Multi-unit Auction over a Social Network

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Abstract

Diffusion auction is an emerging business model where a seller aims to incentivise buyers in a social network to diffuse the auction information thereby attracting potential buyers. We focus on designing mechanisms for multi-unit diffusion auctions. Despite several attempts at this problem, existing mechanisms are unsatisfactory in one way or another. Here, we propose two new mechanisms for multi-unit diffusion auction: (1) MUDAN which satisfies IC and a weakened form of efficiency, and (2) MUDAR which satisfies efficiency and a weakened form of IC. We define the mechanisms for the bottleneck case of the problem, namely when each buyer demands a single item, to which the more general multi-demand setting can be reduced. Our mechanisms are the first multi-unit diffusion auctions that satisfy a range of desirable properties that include truthfulness and efficiency conditions. This closes the gap left open by the previous work.

1 Introduction

Online social networks such as Tiktok, Twitter, and Temu not only enhance our social connectivity, but also provide new business opportunities: A user is able to act as a seller on an online social network, launching sales campaigns through the virtual space [14, 21]. Unlike traditional campaigns, a seller in this virtual market could leverage the social network to diffuse information. By implementing an appropriate marketing strategy, sales information passed to only a few initial individuals may trigger widespread dissemination, reaching a large cohort of potential buyers. Efforts have thus focused on designing mechanisms, termed *diffusion auctions*, that incentivise buyers to reveal not only their hidden valuations, but also their social connections to that sales information diffuses in the network [3, 15].

Diffusion auction differs from traditional auction designs in many aspects. First, classical tools such as Myerson's lemma no longer apply to incentive compatibility (IC) when the buyers are allowed to strategically declare both their valuations and social connections [3]. Then, even though the generic VCG mechanism can be conveniently extended to a diffusion auction, extreme cases exist that result in a large negative revenue for the seller [9]. Last, unlike the traditional auction designs, for diffusion auction no mechanism would simultaneously satisfy IC, individual rationality (IR), non-deficit (ND), as well as optimal social welfare [16]. The fundamental challenge in designing diffusion auctions is to mitigate the intrinsic conflict between the seller's desire to attract more participants to the auction, and buyers' wish to lower competition. Namely, by diffusing auction information to neighbours, a buyer may increase the chance of being out-bidded by others as more buyers may join the auction. Hence new ideas and tools must be developed for designing diffusion auctions.

In recent years, numerous studies have proposed diffusion mechanisms for *single-unit auction*, i.e., where the seller has only one item to sell [10, 8, 20, 19]. For example, the IDM mechanism – one of the starting points of this field [9] – achieves IC using the notion of *critical buyers*, those individuals who have the ability to alter the level of competition, and rewarding critical buyers for their losses due to information diffusion. Moving beyond single-unit case, *multi-unit auctions* study cases when the seller has multiple (homogeneous) items to sell. One would hope that this case, being a natural generalisation of the single-unit counterpart, could be addressed using mechanisms similar to IDM. However, repeated attempts have failed to satisfy the crucial IC property: (1) The GIDM mechanism, proposed in [22], determines the allocation of items and rewards using critical buyers. This mechanism, however, was pointed out to violate the IC property [16]. See Appendix A. (2) The subsequent DNA-MU mechanism, proposed in [6], also utilises critical buyers while further leveraging a priority order based on buyers' distances from the seller. Unfortunately, this mechanism is once again shown to be not IC [4]. See Appendix B. These failures attest the importance and difficulty of finding a truthful multi-unit diffusion mechanism.

Remarkably, two recent mechanisms for multi-unit diffusion auction have claimed to be truthful. First, the SNCA mechanism [17] extends the classical clinching auction to the social network context. Similar to DNA-MU, the mechanism grants a buyer who is closer to the seller in the social network a higher priority when determining the allocation of items. This mechanism, however, relies on the buyers' budgets which is not available in the standard multi-unit diffusion auction. Then, the LDM-Tree mechanism [11] applies a layer-based iterative allocation process, where buyers in the same layer have equal distance to the seller. Essentially, the LDM-Tree alleviates competition by restricting information diffusion. In particular, when applying LDM-Tree to single-unit auctions, this mechanism can produce arbitrarily inferior result as compared to IDM in terms of social welfare. Moreover, as the algorithm uses certain feature about the social network which may not be known *a priori*, the mechanism cannot be applied to the general setting of diffusion auction. See Appendix C. All these suggest that *the problem of designing reasonable mechanisms for multi-unit diffusion auctions is far from being settled*.

Contribution. In this paper, we focus on multi-unit diffusion auction and close the problems left open by previous work. **First,** we introduce a generic mechanism for diffusion auction which iteratively explores the social network starting from the seller s. At each iteration, a part of the network is *explored*. The mechanism chooses a *winner* from the explored buyers while *exhausting* some other buyers. The winner and exhausted buyers are incentivised to diffuse the auction information allowing more buyers to be explored. See Section 3. **Then,** we design two mechanisms that can be embedded into this framework: (1) Our first mechanism, named MUDAN, allocates an item to winners during the graph exploration "on the fly". MUDAN is IC, IR, ND, non-wasteful (NW), while satisfying a weakened version of social welfare optimisation. Moreover, the optimality ratio of MUDAN is tight for any truthful diffusion auction that incentivises buyers without using reward. See Section 4. (2) Our second mechanism, named MUDAR, allocates items after the graph exploration terminates. A winner is either allocated an item or is paid a reward. In this way, MUDAR guarantees to achieve the optimal social welfare, while ensuring a bounded form of IC. See Section 5. **Last,** as our mechanisms are defined for *single-demand* multi-unit diffusion auction, where each buyer is assumed to demand only one item, we extend the study to *multi-demand* multi-unit diffusion auction. We present a reduction from multi-demand multi-unit diffusion auction to the single-demand counterpart. Thus MUDAN and MUDAR can be generalised to the multi-demand case and satisfy all mentioned properties. See Section 6. We summarise the highlights of our achievements:

- The first truthful multi-unit diffusion auction MUDAN that achieves a reasonable social welfare guarantee;
- The first efficient multi-unit diffusion auction MUDAR that achieves a reasonable truthfulness guarantee; and
- A reduction from multi- to single-demand multi-unit diffusion auction which preserves the mechanism properties.

2 Model and problem formulation

We present our model for *single-demand multi-unit* diffusion auction which was addressed by GIDM [22] and DNA-MU [6]. The more general case of *multi-demand multi-unit* diffusion auction will be discussed in Section 6. Our model consists of the following:

- a seller, s, has $m \geq 1$ homogeneous items to sell;
- n buyers $B = \{1, 2, \ldots, n\}$. Each buyer $i \in B$ demands one item and attaches a valuation $v_i \in \mathbb{R}_+$. We often call buyers or the seller the *agents* of the network.
- a social network, represented as a directed graph $G = (B \cup \{s\}, E)$ on the agents, where the edge set $E \subseteq (B \cup \{s\})^2$ represents social connections between agents. The *neighbour set* of an agent $i \in V$ is $r_i \coloneqq \{j \in B \mid (i, j) \in E\}$. In particular, r_s is the set of all neighbours of s. We assume that all buyers v are reachable from s in G via paths.

