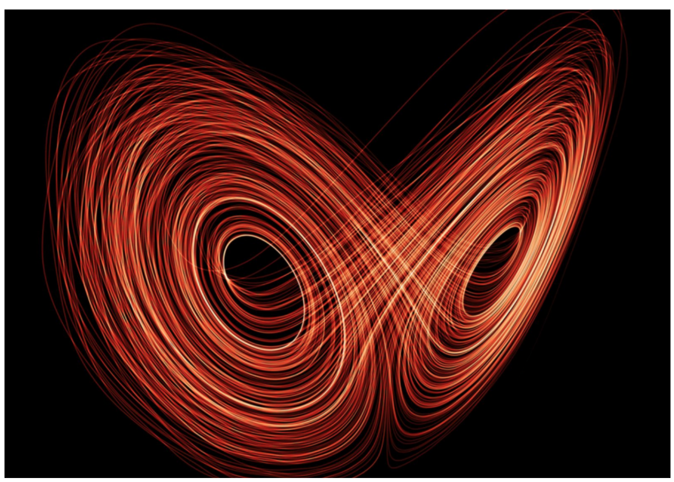
Ordinary Differential Equations

Winter 2025, Math 325 Course Notes

McGill University



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Despite all efforts, there may still be some typos, unclear explanations, etc. If you find potential mistakes, or any suggestions regarding concepts or formats, etc., feel free to reach out to the author at *zhangjohnson729@gmail.com.*

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Chapter 1

Classification and Existence & Uniqueness of ODEs

Definition and Examples of ODEs

Definition

Definition 1. *A differential equation is a relation involving an unknown function and some of its derivatives.*

Classification of Differential Equations

Definition

Definition 2. *An Ordinary Differential Equation (ODE) is a differential equation whose unknown function depends on one variable only. On the other hand, a Partial Differential Equation (PDE) has an unknown function that depends on more than one variable.*

Example: Consider the following differential equations:

• $my''(t) + \gamma y'(t) + mg = 0$ where γ, m, g are constants is an ODE, since the unknown function y depends on t only;

• $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial^2 x}$ (Heat Equation) is a PDE, since the unknown function u depends on two variables t and x.

Definition

Definition 3. *The order of an ODE is the order of the highest derivative appearing in the equation.*

Example: Consider the following differential equations:

• $y''(t) + \sin(t)y'''(t) = \cos(t)$ is a 3rd order ODE, since the order of the highest derivative is 3.

In general, an nth order ODE can be written as

$$F(t, y, y', y'', \cdots, y^{(n)}) = 0 \qquad (*)$$

where $F : \mathbb{R}^{n+2} \to \mathbb{R}$ is a function.

Now we would like to introduce systems of ODEs. Note that if $(*)$ takes the form

$$y^{(n)} = G(t, y, y', y'', \cdots, y^{(n-1)}),$$

then we can define intermediate variables as follows:

$$y_1 = y, y_2 = y', y_3 = y'', \cdots, y_n = y^{(n-1)}.$$

Then we have

$$\begin{aligned} y_1' &= y' = y_2 \\ y_2' &= y'' = y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= y^{(n)} = G(t, y, y', y'', \cdots, y^{(n-1)}). \end{aligned}$$

Define $\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \in \mathbb{R}^n$, then we have

$$\boldsymbol{Y}'(t) = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ g \end{pmatrix}$$

This leads to a system of ODEs of the form

$$\boldsymbol{Y}'(t) = F(y(t), t) \qquad F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

Here F could be general.

Example: If we consider the ODE $y'' + 3y' + 4y = 0$, then let $y_1 = y, y_2 = y'$, we have

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -4y_1 - 3y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which is a system of 1st order linear ODEs.

Definition

Definition 4. *We say that the ODE $F(t, y, y', y'', \cdots, y^{(n)})$ is linear if the map*

$$F : \mathbb{R}^{n+2} \to \mathbb{R}$$

is a linear polynomial in the variables $y, y', \cdots, y^{(n)}$. In this case the ODE can be written as

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = g(t) \qquad (**)$$

where $a_i(t)$ are functions of t. And we say that the ODE is non-linear if it is not of the form $(**)$.

