Functional Analysis

Winter 2025, Math 565 Course Notes

McGill University

Jiajun Zhang

January 25, 2025

Acknowledgments

I would like to extend my deepest thanks and appreciation to the following people, without whose support this note would not have been possible:

[**John Toth**], *Professor, McGill University*
I would like to express my deepest gratitude to Professor Toth, the instructor for this course, whose guidance and expertise were invaluable throughout the development of my notes.

Despite all efforts, there may still be some typos, unclear explanations, etc. If you find potential mistakes, or any suggestions regarding concepts or formats, etc., feel free to reach out to the author at *zhangjohnson729@gmail.com.*

Contents

Chapter 1

Differentiation of Measures

Signed Measures

Definition

Definition 1. *Let* $(X, \mathcal{M})$ *be a measurable space, then* $\nu : \mathcal{M} \to \mathbb{R}$ *is a signed measure if*

(i) $\nu(\varnothing) = 0$*;*

(ii) The range of ν *is* $\mathbb{R}/\{+\infty\}$ *or* $\mathbb{R}/\{-\infty\}$*, not both.*

(iii) For a sequence of disjoint sets $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$*, we have*

$$\nu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \nu(E_j).$$

where R.H.S converges absolutely if $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) < +\infty$.

Examples:

(i) Given two standard measures μ_1, μ_2 on $(X, \mathcal{M})$, then $\nu = \mu_1 - \mu_2$ is a signed measure.

(ii) Suppose $f : X \to (-\infty, +\infty]$ is $\mu-$measurable and either $\int_X f^+ d\mu < +\infty$ or $\int_X f^- d\mu < +\infty$. Let $f = f^+ - f^-$, then

$$\nu(E) = \int_E f d\mu$$

is a signed measure. Where $d\nu$ means that the integral is with respect to an arbitrarily measure, not necessarily the Lebesgue measure.

Remark

We will see that all signed measures are of this form. In the following, we'll assume that $\nu : \mathcal{M} \to (-\infty, +\infty]$.

Corollary

Corollary 1. ***(Continuity)***

Let $\nu : \mathcal{M} \to (-\infty, +\infty]$ be a signed measure on $(X, \mathcal{M})$.

(i) If a sequence $\{E_j\}_{j=1}^{+\infty}$ is increasing and $E_j \in \mathcal{M}$, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

(ii) If a sequence $\{E_j\}_{j=1}^{+\infty}$ is decreasing and $E_j \in \mathcal{M}$, $\nu(E_1) < +\infty$, then

$$\nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

Proof. For (i): If we set $N_j = E_j \Big\backslash \bigcup_{i=1}^{j-1} E_i$, then we claim that the sequence of sets $\{N_j\}_{n=1}^{\infty}$ are disjoint, and also $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} N_j$, $E_j = \bigcup_{i=1}^{j} N_i$. So we have

$$\begin{aligned}
\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \nu\left(\bigcup_{j=1}^{\infty} N_j\right) &= \sum_{j=1}^{\infty} \nu(N_j) && \text{by countable additivity} \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \nu(N_j) \\
&= \lim_{N\to\infty} \nu\left(\bigcup_{j=1}^{N} N_j\right) \\
&= \lim_{N\to\infty} \nu(E_N).
\end{aligned}$$

For (ii), set $N_j = E_1 \backslash E_j$, then the sequence of sets $\{N_j\}_{j=1}^{\infty}$ is increasing, and

$$\bigcup_{j=1}^{M} N_j = \bigcup_{j=1}^{M} E_1 \backslash E_j = E_1 \Big\backslash \bigcap_{j=1}^{M} E_j.$$

Given the fact that $\nu(E_1) < +\infty$, we have

$$\nu\left(E_1 \Big\backslash \bigcap_{j=1}^{\infty} E_j\right) = \nu(E_1) - \nu\left(\bigcup_{j=1}^{\infty} E_j\right)$$

