General Mathematics

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Contents

Chapter 1

Exponential Function

1.1 Definition of an exponential function

Definition 1.1.1

Let $a > 0$ and $x \in \mathbb{R}$. The function $f(x) = a^x$ is called the exponential function with base a.

Based on the above definition we have:

1. $f(x) = 2^x$ is the exponential function with base 2.
2. $f(x) = e^x$ is the exponential function with base e.

Definition 1.1.2

Consider the exponential function with the form $f(x) = b*a^{x-h}+k$ with $f : \mathbb{R} \longrightarrow \mathbb{R}$; f is an exponential function if and only if $a \in]0,1[$ or $a \in]1,\infty[$.

Some exponential functions with the form $f(x) = b * a^{x-h} + k$ are:

- $f(x) = 2^{x-1} + 1$
- $f(x) = \frac{1}{2}^{x-1} - 8$

1.2 Properties

Theorem 1.2.1

If the base is negative, the function is not defined for all real numbers.
Let's consider a function $f(x) = (-3)^x$.

At $x = \frac{1}{2}$**.**

$$f\left(\frac{1}{2}\right) = (-3)^{\frac{1}{2}} = \sqrt{-3} = \pm i\sqrt{3} \tag{1.1}$$

Hence, the base of an exponential function cannot be negative.

Theorem 1.2.2

If the function is in the form $f(x) = a^x$, then the function is not transformed, and doesn't have a vertical or horizontal shift.
This also means that the function doesn't have an intersection with the x-axis and the asymptoten is $y = 0$.

Theorem 1.2.3

When $x = 0$ or $x = 1$ the function is constant. It means that isn't an exponential function at these x

values. For this is why the domain of the exponential function is defined like this: $D_f :]0, 1[\cup]1, \infty[$

- If the function base is 1, then the function is constant and $y = 1$.

- If the function base is 0, then the function is constant and equal to 0.

1.3 Graphs

Consider the graph of the function $f(x) = a^x$, $f : \mathbb{R} \to \mathbb{R}^+$.

- The figure 1.1 shows the graph of the function, where a is defined in the interval $]0, 1[$.
- The figure 1.2 shows the graph of the function, where a is defined in the interval $]1, \infty[$.

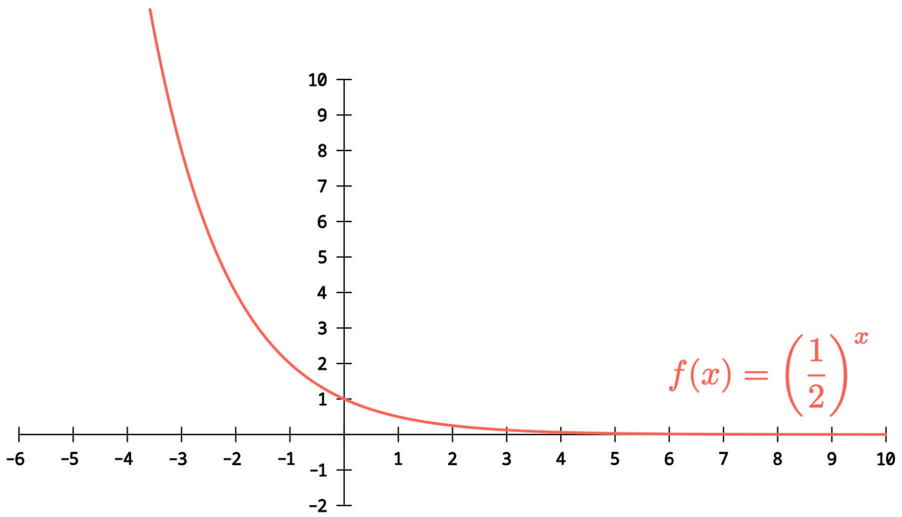

Figure 1.1: Case in which $0 < a < 1$.

