

Question 1

Part (a) :

Since $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$, note that there are no "holds" on variables x, y so $x, y \in \mathbb{R}$ and the domain of \mathbf{F} is \mathbb{R}^2 .

Part (b) :

If \mathbf{F} is conservative, then there exists $\phi(x, y) \in U$ such that $\frac{\partial \phi}{\partial x} = x^2 + y^2$ and $\frac{\partial \phi}{\partial y} = -2xy$. Then

$$\frac{\partial \phi}{\partial x \partial y} = 2y, \quad \frac{\partial \phi}{\partial y \partial x} = -2$$

Suppose such a function $\phi(x, y)$ exists, and both ϕ_{xy} and ϕ_{yx} are functions in C^1 , then ϕ itself must have continuous first and second order partial derivatives. But clearly $\frac{\partial \phi}{\partial x \partial y} \neq \frac{\partial \phi}{\partial y \partial x}$. Which violates the equality of mixed partials (*Schwarz Theorem*). Thus, this cannot be the case, such a function $\phi(x, y)$ doesn't exist. So, \mathbf{F} is not a conservative vector field.

Question 2

Part (a) :

$$\text{Note that } \mathbf{F} = (xy - \sin z)\mathbf{i} + \left(\frac{x^2}{2} - \frac{e^y}{z}\right)\mathbf{j} + \left(\frac{e^y}{z^2} - x \cos z\right)\mathbf{k}$$

Since the denominator cannot be zero, which implies $z \neq 0$ and $z^2 \neq 0$. So $z \neq 0$.

In this case, the domain U is $\mathbb{R}^3 \setminus \{z = 0\}$.

Part (b) :

By definition,

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy - \sin z & \frac{x^2}{2} - \frac{e^y}{z} & \frac{e^y}{z^2} - x \cos z \end{bmatrix} \\ &= \left[\frac{\partial}{\partial y} \left(\frac{e^y}{z^2} - x \cos z \right) - \frac{\partial}{\partial z} \left(\frac{x^2}{2} - \frac{e^y}{z} \right) \right] \mathbf{i} + \left[\frac{\partial}{\partial x} \left(\frac{e^y}{z^2} - x \cos z \right) - \frac{\partial}{\partial z} (xy - \sin z) \right] \mathbf{j} + \\ &\quad \left[\frac{\partial}{\partial x} \left(\frac{x^2}{2} - \frac{e^y}{z} \right) - \frac{\partial}{\partial y} (xy - \sin z) \right] \mathbf{k} \\ &= \left(\frac{e^y}{z^2} - \frac{e^y}{z^2} \right) \mathbf{i} + (-\cos z + \cos z) \mathbf{j} + (x - x) \mathbf{k} = 0 \end{aligned}$$

So $\nabla \times \mathbf{F} = 0$, according to the calculation.

Part (c) :

To find the function $V(x, y, z)$ such that $\mathbf{F} = \nabla V$, we can do the following:

For the first term, we know that $\frac{\partial V}{\partial x} = xy - \sin z$, so $V = \frac{1}{2}x^2y - x \sin z + h(y, z)$, where $h(y, z)$ is a function only depend on variables y, z .

We can do the same on the second term, we find that $\frac{\partial V}{\partial y} = \frac{x^2}{2} - \frac{e^y}{z}$, and compare to the V we obtained in the first term, we find that

$$\frac{\partial V}{\partial y} = \frac{\partial \left(\frac{1}{2}x^2y - x \sin z \right)}{\partial y} + \frac{\partial h(y, z)}{\partial y} = \frac{1}{2}x^2 + \frac{\partial h(y, z)}{\partial y} = \frac{1}{2}x^2 - \frac{e^y}{z}$$

So $\frac{\partial h(y, z)}{\partial y} = -\frac{e^y}{z}$, which means that $h(y, z) = -\frac{e^y}{z} + p(z)$.

Now, do the same operation on the third term, we get

$$\frac{\partial V}{\partial z} = \frac{\partial \left(\frac{1}{2}x^2y - x \sin z - \frac{e^y}{z} + p(z) \right)}{\partial z} = -x \cos z + \frac{e^y}{z^2} + \frac{\partial p(z)}{\partial z} = \frac{e^y}{z^2} - x \cos z$$

So, we get $\frac{\partial p(z)}{\partial z} = 0$, so $p(z) = C$, where C is a constant.

Then, we find that $V(x, y, z) = \frac{1}{2}x^2y - x \sin z - \frac{e^y}{z} + C$.