Also

$$\nu\left(E_1 \Big\backslash \bigcap_{j=1}^{\infty} E_j\right) = \nu\left(\bigcup_{j=1}^{\infty} N_j\right) = \lim_{j\to\infty} \nu(N_j) = \lim_{j\to\infty} \nu(E\backslash E_j).$$

Thus we have

$$\nu(E_1) - \nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to+\infty} (\nu(E_1) - \nu(E_j)),$$

and thus

$$\nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

■

Definition

Definition 2. *Let $\nu : \mathcal{M} \to (-\infty, +\infty]$ be a signed measure, then $E \in \mathcal{M}$ is positive, written as $E \geq 0$, if $\forall F \subseteq E, F \in \mathcal{M}$, $\nu(F) \geq 0$; and $E \in \mathcal{M}$ is negative, written as $E \leq 0$, if $\forall F \subseteq E, F \in \mathcal{M}, \nu(F) \leq 0$; $E \in \mathcal{M}$ is null if $\nu(F) = 0$ for all $F \subseteq E$.*

Lemma

Lemma 1. *Let $F \geq 0$, $E \subseteq F, E \in \mathcal{M}$, then*

(i) $E \geq 0$;

(ii) If a sequence $\{E_j\}_{n=1}^{\infty} \subseteq \mathcal{M}$, $E_j \geq 0$, then

$$\bigcup_{j=1}^{+\infty} E_j \geq 0.$$

Proof. (i)

This proof is obvious from the definition of positivity.

■

Proof. (ii)

Set

$$Q_n = E_n \Big\backslash \bigcup_{j=1}^{n-1} E_j,$$

then $\bigcup E_j = \bigcup Q_n$ and $\{Q_n\}_{n=1}^{\infty}$ are disjoint, moreover since $Q_n \subseteq E_n$, $E_n \geq 0$, then by (i) we have $Q_n \geq 0$. Let $\widetilde{E} \subseteq \bigcup E_j = \bigcup Q_n$, then by countable additivity of ν, we have

$$\nu(\widetilde{E}) = \sum_{k=1}^{\infty} \nu(Q_k \cap \widetilde{E}).$$

Since $Q_k \cap \widetilde{E} \subseteq Q_k$ and $Q_k \geq 0$, so $\nu(Q_k \cap \widetilde{E}) \geq 0$.

■

Theorem

Theorem 1. *(Hahn Decomposition)*

Let $\nu : (X, \mathcal{M}) \to (-\infty, +\infty]$ be a signed measure, then $\exists P \geq 0, N \leq 0$ where $P, N \in \mathcal{M}$ such that $P \bigcup N = X$ and $P \bigcap N = \varnothing$.

Proof. W.L.O.G, we may assume that ν does not assign the value $-\infty$. Let m be the supremum of $\nu(E)$ as E ranges over all positive sets, thus there is a sequence $\{P_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \nu(P_j) = m$. Now let $P = \bigcup_{j=1}^{\infty} P_j$, we know that $P \geq 0$, $\nu(P) = m$; in particular $m < +\infty$. We claim that $N \; X/P$ is negative, to see this we assume it's not negative and derive a contradiction.

Notice that N cannot contain any non-null positive sets, that is because if $E \subseteq N$ is positive and $\nu(E) > 0$, then $E \bigcup P$ is positive and $\nu(E \bigcup P) = \nu(E) + \nu(P) > m$ which is impossible.

If N is not non-negative, then we can specify a sequence of subsets $\{A_j\}$ of N and a sequence $\{n_j\}$ of positive integers as follows: n_1 is the smallest integer for which $\exists B \subseteq N$ such that $\nu(B) > \frac{1}{n_1}$ and A_1 is such a set. Proceeding inductively, n_j is the smallest integer for which $\exists \; B \subseteq A_{j-1}$ with $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$, and A_j is such a set.

