Chapter 1

Intro, Classification, Existence and Uniqueness

Differential equations are a cornerstone of scientific inquiry, providing a powerful framework to model and understand changes in the natural world. From predicting planetary motions and describing the spread of diseases to modeling population dynamics and the behavior of electric circuits, these equations bridge abstract mathematics and real-world phenomena. Their ability to capture relationships between variables that evolve over time or space makes them indispensable across disciplines, fostering deeper insights into the mechanisms driving complex systems.

In this course, we explore the foundational concepts and methods needed to analyze ordinary differential equations (ODEs) and their applications. The selected topics provide a balanced mix of theoretical understanding, computational techniques, and practical problem-solving tools essential for both further study and real-world applications. The intended content includes:

1. **Examples, Classification and Theorem of Existence and Uniqueness**
2. **First-Order Scalar Equations**
    - First-order linear equations.
    - Separable equations.
    - Exact equations and integrating factors.
3. **Systems of Linear Equations**
    - Transition from higher-order equations to systems of linear equations.
    - Theory of first-order linear systems.
    - Constant coefficient homogeneous systems, including non-diagonalizable matrices and matrix exponentials. Non-homogeneous systems and solution sets.

4. **Stability and Phase Portraits**
   - Stability theory and phase portraits for linear and nonlinear systems.
   - Orbits and examples.
5. **Laplace Transform**
   - Definitions and examples.
   - Solving initial value problems.
   - Computational methods, step functions, and impulse functions.
6. **Power Series Solutions and Numerical Methods**
   - Power series solutions and Taylor approximations.
   - Numerical methods such as Euler's method and higher-order Taylor methods.
7. **Computer-Assisted Proofs in ODEs: A Brief Introduction**

This curriculum is designed to equip students with both a solid theoretical foundation and practical computational tools, preparing them for advanced studies or applications in science and engineering.

1.1 Definition and Examples

We begin this section by defining a differential equation, presenting examples, and classifying the various types of differential equations.

Definition 1.1.1. A *differential equation* is a relation involving an unknown function and some of its derivatives.

The unknown function in a differential equation may for instance represent the position of a particle, the number of individuals in a population, the concentration of a given chemical, the velocity of a fluid or the distribution of temperature of a metal beam.

Example 1.1.2 (Newton's Second Law). One of the greatest scientific achievements of the seventeenth century is the discovery by sir Isaac Newton (1643–1727) of its *Laws of Motion*, a discovery which he formalized in 1687 in his three famous volumes *Philosophiæ Naturalis Principia Mathematica*. A simple application of its laws describe the position (the height) $y(t)$ at time t of a body of mass m which moves vertically in the air, subject to gravitational and damping forces. The resulting differential equation is given by

$$my''(t) = -mg - \gamma y'(t), \tag{1.1}$$

where $\gamma \geq 0$ is a constant related to the density of the air and the shape of the object and g is the gravitational acceleration.

Example 1.1.3 (Standard Model Describing the Growth of a Single Population). Denote by $N(t)$ the number of individuals in a single population. The population could be representing for instance the number of animals, bacteria or cells. A simple observation from population dynamics in biology is that the rate of change of the population at time t is proportional (with constant of proportionally r) to the number of individuals at time t. Mathematically, this leads to the differential equation

$$N'(t) = rN(t). \tag{1.2}$$

The constant $r \in \mathbb{R}$ is called a *growth rate* if the population increases and a *decay rate* if it decreases. Assume that $N(t) > 0$ for all time. This is a reasonable assumption, since if not, there exists a time t_0 such that $N(t_0) = 0$ and in that case no reproduction or decay can occur, and the population stays null forever. Hence, if $N(t) > 0$, then by the chain rule

$$\frac{d}{dt}\ln N(t) = \frac{d(\ln N)}{dN} \cdot \frac{dN}{dt} = \frac{N'(t)}{N(t)} = r.$$

Integrating on both sides with respect to t leads to

$$\ln N(t) = rt + C, \quad C \in \mathbb{R}$$

where C is the constant of integration. Taking the exponential on both sides leads to

$$e^{\ln N(t)} = e^{rt+C} \Longleftrightarrow N(t) = e^C e^{rt} = ke^{rt},$$

where $k = e^C$ is any positive constant. Hence, the *general solution* of (1.2) is given by

$$N(t) = ke^{rt}, \quad k \in \mathbb{R}. \tag{1.3}$$

In case an initial population N_0 is given at time $t = 0$ (that is $N(0) = N_0$), then the constant k can be determined uniquely. Indeed,

$$N_0 = N(0) = ke^{r0} = k \Longrightarrow k = N_0.$$

Hence, the *unique* solution of the differential equation (1.2) which satisfies the initial condition $N(0) = N_0$ is given by

$$N(t) = N_0 e^{rt}, \quad t \in \mathbb{R}. \tag{1.4}$$

For $r > 0$, the solution (1.4) describes an exponential growth, which depends on the growth rate r. For obvious reasons, the growth rate of rabbits is larger than one of elephants. In case $r < 0$, the model could be used to describe the radioactive decay of some isotopes, which is what we consider in the Example 1.1.5.

Example 1.1.4 (Incorporating the Carrying Capacity of the Environment). In Example 1.1.3, we concluded that the growth of the population subject to the law (1.2) was exponential, that is $N(t) = N_0 e^{rt}$. In reality, the ressources (e.g. food, lands) are finite (limited) and while an exponential growth might be plausible in the short term, it is obviously not possible in the long term. To incorporate the limitation of the environment, we may introduce the notion of a *carrying capacity*, which is a given number K such that above that critical number, the population begins decreasing. In other words, for $N(t) < K$, there is a growth rate while for $N(t) > K$, there is a decay rate. We may therefore assume that the rate is no longer constant and is a function of N, which we now denote by $R = R(N)$. For N close to zero, we may still assume exponential growth, that is $R(0) = r$ while when the population reaches the carrying capacity K, $R(K) = 0$. The simplest continuous function which satisfies both requirements is given by

$$R(N) = r\left(1 - \frac{N}{K}\right).$$

In this case, the new growth model is given by $N'(t) = R(N)N(t) = r\left(1 - \frac{N(t)}{K}\right)N(t)$ which leads to the *logistic equation*

$$N'(t) = rN\left(1 - \frac{N}{K}\right). \tag{1.5}$$

Example 1.1.5 (Radioactive Decay). Assume that $N(t)$ denotes the amount of a radioactive material (for instance an isotope). The *radioactive decay law* of Rutherford is then given by (1.2), where $r < 0$ is the decay rate. Note that different isotopes have different decay rates. For instance, denote by $r_{U^{238}}$, $r_{C^{14}}$, $r_{C^{60}}$ and r_{H^7}, the decay rate of the Uranium-238 (the most common isotope of uranium found in nature which most modern nuclear weapons utilize), Carbon-14 (its presence in organic materials is the basis of the radiocarbon dating method), Cobalt-60 (a high intensity gamma-ray emitter) and Hydrogen-7 (one of the fastest decaying known isotope), respectively. Then,