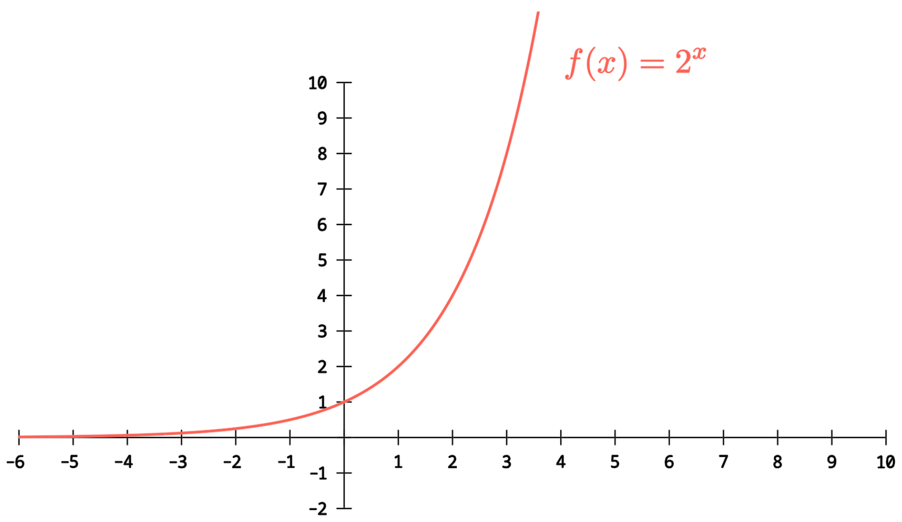

Figure 1.2: Case in which $a > 1$.

1.4 Intercepts and asymptotes

Theorem 1.4.1

The function $f(x) = a^x$ has no intercepts with the x-axis and the asymptoten is $y = 0$.

The function $f(x) = b * a^{x-h} + k$ has an asymptoten in $y = k$, and if k is negative the function has an intersection with the x-axis.

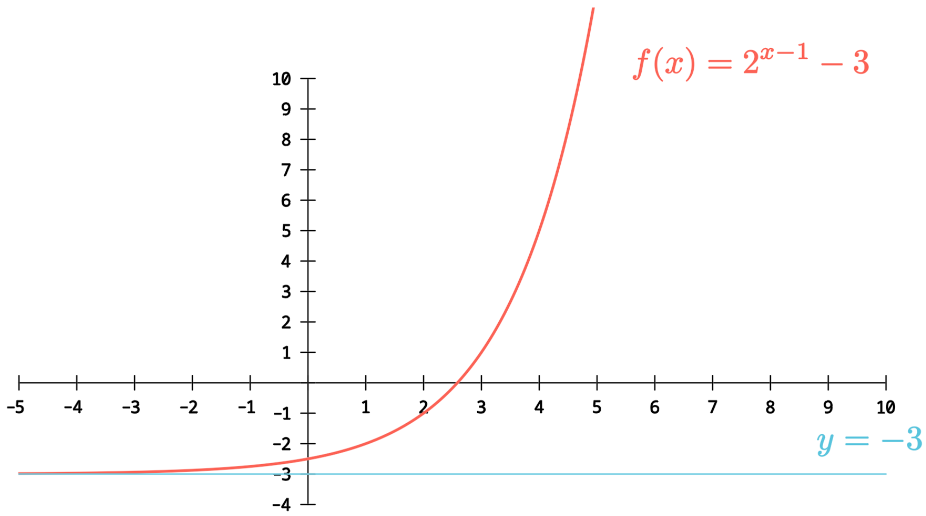

Figure 1.3: Asymptote in $y = -3$ for $f(x) = 2^{x-1} - 3$.

1.5 Solved exercises

Chapter 2

Logarithmic Function

2.1 Definition of a logarithmic function

Definition 2.1.1

Let $a > 0$ and $x \in \mathbb{R}$. The function $f(x) = \log_a(x)$ is called the logarithmic function with base a.

