Question 1

We can decompose $\boldsymbol{F}$ into $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ where $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$, like:

$$\boldsymbol{F}_1 = e^x \sin(y)\boldsymbol{i} + e^x \cos(y)\boldsymbol{j} \quad , \boldsymbol{F}_2 = 3y\boldsymbol{i} + (2x - 2y)\boldsymbol{j}$$

And by obversation, $\boldsymbol{F}_1$ is also a conservative field, meaning there exists a function V such that $\boldsymbol{\nabla} V = \boldsymbol{F}$. Such a function is not hard to find, the answer is $V(x, y) = e^x \sin(y) + C$ for some $C \in \mathbb{R}$. Then, we come back into our problem, we may rewrite the intergal as:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F_1} \cdot d\boldsymbol{s} + \int_C \boldsymbol{F_2} \cdot d\boldsymbol{s}$$

Since $\boldsymbol{F}_1$ is conservative, then by the theorem,

$$\int_C \boldsymbol{F}_1 \cdot d\boldsymbol{s} = V(B) - V(A)$$

In this problem, we may parametrize the ellipse as $\boldsymbol{c}(t) = \langle \cos(t), 2\sin(t) \rangle$ where t ranges from 0 to 2π, counter-clockwise. So it means that

$$\int_C \boldsymbol{F_1} \cdot d\boldsymbol{s} = (e^{\cos(t)} \sin(2\sin(t)) + C)\Big|_0^{2\pi} = 0$$

Then, back to $\boldsymbol{F}_2$, by the parametrization of $\boldsymbol{c}$, we know that

$$\begin{aligned}
\int_C \boldsymbol{F}_2 \cdot d\boldsymbol{s} &= \int_0^{2\pi} \left[(6\sin(t))\,\boldsymbol{i} + (2\cos(t) - 4\sin(t))\,\boldsymbol{j}\right] \cdot \left[-\sin(t)\boldsymbol{i} + 2\cos(t)\boldsymbol{j}\right] dt \\
&= \int_0^{2\pi} -6\sin^2(t) + 4\cos^2(t) - 8\sin(t)\cos(t) dt \\
&= \int_0^{2\pi} 5\cos(2t) - 4\sin(2t) - 1 dt \\
&= \frac{5}{2}\sin(2t) + 2\cos(2t) - t\Big|_0^{2\pi} \\
&= 1 - 2\pi \quad \text{which is the final answer}
\end{aligned}$$

Question 2

Proof. By the properties of the intergal, we may write the origonal formula as:

$$\int_C f\nabla g + g\nabla f \cdot d\boldsymbol{s}$$

Then, we observe that, by *Chain Rule*, $\displaystyle\int_C f\nabla g + g\nabla f \cdot d\boldsymbol{s} = f \cdot g\Big|_a^b$. Then, since $\mathcal{P} := \boldsymbol{c}(a)$, $\mathcal{Q} := \boldsymbol{c}(b)$, so we say that

$$f \cdot g\Big|_a^b = f(\mathcal{Q})g(\mathcal{Q}) - f(\mathcal{P})g(\mathcal{P})$$

So the equation we want to prove is correct.

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Question 3

If we project the region on the xy plane, then it is a circle centered at $(0, a)$ with radius a, and its equation can be described as $x^2 + (y-a)^2 = a^2$. Now, by using the polar coordinates, we may describe the region $S := \{(x, y) : x^2 + (y-a)^2 \leq a^2\}$ as $x = r\cos\theta$, $y = r\sin\theta$ where $0 \leq r \leq 2a$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Now, we may consider a map from the region S to the surface we would like to intergrate. The function can we written as

$$\mathbf{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{4a^2 - r^2\cos^2\theta - r^2\sin^2\theta})$$

Then the cross product given by $T_r \times T_\theta$ is:

$$T_r \times T_\theta = \det\begin{bmatrix} i & j & k \\ \cos\theta & \sin\theta & -\frac{r}{\sqrt{4a^2 - r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$$= (r\cos^2\theta + r\sin^2\theta)\boldsymbol{k} + (\frac{r^2}{\sqrt{4a^2 - r^2}}\cos\theta)\boldsymbol{i} + (\frac{r^2}{\sqrt{4a^2 - r^2}}\sin\theta)\boldsymbol{j}$$

So, the norm $\|T_r \times T_\theta\| = \frac{2ar}{\sqrt{4a^2 - r^2}}$. (Assume a is positive). In this case, the double intergal

$$\iint_S 1 ds = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2a} 1 \times \frac{2ar}{\sqrt{4a^2 - r^2}} dr d\theta$$

$$\begin{aligned} &= \pi \int_0^{2a} \frac{2ar}{\sqrt{4a^2 - r^2}} dr \\ &= -a\pi \int_{2a}^{0} \frac{1}{\sqrt{u}} du \quad \textit{By u-substitution} \\ &= 2a\pi u^{\frac{1}{2}} \Big|_0^{2a} \\ &= 2\sqrt{2}a^{\frac{3}{2}}\pi \quad \text{, which is the final answer.} \end{aligned}$$