Examples : Consider the following ODEs:

- $y''(t) + 7ty(t) = \cos(t)$ is a linear ODE, and this is a 2nd order scalar linear ODE.

- $(y'(t))^2 + \cos(t)y = 3$ is not a linear ODE because of the term $(y'(t))^2$.

Definition

Definition 5. *If a first order ODE can be written as*

$$y'(t) = F(y(t)),$$

we say that the ODE is autonomous, otherwise it is non-autonomous.

Example : Consider the following ODEs:

- $y'(t) + \cos(t)y(t) = \sin(t)$ is non-autonomous.

- $my'' + \gamma y' + mg = 0$ is autonomous, since we have a linear system

$$\begin{pmatrix} y_1' \\ \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ \\ -\dfrac{\gamma}{m} y_2 - g \end{pmatrix}$$

which has the form

$$F(y(t), t) = \boldsymbol{A}(t)y(t) + \boldsymbol{b}(t).$$

- $\dfrac{dN}{dt} = RN$ where R is a constant is also autonomous.

Definition

Definition 6. *A solution to the ODE $y'(t) = F(y(t), t)$ on an interval $J \subseteq \mathbb{R}$ is a differentiable function $y : J \to \mathbb{R}^n$ that satisfies the ODE $y'(t) = F(y(t), t)$ for all $t \in J$, here t is the independent variable and y is the dependent variable.*

Example : $y(t) = e^{-t} + 1$ is a solution to the ODE $y' + y = 1$, in this case $y : \mathbb{R} \to \mathbb{R}$ and $J = \mathbb{R}$ is the maximum interval of existence.

However, if $y(t) = \dfrac{1}{t-1}$ is a solution to some ODE, then the maximum interval of existence is not $\mathbb{R}$ since at $t = 1$ the function is not differentiable. But we can pick $J = [2, +\infty)$ (is not a maximum interval of existence).

Existence and Uniqueness Theorem

Definition

Definition 7. *Suppose we have an ODE $y'(t) = F(y(t), t)$ and $y(t_0) = y_0 \in \mathbb{R}^n$ is given, we call the system*

$$\begin{cases} y'(t) = F(y(t), t) \\ \\ y(t_0) = y_0 \end{cases}$$

an initial value problem (IVP).

Before introducing the existence and uniqueness theorem, we need some analysis background.

Definition

Definition 8. *Let $D \subseteq \mathbb{R}^n$ under any norm $\|\cdot\|$ on $\mathbb{R}^n$, a function $f : D \to \mathbb{R}^n$ is Lipschitz continuous (LC) if $\exists L > 0$, such that $\forall x, y \in D$,*

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad (*)$$

The smallest L which satisfies condition $(*)$ is called the Lipschitz constant.

Example : Consider the following functions:

- $f(x) = 10x - 3$ is LC, since

$$\|f(x) - f(y)\| = 10\|x - y\|,$$

and the Lipschitz constant is 10.

- $f(y) = \dfrac{1}{y-1}$ is not LC, but it is LLC on $(1, +\infty)$. See the definition below for LLC functions.

Definition

Definition 9. *Consider $D \subseteq \mathbb{R}^n$ to be an open set, a function $f : D \to \mathbb{R}^n$ is locally Lipschitz continuous (LLC) if for every compact set $K \subseteq D$, the function $f : K \to \mathbb{R}^n$ is Lipschitz continuous.*

Proposition

Proposition 1. *If a function $f : D \to \mathbb{R}^n$ is continuous differentiable and D is open, then f is LLC.*

Proof. By Mean-Value Theorem (MVT). ■

Now we would like to introduce the existence and uniqueness theorem:

Theorem

Theorem 1. *(Existence and Uniqueness Theorem)*

Consider $D \subseteq \mathbb{R}^n$ open and an open interval $I = (a, b) \subseteq \mathbb{R}$. Assume as well that $f : D \times (a, b) \to \mathbb{R}^n$ is continuous and that for all compact $K \subseteq D \times (a, b) \subseteq \mathbb{R}^{d+1}$, $\exists L(K)$ such that $\forall x, y \in K$

$$\|f(x, t) - f(y, t)\| \leq L\|x - y\| \qquad (**)$$

Then $\exists$ open interval $J \subseteq \mathbb{R}$, $t_0 \in J$, over which the solution to the IVP

$$\begin{cases} y'(t) = f(y(t), t) \\ \\ y(t_0) = y_0 \end{cases}$$

is defined. Furthermore any two solutions to that IVP agrees on the domain of their intersection.

To prove the theorem, we need to establish some lemmas and intermediate steps to help us.

Lemma

Lemma 1. *y solves the IVP*

$$\begin{cases} y' = f(y, t) \\ \\ y(t_0) = y_0 \end{cases}$$

if and only if

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds \qquad (***)$$

Proof. $(\Longrightarrow)$: Assume y solves the IVP, then

$$\int_{t_0}^{t} y'(t)dt = \int_{t_0}^{t} f(y(s), s)ds,$$

which implies $y(t) - y(t_0) = y(t) - y_0$.

($\Longleftarrow$) Assume that the equation $(***)$ holds, then note that

$$y(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s), s)ds = y_0$$

which solves the second equation of the IVP. Also

$$\begin{aligned} y'(t) &= \frac{d}{dt}y_0 + \frac{d}{dt}\int_{t_0}^{t} f(y(s), s)ds \\ &= 0 + f(y(t), t)). \end{aligned}$$

Which solves the IVP.

$\blacksquare$

So now our goal is to show that $(***)$ has a solution. Since $(y_0, t_0) \in D \times (a, b)$ open, then $\exists \alpha > 0, \delta > 0$, such that the compact cylinder

$$D_{\alpha,\delta} := \{(y, t) \in \mathbb{R}^{n+1} \Big| \|y - y_0\| \leq \alpha, |t - t_0| \leq \delta\} \subseteq D \times (a, b).$$

Let

$$M_{\alpha,\delta} := \sup_{(y,t)\in D_{\alpha,\delta}} \|f(y, t)\| < +\infty,$$

we then define the Picard operator:

Lemma

Lemma 2. *Let $\varepsilon > 0$ be defined by*

$$\varepsilon := \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right),$$

then $\varepsilon \leq \delta, D_{\alpha,\varepsilon} \subseteq D_{\alpha,\delta}$, and let $J = (t_0 - \varepsilon, t_0 + \varepsilon)$, then for any function $y(t)$ which satisfies $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha,\varepsilon}, \forall t \in J$, the function $T(y) : T \to \mathbb{R}^n$ defined by

$$T(y)(t) \equiv y_0 + \int_{t_0}^{t} f(y(s), s)ds$$

is called the Picard operator, and it satisfies the followings:

$$(1) \quad T(y_0)(t_0) = y_0;$$

$$(2) \quad (T(y)(t), t) \in D_{\alpha,\varepsilon}, \forall t \in J.$$

Before proving this lemma, below is a graph to illustrate $D_{\alpha,\delta}$ and $D_{\alpha,\varepsilon}$.