Based on the above definition we have:

1. $f(x) = \log_2(x)$ is the logarithmic function with base 2.
2. $f(x) = \log_e(x)$ is the logarithmic function with base e.

2.2 Properties

Theorem 2.2.1

The logarithmic function is the inverse function of the exponential function.

$$\log_a(x) = y \Leftrightarrow a^y = x \tag{2.1}$$

Example 2.2.1

$$\begin{aligned} \log_2(8) &= y \\ 2^y &= 8 \\ 2^y &= 2^3 \\ y &= 3 \end{aligned} \tag{2.2}$$

$$\therefore \log_2(8) = 3$$

2.3 Domain and Range

Theorem 2.3.1

The domain of the logarithmic function is defined like this: $D_f : \mathbb{R}^+$
The range of the logarithmic function is defined like this: $D_f : \mathbb{R}^+$

2.4 Solved exercises

Question 1

Simplify the following expression:

$$\log_3(5) * \log_2(3) * \log_5(6) * \log_2(6)$$

Solution:

$$\begin{aligned}
\log_3(5) * \log_2(3) * \log_5(6) * \log_2(6) &= \frac{\log(5)}{\log(3)} * \frac{\log(3)}{\log(2)} * \frac{\log(6)}{\log(5)} * \frac{\log(6)}{\log(2)} \\
&= \frac{\log(6)}{\log(2)} * \frac{\log(6)}{\log(2)} \\
&= \frac{\log^2(6)}{\log^2(2)} \\
&= \left[\frac{\log(6)}{\log(2)}\right]^2 \\
&= [\log_2(6)]^2 \\
&= \log_2^2(6)
\end{aligned}$$

☺

Question 2

Check the identity

$$\ln(x * \sqrt[3]{2x+1} * \sqrt[5]{(3x+1)^2}) = \ln(x) + \frac{1}{3}\ln(2x+1) + \frac{2}{5}\ln(3x+1)$$

Solution:

$$\begin{aligned}
\ln(x * \sqrt[3]{2x+1} * \sqrt[5]{(3x+1)^2}) &= \ln(x) + \ln(\sqrt[3]{2x+1}) + \ln(\sqrt[5]{(3x+1)^2}) \\
&= \ln(x) + ln((2x+1)^{\frac{1}{3}}) + ln((3x+1)^{\frac{2}{5}}) \\
&= \ln(x) + \frac{1}{3}\ln(2x+1) + \frac{2}{5}\ln(3x+1)
\end{aligned}$$

☺

Question 3

Check the identity

$$\frac{1}{4}\log(x^2+4x+3) + \frac{1}{4}\log\left(\frac{x+1}{x+3}\right) = \log\sqrt{x+1}$$

Solution:

$$\begin{aligned}
\frac{1}{4}\log(x^2+4x+3) + \frac{1}{4}\log\left(\frac{x+1}{x+3}\right) &= \frac{1}{4}\log\left[(x+1)(x+3)\right] + \frac{1}{4}\log\left(\frac{x+1}{x+3}\right) \\
&= \frac{1}{4}\log\left[(x+1)(x+3)\right] + \frac{1}{4}\left[\log(x+1) - \log(x+3)\right] \\
&= \frac{1}{4}\log(x+1) + \frac{1}{4}\log(x+3) + \frac{1}{4}\log(x+1) - \frac{1}{4}\log(x+3) \\
&= \frac{1}{4}\log(x+1) + \frac{1}{4}\log(x+1) \\
&= \frac{1}{2}\log(x+1) \\
&= \log\sqrt{x+1}
\end{aligned}$$

☺

Question 4

Check the identity

$$\log\left(\frac{x\sqrt{x+1}}{x^2-1}\right) = \log(x) - \frac{1}{2}\log(x+1) - \log(x-1)$$