$$r_{H^7} < r_{C^{60}} < r_{C^{14}} < r_{U^{238}} < 0.$$

To determine these decay rates, one can use the notion of *half-life*, which is the time required for a quantity to reduce to half its initial value. Denoting by N_0 the initial value at time $t = 0$ of a given isotope, the half-life is the time $t_{1/2}$ such that

$$N(t_{1/2}) = \frac{N_0}{2}.$$

Using (1.4), we get that

$$N(t_{1/2}) = N_0 e^{r \cdot t_{1/2}} = \frac{N_0}{2} \Longrightarrow e^{r \cdot t_{1/2}} = \frac{1}{2} \Longrightarrow r \cdot t_{1/2} = \ln\frac{1}{2} = -\ln 2,$$

and therefore the decay rate can be determined from the half-life $t_{1/2}$ via the formula

$$r = -\frac{\ln 2}{t_{1/2}} < 0. \tag{1.6}$$

Assuming that the time t is in years and considering the fact that the half-lives of Cobalt-60, Carbon-14 and Uranium-238 are given by 5.2714, 5730 and 4.468×10^9 years, respectively, then their corresponding decay rates are given by

$$\begin{aligned} r_{C^{60}} &= -\frac{\ln 2}{5.2714} \approx -0.1315 \\ r_{C^{14}} &= -\frac{\ln 2}{5730} \approx -1.21 \times 10^{-4} \\ r_{U^{238}} &= -\frac{\ln 2}{4.468 \times 10^9} \approx -1.55 \times 10^{-10}. \end{aligned}$$

The half-life of the Hydrogen-7 is 23×10^{-24} seconds which corresponds roughly to 7.3×10^{-31} years. Its corresponding decay rate is given by

$$r_{H^7} = -\frac{\ln 2}{7.3 \times 10^{-31}} \approx -9.4 \times 10^{29}.$$

1.2 Classification

In the study of differential equations, classification plays a crucial role in determining the appropriate techniques for solving them. Once we identify the class of a given equation, we often know which methods are best suited for finding a solution. This process of classification can be thought of as recognizing patterns: different types of equations have well-established solution techniques that are tailored to their specific forms.

Differential equations can be classified in several ways. One of the first distinctions is between ordinary differential equations (ODEs) and partial differential equations (PDEs). ODEs involve functions of a single variable, whereas PDEs involve functions of multiple variables. Another important classification is based on the order of the equation, which refers to the highest derivative present. Equations can also be categorized as linear or nonlinear, with linear equations often being more straightforward to solve using well-known methods like separation of variables or the Laplace transform. Additionally, the equation may be classified as autonomous, where the independent variable (typically time) does not explicitly appear in the equation, or non-autonomous, where it does.

By recognizing these classifications, we can efficiently apply the correct solution methods and proceed systematically. The more familiar we become with the patterns that emerge in these different types of differential equations, the more intuitive the process of solving them becomes.

Let us now formalize these concepts.

1.2.1 ODEs vs PDEs

Definition 1.2.1. An *ordinary differential equation* (ODE) is a differential equation whose unknown function depends only on one independent variable, while a *partial differential equation* (PDE) is a differential equation whose unknown function depends on more than one independent variable.

Example 1.2.2. Models (1.1), (1.2) and (1.5) are ODEs, while the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is a PDE, as the unknown function $u = u(x,t)$ (the temperature) depends on two independent variables: t (time) and x (space).

In this class, we focus solely on studying ODEs.

1.2.2 Order of an ODE

Definition 1.2.3. The *order* of an ODE is the order of the highest derivative that appears in the equation.

Example 1.2.4. (1.1) is a second order ODE, while (1.2) and (1.5) are first order ODEs. The equation

$$y'''(t) - y(t)y'(t) = \sin(t)$$

is a third order ODE.

More generally, a *scalar n^{th} order ODE* can be written as

$$F[t, y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t), y^{(n)}(t)] = 0, \tag{1.7}$$

where $F : \mathbb{R}^{n+2} \to \mathbb{R}$ is a mapping and

$$y^{(k)}(t) \overset{\text{def}}{=} \frac{d^k y}{dt^k}(t)$$

denotes the k^{th} derivative of $y(t)$ with respect to t, for $k = 1, \ldots, n$.

1.2.3 Systems of ODEs

First note that a scalar n^{th} order ODE of the form

$$y^{(n)}(t) = G[t, y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)] \tag{1.8}$$

can always be written as a *system* of first order ODEs. Indeed, let $y_1 \stackrel{\text{def}}{=} y$ and $y_i \stackrel{\text{def}}{=} y^{(i-1)}$ for $i = 2, \ldots, n$, and observe that

$$y_i'(t) = y^{(i)}(t) = y_{i+1}(t), \text{ for } i = 1, \ldots, n-1$$

and

$$y_n'(t) = y^{(n)}(t) = G[t, y(t), y'(t), y''(t), \ldots, y^{(n-1)}(t)] = G[t, y_1(t), y_2(t), y_3(t), \ldots, y_n(t)]$$

and therefore

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_{n-1}'(t) \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \\ G[t, y_1(t), y_2(t), y_3(t), \ldots, y_n(t)] \end{pmatrix}.$$

Denote

$$\mathbf{y}(t) \stackrel{\text{def}}{=} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \quad \text{and} \quad f(\mathbf{y}(t), t) \stackrel{\text{def}}{=} \begin{pmatrix} y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \\ G[t, y_1(t), y_2(t), y_3(t), \ldots, y_n(t)] \end{pmatrix},$$

we obtain the *first order system* of ODEs

$$\mathbf{y}'(t) = f(\mathbf{y}(t), t).$$

Example 1.2.5. The equation $y''(t) + 3y'(t) + 4y(t) = 0$ is a second order ODE of the form $y''(t) = -3y'(t) - 4y(t)$. Letting $y_1(t) = y(t)$ and $y_2(t) = y'(t)$ yields the system of first order ODEs

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = f(\mathbf{y}(t)) \stackrel{\text{def}}{=} \begin{pmatrix} y_2(t) \\ -4y_1(t) - 3y_2(t) \end{pmatrix}.$$

As a consequence of the previous discussion, many ODEs can be written as systems of first order ODEs, which in general take the form

$$y'(t) = f(y(t), t), \tag{1.9}$$

where $f : D \times (a, b) \to \mathbb{R}^n$, with $D \subset \mathbb{R}^n$ an open set and $(a, b) \subset \mathbb{R}$ is a *time* interval. Here $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$.

