# Problem 1

Recall the formula:

$$\text{Say } \boldsymbol{x}, \boldsymbol{y} \text{ are vectors in } \mathbb{R}^n \text{ then the angle between } \boldsymbol{x}, \boldsymbol{y} \text{ is } \cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}$$

In this problem, we are given that $\boldsymbol{v}_1 = \left(4\sqrt{6},\quad 8,\quad 4\sqrt{6}\right)$ and $\boldsymbol{v}_2 = \left(\frac{15\sqrt{2}}{2},\quad 0,\quad \frac{15\sqrt{2}}{2}\right)$. So

$$\begin{aligned}
\cos\theta &= \frac{\left(4\sqrt{6} \times \frac{15\sqrt{2}}{2}\right) \times 2}{\sqrt{256} \times \sqrt{15}} \\
&= \frac{120\sqrt{3}}{160} \\
&= \frac{\sqrt{3}}{2} \\
&= \cos\left(\frac{\pi}{6}\right)
\end{aligned}$$

So we say that the angle between $\boldsymbol{v}_1, \boldsymbol{v}_2$ is $\frac{\pi}{6}$.

# Problem 2

*Proof.* In order to show that $\|\boldsymbol{x}-\boldsymbol{y}\|\|\boldsymbol{z}\| \leq \|\boldsymbol{y}-\boldsymbol{z}\|\|\boldsymbol{x}\| + \|\boldsymbol{z}-\boldsymbol{x}\|\|\boldsymbol{y}\|$, we can devide both $L.H.S$ and $R.H.S$ by a same positive value $\|\boldsymbol{x}\|\|\boldsymbol{y}\|\|\boldsymbol{z}\|$, so it becomes

$$\begin{aligned}\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} &\leq \frac{\|\boldsymbol{y}-\boldsymbol{z}\|}{\|\boldsymbol{y}\|\|\boldsymbol{z}\|} + \frac{\|\boldsymbol{z}-\boldsymbol{x}\|}{\|\boldsymbol{z}\|\|\boldsymbol{x}\|} \\ &= \frac{1}{\|\boldsymbol{z}\|}\left(\frac{\|\boldsymbol{y}-\boldsymbol{z}\|}{\|\boldsymbol{y}\|} + \frac{\|\boldsymbol{z}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right)\end{aligned}$$

$$\text{By trangle inequality, we know that } \frac{1}{\|\boldsymbol{z}\|}\left(\frac{\|\boldsymbol{y}-\boldsymbol{z}\|}{\|\boldsymbol{y}\|} + \frac{\|\boldsymbol{z}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right) \geq \frac{1}{\|\boldsymbol{z}\|}\|\frac{\boldsymbol{y}-\boldsymbol{z}}{\boldsymbol{y}} + \frac{\boldsymbol{z}-\boldsymbol{x}}{\boldsymbol{x}}\|$$

$$\text{Which is } \frac{1}{\|\boldsymbol{z}\|}\left(\frac{\|\boldsymbol{y}-\boldsymbol{z}\|}{\|\boldsymbol{y}\|} + \frac{\|\boldsymbol{z}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right) \geq \frac{1}{\|\boldsymbol{z}\|}\frac{\|\left(\boldsymbol{y}-\boldsymbol{z}\right)\boldsymbol{x} + \left(\boldsymbol{z}-\boldsymbol{x}\right)\boldsymbol{y}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{So } L.H.S \geq \frac{1}{\|\boldsymbol{z}\|}\frac{\|\boldsymbol{y}\boldsymbol{x}-\boldsymbol{z}\boldsymbol{x}+\boldsymbol{z}\boldsymbol{y}-\boldsymbol{x}\boldsymbol{y}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{Which is } L.H.S \geq \frac{1}{\|\boldsymbol{z}\|}\frac{\|\boldsymbol{z}\boldsymbol{y}-\boldsymbol{z}\boldsymbol{x}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{That is } L.H.S \geq \frac{1}{\|\boldsymbol{z}\|}\frac{\|\boldsymbol{z}\|\|\boldsymbol{y}-\boldsymbol{x}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{Then, we conclude that } L.H.S \geq \frac{\|\boldsymbol{y}-\boldsymbol{x}\|}{\|\boldsymbol{x}\boldsymbol{y}\|} = \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{Then, it is the same as saying } \frac{1}{\|\boldsymbol{z}\|}\left(\frac{\|\boldsymbol{y}-\boldsymbol{z}\|}{\|\boldsymbol{y}\|} + \frac{\|\boldsymbol{z}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right) \geq \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\boldsymbol{y}\|}$$