Question 3

By definition, we know that

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \quad \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \\ \nabla f \times \nabla g &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{bmatrix} \\ &= \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) \mathbf{i} - \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \mathbf{z}\end{aligned}$$

So, in this case,

$$\nabla \cdot (\nabla f \times \nabla g) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial y} \right) \mathbf{i} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} \right) \mathbf{j} + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \mathbf{z}$$

Clearly, all three terms are zero. So we say that $\nabla \cdot (\nabla f \times \nabla g) = 0$.

Question 4

Part (a) :

By showing that $L(t)$ is monotone increasing, we may show that $L'(t) \geq 0$. By differentiate both sides of $L(t) = V(\mathbf{c}(t))$, we get

$$L'(t) = \frac{d}{dt}V(\mathbf{c}(t)) = \nabla V \cdot \frac{d\mathbf{c}}{dt}$$

By knowing that $F = \nabla V$, we can say that $L'(t) = F \cdot \frac{d\mathbf{c}}{dt}$. If we denote the angle between F and $\frac{d\mathbf{c}}{dt}$ to be θ , then

$$L'(t) = \|F\| \left\| \frac{d\mathbf{c}}{dt} \right\| \cos \theta$$

Then, since F is a conservative vector field and $\mathbf{c}(t)$ is a flow line, we know that $F(\mathbf{c}(t)) = \mathbf{c}'(t)$, which is saying that, the angle between F and $\frac{d\mathbf{c}}{dt}$ is 0, i.e they are parallel. So in this case $\cos \theta = 1$.

Then, we find that

$$L'(t) = \|F\| \left\| \frac{d\mathbf{c}}{dt} \right\| \text{ which is of course, non-negative.}$$

So, we have shown that $L'(t) \geq 0$, saying that $L(t)$ is monotone increasing.

Part (b) :

Since $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 5z\mathbf{k}$, we need to find $V(x, y, z)$ such that $\mathbf{F} = \nabla V$. We can see that x, y, z are the only dependent variables in $\mathbf{i}, \mathbf{j}, \mathbf{k}$ terms, so We can just intergrate each part and get the result. Say $V(x, y, z) = \int xdx + \int 2ydy + \int 5zdz$, so we get

$$V(x, y, z) = \frac{1}{2}x^2 + y^2 + \frac{5}{2}z^2 + C$$

.

Part (c) :

Recall the definition, the flow line $\mathbf{c}(t)$ satisfies $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. In this problem, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ where $P = x$, $Q = 2y$, $R = 5z$. The flow line $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ also satisfies:

$$\begin{cases} x'(t) = P(x(t), y(t), z(t)) \\ y'(t) = Q(x(t), y(t), z(t)) \\ z'(t) = R(x(t), y(t), z(t)) \end{cases}$$

by plug in those values, we get

$$\begin{cases} x'(t) = x(t) \\ y'(t) = 2y(t) \\ z'(t) = 5z(t) \end{cases} \implies \begin{cases} x(t) = e^t + C_1 \\ y(t) = e^{2t} + C_2 \\ z(t) = e^{5t} + C_3 \end{cases}$$

And we know that $\mathbf{c}(0) = (1, 15, 7)$, then solve for C_1, C_2, C_3 we get $C_1 = 0$, $C_2 = 14$, $C_3 = 6$.

So the folw line is $\mathbf{c}(t) = (e^t, e^{2t} + 14, e^{5t} + 6)$.

Part (d) :

As we have shown in part (a), $\|\mathbf{F}\| = \sqrt{x^2 + 4y^2 + 25z^2}$, $\left\| \frac{d\mathbf{c}}{dt} \right\| = \sqrt{e^{2t} + 4e^{2t} + 25e^{10t}}$, of course their multiple is non-negative.

Question 5

In this problem, we may separate the area into smaller parts in order to simplify our integration.

Let's define the following sub-areas:

$$\begin{cases} D_1 := \{(x, y) \mid 0 \leq x \leq 4, y \leq x\} \\ D_2 := \{(x, y) \mid 4 \leq x \leq 8, 0 \leq y \leq 2\} \\ D_3 := \{(x, y) \mid 4 \leq x \leq 8, 2 \leq y \leq \frac{16}{x}\} \end{cases}$$

Clearly, $D = D_1 + D_2 + D_3$. So by the definition,

$$\iint_D x^2 dx dy = \iint_{D_1} x^2 dx dy + \iint_{D_2} x^2 dx dy + \iint_{D_3} x^2 dx dy$$

Note that we may need to change the integration variable, so we do the following. This problem can also be solved by using Fubini's Theorem, but in this case both methods are almost the same.