Example 1.2.6 (Lorenz equations). A well-studied first order system of ODEs is given by

$$\begin{aligned} y_1' &= \sigma(y_2 - y_1) \\ y_2' &= \rho y_1 - y_2 - y_1 y_3 \\ y_3' &= y_1 y_2 - \beta y_3 \end{aligned}$$

which are called the *Lorenz equations*, and they were derived by the meteorologist Edward Lorenz in the early 1960's as a simple model for weather prediction (more precisely as a model of a pair of coupled convection cells in the atmosphere). In the model $\sigma, \beta, \rho > 0$ are positive real parameters.

1.2.4 Linear vs Nonlinear

Definition 1.2.7. The ODE (1.8) is said to be *linear* if the mapping F in (1.7) is a linear polynomial in $y, y', \ldots, y^{(n)}$, that is it is of the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t), \tag{1.10}$$

where $a_0, \ldots, a_n$ and g are functions of t. The ODE (1.8) is said to be *nonlinear* if it is not linear, that is, if it is not of the form (1.10).

Example 1.2.8. The ODE $ty''(t) + \sin(t)y'(t) + y(t) = e^t$ is linear (second order) while the ODEs $y(t)y'(t) = 1$ (first order) and $y^{(3)} = y^2$ (third order) are nonlinear.

1.2.5 Autonomous vs Non-autonomous

Note that when the right-hand side f of the ODE (1.9) does not depend explicitly on t, i.e. it is of the form

$$y'(t) = f(y(t))$$

the ODE is called *autonomous*, while if it does, the ODE is called *non-autonomous*.

Example 1.2.9. $y'(t) = ty(t)$ is non-autonomous while $y'(t) = y^2(t)\sin y(t)$ is autonomous.

1.2.6 Solution of an ODE

Definition 1.2.10. A *solution* to the ODE (1.9) on an interval $J \subset \mathbb{R}$ is a differentiable function $y\colon J \to D$ such that

$$\frac{dy}{dt}(t) = f(y(t), t), \quad \text{for all } t \in J.$$

In this case, t is called the *independent* variable and y is called the *dependent* variable.

Verifying that a given function solves a differential equation is often a simple process of substitution. Below, we illustrate this with two examples: one where the solution is given explicitly and another where the solution is given implicitly.

Example 1.2.11 (Explicit solution). Let us verify that $y(t) = e^{-t} + 1$ is as a solution for the differential equation

$$y'(t) + y(t) = 1. \tag{1.11}$$

First, compute the derivative of $y(t)$, that is $y'(t) = -e^{-t}$, and then substitute $y(t)$ and $y'(t)$ into the differential equation to obtain

$$y'(t) + y(t) = \big(-e^{-t}\big) + \big(e^{-t} + 1\big) = 1.$$

Since the equality holds, $y(t) = e^{-t} + 1$ is indeed a solution to the ODE (1.11).

Example 1.2.12 (Implicit solution). Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} \tag{1.12}$$

and the candidate *implicit solution* $x^2 - y^2 = C$, where C is a constant. In this example, x is treated as the independent variable, and y is regarded as the dependent variable. Differentiating the implicit solution with respect to x using the chain rule, we obtain

$$\frac{d}{dx}\big(x^2 - y^2\big) = 0,$$

which expands to

$$2x - 2y\frac{dy}{dx} = 0.$$

Simplifying this expression gives

$$\frac{dy}{dx} = \frac{x}{y}.$$

Therefore, the implicit solution $\psi(x, y) = x^2 - y^2 = C$ describes a family of solutions to the original ODE.

1.2.7 Initial value problems (IVPs)

In most applications, a condition of the form $y(t_0) = y_0 \in \mathbb{R}^n$ is supplemented to the ODE (1.9), where $y_0 \in \mathbb{R}^n$ is called the *initial condition.* This leads to what we call an *initial value problem* (IVP)

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0. \tag{1.13}$$

Having introduced the notion of an IVP the questions of existence and uniqueness of solutions natural arise. This is the subject of the next Section 1.3. Before that, it might be useful for some readers to read some notions from real analysis in $\mathbb{R}^n$ in Chapter 9.

1.3 Existence and Uniqueness of Solutions to IVPs

In this section, we provide fundamental results concerning the existence and uniqueness of solutions to ODEs. The results of this section are classical and can be found in many text on ordinary differential equations. See [1, Section 1.12], [2, Chapter 1] or [3, Section V.5.3] for alternative and/or more general proofs of existence and uniqueness.

In this section we focus on solutions to the IVP (1.13), that is the existence of a solution $y\colon J \to D$ such that $t_0 \in J$ and $y(t_0) = y_0$.

Definition 1.3.1. Consider $D \subset \mathbb{R}^n$ an open set and any norm $\|\cdot\|$ on $\mathbb{R}^n$. A function $f\colon D \to \mathbb{R}^n$ is *Lipschitz continuous* (LC) if there exists a real constant $L \geq 0$ such that, for all $y_1, y_2 \in D$,

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\|.$$

The smallest L satisfying this inequality is denoted by $Lip(f) \stackrel{\text{def}}{=} L$ and is called the *Lipschitz constant of* f.

Example 1.3.2. Let $n = 1$ and $D = \mathbb{R}$. The function $f\colon D \to \mathbb{R}$ defined by $f(y) = 2y - 10$ is LC with Lipschitz constant $Lip(f) = 2$.

Example 1.3.3. Let $n = 1$ and let $f(y) \stackrel{\text{def}}{=} \frac{1}{y-1}$. Then $f\colon (1, \infty) \to \mathbb{R}$ is not LC. However, for any $\delta > 1$, the function $f\colon (\delta, \infty) \to \mathbb{R}$ is LC. In this case, let us determine its Lipschitz constant. Denote $D_\delta \stackrel{\text{def}}{=} (\delta, \infty)$ with $\delta > 1$. Since $f\colon D_\delta \to \mathbb{R}$ is C^1, then by the Mean Value Theorem, for any $y_1 < y_2 \in D_\delta$, there exists $z \in (y_1, y_2)$ such that

$$f(y_1) - f(y_2) = f'(z)(y_2 - y_1) = -\frac{1}{(z-1)^2}(y_2 - y_1).$$

Denote

$$L(\delta) \stackrel{\text{def}}{=} \sup_{z \in D_\delta} \left| -\frac{1}{(z-1)^2} \right| = \sup_{z \in (\delta, \infty)} \frac{1}{(z-1)^2} = \frac{1}{(\delta-1)^2} < \infty.$$

For instance, for $\delta = 2$, $Lip(f) = \frac{1}{(2-1)^2} = 1$.