Solution:

$$\begin{aligned}
\log\left(\frac{x\sqrt{x+1}}{x^2-1}\right) &= \log\left(\frac{x\sqrt{x+1}}{(x+1)(x-1)}\right)\\
&= \log(x\sqrt{x+1}) - \log((x+1)(x-1))\\
&= \log(x\sqrt{x+1}) - \log(x+1) - \log(x-1)\\
&= \log(x) + \log((x+1)^{\frac{1}{2}}) - \log(x+1) - \log(x-1)\\
&= \log(x) + \frac{1}{2}\log(x+1) - \log(x+1) - \log(x-1)\\
&= \log(x) - \frac{1}{2}\log(x+1) - \log(x-1)
\end{aligned}$$

😄

Question 5

Solve in $\mathbb{R}$ the following equation applying power laws:

$$2^x\left(\frac{2^{2x}}{4^{-3+x}}\right)^{x+1} = \sqrt{4^{6-x}}$$

Solution:

$$\begin{aligned}
2^x\left(\frac{2^{2x}}{4^{-3+x}}\right)^{x+1} &= \sqrt{4^{6-x}}\\
2^x\left(\frac{2^{2x}}{(2^2)^{-3+x}}\right)^{x+1} &= \sqrt{(2^2)^{6-x}}\\
2^x\left(\frac{2^{2x}}{2^{-6+2x}}\right)^{x+1} &= \sqrt{2^{12-2x}}\\
2^x(2^{2x-(-6+2x)})^{x+1} &= 2^{\frac{12-2x}{2}}\\
2^x(2^6)^{x+1} &= 2^{6-x}\\
2^x(2^{6x+6}) &= 2^{6-x}\\
2^{x+6x+6} &= 2^{6-x}\\
x+6x+6 &= 6-x\\
7x &= -x\\
8x &= 0\\
x &= 0
\end{aligned}$$

😄

Question 6

Solve in $\mathbb{R}$ the following equation applying power laws:

$$2^{2x} - 6 * 2^x + 8 = 0$$

Solution: With $u = 2^x$ we have

$$\begin{aligned}
(2^x)^2 - 6 * 2^x + 8 &= 0\\
u^2 - 6u + 8 &= 0\\
(u-2)(u-4) &= 0
\end{aligned}$$

For $u = 2$:

$$\begin{aligned} 2^x &= 2 \\ x &= 1 \end{aligned}$$

For $u = 4$:

$$\begin{aligned} 2^x &= 4 \\ 2^x &= 2^2 \\ x &= 2 \end{aligned}$$

Therefore, for $u = 2$ we have $x = 1$ and for $u = 4$ we have $x = 2$. 😄

Question 7

Solve in $\mathbb{R}$ the following equation:

$$e^{5x-2} - 7^{1-x} = 0$$

Solution:

$$\begin{aligned} e^{5x-2} &= 7^{1-x} \\ \ln(e^{5x-2}) &= \ln(7^{1-x}) \\ 5x - 2 * \ln(e) &= (1-x)\ln(7) \\ 5x - 2 &= (1-x)\ln(7) \\ 5x - 2 &= \ln(7) - x\ln(7) \\ 5x + x\ln(7) &= \ln(7) + 2 \\ x(5 + \ln(7)) &= \ln(7) + 2 \\ x &= \frac{\ln(7) + 2}{5 + \ln(7)} \end{aligned}$$

$\therefore x = \frac{\ln(7)+2}{5+\ln(7)}$ 😄

Question 8

Solve in $\mathbb{R}$ the following equation:

$$3 * 2^{2x} - 29 * 2^x = -40$$

Solution: With $u = 2^x$ we have:

$$\begin{aligned} 3 * 2^{2x} - 29 * 2^x &= -40 \\ 3u^2 - 29u &= -40 \\ 3u^2 - 29u + 40 &= 0 \\ (3u-5)(u-8) &= 0 \end{aligned}$$

