Question 1

What it means, is to find a curve $\boldsymbol{c(t)}$ such that $\nabla \boldsymbol{f} \cdot \boldsymbol{c'(t)} = 0$. Only in this case, $\boldsymbol{c(t)}$ may interset with the level curves of f everywhere with the same angles. So what we need to do is calulate $\nabla \boldsymbol{f}$, and we get $\nabla \boldsymbol{f} = \langle 4x^3, 2y \rangle$. So we need to solve for $\langle 4x^3, 2y \rangle \cdot \langle \frac{t}{dt}, \frac{g(t)}{dt} \rangle = 0$. Then, we may substract $x = t$ and $y = g(t)$, then we have

$$\langle 4t^3, 2g(t) \rangle \cdot \langle 1, g'(t) \rangle = 0$$

$$\text{Which is } \ 4t^3 + 2g(t)g'(t) = 0$$

Then, it is an ordinary differential equation, we can solve it by seperate variables. To make it easier, we do $g(t) = y$, $g'(t) = \frac{\mathrm{d}\, y}{\mathrm{d}\, t}$, then we have

$$2y\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = -4t^3, \ \text{ which is}$$

$$y \,\mathrm{d}\, y = -2t^3 \,\mathrm{d}\, t$$

Then, we can intergrate both sides and get

$$\int y \,\mathrm{d}\, y = -\int 2t^3 \,\mathrm{d}\, t$$

Solve for it and get the general solutions: $\frac{1}{2}y^2 + C_1 = -\frac{1}{2}t^4 + C_2$. Then we use $(1, 1)$ as a special case for this solution, then we can solve for the constant C_1, C_2. Hence we have $\frac{1}{2}y^2 = -\frac{1}{2}t^4 + 1$. Then, we solve for y, and we get $y = \pm\sqrt{-t^4 + 2}$.

So the function $g(t)$ we are looking for is: $g(t) = \pm\sqrt{-t^4 + 2}$.

Question 2

First of all, we can calculate $\nabla \boldsymbol{f}$ at the given point $(1, 1, \sqrt{3})$ where f is given by the surface $x^2 + y^2 + z^2 + 1 = 0$. So $\nabla \boldsymbol{f}\Big|_{(1,1,\sqrt{3})} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2x, 2y, -2z \rangle = \langle 2, 2, -2\sqrt{3} \rangle$.

Now we may normalize $\nabla \boldsymbol{f}$, say $\nabla \boldsymbol{F} = \frac{\nabla \boldsymbol{f}}{\|\nabla \boldsymbol{f}\|}$. In this case, we can get $\nabla \boldsymbol{F} = \langle \frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, -\frac{\sqrt{15}}{5} \rangle$. Then, we can define a function $\boldsymbol{c}(t)$, which indicates the path of that object. We know that the starting point of that object is $(1, 1, \sqrt{3})$, which means $\boldsymbol{c}(0) = (1, 1, \sqrt{3})$. And it travels at a speed of 10 unit length per second in the direction of $\nabla \boldsymbol{F}$, where $\|\nabla \boldsymbol{F}\| = 1$. So the function can be defined as:

$$\boldsymbol{c}(\boldsymbol{t}) = \langle 1, 1, \sqrt{3} \rangle + 10\boldsymbol{t} \langle \frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, -\frac{\sqrt{15}}{5} \rangle$$

Since we need to find the time and position when the object intersets with the xy plane, we just need to find the time $\boldsymbol{t}$ such that the z coordinate at this certain time is 0, i.e, $\sqrt{3} - 10\boldsymbol{t}\frac{\sqrt{15}}{5} = 0$. Solve for $\boldsymbol{t}$, we get $\boldsymbol{t} = \frac{\sqrt{5}}{10}$. Now, we plug into $\boldsymbol{c}(\boldsymbol{t})$ with the exact value of $\boldsymbol{t}$, we get

$$\boldsymbol{c}(\frac{\sqrt{5}}{10}) = \langle 2, 2, 0 \rangle$$

So, after $\frac{\sqrt{5}}{10}$ seconds, the object will cross the xy plane, and the intersection point is $(2, 2, 0)$.

Question 3

(a)

We know that $f(x,y) = (y-3x^2)(y-x^2) = y^2 - 4x^2y + 3x^4$. In order to find the critical points, we need to find the points such that $\bigtriangledown \boldsymbol{f} = \mathbf{0}$. Since we know that $\bigtriangledown \boldsymbol{f} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$, by calculating $\frac{\partial f}{\partial x} = -8yx + 12x^3 = 0$ and $\frac{\partial f}{\partial y} = 2y - 4x^2 = 0$, we get $x = 0, y = 0$. Then we may say that the origion $(0,0)$ is a critical point.