Then let $A = \bigcap_{n=1}^{\infty} A_j$, then

$$\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) > \sum_{j=1}^{\infty} \frac{1}{n_j}$$

and $n_j \to \infty$ as $j \to \infty$. But since $\exists B \subseteq A$ with $\nu(B) > \nu(A) + \frac{1}{n}$ for some n, for j sufficiently large we have $n < n_j$ and $B \subseteq A_{j-1}$, which contradicts the construction of n_j and A_j.

■

Definition

Definition 3. *Let $\nu, \mu : (X, \mathcal{M}) \to (-\infty, +\infty]$ be two signed measures, then ν is singular relative to μ, written as $\nu \perp \mu$, if $\exists E, F \in \mathcal{M}, E \bigcup F = X$, such that*

$$\nu(E) = 0 \text{ and } \mu(F) = 0.$$

Theorem

Theorem 2. *(Jordan)*

Let $\nu : (X, \mathcal{M}) \to (-\infty, +\infty]$ be a signed measure. Then there exists unique standard measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Here $\nu^\pm \geq 0$.

Proof. By Hahn, $X = P \bigcup N; P \cap N = \varnothing; P, N \in \mathcal{M}$. Given $E \in \mathcal{M}$, define

$$\nu^+(E) := \nu(E \cap P); \nu^-(E) = -\nu(E \cap N).$$

Then $\nu^\pm \geq 0$ and

$$\nu(E) = \nu(E \cap P) + \nu(E \cap N) = \nu^+(E) - \nu^-(E).$$

■

Remark

(i) The decomposition in Jordan is unique.

(ii) If ν is a signed measure, the associated total variation measure is defined by $|\nu| = \nu^+ + \nu^-$, which is a standard measure. Let μ to be another standard measure, then

$$\nu \perp \mu \Longleftrightarrow |\nu| \perp \mu \Longleftrightarrow \nu^\pm \perp \mu.$$

Definition

Definition 4. *Given a signed measure $\nu : (X, \mathcal{M}) \to (-\infty, +\infty]$, then $L^1(\nu) := L^1(\nu^+) \bigcap L^1(\nu^-)$ and for any $f \in L^1(\nu)$,*

$$\int_X f d\nu = \int_X f d\nu^+ - \int_X f d\nu^-.$$

Definition

Definition 5. *Given signed measures $\mu, \nu : (X, \mathcal{M}) \to (-\infty, +\infty]$, we say $\nu << \mu$, (meaning ν is absolutely continuous with respect to μ) if $\mu(E) = 0 \implies \nu(E) = 0, E \in \mathcal{M}$.*

Remark

Fix a measure μ, then if $\nu \perp \mu$ and $\nu << \mu$, then $\nu = 0$.

Theorem

Theorem 3. ***(Absolute Continuity of Measure)***
Let ν be a finite signed measure, $\nu : (X, \mathcal{M}) \to (-\infty, +\infty)$ and $\nu \geq 0$ is a standard finite measure. Then $\nu << \mu$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$, such that $\mu(E) < \delta \implies |\nu|(E) < \varepsilon$.

Proof. ($\Longleftarrow$) Suppose $\varepsilon - \delta$ condition is satisfied, then $|\nu|(E) = 0$ since the choice of ε is arbitrarily small. So by definition 5 we say that $\nu << \mu$.

($\Longrightarrow$) Suppose $\nu << \mu$, then we prove by contradiction, that is $\exists\varepsilon > 0$, and a sequence $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with $\mu(E_n) < 2^{-n}$, $|\nu|(E_n) \geq \varepsilon$. Set

$$F = \limsup_n E_n = \bigcup_{k=1}^{\infty} F_k$$

where $F_k = \bigcup_{n=k}^{\infty} E_n$. Then we have

$$\mu(F_k) \leq \sum_{n=k}^{\infty} 2^{-n} = 2^{-k+1},$$

and by assumption $\mu(F_1) < +\infty$. Thus

$$\mu(F) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\to+\infty} \mu(F_k) = 0.$$

Furthermore, we have

$$|\nu|(F_k) = |\nu|\left(\bigcup_{n=k}^{\infty} E_n\right) \geq |\nu|(E_k) \geq \varepsilon, k = 1, 2, \cdots$$

$$|\nu|(F) = |\nu|\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_k |\nu|(F_k) \geq \varepsilon.$$

