$$

$$
\begin{aligned}
\sqrt{2+t} &= 2\cos\left(\frac{\pi}{2^{n+2}}\right) \\
&= 2\cos\left(\frac{\pi}{2^{n+1}} \times \frac{1}{2}\right) \\
2+t &= 4\cos^2\left(\frac{\pi}{2^{n+1}} \times \frac{1}{2}\right) \\
2+t &= 4\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2^{n+1}}\right)\right) \\
2+t &= 2 + 2\cos\left(\frac{\pi}{2^{n+1}}\right) \\
t &= 2\cos\left(\frac{\pi}{2^{n+1}}\right) \qquad \blacklozenge
\end{aligned}
$$

Since

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{\text{n nested sqyare roots}} = t$$

According to the bease case, we know that the above equation $\blacklozenge$ is correct

In this case, we can also say that it is correct for $n+1$

- **In conclusion**

So, according to the base case and inductive case, it is correct for all $n \in \mathbb{N}$ such that

$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{\text{n nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

$\square$

Problem 6

6.1 part (a)

Proof. In order to prove

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

by induction, we know the range for n is $n \in \mathbb{N}_0$

- **The Base Case**

  when $n = 0$, it is clear that $L.H.S = R.H.S = 1$, which is correct

- **The Inductive Case**

  We suppose the equation is correct for one $n \in \mathbb{N}_0$, such that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Then we need to show that it is also correct for

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1} \qquad \blacktriangle$$

By expanding the $R.H.S$ of the above equation, we get

$$R.H.S = 2 \times \left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \right] \qquad \blacklozenge$$

According to the proporty

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} : n \in \mathbb{N}_0; 1 \leq k \leq n$$

In this case,

•If n is even, then

$$R.H.S = 2 \times \left[\underbrace{\binom{n+1}{1} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1}}_{\text{Align the first n -1 terms into pairs and apply the proporty}} + \binom{n}{n} \right] \qquad \textit{Equation(a)}$$

$$R.H.S = 2 \times \left[\underbrace{\binom{n+1}{2} + \binom{n+1}{4} + \cdots + \binom{n+1}{n}}_{\text{Align the first n -1 terms into pairs and apply the proporty}} + \binom{n}{0} \right] \qquad \textit{Equation(b)}$$

Since we know from the proporty that $\binom{n}{n} = \binom{n}{0} = 1$

So from *Equation(a)*, *Equation(b)*, we conclude that

$$\binom{n+1}{1} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1} = \binom{n+1}{2} + \binom{n+1}{4} + \cdots + \binom{n+1}{n}$$

Now apply *Equation(a)* + *Equation(b)*, we get

$$R.H.S = \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n}{0} + \binom{n}{n}$$

Notice that for $n+1$, it also holds that $\binom{n+1}{n+1} = \binom{n+1}{0} = 1$

In this case,

$$R.H.S = \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n+1}{0} + \binom{n+1}{n+1}$$

Now by the proporty, it is pretty clear that

$$R.H.S = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

Compare to equation ▲, we find that $L.H.S = R.H.S$, that is what we need to prove

So the equality is true for even n

•If n is odd, then

$$R.H.S = 2 \times \left[\binom{n+1}{1} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-2} + \binom{n}{n}\right]$$

Since we know that from even cases,

$$\binom{n+1}{1} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1} = \binom{n+1}{2} + \binom{n+1}{4} + \cdots + \binom{n+1}{n}$$

So, it is obvious that

$$R.H.S = \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \cdots + \binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n+1}{0} + \binom{n+1}{n+1}$$

$$R.H.S = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

Then back to equality ▲,it is clear that $L.H.S = R.H.S$

So the equality is true for odd n

•In Conclusion

So according to the base case and inductive case, $\forall n \in \mathbb{N}_0, 1 \leq k \leq n$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{is true.}$$

$\square$