We assume that information regarding the auction is not publicly known and the seller relies on buyers to spread this information to attract potential buyers. Initially, the auction information only reaches buyers in r_s. During an auction, the buyers who have the auction information are asked to report their neighbours and valuations. Formally, for buyer i:

- the *true profile* $\theta_i := (v_i, r_i)$ is private to the buyer i only;
- the *reported profile* $\theta'_i := (v'_i, r'_i)$ where $v'_i \in \mathbb{R}^m_+$ and $r'_i \subseteq B$ are the *reported* valuation and neighbour set by buyer i.

The reported profile θ'_i does not have to be θ_i. The idea is that the buyer i might try to benefit from the auction by strategically reporting θ'_i. By reporting r'_i, buyer i diffuses the auction information to all $j \in r'_i$. Following standard convention [9], we assume that $r'_i \subseteq r_i$. Some buyers may not be able to participate in the auction had i misreported her neighbours (i.e., $r'_i \subsetneq r_i$).

Fix the true profiles $\theta := (\theta_1, \ldots, \theta_n)$. The *global profile* is the reported profiles of all buyers $\theta' := (\theta'_1, \ldots, \theta'_n)$. Given θ', we build the following directed graph, we we call the *profile graph* $G_{\theta'}$: The nodes are $V_{\theta'} := \{s\} \cup B$; put a directed edge from i to j in the edge set $E_{\theta'}$ if $j \in r'_i$. A buyer i is *reachable* if there is a path from the seller s to i in this directed graph. Only reachable buyers can get the auction information. Technically, any buyer j that is not reachable should not have a reported profile, but for convenience we assume that they have the *silent report* (v'_j, r'_j) where $v'_j = 0$, $r'_j = \varnothing$, indicating the agent j's absence from the auction.

Given a global profile θ', a mechanism returns payment and allocation rules to buyers in $G_{\theta'}$. Let Θ denote the set of possible global profiles.

Definition 1. *A* mechanism $\mathcal{M}$ *consists of two functions* $(\pi(\cdot), p(\cdot))$*, where the mapping* $\pi\colon \Theta \to \{0,1\}^n$ *is the* allocation rule *and* $p\colon \Theta \to \mathbb{R}^n$ *is the* payment rule. *For a global profile* θ'*, the* allocation result $\pi(\theta')$ *is written as* $(\pi_1(\theta'), \ldots, \pi_n(\theta'))$ *and the* payment result $p(\theta')$ *as* $(p_1(\theta'), \ldots, p_n(\theta'))$.

For buyer $i \in B$, when $\pi_i(\theta') = 1$, i *wins* an item by paying $p_i(\theta')$ and is thus a *winner*. When $\pi_i(\theta') = 0$, i gets no item. The value $p_i(\theta')$ can be either positive or negative, denoting either cost or reward of buyer i, respectively. When the context is clear, we write π_i for $\pi_i(\theta')$ and p_i for $p_i(\theta')$.

An ideal mechanism should meet a number of requirements: First, it should incentivise buyers to participate in the auction, and truthfully report their neighbours and valuations. Then, it should maximally allocate items without causing deficit to the seller. Last, it should achieve a target level of social welfare. To formally define these properties, we introduce the following notions:

- The *utility* $u_i(\theta')$ of the buyer i is defined as $v_i\pi_i - p_i$.
- The *social welfare* $\mathsf{SW}(\theta')$ of the mechanism $\mathcal{M}$ is the sum of the utilities of all the agents, i.e., $\sum_{i=1}^n v_i\pi_i$.
- The *optimal social welfare* SW_{opt} is the sum of the top-m valuations in θ.
- The *revenue* $\mathsf{RV}(\theta')$ is the sum of the payment of all buyers, i.e., $\sum_{i=1}^n p_i$.

In the next definition, let $\theta'_{-i} := (\theta'_1, \ldots, \theta'_{i-1}, \theta'_{i+1}, \ldots, \theta'_n)$ denote the profiles of all buyers but i.

Definition 2. *Let $\mathcal{M}$ be a mechanism.*

1. *$\mathcal{M}$ is* incentive compatible *(IC) if for any buyer reporting truthfully is a dominant strategy: for all $i \in B$, all global profiles θ' and θ'', we have $u_i(\theta_i, \theta'_{-i}) \geq u_i(\theta''_i, \theta''_{-i})$* [1].
2. *$\mathcal{M}$ is* individually rational *(IR) if any buyer by reporting truthfully receives non-negative utility.*
3. *$\mathcal{M}$ is* non-deficit *(ND) if for any global profile θ', the revenue is non-negative, i.e.,* $\mathsf{RV}(\theta') \geq 0$.
4. *$\mathcal{M}$ is* non-wasteful *(NW) if all items are allocated to buyers (up to the number of reachable buyers), i.e., for any global profile θ', $\sum_{i \in V_{\theta'} \setminus \{s\}} \pi_i(\theta') = \min\{m, |V_{\theta'}| - 1\}$.*
5. *$\mathcal{M}$ is* efficient *if it achieves optimal social welfare, i.e., for any θ',* $\mathsf{SW}(\theta') = \mathsf{SW}_{\text{opt}}$.

Def. 2 lays out ideal properties for individuals (e.g., IC and IR) and for the entire network (e.g., ND and NW). For standard auctions (without social network), mechanisms are expected to be efficient. This, however, is not possible for diffusion auctions as no diffusion auction mechanism can simultaneously satisfy IC, IR, ND and efficiency [16]. In subsequent sections, we present two mechanisms for multi-unit diffusion auctions. The first mechanism, MUDAN, ensures IC, IR, ND, NW, and a weakened form of efficiency. The second mechanism, MUDAR, achieves efficiency, IR, ND, NW, and a weakened notion of IC.

[1] As r''_i may be different from r'_i, some agents j who are reachable in $G_{\theta'}$ may become unreachable had we replace θ'_i with θ''_i. θ''_{-i} is obtained from θ'_{-i} by replacing θ'_j with the silent profile for all such agents j.

3 The generic mechanism

Earlier mechanisms. We define a generic mechanism for multi-unit diffusion auction. Our design draws lessons from earlier attempts to the problem.

GIDM [22] and DNA-MU [6] are two prominent mechanisms defined for multi-unit diffusion auction. Both of these mechanisms fail to ensure the IC property as they potentially grant a buyer the opportunity to manipulate the auction outcome: (1) The mechanisms select winners using a tree structure, called the *diffusion critical tree*, which encodes information flow in the graph $G_{\theta'}$. (2) A misreport by a node down a path (e.g. f in Fig. 1) may affect the decisions of the mechanism made for nodes that are higher up in the tree (e.g. a in Fig. 1). (3) This changes the level of competition *globally*, which in turn presents an unfair advantage to the untruthful node. Appendix A and Appendix B contain detailed descriptions of these mechanisms along with proofs that they are not IC. To mitigate the problem above, we need therefore to (a) confine our decisions to buyers within a local area of the social network, and (b) mitigate the competitions within this local area so that a buyer cannot affect other parts of the network. We call this idea *competition localisation*.

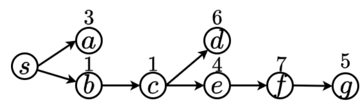

Figure 1: A social network with a seller s and seven buyers (shown in circles). The numbers are the valuations of the circled buyers. GIDM and DNA-MU both fail to ensure truthful reporting here.