Which is a contradiction. ■

Radon-Nikodym Theorem

Definition

Definition 6. *If a function $f \in L^1(\mu)$ and $\nu(E) = \int_E f d\mu$ is a finite signed measure, then we define the Radon-Nikodym derivative of ν with respect to μ, given by*

$$f := \frac{d\nu}{d\mu} \quad \text{or } d\nu = f d\mu.$$

Lemma

Lemma 2. *Let $\mu, \nu : (X, \mathcal{M}) \to [0, +\infty)$, then either $\nu \perp \mu$, or there exist $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) \geq 0$ and $\nu \geq \varepsilon\mu$ on E (E is a positive set for $\nu - \varepsilon\mu$).*

Proof. Let $X = P_n \bigcup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$, $P_n \geq 0, N_n \leq 0$ and let $P = \bigcup P_n, N = \bigcap N_n$. Then n is a negative set for $\nu - \frac{1}{n}\mu$ for all n, i.e $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for all n, so $\nu(N) = 0$. If $\mu(P) = 0$ then $\nu \perp \mu$. If $\mu(P) > 0$ the $\mu(P_n) > 0$ for some n and P_n is a positive set for $\nu - \frac{1}{n}\mu$. ■

Theorem

Theorem 4. ***(Radon-Nikodym)***

Let ν be a σ-finite signed measure, μ be a σ-finite measure ($\mu \geq 0$) on $(X, \mathcal{M})$. Then there exists unique σ-finite measures λ, ρ such that

$$\lambda \perp \mu, \;\; \rho << \mu, \;\; \nu = \rho + \lambda$$

Moreover, there exists $f \in L^1(\mu)$ such that $d\rho = f d\mu$, i.e

$$\rho(E) = \int_E f d\mu, \qquad d\nu = d\lambda + f d\mu.$$

Proof. Suppose $\nu, \mu \geq 0$ and finite, $\nu(X) < +\infty$, $\mu(X) < \infty$. Let $E \in \mathcal{M}$, we write $\nu(E) = \lambda(E) + \displaystyle\int_E f d\mu$, and we maximizes the choice of f, although we don't know λ at this moment, we must have

$$\int_E f d\mu \leq \nu(E).$$

Now consider

$$\mathcal{F} := \left\{ f : X \to [0, +\infty]; f \in L^1(\mu); \int_E f d\mu \leq \nu(E) \right\},$$

clearly $0 \in \mathcal{F}$. Now set

$$0 \leq a := \sup_{f \in \mathcal{F}} \int_X f d\mu < +\infty,$$

and we introduce a technical lemma:

Lemma

Lemma 3. *Suppose $f_1, f_2, \cdots, f_n \in \mathcal{F}$ and set $g_n = \max\{f_1, f_2, \cdots, f_n\}$, then $g_n \in \mathcal{F}$.*

Proof. Suppose $f, g \in \mathcal{F}$, set $h = \max\{f, g\}$, let $E \bigcap A = \{f \geq g\} \bigcap E$ and $E/A = \{f < g\} \bigcap E$, where

$$E = (A \cap E) \cup (E/A).$$

Then

$$\begin{aligned}\int_E h d\mu &= \int_{A \cap E} h d\mu + \int_{E/A} h d\mu \\ &\equiv \int_{A \cap E} f d\mu + \int_{E/A} g d\mu\end{aligned}$$

Since $f, g \in \mathcal{F}$, thus

$$\leq \nu(A \cap E) + \nu(E/A) = \nu(E).$$

Therefore, $h = \max\{f, g\} \in \mathcal{F}$ and by induction, $g_n = \max\{f_1, \cdots, f_n\} \in \mathcal{F}$. ■

Choose $\{f_n\}_{n=1}^{\infty} \subseteq \mathbb{F}$ with $\lim_{n\to\infty} \int_X f_n d\mu = a$. Since this is not monotone is general, the idea is to replace f_n by g_n. Set $F = \sup_{n\geq 1} f_n$, since $g_n = \max\{f_1, f_2, \cdots, f_n\} \implies g_n \in \mathcal{F}$ by lemma and g_n is increasing by construction.