Question 4

By our parametrization $\mathbf{\Phi}$, we know that the area at the given surface is given by:

$$\iint_D 1 \cdot \|T_u \times T_v\| dS$$

In this case, the intergration part is given by $0 < u < 2\pi$ and $0 < v < 2\pi$. Now, the cross product can be calculated as:

$$T_u \times T_v = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -r \sin u \cos v & -r \sin u \sin v & r \cos u \\ -(a + r \cos u) \sin v & (a + r \cos u) \cos v & 0 \end{bmatrix}$$

$$= r(a + r \cos u) \cdot \left[-\cos u \cos v \boldsymbol{i} - \cos u \sin v \boldsymbol{j} + \sin u \boldsymbol{k} \right]$$

Then, by calculating the norm of it, we get $\|T_u \times T_v\| = r(a + r \cos u)$. In this case, the surface intergal can be written as:

$$\iint_D 1 \cdot \|T_u \times T_v\| dS = \int_0^{2\pi} \int_0^{2\pi} r(a + r \cos u) du dv$$

Since this intergal is independent of v, we may multiply 2π in front of it and hence we get $2\pi \int_0^{2\pi} ar + r^2 \cos u du$, then it becomes

$$2\pi \left[aru + r^2 \sin u \right]_0^{2\pi} = 4\pi^2 ar$$

Question 5

By definition, the flux is given by the surface intergal over the vector field $\boldsymbol{F} = 2x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and the surface $\boldsymbol{\Phi}(u, v) = (u^2v, uv^2, v^3)$. The flux $\boldsymbol{\Omega}$ with specific orentation can be defined as

$$\boldsymbol{\Omega} = \iint_D \boldsymbol{F} \boldsymbol{\cdot} (\boldsymbol{T}_u \times \boldsymbol{T}_v) dudv$$

In this case, the cross product $\boldsymbol{T}_u \times \boldsymbol{T}_v$ is given by

$$\det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2uv & v^2 & 0 \\ u^2 & 2uv & 3v^2 \end{bmatrix} = 3v^4\boldsymbol{i} - 6uv^3\boldsymbol{j} + 3u^2v^2\boldsymbol{k}$$

Then, the surface intergal can be written As

$$\iint_D \boldsymbol{F} \boldsymbol{\cdot} (\boldsymbol{T}_u \times \boldsymbol{T}_v) dudv = \int_0^1 \int_0^1 (2u^2v\boldsymbol{i} + uv^2\boldsymbol{j} + v^3\boldsymbol{k}) \cdot (3v^4\boldsymbol{i} - 6uv^3\boldsymbol{j} + 3u^2v^2\boldsymbol{k}) dudv$$

$$= \int_0^1 \int_0^1 6u^2v^5 - 6u^2v^5 + 3u^2v^5 dudv$$

$$= 3 \int_0^1 \int_0^1 u^2v^5 dudv$$

$$= 3 \int_0^1 u^2 du \int_0^1 v^5 dv$$

$$= 3 \times \frac{1}{3} \times \frac{1}{6}$$

$$= \frac{1}{6}, \text{ which is the final answer.}$$

Question 6

In this problem, the surface intergral in the vector field $\boldsymbol{F} = z^2\boldsymbol{k}$ can be described as

$$\iint_{\phi} \boldsymbol{F} \cdot dS = \iint_{D} \boldsymbol{F} \cdot (T_u \times T_v) dudv$$

In this problem, the function $\boldsymbol{\Omega}(\theta, \phi)$ can be parametrized as $\boldsymbol{\Omega}(\theta, \phi) = (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a\cos\phi)$. Where we know that $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$. Now, the cross product of T_θ and T_ϕ is given by

$$T_\theta \times T_\phi = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{bmatrix}$$

$$= (-a^2\sin^2\phi\cos\theta)\boldsymbol{i} + (-a^2\sin^2\phi\sin\theta)\boldsymbol{j} + (-a^2\sin\phi\cos\phi)\boldsymbol{k}$$

Then, by using $z = a\cos\phi$, we get the surface intergal as:

$$\iint_{D} \boldsymbol{F} \cdot (T_u \times T_v) dudv = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (a^2\cos^2\phi)\boldsymbol{k} \cdot (-a^2\sin^2\phi\cos\theta\boldsymbol{i} - a^2\sin^2\phi\sin\theta\boldsymbol{j} - a^2\sin\phi\cos\phi\boldsymbol{k}) d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} -a^4\sin\phi\cos^3\phi d\theta d\phi$$

$$= -\frac{a^4\pi}{2} \int_0^{\frac{\pi}{2}} \sin\phi\cos^3\phi d\phi$$

$$= \frac{a^4\pi}{2} \int_1^0 u^3 du \ \textit{ By u-Substitution}$$

$$= -\frac{a^4\pi}{8}$$

Since the norm $\boldsymbol{n}$ of the surface is of the "same" direction as the vector field, i.e $\boldsymbol{F} \cdot \boldsymbol{n} > 0$. Hence the value should be positive, so the fulx is $\frac{a^4\pi}{2}$.