The *LDM-Tree* mechanism [11] applies a form of competition localisation. Specifically, it localise competition within each *layer* of the diffusion critical tree, where layer L_i contains agents whose distance from the seller is i. The auction runs several rounds. In round i, the mechanism only considers nodes in L_i and those nodes in L_{i+1} that do not pose a potential competition to nodes in L_i. However, LDM-Tree has some obvious flaws: First it requires a structural feature of the social network (i.e., μ in Appendix C) which is usually not given. More importantly, it severely restricts information diffusion. For instance, if buyers in the first layer win all items in the first-layer auction, the outcome of the auction would coincide with the standard VCG mechanism applied *only to neighbours of* s. This defeats the purpose of diffusion auction. Even worse, when applied to single-unit auction (i.e., $m = 1$), LDM-Tree may produce outcomes with arbitrarily inferior social welfare than IDM, one of the earliest diffusion auction mechanisms. Appendix C contains detailed descriptions and discussions of LDM-Tree.

The generic graph exploration mechanism. We apply a different form of competition localisation. The idea is to explore the graph $G_{\theta'}$ from the seller s, iteratively building a set of *explored buyers*. At each iteration, competition is localised to within the explored buyers: A *winner* is chosen from the explored buyers while some other buyers are *exhausted*. The winner and exhausted buyers are incentivised to report their neighbours, which enables more nodes to be explored. Below we explain terms used in a given iteration.

- **Explored buyers** A: Initially, the set of explored buyers $A = r_s$, i.e., neighbours of s. Then at each iteration the set A is updated through *exhausted* and *winner* agents (introduced below) using the following procedures:
  - Repeatedly adding reported neighbours of exhausted agents until no more buyer can be added.
  - Adding the reported neighbours of the chosen winner.
- **Potential winner set** P: At the given iteration, a buyer $i \in A$ is a *potential winner* if i is already selected as a winner, i.e., $i \in W$, or may be selected as a winner in the future. The exact definition of P depends on the mechanism and will be made clear in the next sections.
- **Exhausted agent**: At the given iteration, a buyer in $A \setminus P$ is called an *exhausted agents*. The mechanism will ensure that an exhausted agent stays exhausted.
- **Priority** σ_i: The algorithm uses *priority scores* σ_i, $i \in P$, to select a winner. the priority σ_i of agent i must satisfy the following: *The value of* σ_i *should be independent of* v_i *and does not decrease as* $|r'_i|$ *increases*. A straightforward σ_i that meets this condition is $\sigma_i := |r'_i|$.[2]
- **Winner set** W: The buyer with the highest priority score in P is selected as the winner and is added to the set W.

[2]We introduce other possible σ_i and evaluate them empirically in Appendix G.

- **Tentative payment** $\hat{p}_w$: When a winner w is selected by the mechanism, a *tentative payment* $\hat{p}_w$ is assigned. The tentative payment $\hat{p}_w$ will be used to determine the payment p_w of w. The exact definition of $\hat{p}_w$ and p_w will be made clear for each mechanism.

- **Termination condition**: Terminate the graph exploration if all explored buyers are either winners or exhausted, i.e., $P \setminus W \neq \varnothing$.

Given the ingredients above, Algorithm 1 describes the generic graph exploration mechanism. Our MUDAN and MUDAR mechanisms can be embedded into this framework. Note that the lines in *italic* need to be instantiated.

Algorithm 1 The generic graph exploration mechanism

Initialise $W \leftarrow \varnothing$, $A \leftarrow r_s$
Initialise P
while $P \setminus W \neq \varnothing$ **do** ▷ Termination condition
    while A contains an unmarked agent $i \in W \cup (A \setminus P)$ **do**
        Update $A \leftarrow A \cup r'_i$, mark agent i
    end while
    Update set P
    Assign a priority σ_i *to each* $i \in P$
    Add agent $w \in P \setminus W$ who has the highest priority in W
    Record tentative payment $\hat{p}_w$
end while
Determine the allocation and payment results using W *and tentative payments*

4 The MUDAN mechanism

We now describe our MUDAN (*Multi-Unit Diffusion Auction with No reward*) mechanism for single-demand multi-unit diffusion auction. MUDAN implements the generic mechanism (Alg. 1) as follows: Maintain a variable $m' \geq 0$ which records the remaining number of items to be sold. Initially, $m' := m$, and is decremented every time a winner is selected. Hence $m - m'$ buyers are selected as winners. At a given iteration, the mechanism sets the following:

- **Potential winner set** P: Rank the buyers in $A \setminus W$ by their valuations such that $v'_{i_1} \geq v'_{i_2} \geq \ldots$. If $|A \setminus W| \leq m'$, then set $P := A$; otherwise, add in P the buyers with the top-m' valuations in $A \setminus W$, i.e., $P := W \cup \{i_1, \ldots, i_{m'}\}$.

- **Tentative payment** $\hat{p}_w$: When a winner w is selected, set $\hat{p}_w$ as the current $(m'+1)$th highest valuation in $A \setminus W$, i.e., $\hat{p}_w := v_{m'+1}$.

After the last iteration, we allocate to each winner $w \in W$ an item, i.e., set $\pi_w := 1$, and set $p_w := \hat{p}_w$. No other buyer will receive an item and the payment to them is 0. Table 1 provides a run-through example of MUDAN. See detailed run-through in Figure 4 of Appendix D.

Table 1: Running MUDAN on the network in Fig. 1 with $m = 4$ assuming all buyers report truthfully. Set the priority $\sigma_i := |r_i|$, where buyers with more neighbours get higher priority. 'Iter.' shows the iteration number. 'Incr. to A' column shows the nodes to be added to A in each iteration. The 'P' column lists potential winners in descending order of v_i. The winners are b, c, e, f.

Iter.	m'	Incr. to A	P	π, p
1	4	a, b	a, b	$\pi_b = 1, p_b = 0$
2	3	c	a, c	$\pi_c = 1, p_c = 0$
3	2	d, e	d, e	$\pi_e = 1, p_e = 3$
4	1	f	f	$\pi_f = 1, p_f = 4$

By definition of P, the algorithm terminates when $m' = 0$ (after m iterations). We now show that MUDAN has desirable properties. The next lemma is straightforward (See App. D).

Lemma 3. *MUDAN satisfies IR, ND, and NW.* □

Lemma 4. *The MUDAN mechanism satisfies IC.*

Proof. We prove two statements: *1. A buyer cannot benefit from misreporting her valuation*. Our argument is the following. Consider an iteration and suppose that w, if reporting her profile truthfully, will be selected a winner. We prove that w cannot benefit from misreporting her valuation:

- If $v'_{m'+1} \leq v'_w < v_w$ or $v'_{m'+1} \leq v_w < v'_w$, then w is allocated an item, pays the $(m'+1)$th highest valuation, and the utility $u_w((v'_w, r_w), \theta_{-w}) = u_w((v_w, r_w), \theta_{-w})$.
- If w reports $v'_w \leq v'_{m'+1} \leq v_w$, then w loses the item and her utility is 0.