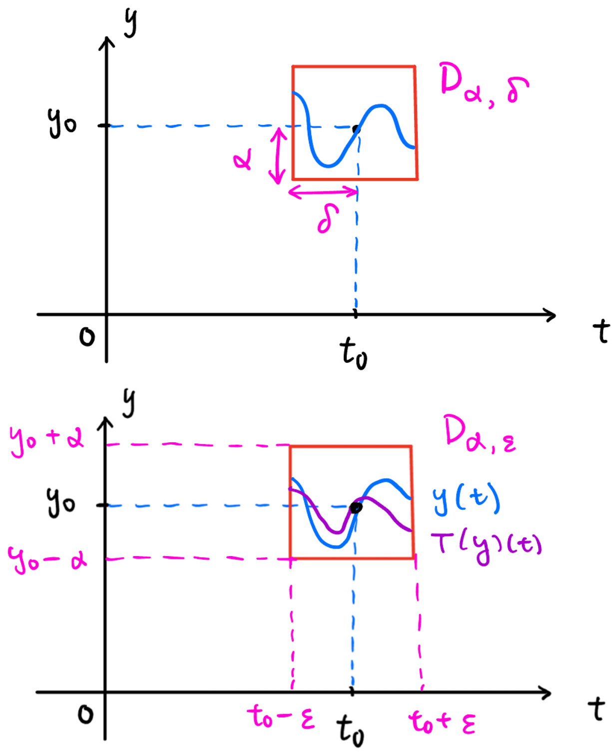

Figure 1.1: $D_{\alpha,\delta}$ and $D_{\alpha,\varepsilon}$

Proof. First we show that $T(y)(t_0) = y_0$. By definition,

$$T(y)(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s), s)dx = y_0,$$

which satisfies the equation.

Then we will show that

$$\|T(y)(t) - y_0\| \leq \alpha, |t - t_0| \leq \delta,$$

where the latter one is trivial since $\varepsilon \leq \delta$. Now

$$\begin{aligned} \|T(y)(t) - y_0\| &= \left\| y_0 + \int_{t_0}^{t} f(y(s), s)ds - y_0 \right\| \\ &= \left\| \int_{t_0}^{t} f(y(s), s)ds \right\| \\ &\leq \left| \int_{t_0}^{t} \|f(y(s), s)\| ds \right| \\ &\leq M_{\alpha,\delta} \cdot \left| \int_{t_0}^{t} ds \right| \\ &\leq M_{\alpha,\delta} \cdot \varepsilon \\ &\leq \alpha. \end{aligned}$$

■

The proof of existence relies on the process of Picard operators. We start by defining a sequence of functions $\{y_k\}$ with $y_0(t) = y_0 \in \mathbb{R}^n$ to be a constant function, and we define

$$y_k(t) \equiv T(y_{k-1})(t) = y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)dx, k \geq 1.$$

Lemma

Lemma 3. *$\forall t \in J = (t_0 - \varepsilon, t_0 + \varepsilon)$, $\{y_k(t)\}$ is Cauchy, and furthermore $y(t) = \lim_{k\to\infty} y_k(t)$ exists, and solves the IVP.*

Proof. We first have

$$\begin{aligned}\|y_1(t) - y_0(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(y_0(s), s)ds - y_0 \right\| \\ &= \left\| \int_{t_0}^{t} f(y_0(s), s)ds \right\|\end{aligned}$$

W.L.O.G, we assume that $t \in [0, t_0 + \varepsilon)$, hence

$$\begin{aligned}&\leq \int_{t_0}^{t} \|f(y_0(s), s)\| ds \\ &\leq M_{\alpha,\delta}(t - t_0)\end{aligned}$$

We further claim that

$$\|y_m(t) - y_{m-1}(t)\| \leq M_{\alpha,\delta} \cdot L^{m-1} \frac{(t - t_0)^m}{m!}.$$

We will prove it using induction. Assume it holds for all $N \leq m$, and we prove it on $m + 1$. We have

$$\begin{aligned}\|y_{m+1}(t) - y_m(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(y_m(s), s)dx - y_0 - \int_{t_0}^{t} f(y_{m-1}(s), s)ds \right\| \\ &\leq \int_{t_0}^{t} \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds \\ (\textit{By LLC}\ ) &\leq \int_{t_0}^{t} L\|y_m(s) - y_{m-1}(s)\| ds \\ (\textit{By induction hypothesis}\ ) &\leq \int_{t_0}^{t} L \cdot M_{\alpha,\delta} \cdot L^{m-1} \frac{(s - t_0)^m}{m!} ds \\ &= \frac{L^m \cdot M_{\alpha,\delta}}{(m + 1)!}(t - t_0)^{m+1},\end{aligned}$$

which finishes the induction. Then $\forall l > 1$, we have

$$\begin{aligned}\|y_l(t) - y_{l-1}(t)\| &\leq M_{\alpha,\delta} \cdot L^{t-1} \frac{(t - t_0)^l}{l!} \\ &\leq \frac{M_{\alpha,\delta}}{L} \frac{(L\varepsilon)^l}{l!}.\end{aligned}$$