(b)

We know that $g(t) = f(at, bt) = (bt - 3a^2t^2)(bt - a^2t^2) = 3a^4t^4 - 4a^2bt^3 + b^2t^2$. Since $g(t)$ is a single-variable function, we can calculate the derivative of g to check if $(0,0)$ is a local minimum. By doing so, we have $g'(t) = 12a^4t^3 - 12a^2bt^2 + 2b^2t$; $g''(t) = 36a^4t^2 - 24a^2bt + 2b^2$. By using the properties, $g''(t)$ has two roots x_1, x_2 and $x_1 < x_2$. So $g'(x)$ is increasing on $(-\infty, x_1)$ and decreasing on (x_1, x_2). Then compare $g'(t), g(t)$, we may conclude that $(0,0)$ is a local minimum.

(c)

By calculation, $f_{xx} = -8y + 36x^2$; $f_{yy} = 2$; $f_{xy} = f_{yx} = -8x$. Then, by second derivative test, the matrix $\mathcal{D}$ at critical point $(0,0)$ is given by:

$$\mathcal{D}\Big|_{(0,0)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Hence $\mathcal{D}_{(1,1)} = 0$, according to the second derivative test, no conclusion can be drawn at this point. So in this case $(0,0)$ is not a local minimum.

Question 4

First by computing $\nabla \boldsymbol{f}$, we get $\nabla \boldsymbol{f} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$, and let $\nabla \boldsymbol{f} = 0$, we get

$$\nabla \boldsymbol{f} = \langle ye^{-\frac{1}{2}(x^2+y^2)} - x^2ye^{-\frac{1}{2}(x^2+y^2)}, xe^{-\frac{1}{2}(x^2+y^2)} - y^2xe^{-\frac{1}{2}(x^2+y^2)} \rangle = 0, \text{ which is}$$

$$\begin{cases} x^2 = 1 \\ y^2 = 1 \end{cases} \implies ① \begin{cases} x = 1 \\ y = 1 \end{cases}, ② \begin{cases} x = -1 \\ y = 1 \end{cases}, ③ \begin{cases} x = 1 \\ y = -1 \end{cases}, ④ \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Then, we shall do the second derivative test to see which ones are maximum, minimum and saddle points. By calaulating the second derivatives, we get:

$$\begin{cases} \boldsymbol{f}_{xx} = ye^{-\frac{1}{2}(x^2+y^2)}(x^3 - 3x) \\ \boldsymbol{f}_{yy} = xe^{-\frac{1}{2}(x^2+y^2)}(y^3 - 3y) \\ \boldsymbol{f}_{xy} = \boldsymbol{f}_{yx} = (1 - x^2)(1 - y^2)e^{-\frac{1}{2}(x^2+y^2)} \end{cases}$$

Then, by the second derivative test, the matrix $\mathcal{D}$ is given by:

$$\mathcal{D} = \begin{bmatrix} \boldsymbol{f}_{xx} & \boldsymbol{f}_{xy} \\ \boldsymbol{f}_{yx} & \boldsymbol{f}_{yy} \end{bmatrix}$$ Then, plug in all the critical points we obtained earlier, we get

$$\mathcal{D}_{(1,1)} = \begin{bmatrix} -\frac{2}{e} & 0 \\ 0 & -\frac{2}{e} \end{bmatrix}, \mathcal{D}_{(-1,1)} = \begin{bmatrix} \frac{2}{e} & 0 \\ 0 & \frac{2}{e} \end{bmatrix} \mathcal{D}_{(1,-1)} = \begin{bmatrix} \frac{2}{e} & 0 \\ 0 & \frac{2}{e} \end{bmatrix} \mathcal{D}_{(-1,-1)} = \begin{bmatrix} -\frac{2}{e} & 0 \\ 0 & -\frac{2}{e} \end{bmatrix}$$

For cases like $(1, 1)$ and $(-1, -1)$, since $\mathcal{D}_{(1,1)} < 0$, hence no conclusion can be drwan. For cases like $(-1, 1)$ and $(1, -1)$, since $\mathcal{D}_{(1,1)} > 0$ and $\det \mathcal{D} > 0$, hence they are local minimums.

Question 5

Firstly, we locate all the critical points in the domain of $\mathcal{T}$. By calculating the partical derivatives,

$$\frac{\partial f}{\partial x} = \sin y(\cos x \sin(x+y) + \sin x \cos(x+y)) = \sin y \sin(2x+y)$$

$$\frac{\partial f}{\partial y} = \sin x(\cos y \sin(x+y) + \sin y \cos(x+y)) = \sin x \sin(2y+x)$$

Then, by making both partial derivatives to be 0, solve for:

$$\begin{cases} \sin y \sin(2x+y) = 0 \\ \sin x \sin(2y+x) = 0 \\ (x,y) \in \mathcal{T} \end{cases}$$