Now consider another buyer $i \in P \setminus W$, $i \neq w$. We prove that i also cannot benefit from misreporting her valuation:

- if i reports valuation v'_i such that $v'_{m'+1} \leq v_i \leq v'_i$ or $v'_{m'+1} \leq v'_i < v_i$, i would still be a potential winner in this iteration, her priority would stay unchanged.
- If i reports valuation $v'_i < v'_{m'+1} < v_i$, she would not be a potential winner and her utility is 0.

Lastly, consider a buyer $i \in A \setminus P$. We prove that i cannot benefit from misreporting her valuation:

- if i reports her valuation v'_i such that $v_i < v'_{m'} \leq v'_i$, then: (i) If she has the highest priority, then her utility is $u_i((v'_i, r_i), \theta_{-i}) = v_i - v'_{m'+1} < 0 = u_i((v_i, r_i), \theta_{-i})$. (ii) Otherwise, her utility remains 0.
- If she reports $v_i < v'_i < v'_{m'}$ or $v'_i < v_i < v'_{m'}$, her utility remains 0.

2. A buyer cannot benefit from misreporting her neighbours. Our argument is the following. Take $i \in A$. If i hides any neighbour, her priority cannot increase. Consider winner w in a certain iteration. If w hides some of her neighbours and her priority is still the highest, her allocation and payment do not change, so her utility $u_w((v_w, r'_w), \theta_{-w}) = u_w((v_w, r_w), \theta_{-w})$. If her priority is not the highest, she loses some items, her utility decreases, i.e., $u_w((v_w, r'_w), \theta_{-w}) < u_w((v_w, r_w), \theta_{-w})$. Now consider agent $i \in A \backslash \{w\}$. If i hides any neighbour, i's priority would not increase and hence, she is still not allocated an item and $u_i((v_i, r'_i), \theta_{-i}) = u_i((v_i, r_i), \theta_{-i}) = 0$. □

Social welfare. We now analyse the social welfare achieved by MUDAN. Note that MUDAN sets the payment $p_i \geq 0$ for any buyer $i \in B$. This condition means that critical buyers are not incentivised to diffuse the auction information using reward, and thus we call it *no-reward* condition. We will show that MUDAN achieves the highest possible social welfare guarantee among IC diffusion auctions with no-reward. Let $w^* \in W$ denote the winner selected in the last iteration. We say that w^* is *critical* for a buyer i if all paths from s to i pass through w^*. The next lemma characterises buyers that are explored by the mechanism.

Lemma 5. *A buyer i is explored if and only if w^* is not critical for i.*

Proof. Suppose i is explored by the mechanism at the jth iteration. One can easily prove by induction on j that a path exists from s to i without passing through w^*.

Conversely, suppose a path exists from s to i without passing through w^*. Let d_i denote the length of the shortest such path. Suppose further i is a node with the smallest d_i that is not explored. Note that $d_i > 1$ as all nodes in r_s are explored. Now take the node j that immediately precedes i in the shortest path from s to i without passing w^*. Note that $j \in A$ and $i \in r_j$ by Lemma 4. If j is selected as a winner by the mechanism, then i is explored. Thus j will not be selected as a winner. This will happen only when $v_j \leq v_{w^*}$, which means that j will be exhausted eventually. When j is exhausted, i will be added in A. Contradiction. □

Let B^* denote the set of buyers for whom w^* is not critical.

Lemma 6. *Suppose a buyer $y \in B$ has a higher valuation than all winners, i.e., $v_y > v_w$ for all $w \in W$. Then $y \notin B^*$.*

Proof. Take such a buyer y that has the highest valuation. Suppose for a contradiction that a path exists from s to y without passing through w^*. By Lemma 5, y will eventually be added to A. Consider the last iteration before w^* is chosen as the winner. Since $v_y > v_{w^*}$, w^* would not be an element of P. Contradiction. □

Lemma 6 describes how MUDAN may fail to achieve optimal social welfare: There exists a buyer $i \in B \backslash B^*$ who has a high (top-m) valuation. This motivates us to define the following weakened notion of efficiency.

Definition 7. *Let* $\mathsf{SW}_{\rm wopt}$ *denote the sum of the top-*m *valuations among buyers in* B^**. A mechanism* $\mathcal{M}$ *is* ϵ-weakly efficient *if for any global profile* θ'*, we have* $\mathsf{SW}(\theta') \geq \epsilon \mathsf{SW}_{\rm wopt}$.

Weak efficiency means achieving the highest social welfare among the explored buyers. By Lemma 6, MUDAN selects the buyer with the highest valuation from A as a winner, thus achieving $1/m$-weak efficiency. For the case of single-unit auction (i.e., $m = 1$), MUDAN achieves social welfare at least as high as the classic IDM mechanism [9]. On the contrary, the only currently-known IC multi-unit diffusion auction mechanism, LDM-Tree, may produce outcomes for the single-unit case that are arbitrarily inferior than IDM in terms of social welfare. See Appendix C. The theorem below summarises the results above.

Theorem 8. *MUDAN terminates within time* $O(n^2 + |E|)$*, satisfies IC, IR, ND, NW, and* $1/m$*-weak efficiency.* □

Lastly, we show that $1/m$-weak efficiency is as good as it can be for IC diffusion auction with no-reward. The proof is given in Appendix D.

Theorem 9. *For any* $m \geq 1$ *and any constant* $\lambda > 0$*, there exists profile* θ *where no* m*-unit IC diffusion auction with no-reward achieves* $(1/m + \lambda)$*-weak efficiency.* □

5 The MUDAR mechanism

A question naturally arises as to whether a mechanism exists for multi-unit diffusion auction that achieves efficiency while satisfying other desirable properties. We now provide a positive answer: Our MUDAR (*Multi-Unit Diffusion Auction with Reward*) mechanism achieves optimal social welfare by ensuring a weakened form of IC. MUDAR also implements the generic mechanism (Alg. 1). The difference between MUDAN lies in how they incentivise winners. Once a winner w is chosen, the MUDAN mechanism commits to allocating w an item, thereby incentiving w to diffuse the auction information. In this way, one may view the allocation result as being determined "on the fly" during the graph exploration. On the other hand, MUDAR may either allocate an item to w or give w a reward (i.e., a negative payment), which equals to the utility of w had she been allocated an item. The allocation result is determined after the graph exploration is completed, when the buyers' connections are fully revealed. In this way, MUDAR can identify buyers that report the m-highest valuations globally. At a given iteration:

- **Potential winner set** P: A locally accessible buyer i is a *potential winner* if her valuation v'_i is among the top-m valuations in A.
- **Tentative payment** $\hat{p}_w$: For the winner w selected at the given iteration, $\hat{p}_w$ records how much $w \in W$ should be paid if she is allocated an item. Set $\hat{p}_w := v'_{m+1}$.

The exploration terminates when the explored buyers with the top-m valuations are all chosen as winners. This happens only when all nodes are explored (either in W or exhausted). The mechanism then partitions the winner set W into two subsets W_{A} and W_{R}:

1. W_{A} contains the buyers that have top-m valuations. Each winner $w \in W_{\mathsf{A}}$ is allocated an item. Set $\pi_w := 1$ and $p_w := \hat{p}_w$.
2. W_{R} contains the winners $W \setminus W_{\mathsf{A}}$. They only get a reward but no item. If $w \in W_{\mathsf{R}}$, the mechanism sets $\pi_w := 0$ and gives a reward which equals to her utility had she obtained the item, i.e., $p_w := \hat{p}_w - v'_w < 0$.