The following definition generalizes the notion of Lipschitz continuity.

Definition 1.3.4. Consider $D \subset \mathbb{R}^n$ an open set and any norm $\|\cdot\|$ on $\mathbb{R}^n$. A function $f\colon D \to \mathbb{R}^n$ is *locally Lipschitz continuous* (LLC) if for every compact set $K \subset D \subset \mathbb{R}^n$, there exists a real constant $L = L(K) \geq 0$ such that

$$\|f(y_1) - f(y_2)\| \leq L\|y_1 - y_2\|, \quad \text{for all } y_1, y_2 \in K.$$

Example 1.3.5. The function $f(y) \stackrel{\text{def}}{=} \frac{1}{y-1}$ with $D = (1, \infty)$ is LLC.

The following proposition, whose proof follows form the mean value inequality, indicates that smooth functions are locally Lipschitz continuous.

Proposition 1.3.6. *Let $D \subset \mathbb{R}^n$ be open and $f\colon D \to \mathbb{R}^n$. If $f \in C^1(D)$, then f is LLC.*

The first goal of this section is the proof of the following fundamental theorem which guarantees local existence and uniqueness of solutions.

Theorem 1.3.7 (Existence and Uniqueness). *Consider an open set $D \subset \mathbb{R}^n$ and an open interval $(a, b) \subset \mathbb{R}$ which contains t_0. Assume that $f\colon D \times (a, b) \to \mathbb{R}^n$ is continuous and that for all compact set $K \subset D \times (a, b) \subset \mathbb{R}^{n+1}$, there exists $L = L(K) > 0$ such that*

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|, \quad \text{for all } (x,t), (y,t) \in K. \tag{1.14}$$

If $y_0 \in D$, then there exists an open interval $J \subset \mathbb{R}$, containing t_0, over which a solution to the IVP (1.13) is defined. Furthermore, any two solutions to the IVP agree on the intersection of their domains of definition.

Remark 1.3.8. Note that if $f(y,t) = f(y)$, that is f does not depend explicitly on t (i.e. the ODE is autonomous), then condition (1.14) is equivalent to f being LLC.

The proof of Theorem 1.3.7 is obtained via a series of propositions and lemmas. The existence will follow from Proposition 1.3.12 while uniqueness will follow from Proposition 1.3.13. The first step in the proof is the observation that by the fundamental theorem of calculus a solution to an ODE can be recast as a solution to an integral equation.

Lemma 1.3.9. *Assume $f\colon D \times (a,b) \to \mathbb{R}^n$ is a continuous function with $t_0 \in (a, b)$, and let $y_0 \in D$. Let $\varepsilon > 0$ such that $J = (t_0 - \varepsilon, t_0 + \varepsilon) \subset (a, b)$. A continuous function $y\colon J \to D$ solves the IVP (1.13) on J if and only if*

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\, ds$$

for all $t \in J$.

Since $(y_0, t_0) \in D \times (a, b) \subset \mathbb{R}^{n+1}$ with $D \times (a, b)$ an open subset of $\mathbb{R}^{n+1}$, there exists $\alpha > 0$ and $\delta > 0$ such that the *compact cylinder*

$$D_{\alpha,\delta} \stackrel{\text{def}}{=} \left\{(y,t) \in \mathbb{R}^{n+1} : \|y - y_0\| \leq \alpha \text{ and } |t - t_0| \leq \delta\right\} \subset D \times (a, b). \tag{1.15}$$

Let

$$M_{\alpha,\delta} \stackrel{\text{def}}{=} \sup_{(y,t) \in D_{\alpha,\delta}} \|f(y,t)\| < \infty \tag{1.16}$$

since the supremum of a continuous function taken over a compact set is attained (e.g. see Theorem 9.0.11) and therefore is finite.

Lemma 1.3.10 (Picard Operator). *Let $\varepsilon > 0$ be defined by*

$$\varepsilon \stackrel{\text{def}}{=} \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right) \tag{1.17}$$

and let

$$J \stackrel{\text{def}}{=} (t_0 - \varepsilon, t_0 + \varepsilon). \tag{1.18}$$

Then for any function $y(t)$ which satisfies

$$y(t_0) = y_0 \quad \textit{and} \quad (y(t), t) \in D_{\alpha,\varepsilon}, \ \textit{for all } t \in J, \tag{1.19}$$

the function $T(y) : J \to \mathbb{R}^n$ defined by

$$T(y)(t) \stackrel{\text{def}}{=} y_0 + \int_{t_0}^{t} f(y(s), s)ds \tag{1.20}$$

also satisfies the conditions in (1.19), *that is*

$$T(y)(t_0) = y_0 \quad \textit{and} \quad (T(y)(t), t) \in D_{\alpha,\varepsilon}, \ \textit{for all } t \in J.$$

Definition 1.3.11. The operator T defined in (1.20) is called the *Picard operator.*

Proof. (*of Lemma 1.3.10) Let $y(t)$ be any function which satisfies $y(t_0) = y_0$ and $(y(t), t) \in D_{\alpha,\varepsilon}$, for all $t \in J$. Such a function exists (for instance $y(t) = y_0$, for all $t \in J$ works). First, note that*

$$T(y)(t_0) = y_0 + \int_{t_0}^{t_0} f(y(s), s)ds = y_0 + 0 = y_0.$$

To show that $(T(y)(t), t) \in D_{\alpha,\varepsilon}$, we have to show that $\|T(y)(t) - y_0\| \le \alpha$ and $|t - t_0| \le \varepsilon$. Since $t \in J$, then $|t - t_0| < \varepsilon$. Finally,

$$\begin{aligned}
\|T(y)(t) - y_0\| &= \left\| \int_{t_0}^{t} f(y(s), s)ds \right\| \\
&\le \left| \int_{t_0}^{t} \|f(y(s), s)\| ds \right| \\
&\le \sup_{(y,s) \in D_{\alpha,\varepsilon}} \|f(y(s), s)\| \left| \int_{t_0}^{t} ds \right| \\
&\le \sup_{(y,s) \in D_{\alpha,\delta}} \|f(y(s), s)\| \, |t - t_0| \\
&\le M_{\alpha,\delta} \cdot \varepsilon \\
&= M_{\alpha,\delta} \cdot \min\left(\delta, \frac{\alpha}{M_{\alpha,\delta}}\right) \\
&\le M_{\alpha,\delta} \cdot \frac{\alpha}{M_{\alpha,\delta}} = \alpha,
\end{aligned}$$

where in the first inequality, we used Theorem 9.0.4. □

A consequence of Lemma 1.3.10 is that the Picard operator T maps the set of functions satisfying (1.19) into itself. In other words, it maps a given function space into itself. This allows iterating the operator, and yields a sequence of functions $\{y_k\}_{k\geq 0}$ called the *Picard iterations*. This process begins by setting $y_0(t) \equiv y_0$ and then

$$y_k(t) \stackrel{\text{def}}{=} y_0 + \int_{t_0}^{t} f(y_{k-1}(s), s)ds, \quad \text{for all } k \geq 1 \text{ and } t \in J. \tag{1.21}$$

Proposition 1.3.12 (Existence). *Under the assumptions of Theorem 1.3.7 and recalling* (1.17) *and* (1.18)*, the Picard iterations* $\{y_k\}_{k\geq 0}$ *defined in* (1.21) *converge uniformly to a function* $y : J \to D$ *which solves the IVP* (1.13).