$$\text{Then, we have proved that } \|\boldsymbol{x}-\boldsymbol{y}\|\|\boldsymbol{z}\| \leq \|\boldsymbol{y}-\boldsymbol{z}\|\|\boldsymbol{x}\| + \|\boldsymbol{z}-\boldsymbol{x}\|\|\boldsymbol{y}\|$$

□

# Problem 3

*Proof.* We need to show that $\forall \epsilon > 0, \exists \delta(\epsilon, \boldsymbol{x}_0) | \quad \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \Longrightarrow \|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)\| < \epsilon$

In this problem, we know that $\|\boldsymbol{x} - \boldsymbol{x}_0\| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$

Also, $\|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)\| = |a(x - x_0) + b(y - y_0) + c(z - z_0) + d - d|$. Then by *Cauchy-Schwarz Inequality*,

$$|a(x - x_0) + b(y - y_0) + c(z - z_0) + d - d| \leq \sqrt{a^2 + b^2 + c^2}\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$\text{If we choose} \quad \epsilon = \delta\sqrt{a^2 + b^2 + c^2} \quad \text{then, since} \quad a^2 + b^2 + c^2 \neq 0,$$

$$\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \Longrightarrow \|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)\| < \epsilon$$

$$\text{So} \quad \lim_{\boldsymbol{x} \to \boldsymbol{x}_0} (ax + by + cz + d) = ax_0 + by_0 + cz_0 + d$$

$\square$

# Problem 4

In order to show that $f(x, y)$ is continuous at $(0, 0)$, we need to show $\lim_{\boldsymbol{x}\to\boldsymbol{0}} = f(0, 0) = 0$

$$\text{Which is to show } \lim_{\boldsymbol{x}\to\boldsymbol{0}} \frac{x^{1+a}y^{1+b}}{x^2+y^2} = 0$$

Since we know that $\forall x, y \in \mathbb{R}, (x-y)^2 \geq 0$, That is the same as $x^2 + y^2 \geq 2xy$, also true for $x^2 + y^2 \geq |2xy|$, and finally the statement

$$|\frac{xy}{x^2+y^2}| \leq \frac{1}{2}$$

$$\begin{aligned} \text{Then, } \frac{x^{1+a}y^{1+b}}{x^2+y^2} &= \frac{xy}{x^2+y^2}x^a y^b \\ &\leq \frac{1}{2}x^a y^b \end{aligned}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} f(x, y) \leq \frac{1}{2} \lim_{(x,y)\to(0,0)} x^a y^b = \frac{1}{2} \lim_{x\to 0} x^a \lim_{y\to 0} y^b$$

If $|x - 0| < \delta$, we pick $\epsilon = \delta^{a-1}$ , then for the function $f(x) = x^a$, we can show that $|x - 0| < \delta$, then $|x^a - 0| < \epsilon$. So, $\lim_{x\to 0} x^a = 0$, We can also use the same process to show that $\lim_{y\to 0} y^b = 0$ by taking $\epsilon = \delta^{b-1}$

Since the limits are both zero, we can conclude that the origional limit is also zero. Since $\lim_{\boldsymbol{x}\to\boldsymbol{0}} f(x, y) = f(0, 0) = 0$, we say $f(x)$ is continuous at $(0, 0)$