Now let $p, m \geq 1$, we rewrite

$$\|y_{m+p}(t) - y_{m+1}(t)\| = \|y_{m+p}(t) - y_{m+p-1}(t) + y_{m+p-1}(t) + \cdots + y_{m+2}(t) - y_{m+1}(t)\|$$

Hence

$$\begin{aligned} L.H.S &\leq \sum_{k=1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\| \\ &\leq \sum_{k=1}^{p-1} \frac{M_{\alpha,\delta}}{L} \cdot \frac{(L\varepsilon)^{m+k+1}}{(m+k+1)!} \\ &= \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{p+m} \frac{(L\varepsilon)^j}{j!} \overset{m,p\to\infty}{\to} 0. \end{aligned}$$

Thus $\{y_k(t)\}$ is Cauchy. Since $\mathbb{R}^n$ is complete, i.e every Cauchy sequence converges, so $\lim_{k\to\infty} y_k(t)$ exists, and we denote $y(t) = \lim_{k\to\infty} y_k(t)$. We will show that $y(t)$ solves the IVP.

Take $p \to \infty$, we know that

$$\sup_{t\in J} \|y_{m+p}(t) - y_{m+1}(t)\| \overset{p\to\infty}{\to} \|y(t) - y_{m+1}(t)\| \leq \frac{M_{\alpha,\delta}}{L} \sum_{j=m+2}^{\infty} \frac{(L\varepsilon)^j}{j!} \overset{m\to\infty}{\to} 0.$$

Which implies that $y_k(t) \to y(t)$ uniformly, and since $y_k(t)$ is continuous, so $y(t)$ is continuous. Then

$$\lim_{k\to\infty} y_k(t) = \lim_{k\to\infty} y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)ds$$

and by uniform convergence,

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds,$$

which solves the IVP. ∎

Now we will prove the solution is unique, under all constructions above.

Proof. Suppose $y(t), z(t)$ both solves the IVP, and $W.L.O.G$, let $t \in (t_0, t_0 + \varepsilon)$. We know that

$$\begin{aligned} \|y(t) - z(t)\| &\leq \int_{t_0}^{t} \|f(y(s), s)ds - f(z(s), s)ds\| \\ &\leq L \int_{t_0}^{t} \|y(s) - z(s)\| ds. \end{aligned}$$

Let

$$g(t) \equiv \int_{t_0}^{t} \|y(s) - z(s)\| ds,$$

where g is non-negative and $g'(t) \leq Lg(t)$. We multiply both sizes of the inequality by an integrating factor $e^{-L(t-t_0)}$, then

$$L.H.S = \frac{d}{dt}\left(e^{-L(t-t_0)} g(t)\right) = e^{-L(t-t_0)}(g(t) - Lg(t)) \leq 0.$$

The function $t \mapsto e^{-L(t-t_0)}$ is decreasing on the chosen interval $(t_0, t_0 + \varepsilon)$, so

$$0 \leq e^{-L(t-t_0)} g(t) \leq g(t_0) = 0,$$

and g is non-negative, thus $g \equiv 0$, i.e $g'(t) = \|y(t) - z(t)\| = 0$. So we conclude that

$$y(t) = z(t),$$

and thus the solution is unique. Hence we have proved the existence and uniqueness theorem. ■

Example: Consider the IVP given by

$$\begin{cases} y' = y + 1 \\ \\ y(0) = 1 \end{cases},$$

this ODE is continuous, and f does not depend on $t, n = 1$, where $y' = y + 1 = f(y)$ which is LLC, then the theorem of existence and uniqueness applies. We take

$$\varepsilon = \min\left\{\delta, \frac{\alpha}{M_{\alpha,\delta}}\right\}, D_{\alpha,\delta} = \left\{(y,t) \middle| \|y - 1\| \leq \alpha, \|t - 0\| \leq \delta\right\} = [1 - \alpha, 1 + \alpha] \times [-\delta, \delta].$$

