1.

Q Determine whether the scheme is a useful or useless scheme.

useless scheme: Forward-time central-space scheme w $\lambda=0$

useful scheme: Leap frog scheme with $\lambda=0.8$
Upwind scheme with $\lambda=0.8$
Lax-Friedrichs scheme with $\lambda=0.8$

only Forward-time central-space scheme's U_j^n is greater than 5 for any value of m and n. ☒

Q What do you notice about the "blow-up time" for the useless schemes as the mesh size decrease?

⇒ when mesh size decreases, blow up time is getting smaller. This means that if we decrease the mesh size, it blows up faster. ☒

Q pattern?

⇒ Except the useless scheme, errors decrease with smaller meshsize.. Also the peaks are consistence with smaller mesh sizes.

Q For the useful cases, how does the error decreases as the mesh size decrease?

⇒ when h decreases by half,
error of leap frog method decreases by $(h)^2$ second order
error of upwind decreases by $\approx(h)$ first order
Lax friedrichs decreases by $\approx(h)$ first order

2. show that schemes of the form

$$U_j^{n+1} = \alpha U_{j+1}^n + \beta U_{j-1}^n$$

are stable if $|\alpha| + |\beta|$ is less than or equal to 1.
Conclue that the Lax-Friedriches scheme is stable if $|a\lambda|$ is less than of equal to 1 for $U_t + aU_x = 0$ and $\lambda = \frac{k}{h}$.

Answer)

step① $U_j^n = g^n e^{ij\xi h}$

step② plug $U_j^n = g^n e^{ij\xi h}$ into $U_j^{n+1} = a U_{j+1}^n + \beta U_{j-1}^n$

Then, $g^{n+1} e^{ij\xi h} = a g^n e^{i(j+1)\xi h} + \beta g^n e^{i(j-1)\xi h}$ --- (1)

Devide (1) by $g^n e^{ij\xi h}$

Then,

$$g = a\left(e^{i(j+1)\xi h - ij\xi h}\right) + \beta e^{i(j-1)\xi h - ij\xi h}$$

$$\Rightarrow a\left(e^{i\xi h}\right) + \beta\left(e^{-i\xi h}\right) \quad \text{--- (2)}$$

Then, since we know

$$e^{i\xi h} = \cos(\xi h) + i\sin(\xi h) \quad \& \quad e^{-i\xi h} = \cos(\xi h) - i\sin(\xi h)$$

(2) becomes

$$a\left(\cos(\xi h) + i\sin(\xi h)\right) + \beta\left(\cos(\xi h) - i\sin(\xi h)\right)$$

$$\Rightarrow (\alpha+\beta)\left(\cos(\xi h)\right) - (\alpha-\beta)\left(i\sin(\xi h)\right).$$

Then,

$$g = (\alpha+\beta)(\cos(\xi h)) + (\alpha-\beta)(i\sin(\xi h))$$

$$\Rightarrow |g|^2 = (\alpha+\beta)^2(\cos(\xi h))^2 + (\alpha-\beta)^2(i\sin(\xi h))^2$$

$$\Rightarrow |g| = \sqrt{(\alpha+\beta)^2\left(\cos(\xi h)^2 + \sin(\xi h)^2\right)}$$

$$= |\alpha+\beta|\sqrt{\cos(\xi h)^2 + \sin(\xi h)^2}$$

$$= |\alpha+\beta|$$

since $|\alpha+\beta| \le |\alpha|+|\beta|$ by triangle inequality,

if $|\alpha|+|\beta| \le 1$ then $|\alpha+\beta| \le 1$.

So, we get $|g(\xi h)| \le 1$.

Therefore, this schemes of the form are stable when $|\alpha+|\beta|$ is less than or equal to 1.

continue

conclued that the Lax-Fridrichs scheme is stable 4-5
if $|a\lambda|$ is less than or equal to 1 for $\forall t$ all $x = 0$
and $\lambda = \frac{k}{h}$

<u>Proof</u>: For stability, we use van Heumann analysis.
Apply Fourier analysis to the discrete problem.

Lax - Fridrichs

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)}{k} + a\frac{(U_{j+1}^n - U_{j-1}^n)}{2h} = 0.$$

$$\Rightarrow U_j^{n+1} - \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) = -a\left(\frac{k}{2h}\right)(U_{j+1}^n - U_{j-1}^n)$$

$$\Rightarrow U_j^{n+1} = \frac{1}{2}(U_{j-1} + U_{j+1}^n) - \frac{a}{2}\left(\frac{k}{h}\right)(U_{j+1}^n - U_{j-1}^n) \quad \cdots \text{①}$$

Step ① $U_j^n = g^n e^{ij\xi h}$

Step ② play $U_j^n = g^n e^{ij\xi h}$ into ①

Then,

$$g^{n+1} e^{ij\xi h} = \frac{1}{2}\left(g^n e^{i(j-1)\xi h} + g^n e^{i(j+1)\xi h}\right) - \frac{a}{2}\left(\frac{k}{h}\right)\left(g^n e^{i(j+1)\xi h} - g^n e^{i(j-1)\xi h}\right) \quad \cdots \text{②}$$

Devide ② by $g^n e^{ij\xi h}$, then we get

$$g = \frac{1}{2}\left(e^{i(j-1)\xi h - ij\xi h} + e^{i(j+1)\xi h - ij\xi h}\right) - \frac{a}{2}\left(\frac{k}{h}\right)\left(e^{i(j+1)\xi h - ij\xi h} - e^{i(j-1)\xi h - ij\xi h}\right)$$

$$\Rightarrow \frac{1}{2}\left(e^{i\xi h(-1)} + e^{i\xi h(1)}\right) - \frac{a}{2}\left(\frac{k}{h}\right)\left(e^{i\xi h(1)} - e^{i\xi h(-1)}\right)$$

We know

$$e^{i\xi h} = \cos(\xi h) + i\sin(\xi h) \quad \& \quad e^{-i\xi h} = \cos(\xi h) - i\sin(\xi h)$$

$$\Rightarrow \frac{1}{2}\left(2\cos(\xi h) - \frac{a}{2}\left(\frac{k}{h}\right)\left(2\sin(\xi h)\right)\right)$$

$$\Rightarrow \cos(\xi h) - a\left(\frac{k}{h}\right)\left(i\sin(\xi h)\right)$$

then, $|g|^2 = \left|\cos(\xi h) - a\left(\frac{k}{h}\right)\left(i\sin(\xi h)\right)\right|^2$

$$\Rightarrow |g(\xi h)| = \sqrt{\cos(\xi h)^2 + (a\lambda)^2\sin(\xi h)^2}$$

Since $|a\lambda| \leq 1$, $|g(\xi h)| \leq 1$ for any ξ and hence, Lax Friedrichs scheme is <u>stable</u> ☒