And we get the following solutions:

$$① \begin{cases} x = 0 \\ y = 0 \end{cases}, ② \begin{cases} x = \pi \\ y = 0 \end{cases}, ③ \begin{cases} x = 0 \\ y = \pi \end{cases}, ④ \begin{cases} x = \frac{\pi}{3} \\ y = \frac{\pi}{3} \end{cases}, ⑤ \begin{cases} x = \frac{\pi}{2} \\ y = 0 \end{cases}, ⑥ \begin{cases} x = 0 \\ y = \frac{\pi}{2} \end{cases}$$

If we plug in solutions ① $\sim$ ⑥ into $f(x,y) = \sin x \sin y \sin(x+y)$, we get that, except ④ $= \frac{3\sqrt{3}}{8}$, the remaining solutuins are all 0. We then look for the maximum and minimum of f viewed as a function only on $\partial\mathcal{T}$. Since we have noticed that the boundary of $\mathcal{T}$ can be parametrized by the following:

$$\boldsymbol{c}(t) = \begin{cases} (0,t) \quad 0 \le t < \pi \\ (\frac{t}{2}, 2\pi - t) \quad \pi \le t < 2\pi \\ (3\pi - t, 0) \quad 2\pi \le t < 3\pi \end{cases}$$

And if we plug $\boldsymbol{c}(t)$ into $f(x,y)$ as a new function $g(t) = f(\boldsymbol{c}(t))$, we get:

$$g(t) = \begin{cases} 0 & 0 \le t < \pi \text{ and } 2\pi \le t < 3\pi \\ \sin t \sin^2(\dfrac{t}{2}) & 2\pi \le t < 3\pi \end{cases}$$

Then, by calculating the derivative of $g(t)$ at $2\pi \le t < 3\pi$, we get $g'(t) = \frac{1}{2}\cos t - \frac{1}{2}\cos(2t)$, by making $g'(t) = 0$, we get $t = \frac{4}{3}\pi$. Then, we know that $t = \frac{4}{3}\pi$ is a critical point on $\partial\mathcal{T}$, and $g(\frac{4}{3}\pi) = -\frac{3\sqrt{3}}{8}$. Then we compare those critical points, we found that the maximum value is $\frac{3\sqrt{3}}{8}$ at point $(\frac{\pi}{3}, \frac{\pi}{3})$. So at this point the maximum is achieved.

Question 6

We may construct the following:

As for the plane $x+y+z=0$, we do $\mathcal{L} = xy + 2z - \lambda(x+y+z) - \mu(x^2+y^2+z^2-24)$,$\lambda, \mu \in \mathbb{R}$

Then, according to the *Lagrange Multiplier*, we need to solve the following system of equations:

$$\mathcal{L}_x = y - \lambda - 2\mu x = 0 \tag{1}$$

$$\mathcal{L}_y = x - \lambda - 2\mu y = 0 \tag{2}$$

$$\mathcal{L}_z = 2 - \lambda - 2\mu z = 0 \tag{3}$$

$$x + y + z = 0 \tag{4}$$

$$x^2 + y^2 + z^2 = 24 \tag{5}$$

By doing (1) − (2), we have $y - x - 2\mu x + 2\mu y = 0$, which is $(y-x) + 2\mu(y-x) = 0$, so $\mu = -\frac{1}{2}$ or $x = y$.

$\boxed{\textbf{If } \mu = -\frac{1}{2}}$

Then we may plug into (1) and (2) and get $x + y = \lambda$.

By plug into (4) and (5), we get

$$\begin{cases} \lambda + z = 0 \\ 2 - \lambda + z = 0 \end{cases} \implies \begin{cases} z = -1 \\ \lambda = 1 \end{cases}$$

Then, we can solve for x, y. We know from (1) and (2) that $x + y = 1$ since $\lambda = 1$. Using (5), we have

$$\begin{cases} x + y = 1 \\ x^2 + y^2 = 23 \end{cases} \implies \begin{cases} x = \frac{1+3\sqrt{5}}{2} \\ y = \frac{1-3\sqrt{5}}{2} \end{cases} \text{or} \begin{cases} x = \frac{1-3\sqrt{5}}{2} \\ y = \frac{1+3\sqrt{5}}{2} \end{cases}$$

So, by pluging in those two points $\mathcal{Q}_1 = (\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}, -1)$ and $\mathcal{Q}_2 = (\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2}, -1)$, we get $f(\mathcal{Q}_1) = f(\mathcal{Q}_2) = -13$.

$\boxed{\textbf{If } x = y}$

By plugging $x = y$ into equations $(4), (5)$, we get

$$\begin{cases} 2x + z = 0 \\ 2x^2 + z^2 = 24 \\ x = y \end{cases} \implies \begin{cases} x = 2 \\ y = 2 \\ z = -4 \end{cases} \quad \text{or} \quad \begin{cases} x = -2 \\ y = -2 \\ z = 4 \end{cases}$$

So, we plug in two points $(2, 2, -4)$ and $(-2, -2, 4)$ into the function $f(x, y)$, we get $f(2, 2) = -4$, $f(-2, -2) = 12$.

So, compared to the four points, we conclude that the maximum value is 12 and the minimum value is -13.