Proof. Fix $t \in (t_0, t_0 + \varepsilon)$ (the case $t \in (t_0 - \varepsilon, t_0)$ is similar). We begin by showing that the sequence of points $\{y_k(t)\}_{k\geq 0}$ in $\mathbb{R}^n$ is a Cauchy sequence. First, note that

$$\|y_1(t) - y_0(t)\| = \left\| \int_{t_0}^{t} f(y_0(s), s)ds \right\| \leq \sup_{(y,s)\in D_{\alpha,\varepsilon}} \|f(y, s)\| \int_{t_0}^{t} ds \leq M_{\alpha,\delta}(t - t_0),$$

where in the first inequality, we used Theorem 9.0.4. By induction, let us now show that for all $m \geq 1$,

$$\|y_m(t) - y_{m-1}(t)\| \leq M_{\alpha,\delta} L^{m-1} \frac{(t - t_0)^m}{m!}. \tag{1.22}$$

We already verified the case $m = 1$. Let us assume that (1.22) holds for m and show that (1.22) holds for $m + 1$.

Since $D_{\alpha,\delta} \subset \mathbb{R}^{n+1}$ is compact let $L = L(D_{\alpha,\delta}) > 0$ be such that (1.14) holds. Using (1.14) and Lemma 1.3.10, we have

$$\begin{aligned}
\|y_{m+1}(t) - y_m(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(y_m(s), s)ds - y_0 - \int_{t_0}^{t} f(y_{m-1}(s), s)ds \right\| \\
&= \left\| \int_{t_0}^{t} [f(y_m(s), s) - f(y_{m-1}(s), s)]ds \right\| \\
&\leq \int_{t_0}^{t} \|f(y_m(s), s) - f(y_{m-1}(s), s)\| ds \\
&\leq L \int_{t_0}^{t} \|y_m(s) - y_{m-1}(s)\| ds \\
&\leq L \int_{t_0}^{t} M_{\alpha,\delta} L^{m-1} \frac{(s - t_0)^m}{m!} ds \\
&= M_{\alpha,\delta} L^m \frac{(t - t_0)^{m+1}}{(m + 1)!}.
\end{aligned}$$

Hence, for all $\ell \geq 1$,

$$\|y_\ell(t) - y_{\ell-1}(t)\| \leq M_{\alpha,\delta} L^{\ell-1} \frac{(t-t_0)^\ell}{\ell!} < \frac{M_{\alpha,\delta}}{L} \cdot \frac{(L\varepsilon)^\ell}{\ell!}. \tag{1.23}$$

Now let $p, m \geq 1$ be two fixed integers. Hence, using (1.23)

$$\begin{aligned} \|y_{m+p}(t) - y_{m+1}(t)\| &\leq \sum_{k=1}^{p-1} \|y_{m+k+1}(t) - y_{m+k}(t)\| \\ &\leq \sum_{k=1}^{p-1} \frac{M_{\alpha,\delta}}{L} \frac{(L\varepsilon)^{m+k+1}}{(m+k+1)!} \\ &= \frac{M_{\alpha,\delta}}{L} \left(\sum_{j=m+2}^{m+p} \frac{(L\varepsilon)^j}{j!} \right). \end{aligned} \tag{1.24}$$

Since the exponentiel series of $e^{L\varepsilon}$ converges, we get that

$$\|y_{m+p}(t) - y_{m+1}(t)\| \leq \frac{M_{\alpha,\delta}}{L} \left(\sum_{j=m+2}^{m+p} \frac{(L\varepsilon)^j}{j!} \right) \longrightarrow 0 \quad \text{as } m, p \to \infty.$$

This implies that $\{y_k(t)\}$ is a Cauchy sequence for all $t \in (t_0, t_0 + \varepsilon)$. Similarly, we can show that the same holds for $t \in (t_0 - \varepsilon, t_0)$. Therefore, for any $t \in J \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $\{y_k(t)\}$ converges. Denote by $y(t)$ the limit. Letting $p \to \infty$ in (1.24) leads to

$$\|y(t) - y_{m+1}(t)\| \leq \frac{M_{\alpha,\delta}}{L} \left(\sum_{j=m+2}^{\infty} \frac{(L\varepsilon)^j}{j!} \right),$$

and so

$$\sup_{t \in J} \|y(t) - y_{m+1}(t)\| \leq \frac{M_{\alpha,\delta}}{L} \left(\sum_{j=m+2}^{\infty} \frac{(L\varepsilon)^j}{j!} \right) \longrightarrow 0, \quad \text{as } m \to \infty,$$

which shows that the convergence is uniform. Since $y_0 : J \to \mathbb{R}^n$ is continuous (it is a constant function) and by construction of the Picard iterations (1.21), we get that $y_k : J \to \mathbb{R}^n$ is continuous for all $k \geq 1$. Since $y : J \to \mathbb{R}^n$ is the uniform limit of a sequence of continuous functions, it is continuous (e.g. see Theorem 9.0.9). By definition of the Picard iterations (1.21),

$$\begin{aligned} y(t) &= \lim_{k\to\infty} y_{k+1}(t) \\ &= \lim_{k\to\infty} \left(y_0 + \int_{t_0}^t f(y_k(s), s)ds \right) \\ &\overset{(*)}{=} y_0 + \int_{t_0}^t \lim_{k\to\infty} f(y_k(s), s)ds \\ &= y_0 + \int_{t_0}^t f(y(s), s)ds, \end{aligned}$$

where the equality $(*)$ follows from the fact that $f(y_k(t), t)$ converges uniformly to $f(y(t), t)$ (which follows by continuity of f) and from Theorem 9.0.10. We conclude that

$$y(t) = y_0 + \int_{t_0}^t f(y(s), s)ds, \quad t \in J.$$

From Lemma 1.3.9, $y : J \to \mathbb{R}^n$ is a solution of the IVP (1.13). $\square$

Proposition 1.3.13 (Uniqueness). *Under the assumptions of Theorem 1.3.7, the IVP* (1.13) *has only one solution on the time interval* $J = (t_0 - \varepsilon, t_0 + \varepsilon)$.