Since $f : \mathbb{R} \to \mathbb{R}$, so for this example α, β can be whatever as we want, and thus

$$M = \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y,t)\| = 2 + \alpha,$$

so

$$\varepsilon = \min\left\{\delta, \frac{\alpha}{2 + \alpha}\right\}.$$

For instance, if we take $\delta = 1, \alpha = 7$, we will get

$$J = \left(-\frac{7}{9}, \frac{7}{9}\right).$$

Hence by the theorem of existence and uniqueness, there is a solution that solves the IVP which is defined on J. Of course the choice of α, β varies in this example.

Definition

Definition 10. *Let $\varphi : I \to D, \psi : J \to D$ be solutions to $y' = f(y,t)$ where $f : D \times [a,b] \to \mathbb{R}^n$ is satisfying the hypothesis of the theorem of existence and uniqueness. We say that ψ is an extension of φ, if $I \subseteq J$ and $\varphi(t) = J(t), \forall t \in I$. If $I \subsetneq J$, then we say that ψ is a proper extension of φ. A solution is called a maximal solution if it has no proper extension. In this case J is called the maximal interval of existence and it's denoted by $J_{\max}$.*

Chapter 2

First Order Scalar Equations

First Order Linear Equations

In this section we consider the ODE of the form

$$a_0(t)y' + a_1(t)y = g(t)$$

where a_0, a_1, g are all continuous functions in $\mathbb{R}$. If $a_0(t) \neq 0$ then we have an ODE of a better form:

$$y'(t) + p(t)y = q(t)$$

and this is the type of equation we will be solving.

Theorem

Theorem 2. *If $p, q : (a, b) \to \mathbb{R}$ are continuous, $t_0 \in (a, b)$, then the unique solution to the IVP*

$$\begin{cases} y' = f(y, t) \\ \\ y(t_0) = y_0 \end{cases}$$

is such that $J_{\max} = (a, b)$.

To solve this ODE, we will use the notation of integrating factors. We multiply the ODE by a non-zero function $\mu(t)$, chosen in such a way that we can solve the ODE given by

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)q(t), \mu(t) \neq 0.$$

Using the product rule, we chose $\mu(t)$ such that

$$\mu(t)y' + \mu(t)p(t)y = (\mu(t) \cdot y)'$$

thus

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t),$$

then integrating both sides,

$$\mu(t)y(t) = \int \mu(t)q(t)dt + C,$$

Hence we have a solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right].$$

To find $\mu(t)$, we have

$$\frac{\mu'(t)}{\mu(t)} = p(t),$$

by chain rule, we have

$$\frac{d}{dt}\ln|\mu(t)| = \frac{\mu'(t)}{\mu(t)} = p(t),$$

thus

$$\ln|\mu(t)| = \int p(t)dt$$
$$\mu(t) = e^{\int p(t)dt}.$$

Note that without loss of generosity, we may omit the constant $+C$ for $\displaystyle\int p(t)dt$, the reason is that the constant on the exponential will result in a e^C term, and since we finally divide the whole equation by $\mu(t)$, so the two constants will cancel out.

So for the ODE $y' + p(t)y = q(t)$, we have a general formula:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right]$$

where

$$\mu(t) = e^{\int p(t)dt}.$$

Example : Solve the ODE $y' - 2y = 3e^t$.

Here we have $p(t) = -2, q(t) = 3e^t$, so the integrating factor $\mu(t)$ is given by

$$\mu(t) = e^{\int -2dt} = e^{-2t},$$

and thus

$$\begin{aligned} y(t) &= \frac{1}{e^{-2t}}\left[\int e^{-2t}3e^t dt + C\right] \\ &= e^{2t}\left[3\int e^{-t}dt + C\right] \\ &= e^{2t}[-3e^{-t} + C] \\ &= -3e^t + Ce^{2t}, \end{aligned}$$

where $C \in \mathbb{R}$ is a constant.