For $3u = 5$:

$$\begin{aligned} 3u &= 5 \\ u &= \frac{5}{3} \\ 2^x &= \frac{5}{3} \\ \log_2(2^x) &= \log_2(\frac{5}{3}) \\ x &= \log_2(\frac{5}{3}) \end{aligned}$$

For $u = 8$:

$$\begin{aligned} u &= 8 \\ 2^x &= 8 \\ 2^x &= 2^3 \quad 😄 \\ x &= 3 \end{aligned}$$

Question 9

Solve in $\mathbb{R}$ the following equation:

$$\ln(x\sqrt[3]{4-2x}) = \ln(x) + \frac{1}{3}\ln(4-2x)$$

Solution:

$$\begin{aligned}
\ln(x\sqrt[3]{4-2x}) &= \ln(x) + \frac{1}{3}\ln(4-2x) \\
\ln(x) + \ln(\sqrt[3]{4-2x}) &= \ln(x) + \frac{1}{3}\ln(4-2x) \\
\ln(x) + \frac{1}{3}\ln(4-2x) &= \ln(x) + \frac{1}{3}\ln(4-2x) \\
\ln(x) + \frac{1}{3}\ln(4-2x) - \ln(x) - \frac{1}{3}\ln(4-2x) &= 0 \\
0 &= 0
\end{aligned}$$

Domain of $\ln(x)$ is $\mathbb{R}^+$, so $x \in \mathbb{R}^+$.

Domain of $\sqrt[3]{4-2x}$ is $\mathbb{R}^+$, so for $4-2x > 0$ we have $x \in \mathbb{R}^+$.

$$\begin{aligned}
4-2x &> 0 \\
-2x &> -4 \\
x &< 2
\end{aligned}$$

Therefore, $x \in \mathbb{R}^+$ and $x < 2$, so $D_f :]0, 2[$. ☺

Question 10

Solve in $\mathbb{R}$ the following equation:

$$\log_3(x+7) + \log_3(x+1) = 1 + \log_3(11-x)$$

Solution: First, we need the domains of the function:

For $\log_3(x+7)$:

$$\begin{aligned}
\log_3(x+7) &> 0 \\
x+7 &> 0 \\
x &> -7
\end{aligned}$$

For $\log_3(x+1)$:

$$\begin{aligned}
\log_3(x+1) &> 0 \\
x+1 &> 0 \\
x &> -1
\end{aligned}$$

For $\log_3(11-x)$:

$$\begin{aligned}
\log_3(11-x) &> 0 \\
11-x &> 0 \\
x &< 11
\end{aligned}$$

So, $D_f :]-1, 11[$.

$$
\begin{aligned}
\log_3(x+7) + \log_3(x+1) &= 1 + \log_3(11-x) \\
\log_3((x+7)(x+1)) - \log_3(11-x) &= 1 \\
\log_3\left(\frac{(x+7)(x+1)}{11-x}\right) &= 1 \\
3^1 &= \frac{(x+7)(x+1)}{11-x} \\
3(11-x) &= (x+7)(x+1) \\
33 - 3x &= x^2 + 7x + x + 7 \\
33 - 3x &= x^2 + 8x + 7 \\
0 &= x^2 + 8x + 7 + 3x - 33 \\
0 &= x^2 + 11x - 26 \\
0 &= (x-2)(x+13)
\end{aligned}
$$

Now we have $x = 2$ and $x = -13$, but $2 \in]-1, 11[$ and $-13 \notin]-1, 11[$, so $x = 2$ is the only solution. 😄

Question 11

Solve in $\mathbb{R}$ the following equation:

$$\log_2(5y-6) - \log_2(5y+1) = 3$$

Solution: First we need the domains of the function:

For $\log_2(5y-6)$:

$$
\begin{aligned}
\log_2(5y-6) &> 0 \\
5y - 6 &> 0 \\
y &> \frac{6}{5}
\end{aligned}
$$