A run-through example of the MUDAR mechanism is provided in Table 2. See more details in Figure 5 of Appendix E. The next lemma shows that MUDAR has a number of desirable properties, among which, ND is the most non-trivial. We thus only give the proof for ND. The full proof can be found in Appendix E.

Lemma 10. *MUDAR satisfies IR, ND, and NW.*

Proof. **ND:** List all winners in the set W as $w_1, w_2, \ldots, w_{|W|}$ in the order as they are added in W. Below we separately discuss w_k for $k \leq m$ and for $k > m$:

1. $1 \leq k \leq m$: The tentative payment $\hat{p}_{w_k} \geq 0$. If $w_k \in W_{\mathsf{R}}$, the payment $p_{w_k} = \hat{p}_{w_k} - v_{w_k} \geq 0 - v_{w_k}$; if $w_k \in W_{\mathsf{A}}$, $p_{w_k} \geq 0$.

2. $m < k \leq |W|$: The tentative payment $\hat{p}_{w_k}$ is the $(m+1)$th highest valuation in A at iteration k. Let ψ_k be the $(m+1)$th highest valuation in the set W at iteration k. Since $W \subseteq A$, $\hat{p}_{w_k} \geq \psi_k$. If $w_k \in W_{\mathsf{R}}$, the payment is $p_{w_k} = \hat{p}_{w_k} - v_{w_k} \geq \psi_k - v_{w_k}$; if agent $w_k \in W_{\mathsf{A}}$, the payment is $p_{w_k} \geq \psi_k$.

Summarising the above, the revenue $\mathsf{RV}(\theta')$ is $\sum_{k=1}^{|W|} p_{w_k}$, which is the sum

$$\sum_{k=1}^{m}\{\hat{p}_{w_k} - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=1}^{m}\{\hat{p}_{w_k} \mid w_k \in W_{\mathsf{A}}\} + \sum_{k=m+1}^{|W|}\{\hat{p}_{w_k} - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=m+1}^{|W|}\{\hat{p}_{w_k} \mid w_k \in W_{\mathsf{A}}\}.$$

The above is at least

$$\sum_{k=1}^{m}\{0 - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=m+1}^{|W|}\{\psi_k - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=m+1}^{|W|}\{\psi_k \mid w_k \in W_{\mathsf{A}}\} = \sum_{k=m+1}^{|W|}\psi_k - \sum_{w_k \in W_{\mathsf{R}}} v_{w_k} = 0$$

The last equation holds because ψ_k, $m \leq k \leq |W|$, coincide with the valuations of $w_k \in W_{\mathsf{R}}$, $1 \leq k \leq |W|$. $\square$

Table 2: Running MUDAR on the network in Fig. 1 with $m = 4$ assuming all buyers report truthfully. We give priorities to agents by the degree where node with higher degree gets higher priority. 'Iter.' 'Incr. to A' , 'W', and 'p' indicate the iteration number. the nodes to be added to A in each iteration, winners in descending order of v_i, and the payment of the winner in the iteration when she is allocated an item, respectively. The winners are b, c, e, f, d, g with $W_{\mathsf{A}} = \{d, e, f, g\}$ & $W_{\mathsf{R}} = \{b, c\}$.

Iter.	Incr. to A	W	p
1	a, b	$\{b\}$	$p_b = -1$
2	c	$\{b, c\}$	$p_c = -1$
3	d, e	$\{b, c, e\}$	$p_e = 1$
4	f	$\{b, c, e, f\}$	$p_f = 1$
5	g	$\{b, c, e, f, d, g\}$	$p_d = 3$
6	$\varnothing$	$\{b, c, , e, f, d, g\}$	$p_g = 3$

Truthfulness. MUDAR does not satisfy IC. Indeed, a winner in W_{R} may report a higher value than the true valuation to receive a higher reward. Nevertheless, we now prove that MUDAR guarantees a weakened form of truthfulness. This property then ensures MUDAR to be efficient.

Definition 11. *For a positive value $\mu > 0$, a mechanism $\mathcal{M}$ is μ-*bounded incentive compatible *(μ-IC) if no buyer $i \in B$ can benefit from either reporting a lower valuation than the true valuation, or reporting a valuation that is higher than μ: for any $i \in B$, for any global profiles θ' and θ'' such that $v_i'' \geq \mu \geq v_i$ or $v_i'' < v_i$,*

$$u_i(\theta_i, \theta'_{-i}) \geq u_i(\theta''_i, \theta''_{-i}).^3$$

Remark. The notion of μ-bounded incentive compatibility resembles the notion of *one-way truthfulness* [1], which asserts that no buyer will benefit from bidding lower than the true valuation. Definition 11 is stronger than one-way truthfulness as in μ-bounded IC, the range of possible valuations at which a buyer i may benefit from misreporting is a bounded interval $[v_i, \mu)$. In particular, if $v_i \geq \mu$, then μ-bounded IC asserts that buyer i will report the true profile.

Lemma 12. *The MUDAR mechanism achieves μ-IC where μ is the mth highest valuation among all buyers.*

Proof. Let μ denote the mth highest valuation among all buyers. Consider a buyer w and suppose w reports her profile truthfully. We now prove that there is no incentive for w to misreport a lower valuation or a higher valuation $> \mu$.

First, if $w \in W_{\mathsf{A}}$ or $w \in A \setminus W$, truthful reporting of her valuation is w's dominant strategy. This can be proved in a similar way as the IC property in Theorem 4. Then, suppose $w \in W_{\mathsf{R}}$. Consider the three alternative strategies of w:

1. If w reports $v'_w < v'_{m+1} \leq v_w$, then she is exhausted in an iteration and her utility will be 0.

[3] Similar to the def. of IC in Def. 2, θ''_{-i} is obtained from θ'_{-i}.

2. If w reports $v'_{m+1} \leq v'_w < v_w$, then she will receive a lower utility, i.e., $v'_w - v'_{m+1} < v_w - v'_{m+1}$.
3. If she reports $v_{m+1} < v_w < \mu \leq v'_w$, her reported valuation will then be among the top-m valuations. In this case, it must be that $w \in W_{\mathsf{A}}$ and her utility becomes $v_w - v'_{m+1}$. This equals to the utility of truthful reporting.

Last, using essentially the same proof as that for MUDAN in Lemma 4, one can prove that no buyer can benefit from misreporting her connections. $\square$

Social welfare. By Lemma 12, (1) the buyers who have the top-m valuations will not misreport their valuations, and (2) no other buyer will report a valuation that is higher than the top-m valuations. This then proves efficiency of MUDAR. See proof in Appendix E.