# Problem 5

## Part(a)

Recall that the tangent plane for function $z = f(x, y)$ at $(x_0, y_0)$ is

$$\boxed{z - f(x_0, y_0) = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} (x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} (y - y_0)} \qquad \textit{Equation (1)}$$

At the point $(1, 2, f(1, 2))$, we can calculate the following:

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} (x - x_0) = (e^{y^2} - 2xye^{x^2})\bigg|_{(1,2)} (x - 1) = (e^4 - 4e)(x - 1)$$

$$\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} (y - y_0) = (2xye^{y^2} - e^{x^2}\bigg|_{(1,2)} (y - 2)) = (4e^4 - e)(y - 2)$$

$$f(1, 2) = e^4 - 2e$$

Then we can plug in those figures into the formula, and we can get

$$z - e^4 + 2e = (e^4 - 4e)(x - 1) + (4e^4 - e)(y - 2)$$

$$\boxed{z = (e^4 - 4e)x + (4e^4 - e)y + 4e - 8e^4} \qquad \textit{Equation (2)}$$

The boxed equation 2 is exactly what we need to find.

## Part(b)

Firstly find the general tangent plane equation for $z = x^2 - y^2$, at $(x_0, y_0)$ by using equation (1) which was shown in part(a), we get

$$z - x_0^2 + y_0^2 = 2x_0(x - x_0) - 2y_0(y - y_0)$$

$$\boxed{z = 2x_0x - 2y_0y - x_0^2 + y_0^2}$$

If the above equation is parallel to equation (2), then it satisifies

$$\begin{cases} x_0 = (e^4 - 4e) \\ y_0 = (e - 4e^4) \end{cases}$$

The point which tangent plane parallel to the one in part(a) is $(e^4 - 4e, e - 4e^4)$

# Problem 6

*Proof.* To show that the angle $\theta$ between $\boldsymbol{c}(t)$ and $\boldsymbol{c}'(t)$ is a constant number, we only need to show that $\cos\theta$ is a constant. Recall the definition of the angle between two vectors, i.e to show

$$\cos\theta = \frac{\boldsymbol{c}(t) \bullet \boldsymbol{c}'(t)}{\|\boldsymbol{c}(t)\|\|\boldsymbol{c}'(t)\|} \quad \text{is a constant}$$

$\boldsymbol{c}(t) = \langle e^t \cos(t), e^t \sin(t)\rangle$

$\boldsymbol{c}'(t) = \langle e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t))$

$\|\boldsymbol{c}(t)\|^2 = e^{2t}\cos^2(t) + e^{2t}\sin^2 t = e^{2t}$

$\|\boldsymbol{c}'(t)\|^2 = e^{2t}(\cos^2(t) + \sin^2(t) - \cos(t))\sin(t) + e^{2t}(\cos^2(t) + \sin^2(t) + \cos(t))\sin(t) = 2e^{2t}$

So, by pluging in these figures into the formlua above, we get

$$\begin{aligned}
\cos\theta &= \frac{e^t \cos(t) e^t(\cos(t) - \sin(t)) + e^t \sin(t) e^t(\sin(t) + \cos(t))}{\sqrt{2}e^{2t}} \\
&= \frac{e^{2t}}{\sqrt{2}e^{2t}} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\text{So,} \quad \cos\theta = \frac{\pi}{4}$$

$\square$

# Problem 7

Suppose two maps $f, g$ such that $f : (u, v) \mapsto (x, y, z)$ and $g : (x, y, z) \mapsto w$, Then, $g(f(u, v)) = w$.

$$\text{By Chain rule, } \mathcal{D}_{g(f(x))} = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

$$\mathcal{D}_f = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}$$

By matrix multiplication, we get

$$\mathcal{D}_{g(f(x))}\mathcal{D}_f = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}$$