For $\log_2(5y+1)$:

$$
\begin{aligned}
\log_2(5y+1) &> 0 \\
5y + 1 &> 0 \\
y &> -\frac{1}{5}
\end{aligned}
$$

So, $D_f :]\frac{6}{5}, \infty[$. Now, we can solve the equation:

$$
\begin{aligned}
\log_2(5y-6) - \log_2(5y+1) &= 3 \\
\log_2\left(\frac{5y-6}{5y+1}\right) &= 3 \\
2^3 &= \frac{5y-6}{5y+1} \\
8(5y+1) &= 5y - 6 \\
40y + 8 &= 5y - 6 \\
35y &= -14 \\
y &= -\frac{14}{35} \\
y &\approx -0.4
\end{aligned}
$$

But $-0.4 \notin]\frac{6}{5}, \infty[$, so there is no solution.
$\therefore S = \emptyset$ 😄

Question 12

Solve in $\mathbb{R}$ the following equation:

$$\log_2(x) + 4 + \log_2(x+1) = \log_2(x+2) + 5$$

Solution: First we need the domains of the function:

For $\log_2(x)$:

$$\begin{aligned} \log_2(x) &> 0 \\ x &> 0 \end{aligned}$$

For $\log_2(x+1)$:

$$\begin{aligned} \log_2(x+1) &> 0 \\ x+1 &> 0 \\ x &> -1 \end{aligned}$$

For $\log_2(x+2)$:

$$\begin{aligned} \log_2(x+2) &> 0 \\ x+2 &> 0 \\ x &> -2 \end{aligned}$$

So, $D_f :]0, \infty[$. Now, we can solve the equation:

$$\begin{aligned} \log_2(x) + 4 + \log_2(x+1) &= \log_2(x+2) + 5 \\ \log_2(x) + \log_2(x+1) - \log_2(x+2) &= 1 \\ \log_2\left(\frac{x(x+1)}{x+2}\right) &= 1 \\ 2^1 &= \frac{x(x+1)}{x+2} \\ 2(x+2) &= x(x+1) \\ 2x + 4 &= x^2 + x \\ 2x &= x^2 + x - 4 \\ 0 &= x^2 - x - 4 \end{aligned}$$

We need to solve the quadratic equation $x^2 - x - 4 = 0$.

$$\begin{aligned} x^2 - x - 4 &= 0 \\ (x - \frac{1+\sqrt{17}}{2})(x - \frac{1-\sqrt{17}}{2}) &= 0 \end{aligned}$$

Now we have $x = \frac{1+\sqrt{17}}{2}$ and $x = \frac{1-\sqrt{17}}{2}$, but $\frac{1-\sqrt{17}}{2} \notin]0, \infty[$, so $\frac{1+\sqrt{17}}{2}$ is the only solution.

$\therefore \text{S} = \left\{\frac{1+\sqrt{17}}{2}\right\}$ 😃

Chapter 3

Trigonometric Ratios

3.1 Introduction

3.1.1 Pythagorean Theorem

Sea ΔABC un triángulo rectángulo, recto en B, como se muestra en la siguiente figura, entonces se cumple que $h^2 = a^2 + b^2$.

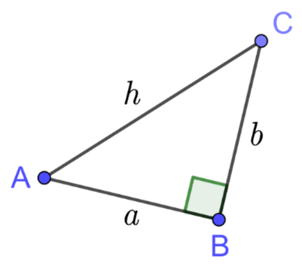

Figure 3.1: Triángulo rectángulo

3.1.2 Fundamental Identities

$$\begin{aligned}
\tan\theta &= \tfrac{\sin\theta}{\cos\theta} \\
\cot\theta &= \tfrac{\cos\theta}{\sin\theta} \\
\cot\theta &= \tfrac{1}{\tan\theta} \\
\sec\theta &= \tfrac{1}{\cos\theta} \\
\csc\theta &= \tfrac{1}{\sin\theta}
\end{aligned}$$