CORSO DI LAUREA MAGISTRALE IN INGEGNERIA INFORMATICA

SOCIAL NETWORKS ANALYSIS A.A. 2021/22

GAME THEORY

Game Theory

- Network Science deals with connectivity and structure of social, natural and technological systems
  - Pattern of connections (graph theory)
  - interdependence in the behaviors of the individuals (game theory)

- Game Theory provides mathematical models and tools to describe behaviours of agents when they take decisions whose output depends also on choices made by the people they are interacting with
  - the outcome for anyone depends at least implicitly on the combined behaviors of all

- Game Theory was born in 30's and developed in 40-50s
  - Studied especially by economists
  - Recently interest on game theory spread in several different areas to model agents' strategic behaviours

Why We Need Game Theory?

- We can use Game Theory to model different common situations
  - Fixing prices for a new product
  - Deciding which social relations to maintain
  - Selecting the path to our destination in a transportation network
  - Deciding which offer to present in an auction
  - Deciding which strategy to follow in a competition

- Game-theoretic approches are also relevant to settings where none is overtly making decisions
  - which behaviors tend to sustain themselves when carried out in a larger population?
  - Es. Evolutionary Biology, Social Sciences
    - mutations are more likely to succeed in a population when they improve the fitness of the organisms that carry the mutation
    - Outside the scope of this course

WORKING HYPOTHESIS OF CLASSICAL GAME THEORY

Game Theory assumes players are selfish and rational and play strategically

- selfish
  - Each player pursue her own personal goal
- rational
  - Each player can recognize what is better to her
- strategic reasoning
  - Players take into account what they know and what are their beliefs about other players' behaviour

What is a Game?

- A Game is any situation where
  - Single individuals have to take decisions
  - Payoff obtained by a player depends on the decisions taken by all the players
- Several examples considered in game theory are real games ...
  - Tick-tack-toe, chess, rock-paper-scissors, penalty game
- ... but this framework applies to much larger contexts
- A game consists of
  - A set of players
  - For each player, a set of actions
  - For each player, a payoff (cost) function that gives the payoff received by the player for each profile of actions taken by all the players

A Classification of Games

In this course we will classify games with respect to three characteristics

- Cooperation among players
  - non-cooperative games: each player decides without any interaction with the other players
  - cooperative games: players cooperate in deciding which action to play
- Information known to players
  - perfect (full) information games: players have perfect (complete) information about the game
    - They know actions and payoffs of other players, and know that other players know
  - imperfect (partial) information games: players have an imperfect (partial) knowledge of the game
- time
  - strategic (normal) games: players decide their strategies before playing the game anc cannot change their decisions
    - A strategy can consist of several actions
  - extensive games: game played in rounds, at each round a player decides her action based on the state of the game and the previous history

Games in Strategic Forms

- A game in strategic (normal) form is a triple $(N, (A_i)_i, (u_i)_i)$
  - N = set of players
  - A_i = set of actions for player *i*
  - $u_i(a_1, a_2, \ldots, a_N)$ = payoff function for player *i*

- Profiles of the game are N-uple of players' actions

- A solution is a profile of the game
  - Players' payoffs depend on the solution

A First Example

- Alice and Bob have to prepare an exam and a (joint) presentation
  - Both want to maximize the global score obtained for the two activities
  - Both have time to work on only one activity and cannot coordinate

- **Exam**
  - If the student studies for the exam takes 28, otherwise takes 20
- **Presentation**
  - If both the students work on the presentation they both take 28
  - If only one student works on the presentation they both take 24
  - If none works on the presentation they both take 22

- What should Alice and Bob decide?
  - They cannot comunicate

Game Formalization

- Two players
  - Alice e Bob
- Two alternatives for each players
  - Exam, presentation

	Exam	*Present*
Exam	**25, 25**	**26, 22**
Present	**22, 26**	**24, 24**

If my colleague works on the presentation, my best move is to study for the exam

What will my colleague decide?

PLAYERS BEHAVIOUR

- Analyze the behaviours of the two players
- Every student has a *strictly dominant strategy*
  - Independently from what her colleague decides, she should study for the exam
- In this case we can predict the outcome of the game
  - Both students will obtain an average score of 25
- Each player could obtain a greater score to detriment of her colleague
  - It's not rational

Prisoner's Dilemma

- Two individuals suspected for a robbery are apprehended by the police for a minor crime
- Each suspected is interrogated in a separate room and offered for a deal
  - a reduction of the fine for a full confession
  - he has to decide how to respond

  - if neither confesses, they are both convicted for 1 year
  - If both confesses, they are both convicted for 4 year
  - If only 1 confesses, he is left free, while the other is convicted to 10 years

- Each suspect has a dominant strategy
  - His best choice is to confess
- His preferred outcome would be that both of them do not confess
  - It's not rational