Theorem 13. *MUDAR terminates within time $O(n^2 + |E|)$, satisfies μ-IC, IR, ND, NW, and efficiency, where μ is the mth highest valuation among all buyers.* $\square$

6 Multi-demand multi-unit diffusion auction

We now generalise the *single-demand* setting in Section 2 to *multi-demand* auction. Here each buyer $i \in B$ may demand more than one items and attaches a valuation $v_{i,j}$ to the jth item she gets, where $1 \leq j \leq m$. The valuation for all items is denoted by a vector $\vec{v_i} := (v_{i,1}, \ldots, v_{i,m}) \in \mathbb{R}^m_{\geq 0}$; we call $\vec{v_i}$ the *valuation vector* of the buyer i. Each additional unit often brings less additional utility than that from the previous unit, which is known as the *law of diminishing marginal utility* in micro-economics [2]. Therefore, we assume that the buyers have *diminishing valuation* towards the items: $v_{i,j} \geq v_{i,j+1}$ for $j = 1, \ldots, m-1$. For simplicity, we omit 0 in this vector. In this setting, we denote the *(multi-demand) profile* of a buyer i as $\eta_i := (\vec{v_i}, t_i)$ where $\vec{v_i}$ is the valuation vector and $t_i \subseteq B$ is the set of neighbours of i. The global profile is $\eta' := (\eta'_1, \ldots, \eta'_n)$ that corresponds to profile graph $G_{\eta'}$. Let H denote the set of all possible (multi-demand) global profiles. A *mechanism* $\mathcal{M}$ in this setting consists of *allocation rule* $\pi\colon H \to \{0,1\}^{n\times m}$ and *payment rule* $p\colon H \to \mathbb{R}^{n\times m}$. Here each $\pi_i(\eta')$, $p_i(\eta')$ are m-dimensional vectors; We write them as $\pi_i(\eta') = (\pi_{i,1}(\eta'), \ldots, \pi_{i,m}(\eta'))$ and $p_i(\eta') = (p_{i,1}(\eta'), \ldots, p_{i,m}(\eta'))$, respectively. All notions regarding utility, social welfare, IC, efficiency, etc., are defined analogously. See Appendix F.

Instead of designing a mechanism for the multi-demand setting from scratch, we reduce the problem to its single-demand counterpart. Given (multi-demand) profiles η of buyers $B = \{1, \ldots, n\}$, our goal is to construct a set of buyers $\widetilde{B}$ with (single-demand) profiles θ in such a way that any mechanism $\widetilde{\mathcal{M}}$ that is applied to $\widetilde{B}$ corresponds to a mechanism $\mathcal{M}$ that is applied to B. The intuition is that, essentially, we may view a buyer $i \in B$ as m buyers, each demanding one item with valuation $v_{i,j}$. More precisely, given a profile η', to define our mechanism $\mathcal{M}$ we perform the following steps.

(1). For each buyer $i \in B$, create m nodes $i_1, \ldots, i_m$ in $\widetilde{B}$, each corresponding to an item $1 \leq j \leq m$, i.e., $\widetilde{B} := \{i_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

(2). Construct the following profiles θ' for buyers in $\widetilde{B}$:

- In the profile graph $G_{\theta'}$, connect all buyers $i_1, \ldots, i_m$ to form a chain using edges $(i_1, i_2), \ldots, (i_{m-1}, i_m)$.
- If an edge exists between the seller s and buyer i in $G_{\eta'}$, then add an edge (s, i_1) in $G_{\theta'}$.
- If an edge exists between buyers i and j in $G_{\eta'}$, then add an edge (i_m, j_1) in $G_{\theta'}$.
- The true and reported valuation of a buyer $i_j \in \widetilde{B}$ are $v_{i,j}$ and $v'_{i,j}$, resp.
- The priority of the buyer i_j, for $1 \leq j \leq m$, is the priority of i in B.

(3). Apply a (single-demand) mechanism $\widetilde{\mathcal{M}} = (\tilde{\pi}, \tilde{p})$ to θ', where $\tilde{\pi}$ is the allocation rule and $\tilde{p}$ is the payment rule. Return the mechanism $\mathcal{M} := (\pi, p)$ where allocation rule π and payment rule p are defined below:

- Define $\pi\colon H \to \{0,1\}^{n\times m}$ by $\pi_{i,j}(\eta') := \tilde{\pi}_{i_j}(\theta')$.
- Define $p\colon H \to \mathbb{R}^{n\times m}$ by $p_{i,j}(\eta') := \tilde{p}_{i_j}(\theta')$.

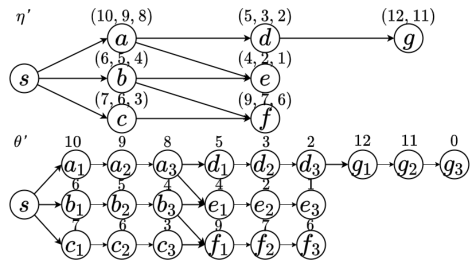

Figure 2: The reduction of an multi-demand auction (above) to the single-demand setting (below). Valuation is on top of each node.

Figure 2 illustrates the construction of θ' given a multi-demand profile η' over seven buyers. When we choose MUDAN (or MUDAR) as $\widetilde{\mathcal{M}}$, the corresponding mechanism $\mathcal{M}$ is called *MUDAN-m* (or *MUDAR-m*).

The next two theorems are proved in Appendix F. IR, ND, NW, efficiency, and $1/m$-weak efficiency in both theorems easily carry over from the single-demand case. The proof of truthfulness (IC and μ-IC), however, requires a non-trivial justification: If a buyer $i \in B$ misreports her valuation vector $\vec{v_i}$, a group of buyers in $\widetilde{B}$, namely $i_1, \ldots, i_m$, may misreport their valuations *together*. This amounts to a case of *collusion* among the buyers in $\widetilde{B}$, which is not accounted for in the single-demand case. Nevertheless, in Lem. 20 and Lem. 21 (App. F), we show that the buyers in B would not violate the truthfulness properties. In particular, our mechanisms ensure that at most one buyer i_j where $1 \leq j \leq m$ may be in the set $P \setminus W$ for any $i \in B$ at any iteration, hence collusion does not give extra incentive for the buyers $i_1, \ldots, i_m$.

Theorem 14. *MUDAN-m is IC, IR, ND, NW, and $1/m$-weakly efficient.* □

Theorem 15. *MUDAR-m is μ-IC, IR, ND, NW, and efficient, where μ is the mth highest valuation among $v_{i,j}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.* □

7 Conclusion and future work

We consider multi-unit diffusion auctions and aim to address important problems left open in previous work. We propose an iterative algorithm that explores the profile graph while determining potential winners. Based on this algorithm, we propose two mechanisms: (1) MUDAN ensures IC, IR, ND, NW, and $1/m$-weak efficiency; We gave a case when the $1/m$-weak efficiency bound is tight; (2) MUDAR ensures μ-IC, IR, ND, NW, and efficiency where μ is the mth highest valuation. These are the first multi-unit diffusion auction that satisfy the properties above. In the case of multi-demand auction, we define a reduction to the single-demand case so that both MUDAN and MUDAR can be employed to this generalised problem. The corresponding MUDAN-m and MUDAR-m mechanisms satisfy all properties in the single-demand case.

As future work, we plan to explore the cases (1) when the buyers can perform *false name attacks*, i.e., reporting agents who are not in their neighbour set, (2) when the buyers may collude in order to benefit from group misreporting, and (3) when the values of the items are not additive.