$$\begin{cases} D_1 := \{(x, y) \mid 0 \leq y \leq 4, y \leq x \leq 4\} \\ D_2 := \{(x, y) \mid 0 \leq y \leq 2, 4 \leq x \leq 8\} \\ D_3 := \{(x, y) \mid 2 \leq y \leq 4, 4 \leq x \leq \frac{16}{y}\} \end{cases}$$

Then, we can calculate the integrals separately.

$$\iint_{D_1} x^2 dx dy = \int_{y=0}^{y=4} \int_{x=y}^{x=4} x^2 dx dy = \frac{1}{3} \int_{y=0}^{y=4} 64 - y^3 dy = 64$$

$$\iint_{D_2} x^2 dx dy = \int_{y=0}^{y=2} 1 dy \int_{x=4}^{x=8} x^2 dx = 2 \int_{x=4}^{x=8} x^2 dx = \frac{896}{3}$$

$$\iint_{D_3} x^2 dx dy = \int_{y=2}^{y=4} \int_{x=4}^{x=\frac{16}{y}} x^2 dx dy = \frac{1}{3} \int_{y=2}^{y=4} \left(\frac{16}{y}\right)^3 - 64 dy = \frac{256}{3}$$

By summing up all the integrals we have calculated, we get $\iint_D x^2 dx dy = 64 + \frac{896}{3} + \frac{256}{3} = 448$.

Question 6

By observation, we can integrate by changing the integration variables. Namely say $u = x^2 - y^2$, $v = xy$. And it is easy to know the range of u, v , where $1 \leq u \leq 9$ and $2 \leq v \leq 4$. Then, by change of variable matrix, we need to find the functions $x(u, v)$ and $y(u, v)$. As we can see, we have the equality $D(T)^{-1} = (D(T))^{-1}$, which is saying that

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{\left| \frac{\partial(u, v)}{\partial(x, y)} \right|}$$

$$\text{So, } \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}} = \frac{1}{\det \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}} = \frac{1}{2x^2 + 2y^2}. \text{ In this case, we get}$$

$$\iint_D (x^2 + y^2) dx dy = \iint_{D^*} (x^2 + y^2) \cdot \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \iint_{D^*} du dv$$

Then, by *Fubini's Theorem*, when we plug in the range of u, v , we get

$$\iint_{D^*} du dv = \int_{u=1}^{u=9} du \cdot \int_{v=2}^{v=4} dv$$

Since both integrals are independent of their integration variables, then

$$\iint_{D^*} du dv = (9 - 1) \times (4 - 2) = 16$$

So we conclude that $\iint_D (x^2 + y^2) dx dy = 16$, with respect to the area bounded by x, y .

Question 7

By sketching the region, we found that the shape is like an "ice cream cone". The equation $x^2 + y^2 + z^2 = a^2$ is a sphere with radius a , and the equation $x^2 + y^2 - (\tan \alpha)^2 z^2 = 0$ is a cone, and we can calculate the "slope" of that cone, which is given by the angle α . Then, by changing the variables, let $x = r \cos \theta$, $y = r \sin \theta$, $z = z$. Then the determinant of the change of variable matrix is given by:

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = \det \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix} = r$$

The range of the new variables are defined by:

$$r \in [0, a \sin \alpha], \theta \in [0, 2\pi], z \in \left[\sqrt{\frac{x^2 + y^2}{\tan^2 \alpha}}, \sqrt{a^2 - x^2 - y^2} \right]$$

And again, $x^2 + y^2 = r^2$, by doing that, the integral can be calculated as:

$$\begin{aligned} V &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a \sin \alpha} \int_{z=\frac{r}{\tan \alpha}}^{z=\sqrt{a^2-r^2}} r dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a \sin \alpha} r \left(\sqrt{a^2 - r^2} - \frac{r}{\tan \alpha} \right) dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \left(-\frac{1}{3}(a^2 - r^2)^{\frac{3}{2}} - \frac{1}{3 \tan \alpha} r^3 \right) \Big|_{r=0}^{r=a \sin \alpha} d\theta \quad (\text{By } u\text{-substitution}) \\ &= \int_{\theta=0}^{\theta=2\pi} -\frac{1}{3}(a \cos \alpha)^3 - \frac{a^3 \sin^3 \alpha}{3 \tan \alpha} + \frac{1}{3} a^3 d\theta = \frac{2\pi a^3}{3} \left(1 - \frac{\sin^3 \alpha}{\tan \alpha} - \cos^3 \alpha \right) \end{aligned}$$

So, the volume of that shape is given By $V = \frac{2\pi a^3}{3} \left(1 - \frac{\sin^3 \alpha}{\tan \alpha} - \cos^3 \alpha \right)$.