	C	*NC*
C	**-4, -4**	**0, -10**
NC	**-10, 0**	**-1, -1**

Best Response Moves

- A rational player always chooses his best move with respect to her belief on the actions of the other players
- Formalize
  - If player 1 chooses strategy *S* and player 2 chooses strategy *T*
  - Player *i* gets a payoff $u_i(S,T)$
- *Def*: *S* is a *best response* with respect to *T* if $u_1(S,T) \geq u_1(S',T)$ for each alternative strategy S' of player 1
  - *S* ia a *strict best response* if $u_1(S,T) > u_1(S',T)$

Dominant Strategies

- *Def:* A *dominant strategy* is a strategy that is a best response with respect to all the possible strategies of the other players
  - Similarly for *strictly dominant strategy*.
- In the Prisoner's Dilemma both the players have a strictly dominant strategy
  - We can easily predict the outcome of the game
- There exist games with no dominant strategies
- What can we say about the outcome of games lacking of dominant strategies?

A Marketing Game

- Two firms are planning to produce and put on the market a new product
  - The new products are in competition
- The population of consumers is formed as follows
  - 60% interested in a low price product
  - 40% interested in a high level product
- Firm R (row) is much more popular
  - If they compete for the same market segment, it gets 80% of all the sales
  - It they cover different market segments, each one gets all the sales in its segment

	low	*high*
low	**.48, .12**	**.60, .40**
high	**.40, .60**	**.32, .08**

- For Firm R low is a dominant strategy
- Firm C should consider the best move of Firm R and move to high

GAMES WITH NO DOMINANT STRATEGIES

- What can we say when no player has a dominant strategy?
  - How can we reason about these games?
  - We know that each player will play his best response with respect to the strategy of the adversaries
    - He cannot be sure about what the adversaries will play
    - But he can reason strategically

- Best responses for Row player
  - A if C plays A
  - B if C plays B
  - C if C plays C
- Best responses for Column player
  - A if R plays A
  - B if R plays C
  - C if R plays B

	A	*B*	*C*
A	4, 4	0, 2	0, 2
B	0, 0	1, 1	0, 2
C	0, 0	0, 2	1, 1

Nash Equilibria

- A solution is a Nash Equilibrium where each player is playing her best response to the strategies of other players
  - Each player has no incentive to change her strategy if other players do not change their strategies

- John Nash (1952) introduced the concept of equilibrium and proved that each finite game has at least a (mixed) Nash Equilibrium

- Each player can reason strategically and predict how her adversaries would react to their moves

Nash Equilibria

	A	B	C
A	4, 4	0, 2	0, 2
B	0, 0	1, 1	0, 2
C	0, 0	0, 2	1, 1

- (A, A) is a Nash Equilibrium
  - A is the the best move for C when R plays A
  - A is the the best move for R when C plays A

- It's the unique Nash Equilibrium

Coordination Games

	A	*B*
A	**1, 1**	**0, 0**
B	**0, 0**	**1, 1**

- Two players have to choose between two alternatives
  - Both prefer to make the same choice
  - They cannot comunicate

- What are the Nash Equilibria for this game?
  - (A, A) and (B, B)
  - What will the players decide?

Battle of Sex

	B	S
B	1, 2	0, 0
S	0, 0	2, 1

- Alice and Bob have to decide what to do for the night
  - They can choose between a basketball match or shopping
  - Alice prefers shopping, Bob prefers basketball
  - Both prefer to stay together

- What are the Nash Equilibria for this game?
  - (B, B) and (S, S)

- Is it possible to predict the outcome of the game?
  - Social conventions can make an outcome better than the other
  - Es. By chivalry, Bob lets Alice to choose

Stag Hunt

	Stag	*Hare*
Stag	**4, 4**	**0, 3**
Hare	**3, 0**	**3, 3**

- Two individuals are hunting
  - They can cooperate and catch a stag
  - Each hunter can decide to not cooperate and catch only a hare
    - The stag is better but more difficult to catch

- Wha are the Nash Equilibria for this game?
  - (Stag, Stag) e (Hare, Hare)

- An equilibrium is much more risky that the other
  - If I hunt the stag but my colleague does not cooperate I'll remain with nothing
  - If I hunt the hare I have a little but certain payoff

HAWK OR DOVE

	Dove	*Hawk*
Dove	**3, 3**	**1, 5**
Hawk	**5, 1**	**0, 0**

- Hawk or Dove(also known as Chicken game)
  - Two animals have to split a pray
  - Each animal can decide to be aggressive (hawk) or submissive (dove)
  - If both are submissive each one takes half of the prey
  - If one is aggressive and the other submissive, the hawk takes the most
  - If both are aggressive, they'll injure each other
- What are the Nash Equilibria for this game?
  - (Hawk, Dove) e (Dove, Hawk)
- This game is used to model relationships among individuals or political relationships

Matching Pennies

	Head	*Tail*
Head	1, -1	-1, 1
Tail	-1, 1	1, -1

- Matching Pennies
  - Each player puts a coin on the table
  - Player Row wins if both the coins have the same side upward
  - Player Column wins if coins have different sides upward
- A Zero-Sum game
  - The sum of the payoffs of the players is 0
  - If a player wins, its adversary looses
- No Nash Equilibria
  - How would you play this game?