Proof. Suppose that $y(t)$ and $z(t)$ are two solutions of the IVP (1.13). Let $t \in (t_0, t_0 + \varepsilon)$ (the case $t \in (t_0 - \varepsilon, t_0)$ is similar). From Lemma 1.3.9,

$$\begin{aligned} \|y(t) - z(t)\| &= \left\| y_0 + \int_{t_0}^t f(y(s), s)ds - y_0 - \int_{t_0}^t f(z(s), s)ds \right\| \\ &\le \int_{t_0}^t \|[f(y(s), s)ds - f(z(s), s)]\| ds \\ &\le L \int_{t_0}^t \|y(s) - z(s)\| ds. \end{aligned}$$

Denote

$$g(t) \overset{\text{def}}{=} \int_{t_0}^t \|y(s) - z(s)\| ds.$$

Hence, $g(t)$ is a non negative function which satisfies

$$g'(t) \le Lg(t).$$

Multiplying both sides of the inequality with the *integrating factor* $e^{-L(t-t_0)} > 0$. This leads to

$$\frac{d}{dt}\left(e^{-L(t-t_0)} g(t)\right) = e^{-L(t-t_0)} g'(t) - Le^{-L(t-t_0)} g(t) = e^{-L(t-t_0)}(g(t) - Lg(t)) \le 0.$$

The function $t \mapsto e^{-L(t-t_0)}g(t)$ is then decreasing on the time interval $(t_0, t_0+\varepsilon)$ which implies that

$$0 \le e^{-L(t-t_0)}g(t) \le g(t_0) = 0, \quad \text{for all } t \in (t_0, t_0+\varepsilon).$$

Therefore $g \equiv 0$ on $(t_0, t_0+\varepsilon)$. This implies that $g'(t) = \|y(t) - z(t)\| = 0$ for all $t \in (t_0, t_0+\varepsilon)$. Hence, $y = z$ on $(t_0, t_0+\varepsilon)$. A similar argument leads to the conclusion that $y = z$ on $(t_0 - \varepsilon, t_0)$. We conclude that

$$y(t) = z(t), \quad \text{for all } t \in J = (t_0 - \varepsilon, t_0 + \varepsilon). \qquad \square$$

Remark 1.3.14. The time interval J given by (1.18) on which we obtained existence and uniqueness of the solution of the IVP (1.13) depends on the domain $D \times (a, b)$ of the function f in a complicated fashion. Therefore, nothing tells us that, if we increase the size of $D_{\alpha,\delta}$ (measured by α and δ), the *radius* ε of the interval J will increase. In general, it is difficult to conclude any valuable quantitative information from the approach based on Picard's iterations. We can however do this in particular cases.

Example 1.3.15. Consider the IVP

$$y' = y + 1, \quad y(0) = 1. \tag{1.25}$$

The function $f(y,t) \stackrel{\text{def}}{=} 1 + y$ is C^1 over $\mathbb{R}$ and therefore it is LLC over $\mathbb{R}$. For any $\alpha, \delta > 0$, the compact cylinder is given by $D_{\alpha,\delta} = [1-\alpha, 1+\alpha] \times [-\delta, \delta]$, and moreover

$$M_{\alpha,\delta} = \sup_{(y,t)\in D_{\alpha,\delta}} \|f(y,t)\| = \sup_{(y,t)\in D_{\alpha,\delta}} |y+1| = \alpha + 2.$$

Hence, recalling (1.17) let

$$\varepsilon = \min\left\{\delta, \frac{\alpha}{M_{\alpha,\delta}}\right\} = \min\left\{\delta, \frac{\alpha}{\alpha+2}\right\}.$$

Note that the function $\alpha \mapsto \frac{\alpha}{\alpha+2}$ is bounded above by 1 (see Figure 1.1) and that δ can be taken arbitrarily.

Hence, the maximum value that ε can attain is $\varepsilon = 1$. By the Theorem of Existence and Uniqueness (Theorem 1.3.7), there is a unique solution $y : (-1, 1) \to \mathbb{R}$ of the IVP (1.25).

Note however that we can solve this IVP explicitly (the ODE is separable and can easily be solved) to get the solution $y(t) = -1 + 2e^t$ which is defined on $J_{\max} = (-\infty, \infty)$ which is **larger** than $J = (-1, 1)$.

In the Example 1.3.15, we obtained a solution defined on the whole real line $\mathbb{R} = (-\infty, \infty)$, but as we shall now see, this is not always possible.

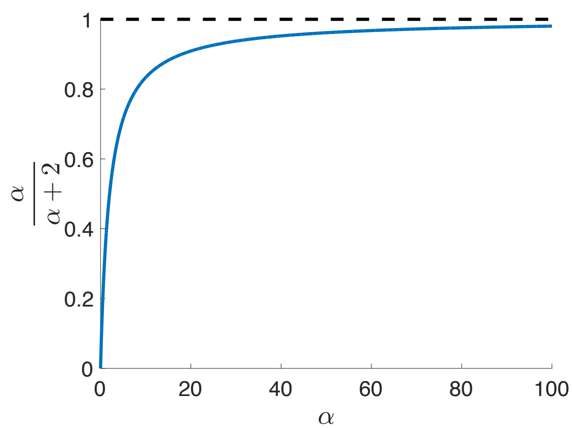

Figure 1.1: The function $\alpha \mapsto \frac{\alpha}{\alpha+2}$.

Example 1.3.16. Consider the IVP

$$y' = y^2, \quad y(0) = 1. \tag{1.26}$$

In this case, $f(y,t) = y^2$, $D_{\alpha,\delta} = [1-\alpha, 1+\alpha] \times [-\delta, \delta]$ and

$$M_{\alpha,\delta} = \sup_{y \in [1-\alpha, 1+\alpha]} |y^2| = (\alpha+1)^2$$

so that we set

$$\varepsilon = \min\left(\delta, \frac{\alpha}{(\alpha+1)^2}\right).$$

The function $\alpha \mapsto \frac{\alpha}{(\alpha+1)^2}$ attains its maximum value at $\alpha = 1$ (see Figure 1.2), which is given by $\frac{1}{(1+1)^2} = 1/4$.

Taking any $\delta \geq 1/4$ and letting $\alpha = 1$, we can set $\varepsilon = \frac{1}{4}$. From the Theorem of Existence and Uniqueness (Theorem 1.3.7), there is a unique solution $y : (-\frac{1}{4}, \frac{1}{4}) \to \mathbb{R}$ of the IVP (1.26).