Remark: In general, if the initial condition is given, we can then solve for C and get an exact solution.

Let's consider some applications of this type of ODE, and how do we solve them.

Example : (Tank Mixing Problem) A tank with capacity $120L$ contains originally $90L$ of brine water in which $90g$ of salt is dissolved. A brine of concentration $2g$ of salt per liter enters the tank at a constant rate of $4L$ per-minute. Assume that the salt water is well-mixed, and it also exists the tank at a rate of $3L$ per-minute. Then what is the quantity of salt in the tank when the tank is full?

Solution : We denote $y(t)$ to be the quantity of salt at time t (minute), then $y' = \dfrac{dy}{dt}$ would be the rate of change of the quantity of salt at time t, measured in minute. The rate of change can be described by

$$y' = R_{in} - R_{out},$$

where R_{in} is the rate of the quantity of salt that enters the tank, which is given by

$$R_{in} = 4L/min \times 2g/L = 8g/min.$$

For R_{out}, we first wish to find the quantity of salt at time t, which is given by $y(t)$ (measured in gram) as the assumption gives. Also at time t there will be $90 + (4L/min - 3L/min)t = 90 + t$ liters of water, thus we can also get the concentration of the salt at time t, given by $\dfrac{y(t)}{90 + (4-3)t}$. Hence

$$R_{out} = 3L/min \times \frac{y(t)}{90+t} g/L.$$

So finally we get an ODE with IVP of the form

$$\begin{cases} y'(t) + \dfrac{1}{90+t} y(t) = 8 \\ \\ y(0) = 90 \end{cases}.$$

So by solving this equation we get

$$y(t) = 180 + 2t - \frac{90^4}{(90+t)^3},$$

where $t \in [0, 30]$, hence $y(30) \approx 202$ grams of salt.

Separable Equations

Definition

Definition 11. *A first order scalar ODE of the form*

$$\frac{dy}{dx} = f(x, y)$$

is called separable, if we can write $f(x, y) = f_1(x) \cdot f_2(y)$.

Here are some examples:

① $\frac{dy}{dx} = e^{xy}$ is not separable;

② $\frac{dy}{dx} = e^{x+y}$ is separable;

③ $\frac{dy}{dx} = \frac{x^4}{\sin(y)}$ is not separable;

④ $\frac{dy}{dx} = x^2 + y^2$ is not separable.

We assume $f_2(y) \neq 0$, then we have the form

$$-f_1(x) + \frac{1}{f_2(y)} \cdot \frac{dy}{dx} = 0,$$

by making substitutions, we have

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (*)$$

We assume that $H_1(x), H_2(y)$ are anti-derivatives of M, N, then $H_1'(x) = M(x), H_2'(x) = N(x)$, so equation $(*)$ becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0 \qquad (**)$$

Think of y as a function of x, then by chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y(x)),$$

now equation $(**)$ becomes

$$\frac{d}{dx}\left[H_1(x) + H_2(y(x))\right] = 0,$$

meaning that $H_1(x) + H_2(y) = C$ where $C \in \mathbb{R}$.

So now we let $H_1(x) = \displaystyle\int M(x)dx$, $H_2(y) = \displaystyle\int N(y)dy = \int \frac{1}{f_2(y)}dy$, we now have a general solution of the form

$$\int \frac{dy}{f_2(y)}dy = \int f_1(x)dx + C$$

where $c \in \mathbb{R}$ is a constant.

Example : Solve the ODE

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}, \quad y \neq \pm 1.$$

We know that this ODE is separable, so its general solution is given by

$$\int (1-y^2)dy = \int x^2 dx + C,$$

thus

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C.$$

We may have noticed that this is not a "strict" function, y is implicitly given. But we can see that the graph is the level curve of the multi-variable function $f(x,y) = \dfrac{1}{3}y^3 = \dfrac{1}{3}x^3 + C$, as the figure below shows.