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Appendix

A The GIDM Mechanism

For any global profile θ' and its profile digraph $G_{\theta'} = (V_{\theta'}, E_{\theta'})$, we define the θ'-*criticality* relation on the buyer set $V_{\theta'}$:

Definition 16. *For any buyers $i, j \in V_{\theta'}$, i is θ'-*critical to j*, denoted by $i \preceq_{\theta'} j$, if all paths from s to j in $G_{\theta'}$ go through i.*

The criticality relation can be visualised as a tree structure $T_{\theta'}$ which we call θ'-*diffusion critical tree* [22]. The root of $T_{\theta'}$ is s and for any buyer $j \in V_{\theta'}$, the parent of j is the node $i \preceq_{\theta'} j$ that has the closest distance to j. $T_{\theta'}$ provides insights on how competitions can be alleviated among the buyers to elicit truthful behaviours: If $i \preceq_{\theta'} j$, then i has the power to block information diffusion to j. A truthful mechanism should thus make i indifferent on whether j participates the auction or not. Intuitively, if i's valuation is sufficiently high so that i may win an item had i misreported her connections, then i should be compensated, in the forms of rewards or allocation.

GIDM allocates items based on the diffusion critical tree $T_{\theta'}$. Essentially, the mechanism treats each subtree of $T_{\theta'}$ as a *sub-market* and allocates items recursively for each sub-market. The number of items allocated to each sub-market depends on the number of top-m bidders that subtree contains. An agent i becomes a winner either if she is one of the top-m bidders, or her reported valuation is high enough to win had she misreported her neighbours. A buyer who does not win any item may also receives a reward. See detailed procedures below.

Detailed procedure description: Given a global profile θ', the mechanism first constructs the tree T'_θ, which includes only the buyers with top m valuations and their critical parents. For each buyer $i \in T'_\theta$, let w_i denote the total number of items that buyer i and her critical children Desc_i can get in the optimal allocation tree. Then the allocation is done with a DFS-like procedure. Let S be a stack that the mechanism used for DFS-like traverse. Initially, S is empty. The seller s first gives w_i items to buyers $\{i \in r_s \mid w_i > 0\}$ and pushes all of the buyers into S. Then the mechanism pops the top buyer i from S and checks whether she can be allocated an item in the optimal allocation. If so, the mechanism allocates her an item and adds her into the winner set W; otherwise, check whether i can receive an item if removing a subset Desc_i^m of her critical children: $\mathsf{Desc}_i^m := \mathsf{Desc}_i(\theta)^m \cup \mathsf{Ance}(\mathsf{Desc}_i(\theta)^m) \cup \mathsf{Desc}(\mathsf{Ance}(\mathsf{Desc}_i(\theta)^m))$, where $\mathsf{Desc}_i(\theta)^m$ contains the top-m critical children of i. If so, she gets an item from her critical children who has the lowest valuation and the mechanism updates the other agents' weight. If not, she passes the item to her critical children C_i according to their weight and adds them into S. At the same time, she gets some reward. Repeat the allocation process until S is empty. See Algorithm 2.

Algorithm 2 The GIDM mechanism

Initialise empty stack S
Construct optimal allocation tree T'_θ and calculate each agent i's weight w_i
Push buyers from $\{i \in r_s \mid w_i > 0\}$ into S
while $S \neq \varnothing$ **do**
  Pop the top buyer $i \in S$ from S
  if i can get an item in optimal allocation **then**
    Allocate i an item with payment $\mathsf{SW}_{-D_i} - (\mathsf{SW}_{-\mathsf{Desc}_i^m} - v_i)$, add her into winner set W
  else
    if i can get an item if removing Desc_i^m **then**
      Allocate i an item from the agent who who has the lowest valuation with payment $\mathsf{SW}_{-D_i} - (\mathsf{SW}_{-\mathsf{Desc}_i^m} - v_i)$
      Update w_i of relative agents
    else
      Pass i' item(s) to her critical children, reward i by $\mathsf{SW}_{-D_i} - \mathsf{SW}_{-\mathsf{Desc}_i^m}$, where $D_i = \mathsf{Desc}_i \cup \{i\}$
    end if
  end if
  Push i's critical children into S
end while

This mechanism is not IC as it enables a buyer with a high valuation to potentially manipulate the outcome to her own benefit [16, 6]. We show it with Example 1 below.

Example 1. *Consider the social network in Fig. 1 with seven buyers. As the network is a tree, G coincides with the diffusion critical tree T_θ. The seller has $m = 4$ items to sell. Suppose we run GIDM on this social network. If all buyers report truthfully, GIDM gives all four items to the sub-market of b as all top-four bidders (i.e., the vip) d, e, f, g are descendents of b. The resulting winners are b, c, d, e. On the other hand, suppose f misreport her connection declaring $r'_f = \varnothing$, then GIDM will only give three items to the sub-market of b, while allocating an item to a. This increases the competitions within the sub-market of b and the allocation result is a, b, c, f. In this way, f manipulates the market to her advantage. The detailed run-through of these two cases are shown in Table 3 and Table 4.*

Table 3: Running GIDM on the network in Fig. 1 with $m = 4$ assuming all buyers report truthfully. The winners are b, c, d, e.

rd	m'	winners	π, p
1	4	$\varnothing$	$\pi_b = 1, p_b = 0$
2	3	$\{b\}$	$\pi_c = 1, p_c = 0$
3	2	$\{b, c\}$	$\pi_d = 1, p_d = 5$
4	1	$\{b, c, d\}$	$\pi_e = 1, p_e = 3$

Table 4: Running GIDM on the network in Fig. 1 with $m = 4$ assuming f reports $r'_f = \varnothing$. The winners are a, b, c, f.

rd	m'	winners	π, p
1	4	$\varnothing$	$\pi_a = 1, p_a = 0$
2	3	$\{a\}$	$\pi_b = 1, p_b = 0$
3	2	$\{a, b\}$	$\pi_c = 1, p_c = 0$
4	1	$\{a, b, c\}$	$\pi_f = 1, p_e = 6$

B The DNA-MU Mechanism

DNA-MU follows a conceptually simpler process. The main difference lies in assigning a priority order to buyers based on their distance from the seller in the social network. Buyers who are closer to the seller get higher priority. In this priority order, the mechanism traverses through all agents while allocating winners.

Detailed procedure description: DNA-MU first initialises the residual supply $m' = m$. For each agent i, DNA-MU extracts the m'th highest valuation, called p_i, among buyers in $T_{\theta'}$ who fall outside of the subtree of i. If i's declared valuation $v'_i \geq p_i$, then i has an incentive to misreport her neighbours as doing so will guarantee her winning an item. In this case, the mechanism compensates i by allocating an item to i with payment p_i, before reducing m' by 1. Otherwise, the mechanism proceeds to the next node in the priority order until $m' = 0$. See Algorithm 3.

Algorithm 3 The DNA-MU mechanism

Initialise residual supply $m' \leftarrow m$, winner set W
for agent $i \in N$ according to the priority **do**
    if $m' = 0$ **then**
        Break
    end if
    p_i is the m'th highest valuation in $N_{-i} \backslash W$
    if $v_i > p_i$ **then**
        Allocate i an item with payment p_i
        Update $W \leftarrow W \cup i$ and $m' \leftarrow m' - 1$
    end if
end for

As shown in an unpublished manuscript [4], DNA-MU also fails in terms of IC. Namely, a similar problem as in Example 1 exists also for DNA-MU. See Example 2.