We can solve this IVP explicitly (the ODE is separable) to get the solution $y(t) = \frac{1}{1-t}$ which is defined on $J_{\max} = (-\infty, 1)$ which is once again quite larger than $J = (-\frac{1}{4}, \frac{1}{4})$. The solution $y : (-\infty, 1) \to \mathbb{R}$ exhibits what we call a *finite time blowup*. See Figure 1.3 for the profile of the solution.

Example 1.3.17. Consider the IVP

$$y' = t^2 + y^2, \quad y(0) = 0. \tag{1.27}$$

In this case, $f(y,t) = t^2 + y^2$ is LLC for any fixed $t \in \mathbb{R}$, $D_{\alpha,\delta} = [-\alpha, \alpha] \times [-\delta, \delta]$ and

$$M_{\alpha,\delta} = \sup_{(y,t) \in [-\alpha,\alpha] \times [-\delta,\delta]} |t^2 + y^2| = \alpha^2 + \delta^2$$

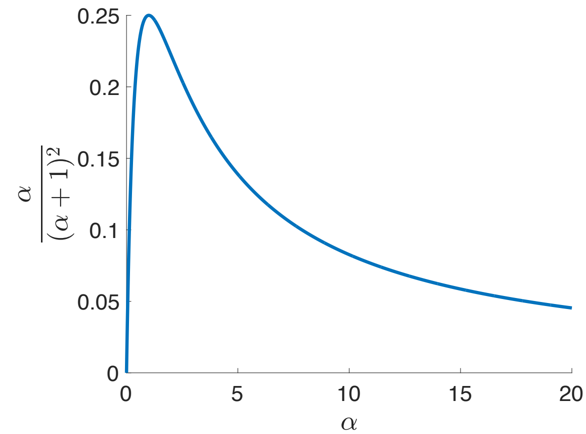

Figure 1.2: The function $\alpha \mapsto \frac{\alpha}{(\alpha+1)^2}$.

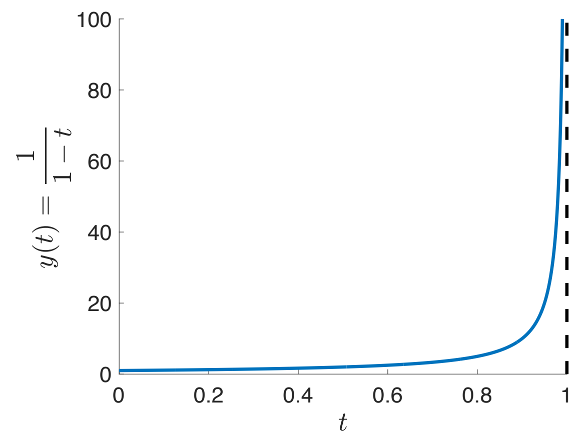

Figure 1.3: The unique solution $y(t) = \frac{1}{1-t}$ of the IVP (1.26).

so that we set

$$\varepsilon = \min\left(\delta, \frac{\alpha}{\alpha^2 + \delta^2}\right).$$

If we fix $\delta = 1$, then we get that $\frac{\alpha}{\alpha^2+\delta^2} = \frac{\alpha}{\alpha^2+1}$ attains its maximum value ($= \frac{1}{2}$) at $\alpha = 1$. Hence in this case $\varepsilon = 1/2$. From the Theorem of Existence and Uniqueness, there is a unique solution $y : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ of the IVP (1.27). The radius ε of $J = (-\varepsilon, \varepsilon) = (-\frac{1}{2}, \frac{1}{2})$ was obtained by fixing $\delta = 1$. Other choices of δ would lead to different resulting ε. It is an interesting exercise to try to get the largest possible ε (see Exercise 1.4.6).

Example 1.3.18. Consider the IVP

$$y' = t^2 + 2y, \quad y(0) = 1, \tag{1.28}$$

and let us construct the corresponding Picard iterations (1.21). They are given by

$$y_0 = 1, \quad y_k(t) \stackrel{\text{def}}{=} 1 + \int_0^t [s^2 + 2y_{k-1}(s)]ds, \quad \text{for all } k \geq 1.$$

Simple calculations lead to

$$\begin{aligned}
y_1(t) &= 1 + 2t + \frac{1}{3}t^3 \\
y_2(t) &= 1 + 2t + 2t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4 \\
y_3(t) &= 1 + 2t + 2t^2 + \frac{5}{3}t^3 + \frac{1}{6}t^4 + \frac{1}{15}t^5 \\
y_4(t) &= 1 + 2t + 2t^2 + \frac{5}{3}t^3 + \frac{5}{6}t^4 + \frac{1}{15}t^5 + \frac{1}{45}t^6 \\
y_5(t) &= 1 + 2t + 2t^2 + \frac{5}{3}t^3 + \frac{5}{6}t^4 + \frac{1}{3}t^5 + \frac{1}{45}t^6 + \frac{2}{315}t^7.
\end{aligned}$$

We can solve the IVP (using the method of integrating factor for first order linear equations) to get that the unique solution is given by $y(t) = -\frac{1}{4} - \frac{1}{2}t - \frac{1}{2}t^2 + \frac{5}{4}e^{2t}$. In Figure 1.4, we superpose some Picard iterations over the exact solution to appreciate the convergence.

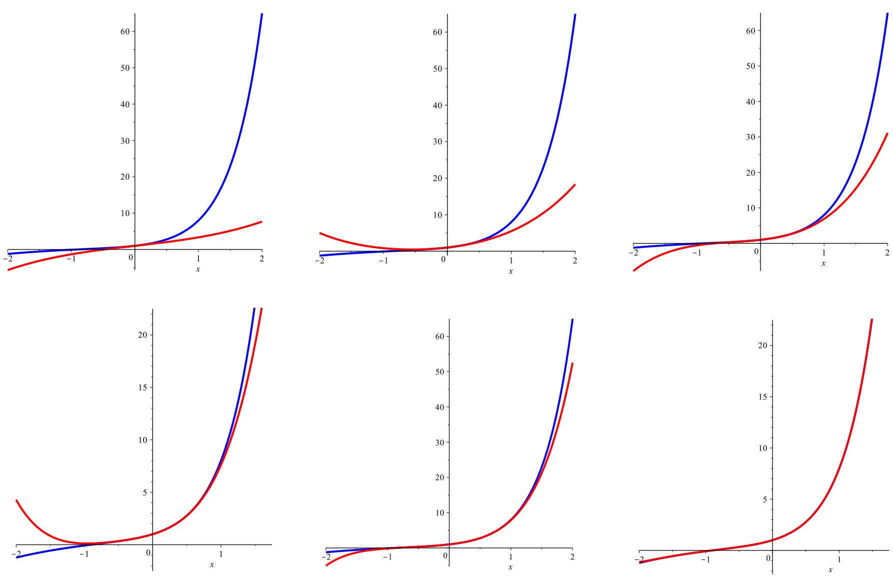

Figure 1.4: In blue, the exact solution $y(t) = -\frac{1}{4} - \frac{1}{2}t - \frac{1}{2}t^2 + \frac{5}{4}e^{2t}$. In red, the Picard iterations $y_k(t)$, for $k = 1, 2, 3, 4, 5, 10$ ordered from top left to bottom right.

