# Question 1

The sequence $(x_n)$ which satisifies the properties as mentioned is:

$$x_n = n + \frac{1}{n}$$

**Now let's prove it**

*Proof.* We will show that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t $\forall n \geq N$, $|x_{n+1} - x_n| < \epsilon$.

It is not hard to observe that $|x_{n+1} - x_n| = \left| \left(n + 1 + \frac{1}{n+1}\right) - \left(n + \frac{1}{n}\right) \right| = \left| \frac{1}{n+1} - \frac{1}{n} + 1 \right|$.

Then, we can do further simplification, $|x_{n+1} - x_n| = \left| \frac{-1}{n(n+1)} + 1 \right|$. By *Triangle Inequality,*

$$\begin{aligned} |x_{n+1} - x_n| &\leq \left| \frac{-1}{n(n+1)} \right| + |1| \\ &= \frac{1}{n(n+1)} + 1 \end{aligned}$$

Since $n$ is always a positive integer, So $n + 1 > n$, conversely $\frac{1}{n+1} < \frac{1}{n}$. So we get

$$|x_{n+1} - x_n| < \frac{1}{n^2} + 1$$

In this case, if we slove for $\frac{1}{n^2} + 1 < \delta$, where $\delta > 1$, it's the same as sloving $\frac{1}{n^2} < \epsilon$. where $\epsilon > 0$. So, we get when $n \geq \sqrt{\frac{1}{\epsilon}}$, it is always true that $|x_{n+1} - x_n| < \frac{1}{n^2} + 1 < \epsilon + 1$.

**However**, as we can see, $(x_n)$ does not converge, because it is a strictly increasing sequence when $n \geq 1$, and $\lim(x_n)$ does not exist. (infinity)

**So, according to the judgements above, this sequence satisities those properties**.

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# Question 2

## Part (a) :

*Proof.* Since we know that $x_1 > 0$, then the base case is automatically correct.Now we assume that the statement is true for a fixed integer $n$. Then we will prove that it still holds for $n+1$. Since we know $x_n > 0$ by our assumption, and according to the definition of this sequence, $x_{n+1} = \dfrac{1}{2+x_n}$, which means that $x_{n+1}$ must also be positive. So the inductive step is also proved. Hence, *WLOG*, we say that $x_n > 0$ for all $\mathbb{N}$. ■

## Part (b) :

*Proof.* To show tht $(x_n)$ is contractive, we need to show that $\forall n \in \mathbb{N}$, $\exists c \in (0,1)$, s.t

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|$$

By the formula $x_{n+1} = \dfrac{1}{2+x_n}$, we get $|x_{n+2} - x_{n+1}| = \left|\dfrac{1}{2+x_{n+1}} - \dfrac{1}{2+x_n}\right|$. Then, by further simplification, we get

$$|x_{n+2} - x_{n+1}| = \left|\frac{x_n - x_{n+1}}{(2+x_{n+1})(2+x_n)}\right| = \frac{1}{|(2+x_{n+1})(2+x_n)|} \cdot |x_{n+1} - x_n|$$

We see that the term $\left|\dfrac{1}{(2+x_{n+1})(2+x_n)}\right|$ is always between 0 and 1 by the property, and it is constant befined by a given sequence, also note that $x_n$ is always positive for any $n \in \mathbb{N}$, which means $\dfrac{1}{(2+x_{n+1})(2+x_n)} < \dfrac{1}{2 \times 2} < 1$. So by the property, we conclude that

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|$$

■

## Part (c) :

*Proof.* By the property, we know that $\lim(x_{n+1}) = \lim(x_n)$, then if we use the relations between $x_{n+1}$ and $x_n$, we get

$$\lim(x_{n+1}) = \lim(\frac{1}{2 + x_n})$$

Then by limit laws, we can simplify as:

$$\lim(x_{n+1}) = \frac{1}{2 + \lim(x_n)}$$

As we mentioned, $\lim(x_{n+1}) = \lim(x_n)$, so we may let $\lim(x_n) = u$, then solve for the equation

$$u = \frac{1}{2 + u}$$

Then we get $u = -1 \pm \sqrt{2}$, but note that by definition, $x_n$ is always positive, so the only solution inserting $u = -1 + \sqrt{2}$, i.e $\lim(x_n) = -1 + \sqrt{2}$. ■