Question 1

Part (a) :

Proof. By Bernoulli's inequality: $\left(1+\frac{1}{\sqrt{n}}\right)^n \geq 1+n\times\frac{1}{\sqrt{n}} = 1+\sqrt{n} > \sqrt{n}$. So it's true for all $n \in \mathbb{N}$. ■

Part (b) :

Proof. In order to prove $\left(1+\frac{1}{\sqrt{n}}\right)^2 > \sqrt[n]{n}$, first by taking the n-th square on both sides, which is equivalant as proving $\left(1+\frac{1}{\sqrt{n}}\right)^{2n} > n$.

$$\begin{aligned}
\left(1+\frac{1}{\sqrt{n}}\right)^{2n} &> n \\
\left(1+\frac{1}{\sqrt{n}}\right)^{n}\left(1+\frac{1}{\sqrt{n}}\right)^{n} &> n \\
&\geq \left(1+\sqrt{n}\right)\left(1+\sqrt{n}\right) \quad \textit{By Bernoulli's inequality} \\
&= 1+n+2\sqrt{n} \\
&> n.
\end{aligned}$$

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Part (c) :

Proof. By definition, we need to show that $\exists N \in \mathbb{N}$, such that $\forall n \geq N$, $|a_n - L| < \epsilon$ for all $\epsilon > 0$. i.e $|\sqrt[n]{n} - 1| < \epsilon$. By part (b), we know that

$$|\sqrt[n]{n}-1| < \left|\left(1+\frac{1}{\sqrt{n}}\right)^2 - 1\right| = \left|1+\frac{1}{n}+\frac{2}{\sqrt{n}}-1\right| = \left|\frac{2}{\sqrt{n}}+\frac{1}{n}\right|$$

For large n, especially $n > 1$, we have $n > \sqrt{n}$, and we can get rid of absolute signs because they are all positive. so

$$\left|\frac{2}{\sqrt{n}}+\frac{1}{n}\right|<\frac{2}{\sqrt{n}}+\frac{1}{\sqrt{n}}=\frac{3}{\sqrt{n}}<\epsilon$$

Now, solve for n, we have $n>\left(\frac{3}{\epsilon}\right)^2$. So in this case, we have proved that when $N=\left(\frac{3}{\epsilon}\right)^2$, for all $n \geq N$, $|a_n - L| < \epsilon$. Hence

$$\lim\left(\sqrt[n]{n}\right)=1$$

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Question 3

Proof. Knowing that $0 < a < b$, and the function $f(x) = \sqrt{x}$ is increasing, we know that it is true for $\sqrt[n]{2a^n} < \sqrt[n]{a^n + b^n} < \sqrt[n]{2b^n}$, i.e $a\sqrt[n]{2} < \sqrt[n]{a^n + b^n} < b\sqrt[n]{2}$. We know that the sequence $(x_n) = \sqrt[n]{2}$ is convergent, because we can rewrite as $x_n = 2^{\frac{1}{n}}$ and $0 < \dfrac{1}{n} < 1$. Since a, b are constant, then the sequences $a_n = a\sqrt[n]{2}$ and $b_n = b\sqrt[n]{2}$ are also convergent, which will make the sequence $x_n = \sqrt[n]{a^n + b^n}$ between them to be convergent. So (x_n) converges.

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