MIXED STRATEGIES

- A *mixed strategy* is a probability distribution (lottery) on the set of all the possible actions
  - Each player has to choose which kind of lottery to use
    - Infinite set
  - A lottery where the player chooses an action with probability 1 is called pure strategy

- In Matching Pennies
  - Player Row chooses to play Head with probability p
  - Player Column chooses to play Head with probability q

- How can we calculate the payoffs?
  - Expected values computed on all the possible profiles of pure strategies

Payoff Computation -- 1

- Player R compares her pure strategies with respect to the mixed strategy (q, 1-q)
  - If she chooses Head, her expected payoff is
  $q + (1\text{-}q)(\text{-}1) = 2q\text{-}1$
  - If she chooses Tail, her expected payoff is
  $(\text{-}1)q + (1\text{-}q) = 1\text{-}2q$

- Which is her best move?
  - Depends on q
  - if $q < ½$ best move is Tail
  - if $q > ½$ best move is Head
  - if $q = ½$ both the moves are equivalent
    - It can randomize between the two

Payoff Computation -- 2

- Player C compares her mixed strategies (q, 1-q) taking into account that his adversary will react with her best move
  - If she chooses $q < ½$ his adversary will play Tail
    - Her expected payoff is $2q-1 < 0$
  - If she chooses $q > ½$ his adversary will play Head
    - Her expected payoff is $1-2q < 0$
  - If she chooses $q = ½$ his adversary has to choose among two equivalent alternatives
    - If the adversary plays mixed strategy (p, 1-p) her expected payoff is
      $1/2\ (-p + (1-p) + p - (1-p)) = 0$
- Every choice different from $q \neq ½$ is not rational

Mixed Strategy Nash Equilibria

- Two mixed strategies are in Nash Equilibria if each one is the best response to the other
  - The adversary has no incentive in changing his strategy
- Nash proved that each finite game has at least a mixed strategy Nash Equilibrium
- Matching Pennies has no Pure Nash Equilibrium
  - In each profile there is a player that wants to change her strategy
- (½, ½) is a mixed Strategy Nash Equilibrium for Matching Pennies
  - Each player has an expected ayoff = 0 and this is the best she can obtain

Interpreting Mixed Strategy Nash Equilibria

- Players use mixed strategies to make difficult to their adversaries to predict her choice
  - playing q=1/2, player C makes both the strategies of her adversary equivalent

- Possible intepretations of mixed strategy Nash Equilibria
  - In sports and other competitions
    - Players randomize their strategies to make them less prevedible
  - Food competition among species
    - Individuals are fitted to play some strategies and cannot change
    - In a population there are different individuals
    - Mixed strategies give the proportions of each type in the whole population
    - A population is a mixed strategy Nash Equilibrium
  - A mixed strategy Nash Equilibrium is best thought as an equilibrium between beliefs
    - If a player thinks his adversary will play a Nash Equilibrium strategy then her best move is to play a Nash Equilibrium strategy

The Run-Pass Game

	Run	*Pass*
Run	**0, 0**	**5, -5**
Pass	**10, -10**	**0, 0**

- The Row team offends, column team defends
  - Offenders have to decide if play a pass-game or a run-game
    - Defenders have to decide on which kind of game to defend
- Defenders decide to defend on a run-game with probability q
  - The two offenders' alternatives are equivalent if
  - $5(1-q) = 10q \rightarrow q = 1/3$
- Offenders decide to play a run-game with probability p
  - The two offenders' alternatives are equivalent if
  - $-10(1-p) = -5p \rightarrow p = 2/3$
- (1/3, 2/3) is a mixed strategy Nash Equilibrium

Penalty-Game

	right	*left*
right	**0,58, -0,58**	**0,95, -0,95**
left	**0,93, -0,93**	**0,70, -0,70**

- The Row player shoots the penalty, the Column player is the goalkeeper
  - The blue values in the payoff matrix are the probabilities to score
- The goalkeeper goes on his left with probability q
  - Stricker's alternatives are equivalent if
  - $(0.58)(q) + (0.95)(1\text{-}q) = (0.93)(q) + (0.70)(1\text{-}q) \rightarrow q = 0.42$
- Similarly, we can compute the probability that the stricker kick on the left

  p=0.39
- (0,39, 0,42) is a mixed strategy Nash Equilibrium
  - Real data very close to what is predicted by the theory

Pareto Optimality

- Even if each player plays her best move the outcome could be not the best outcome for the society
  - Es. Prisoner’s Dilemma

- How can we define a socially good outcome?

- A solution is Pareto Optimal if there is no other solution such that:
  - Each player obtains at least the same payoff
  - There is a player that obtains a strictly greater playoff

Social Optimality

- A solution is socially optimal if it maximizes the social welfare
  - Sum of the payoffs of all the players

	Exam	*Present*
Exam	**25, 25**	**26, 22**
Present	**22, 26**	**24, 24**

- In this game there is a unique Nash Equilibrium that is also socially optimal