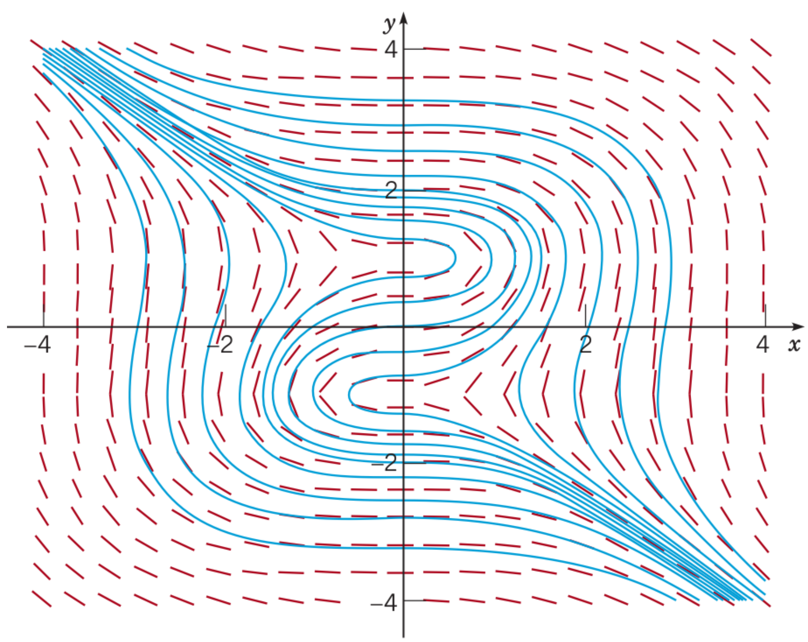

Figure 2.1: The graph of $y = \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$, each C represents a level curve of the function $f(x,y) = y - \frac{1}{3}y^3 - \frac{1}{3}x^3$ where $f(x,y) = C$.

The level curve of the can be defined by $\psi(x,y) = C$, and we denote

$$\Gamma_{x_0,y_0} = \{(x,y) \big| \psi(x,y) = \psi(x_0,y_0) = C\}$$

to be the level curve of height $C = \psi(x_0, y_0)$ containing the point (x_0, y_0).

Example : Solve the IVP

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)} \\ \\ y(0) = -1 \end{cases}$$

This ODE is separable, so we have a general solution of the form

$$\int 2(y-1)dy = \int (3x^2 + 4x + 2)dx + C,$$

where the general solution is given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

where C is a constant. To determine C, we substitute the condition that $x = 0, y = -1$ into the equation, and we get $C = 3$. Now in order to solve the equation explicitly, we need to solve y in terms of x, which in this case we have

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Since the initial condition states that $y(0) = -1$, so this will corresponds to the equation with minus sign above, i.e

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

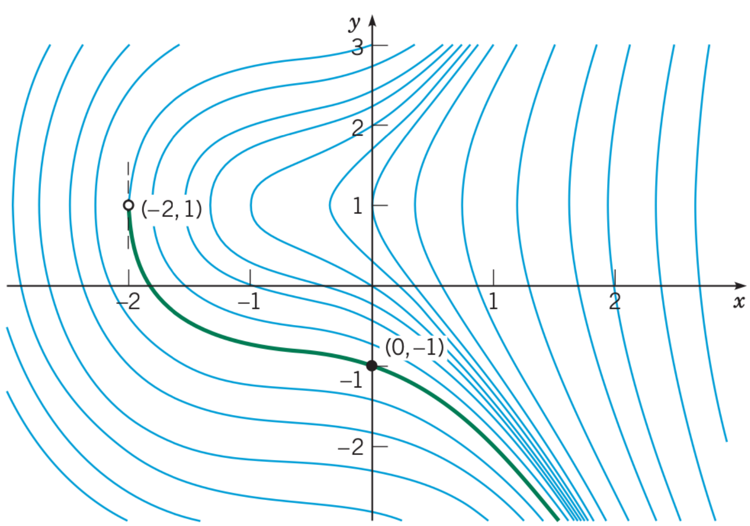

Figure 2.2: The solution satisfying $y(0) = -1$ is shown in green.

Example :