The most significant restriction on the assumptions of Theorem 1.3.7 is that f must satisfy (1.14). As the following example indicates, existence is possible in more general settings, however uniqueness of the solution can no longer be assumed.

Example 1.3.19. Consider the initial value problem

$$y'(t) = 3y^{\frac{2}{3}}, \quad y(0) = 0. \tag{1.29}$$

Observe that

$$\varphi(t) \equiv 0 \quad \text{and} \quad \psi(t) \stackrel{\text{def}}{=} \begin{cases} t^3 & \text{if } t \geq 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

are two (of infinitely many) distinct solutions to (1.29) on $\mathbb{R}$. See Figure 1.5.

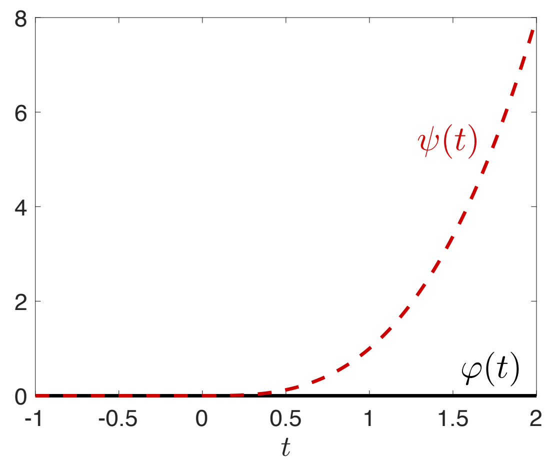

Figure 1.5: Two distinct solutions to (1.29).

Having established existence and uniqueness we turn to the question of the maximal time interval on which a solution is defined.

Definition 1.3.20. Let $\varphi\colon I \to D$ and $\psi\colon J \to D$ be solutions to $y' = f(y,t)$, where $f\colon D \times (a,b) \to \mathbb{R}^n$ is a continuous function defined on an open set $D \times (a,b) \subset \mathbb{R}^{n+1}$. We say that ψ is an *extension* of φ if $I \subset J$ and $\varphi(t) = \psi(t)$ for all $t \in I$. Moreover, if $I \subsetneq J$, then we say that ψ is a *proper extension* of φ.

Definition 1.3.21. A solution $\varphi\colon J \to D$ to $y'(t) = f(y,t)$ is called a *maximal solution* if it has no proper extensions. In this case the interval J is called the *maximal interval of existence* of the solution φ. We denote this interval by $J_{\max}$.

Theorem 1.3.22. *Under the assumption of Theorem 1.3.7, the IVP* (1.13) *has a unique maximal solution* $\varphi\colon J_{\max} \to D$*, and the maximal interval of existence* $J_{\max}$ *is open.*

Proof. Let $J_{\max}$ be the union of all intervals I for which there is a solution $\psi\colon I \to D$ for the initial value problem $y' = f(y,t), \quad y(0) = y_0$. Since all the intervals I contain t_0 we have that $J_{\max}$ is an interval. Define $\varphi\colon J_{\max} \to D$ by $\varphi(t) \stackrel{\text{def}}{=} \psi(t)$ if $t \in I$ and $\psi\colon I \to D$ is a solution. Since any two solutions agree on the intersection of their domains of definition, we have that φ is well defined. The function $\varphi\colon J_{\max} \to \mathbb{R}^n$ just defined is clearly the unique maximal solution to $y' = f(y,t), \quad y(0) = y_0$. It remains to show that the interval $J_{\max}$ is open. Let t_- and t_+ be the left and right end points of $J_{\max}$, respectively. Assume that $J_{\max}$ is closed at t_+. Then, since D is open, we can by Theorem 1.3.7 extend the solution to an interval about t_+, which contradicts the fact that $J_{\max}$ is the maximal interval. Likewise if we assume that $J_{\max}$ is closed at t_-. Therefore $J_{\max}$ is the open interval (t_-, t_+). $\square$

1.4 Exercises

Exercise 1.4.1. Consider the IVP

$$y' = y, \quad y(0) = 1$$

and compute the Picard iterations. Which function are we converging to?

Exercise 1.4.2. Consider the differential equation

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -y_2(t) \\ y_1(t) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}$$

for a given condition (y_1^0, y_2^0). Compute the Picard iterations for the case $(y_1^0, y_2^0) = (1,0)$. Which function are we converging to? Repeating the same with $(y_1^0, y_2^0) = (0,1)$, toward which function are we converging to?

Exercise 1.4.3. Consider the IVP

$$y' = 2t(y+1), \quad y(0) = 0$$

and compute the first five Picard iterations.

Exercise 1.4.4. Show that the IVP

$$y' = y^2 + \cos(t^2), \quad y(0) = 0$$

has a unique solution on the interval $J = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Exercise 1.4.5. Show that the IVP

$$y' = y + e^{-y} + e^{-t}, \quad y(0) = 0$$

has a unique solution, and find an interval of the form $J = (-\varepsilon, \varepsilon)$ on which the solution is defined.

Exercise 1.4.6. Consider the IVP

$$y' = t^2 + y^2, \quad y(0) = 0.$$

In Example 1.3.17, we showed that this IVP had a solution $y : J \to \mathbb{R}$ with $J = \left(-\frac{1}{2}, \frac{1}{2}\right)$, which was obtained by fixing $\delta = 1$. Find the optimal δ for which the ε is the largest.

Exercise 1.4.7. Let $a > 0$ a real number and consider the IVP

$$y' = ty^a, \quad y(0) = 0.$$

Clearly, $y \equiv 0$ solves the IVP. Show that for certain a, the solution y is not the only solution and explain why this does not violate the Theorem of Existence and Uniqueness (Theorem 1.3.7).

Exercise 1.4.8. Let $b, c \in \mathbb{R}$ be two given constants, and consider the second order IVP

$$y'' + by' + cy = 0, \quad y(0) = y_0,\ y'(0) = z_0.$$

(a) Show that we can re-write the second order IVP as

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}.$$

(b) Let

$$A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix},$$

and find a constant L such that

$$\|Ax_1 - Ax_2\| \le L\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in \mathbb{R}^2.$$

(c) Using Picard iterations, show that, for each initial condition (y_0, z_0) the IVP has a unique solution defined on the whole line $\mathbb{R}$.