Note that

$$\int_E g_n d\mu = \int_E \max\{f_1, \cdots, f_n) d\mu \geq \int_E f_n d\mu,$$

so

$$\lim_{n\to\infty} \int_X g_n d\mu \geq \lim_{n\to\infty} \int_X f_n d\mu = a.$$

Also note that $g_n \in \mathcal{F}$, this automatically guarantees that $\lim_{n\to\infty} \int_x g_n d\mu \leq a$.

Here is the key-point: We have a sequence $\{g_n\}_{n=1}^{\infty}$ where $g_n \in L^1(\mu)$, and g_n is increasing, also $\lim_{n\to+\infty} \int_X g_n d\mu = a$, so Monotone Convergence Theorem may apply:

$$\lim_{n\to\infty} \int_X g_n d\mu \overset{MCT}{=} \int_X \lim_{n\to\infty} g_n d\mu \overset{\text{def}}{=} \int_X F d\mu.$$

Recall. $F = \sup_{n\geq 1} f_n$ measurable, moreover $\forall E \in \mathcal{M}$,

$$\int_E F d\mu = lim_{n\to\infty} \int_E g_n d\mu \leq \lim_{n\to\infty} \nu(E) = \nu(E),$$

and thus $F \in \mathcal{F}$. We write $d\nu = d\lambda + F d\mu$, and since $d\nu \geq F d\mu$, so $d\lambda \geq 0$, i.e λ is a (finite) measure.

Hence it remains to show that $d\lambda \perp d\mu$, we argue this by contradiction. Suppose not, there exists $\varepsilon > 0$ and $E \in \mathcal{M}$ with $\mu(E) \geq 0$ and $\lambda(E) \geq \varepsilon\mu(E)$, and $d\nu = d\lambda + F d\mu$. Then $d\nu \geq (\varepsilon\chi_E + F) d\mu$, choosing $\widetilde{F} = \varepsilon\chi_E + F \geq F$ on E, we get that $\widetilde{F} \in \mathcal{F}$ and $\widetilde{F} > F$ which is impossible. So $\lambda \perp \mu$. That proves the Radon-Nikodym for finite measure space, by standard argument, the proof extends for σ-finite measure and signed measure.

■

Corollary

Corollary 2. *Let ν be a σ-finite signed measure on $(X, \mathcal{M})$, μ be a σ-finite measure. Then if $\nu << \mu$, there exists $f \in L^1(\mu)$ such that*

$$d\nu = f d\mu, \quad i.e\ \nu(E) = \int_E f d\mu, \forall E \in \mathcal{M}.$$

If $d\nu = f d\mu$, we write $f = \dfrac{d\nu}{d\mu}$ to be the Radom-Nikodym derivative of ν with respect to μ.

Corollary

Corollary 3. *Let ν be a σ-finite signed measure, μ, λ are σ-finite measures such that $\nu << \mu, \mu << \lambda$, then $\nu << \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$$

Proof. Exercise. ■

Lebesgue Differentiation Theorem

Let's give some motivation before starting this section.

Consider $I = [a,b] \subseteq \mathbb{R}^2, x_0 \in I, I_{x_0}(h) = [x_0, x_0+h], h > 0$. Then by fundamental theorem of calculus we have

$$f \in C^0(I), F(x) = \int_{x_0}^{x} f(t)dt, F \in C^1.$$

and $F'(x) = f(x)$ in I. We further have

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt,$$

by taking $h \to 0^+$, we have

$$\begin{aligned} F'(x_0) &= \lim_{h\to 0}\frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt \\ &= \lim_{h\to 0}\frac{1}{|I_{x_0}(h)|}\int_{I_{x_0}(h)} f(t)dt \\ (\text{in } \mathbb{R}^n) &= \lim_{h\to 0}\frac{1}{|B_{x_0}(h)|}\int_{B_{x_0}(h)} f dx, f \in L^1(\mathbb{R}^n). \end{aligned}$$

