Problem 3(a)

Proof. To prove A is bounded and has supremum and infimum, show that:

- **Bounded from above :**

Noticed that $A := \left\{2 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$, for $x \in A$, we have $x = 2 + \frac{(-1)^n}{n}$.

Since we know that $\frac{(-1)^n}{n} < \frac{1}{n}$, so it is clear that $2 + \frac{(-1)^n}{n} < 2 + \frac{1}{n} \leq 2 + 1 = 3$.

Then we can say that A is bounded from above because $\forall x \in A, x \leq 3$.

- **Bounded from below :**

Similarily, we know that $\frac{(-1)^n}{n} \geq -\frac{1}{n}$, so it is clear that $2 + \frac{(-1)^n}{n} \geq 2 - \frac{1}{n} \geq 2 - 1 = 1$.

Then we say that A is bounded from below because $\forall x \in A, x \geq 1$.

Since $1 \in A$, and $\forall x \in A, x \geq 1$, We may conclude that $\inf(A) = 1$.

- **Supremum :**

Assume that $\sup(A) = \mu$, this means that $\forall x \in A$, $\mu \geq x$.

By the justifications before, we know that $2 + \frac{(-1)^n}{n} < 2 + \frac{1}{n}$, when n is even, $(-1)^n \geq 0$

Since 2 is the minimum even number in $\mathbb{N}$, it applies that $2 + \frac{(-1)^n}{n} = 2 + \frac{1}{n}$, for even numbers n.

Similarily, $2 + \frac{(-1)^n}{n} = 2 - \frac{1}{n}$, for odd numbers n.

In this case, we may conclude that $\sup(A) \geq 2$ because for each n to be even the statement is always true, and if n is odd, the result is always less than 2.

Also, we know that $f(n) = \frac{1}{n}$ is strictlt decreasing, so for even numbers n, $\frac{1}{2} \geq \frac{1}{n}$.

So, $2 + \frac{(-1)^n}{n} \leq 2 + \frac{1}{2} = \frac{5}{2}$. By definition, we then conclude that $\sup(A) = \frac{5}{2}$

- **Infimum :**

As shown in the bounded from below part, we already know that $\inf(A) = 1$.

Finally, we have proved that A is bounded and we know that $\sup(A) = \frac{5}{2}$ and $\inf(A) = 1$.

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Problem 3(b)

Proof. To prove B is bounded and has supremum and infimum, show that:

- **Bounded from above:**

To show that $B := \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\right\}$, for $x \in B$, we have $x = (-1)^n + \frac{1}{n}$.

Since we know that $\frac{1}{n} + (-1)^n < \frac{1}{n} + 1 \leq 2$, so we can say that B is bounded from above, because $\forall x \in B, x \leq 2$. And obviously 2 is an upper bound.

- **Bounded from below :**

Since we know that $(-1)^n \geq -1$,and $\frac{1}{n} > 0$, so $\frac{1}{n} + (-1)^n \geq -1$, then we say that B has a lower bound, because $\forall x \in B, x \geq -1$, thus B is bounded from below.

- **Supremum :**

Since when n is even, $2 > (-1)^n + \frac{1}{n} > 1$, when n is odd, $-1 < (-1)^n + \frac{1}{n} < 0$. It implies that $2 > \sup(B) > 1$. Again the function $f(x) = \frac{1}{x}$ is strictly decreasing, so $\frac{1}{2} \geq \frac{1}{n}$ for all even n.

In this case, $\frac{1}{n} + (-1)^n \leq \frac{1}{2} + 1 = \frac{3}{2}$. So $\sup(B) = \frac{3}{2}$.

- **Infimum**

Similarily, we know that $\frac{1}{n} + (-1)^n > -1$, when n is odd, since $\frac{1}{n} \leq 1$, so $\frac{1}{n} + (-1)^n \leq 0$

Then, we may say that $\sup(B) \leq 0$. Since $f(x) = \frac{1}{x}$ is strictly decreasing, when x gets larger, $f(x)$ will approaching 0.

So in this case, $\frac{1}{n} + (-1)^n \geq -1$, for all $n \in \mathbb{N}$. Then we conclude that $\inf(B) = -1$.

Finally, we have proved that B is bounded and we know that $\sup(A) = \frac{3}{2}$ and $\inf(A) = -1$.

$\blacksquare$

Problem 6

Proof. We know that S is bounded from above, meaning that $\exists \lambda \in \mathbb{R} : \forall s \in S, s \leq \lambda$I in this case, λ is an upper bound of S. If $k \geq 0$, then it is still true that $ks \leq k\lambda$.

By definition, we know that $ks \in kS$, so $\forall t \in kS : t \leq k\lambda$. Then we say kS is bounded from above. As $k\lambda$ is an upper bound of kS.

Then we assume that $\sup(S) = \mu$, meaning that μ is the smallest upper bound of S, i.e, for all upperbounds $\lambda, \lambda \geq \mu$. So $\forall s \in S : s \leq \mu \leq \lambda$.

Similarily, if $k \geq 0$, then $ks \leq k\mu \leq k\lambda$. By definition, $ks \in kS$, so $k\mu$ is the supremum of kS.

Then, $\sup(S) = \mu$ and $\sup(kS) = k\mu$. So we conclude that $\sup(kS) = k\sup(S)$.

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