Suppose we have $(X, \mathcal{M}) = (\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, $\mu = m$ to be the Lebesgue measure, given ν as a Borel measure, let $\nu << m$, also

$$B(x,r) = \{y \in \mathbb{R}^n | \|y - x\| < r\}, r > 0,$$

then we are interested in the ratio $\lim_{r\to 0^+}\frac{\nu(B(x,r))}{|B(x,r)|}$ where $|B(x,r)| = m(B(x,r))$.

Theorem

Theorem 5. *(General Version of Lebesgue Differentiation Theorem (LDT))*

If $\nu << m$, then

$$F(x) := \lim_{r\to 0^+}\frac{\nu(B(x,r))}{|B(x,r)|} \quad \textit{exists a.e},$$

and moreover $F(x) = f(x)$ a.e, where f is the density of ν in Radom-Nikodym, i.e

$$\nu(E) = \int_E f dm, \forall E \in \mathfrak{B}(\mathbb{R}^n).$$

Note that by Radon-Nikodym theorem,

$$\frac{\nu(B(x,r))}{|B(x,r)|} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy, f \in \mathcal{L}^1(\mathbb{R}^n).$$

and

$$F(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy.$$

This is the classical setting of LDT. We would like to show that

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x) \quad \text{for a.e } x \in \mathbb{R}^n.$$

Lemma

Lemma 4. *(Covering Lemma)*

Let $U = \bigcup_{B \in \mathcal{C}} B$ *where* B *are open balls in* $\mathbb{R}^n$*, and assume that* $m(U) > C > 0$ *where* C *is a constant. Then there exists a sub-collection* $\{B_1, B_2, \cdots, B_k\} \subseteq \mathcal{C}$*, such that*

$$\sum_{j=1}^{k} m(B_j) > 3^{-n} \cdot C$$

Proof. By regularity of a measure, we can find compact $K \subseteq U$ such that $m(K) > c$, but $K \subseteq \bigcup_{B \in \mathcal{C}} B$ is compact, then we may find a sub-collection $\{A_1, A_2, \cdots, A_m\} \subseteq \mathcal{C}$ such that $K \subseteq \bigcup_{i=1}^{m} A_i$. We may then define $\{B_i\}$ as follows:

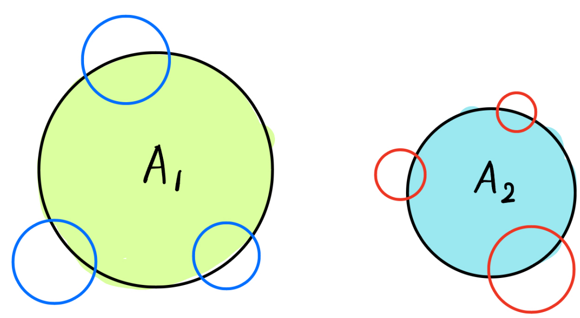

Figure 1.1: A graph of A_1, A_2 and the construction of B_i

$$B_1 = \max\{A_j\}_{j=1}^m; B_2 = \max_{A_j \cap B_1 = \varnothing} \{A_j\}_{j=1}^m; B_3 = \max_{A_j \cap (B_1 \cap B_2) = \varnothing} \{A_j\}_{j=1}^m, \cdots$$

The main point is that to start with $3A_1 = 3B_1$, by triple the size we may cover all of them.

If A_i is not one of B_j's, pick minimal j such that $A_i \cap B_j = \varnothing$, and $radius(A_i) < 3radius(B_j)$, and $|A_1| < 3^n|B_j|$, setting $B_j^* = 3B_j$, we have

$$c < m(K) \leq \sum_{j=1}^{k} m(B_j^*) = 3^n \sum_{j=1}^{k} m(B_j).$$

■

Definition

Definition 7. *We define*

$$L^1_{loc}(\mathbb{R}^n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n); \int_K |f|dx < +\infty, \forall K \subseteq \mathbb{R}^n \text{ bounded} \right\}.$$

If $f \in L^1_{loc}$, then we define

$$A_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy.$$

The first lemma will discuss the continuity of the function

$$(x,r) \in \mathbb{R}^n \times \mathbb{R}^+ \to A_r f(x).$$

Lemma

Lemma 5. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then the function*

$$(x,r) \in \mathbb{R}^n \times \mathbb{R}^+ \to A_r f(x)$$

is continuous.

Proof. By translation invariant, we have $|B(x,r)| = |B(0,r)| = |B(0,1)| \cdot r^n$, then

$$(x,r) \to \frac{1}{|B(x,r)|} = \frac{1}{|B(0,1)|} \cdot \frac{1}{r^n}$$

is continuous for $r > 0$. As for

$$(x,r) \to \int_{B(x,r)} f(y)dy = \int_{\mathbb{R}^n} \chi_{B(x,r)}(y) f(y)dy,$$

we fix $(x_0, r_0) \in \mathbb{R}^n \times \mathbb{R}^+$, then if we set

$$S(x,r) = \{y \in \mathbb{R}^n \Big| |y-x| = r\} = \partial \overline{B(x,r)},$$

then we claim that

$$\lim_{(x,r)\to(x_0,r_0)} \chi_{B(x,r)}(y) = \chi_{B(x_0,r_0)} \cdot |y|, \forall y \notin S(x_0,r_0).$$

Since $f \in L^1_{loc}$, then $\chi_{B(x,r)} \cdot f \in L^1$ and

$$\int_{\mathbb{R}^n} \left| \chi_{B(x,r)} \cdot f \right| = \int_{B(x,r)} |f(y)| < +\infty.$$

Then by dominated convergence theorem,

$$\lim_{(x,r)\to(x_0,r_0)} \int_{B(x,r)} f(y)dy = \int_{B(x_0,r_0)} f(y)dy,$$

which implies that

$$\lim_{(x,r)\to(x_0,r_0)} A_r f(x) = A_{r_0} f(x_0).$$

■

Definition

Definition 8. *(Maximal Function)*

Let $f \in L^1_{loc}(\mathbb{R}^n)$ and we define

$$\begin{aligned} Hf(x) &:= \sup_{r>0} A_r|f|(x) \\ &= \sup_{r>0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)|dy. \end{aligned}$$

Theorem

Theorem 6. *(Maximal Theorem)*

There exists some $C > 0$ such that $\forall f \in L^1_{loc}(\mathbb{R}^n)$,

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f|dx.$$

Proof. Let $E_\alpha = \{x \in \mathbb{R}^n | Hf(x) > \alpha\}$, since $Hf(x) = \sup_{r>0} A_r|f|(x)$, then $x \in E_\alpha \implies \sup_{r>0} A_r|f|(x) > \alpha$. Then $\forall x \in E_\alpha, \exists r_x > 0$, such that $A_{r_x}|f|(x) > \alpha$. Clearly,

$$E_\alpha \subseteq \bigcup_{x \in E_\alpha} B(x, r_x), m(E_\alpha) > C > 0.$$

Then by applying the covering lemma, we have

$$\sum_{j=1}^{k} |b_j| > 3^{-n} C.$$

Note that we have $A_{r_j}|f|(x_j) > \alpha, \forall j = 1, 2, \cdots, k, r_j = r_{x_j}$, this means $\dfrac{1}{|B_j|}\displaystyle\int_{B_j} |f| > \alpha$, so $\dfrac{1}{\alpha}\displaystyle\int_{B_j} |f| > |B_j|$, hence

$$c < 3^n \sum_{j=1}^{k} |B_j| < \frac{3^n}{\alpha} \sum_{=1}^{k} \int_{B_j} |f| \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f|.$$

■