Example 2. *This example is from [4]. Let us run DNA-MU on the social network from Figure 1. When all reports are truthfully DNA-MU does not allocate an item to a as a's valuation $v_a = 3$ is not larger than the 4-th highest valuations $v_e = 4$. Hence, the winners are b, c, d, e. Suppose, on the other hand, f misreports her connection $r'_f = \varnothing$. Then DNA-MU would allocate an item to a, thus the elements in the other subtree face increased competition. This will result in the winners a, b, c, f. As in Example 1, f is able to manipulate the outcome to her advantage. The detailed run-through of these two cases are presented in Table 5 and Table 6.*

Table 5: Running example of DNA-MU in Fig. 1 with $m = 4$ assuming all buyers are truthful. The winners are b, c, d, e.

rd	m'	agent	π, p
1	4	b	$\pi_b = 1, p_b = 0$
2	3	c	$\pi_c = 1, p_c = 0$
3	2	d	$\pi_d = 1, p_d = 5$
4	1	e	$\pi_e = 1, p_e = 3$

Table 6: Running example of DNA-MU in Fig. 1 with $m = 4$ assuming all buyers are truthful. The winners are a, b, c, f.

rd	m'	agent	π, p
1	4	a	$\pi_a = 1, p_a = 1$
2	3	b	$\pi_b = 1, p_b = 0$
3	2	c	$\pi_c = 1, p_c = 0$
4	1	f	$\pi_f = 1, p_e = 6$

C The LDM-Tree mechanism

The *LDM-Tree mechanism* [11] utilises *local* information given by *layers* of the tree, where layer L_i contains agents whose distance from the seller is i. The auction runs several rounds. In round i, the mechanism only involves nodes in L_i and those nodes in L_{i+1} that do not pose a potential competition to nodes in L_i. Example 3 shows that for single-unit auction (i.e., $m = 1$) the social welfare of LDM-Tree may be arbitrarily worse than that of IDM, one of the earliest diffusion auction baselines.

Detailed procedure description. The LDM-Tree mechanism allocates items layer by layer. When the mechanism is considering allocation of a certain layer, it guarantees the buyers report their neighbours truthfully by removing the buyers below layer L_i who are potential competitors of layer L_i. These competitors are divided into two parts. One part is the buyers who diffuse information to potential winners, these buyers can misreport her valuation to take more items from the high-priority layer and get more rewards through resale. The other part contains those buyers who are potential winners. So for each agent $j \in L_i$, the first part is a set of buyers $C_j^P(\theta')$ that contains all children of j who have a child. Then for another part, in the remaining children of j, the buyers who are potential winners need to be removed. If only remove the top-m ranked buyers in j's remaining children according to their valuation report for the first item is not IC. If the agent in $C_j^P(\theta')$ has high valuation but does not report her neighbours, she will be removed in top-m buyers. The buyers to be replaced will compete with the high-priority layers to get more items. So the mechanism needs to remove the buyers $C_j^W(\theta')$ which includes the top $m + \mu - |C_j^P(\theta')|$ ranked buyers in j's remaining children where μ is the upper bound of $|C_i^P(\theta')|$. The children to be removed of agent j is denoted by R_j and the children be removed by layer L_i is denoted by R_{L_i}. After removing all these competitors of layer L_i, the mechanism performs the optimal allocation in the remaining buyers. If an agent $j \in L_i$ can be allocated k items, she pays them for $SW_{-D_i} - (SW_{-R_{L_i}} - (v_1 + \ldots + v_k))$. If $j \in L_i$ cannot be allocated an item, she gets a reward $\mathsf{SW}_{-D_i} - \mathsf{SW}_{-R_{L_i}})$. Then the remaining items are passed to the next layer L_{i+1}. See Algorithm 4.

Example 3. *Let us run LDM-Tree on the same network as in Figure 1 with $m = 1$. In the first round, the mechanism considers only agents in layer 1, and allocates the item to a, whereas IDM allocates the item to the highest bidder f. Note that this holds when v_f is arbitrarily high.*

Algorithm 4 The LDM-Tree mechanism

```
Calculate each agent's layer
Initialise the residual supply m' ← m
for layer ℓ ← 1, 2, ..., ℓ^max do
    Remove the competitor set R_ℓ and calculate the optimal allocation in the remaining buyers denoted by π_i^ℓ
    for agent i ∈ ℓ do
        if π_i^ℓ > 0 then
            Allocate π_i^ℓ items to i with payment
                SW_{-D_i} − (SW_{-R_{L_i}} − (v_1 + ... + v_k))
            Update m' ← m' − π_i^ℓ
        else
            Reward i by SW_{-D_i} − SW_{-R_{L_i}}
        end if
    end for
    if m' = 0 then
        Break
    end if
end for
```

D The MUDAN mechanim

Lemma 3. *MUDAN satisfies IR, ND, and NW.*

Proof. **IR:** By our payment rule, suppose w is selected as a winner at an iteration. The payment $p_w = v'_{m'+1}$ where $v'_{m'+1}$ is the $(m'+1)$th highest valuation in $A \setminus W$ while v'_w is among the top-m' valuations in $A \setminus W$. If w reports the true valuation, then $v_w = v'_w \geq v'_{m'+1}$, which ensures that the utility $u_w = v_w - p_w \geq v_{m'+1} - v'_{m'+1} = 0$. The utility u_i of any other buyer $i \in A \backslash \{w\}$ is 0.

ND: This condition trivially holds as no buyer will have a negative payment (i.e., receiving a reward) in MUDAN.

NW: When the termination condition is met, W either contains exactly m buyers, or less than m buyers while all explored buyers are winners. Thus NW easily holds. □

Theorem 8. *MUDAN terminates within time $O(n^2 + |E|)$, satisfies IC, IR, ND, NW, and $1/m$-weak efficiency.*

Proof. We only need to prove the time complexity of MUDAN. The other properties are handled by lemmas above.

Termination. As described above (in the main paper), the algorithm terminates in exactly m iterations.

Complexity. For running time analysis, we describe an implementation of the algorithm: Maintain a sorted list of buyers in A. The buyers in this list are sorted in descending order of their valuations. In this way, the set P can be accessed in constant time. When a node is added in A, scan the list and add the node to the appropriate location. With each buyer in A, put a label indicating whether the node has been chosen as a winner, i.e., added in W. At each iteration, scan the list to find the node in $P \setminus W$ with the highest priority and label it as a winner.

- To explore the graph involves potentially checking all edges and nodes. This takes $O(n + |E|)$.
- To add all nodes in A involves performing an insertion sort to the set of buyers. This takes $O(n^2)$.
- To choose m winners from P involves m iterations, each iteration scans the current list once. This takes another $O(n^2)$.

Therefore in summary the algorithm runs in time $O(n^2 + |E|)$. □

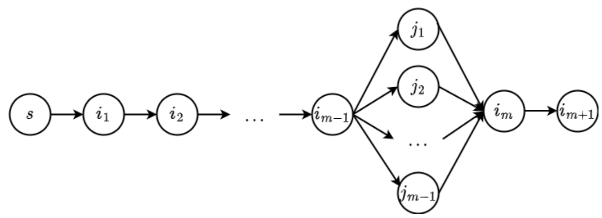

Figure 3: A social network with a seller s and $2m$ buyers..

Theorem 9. *For any $m \geq 1$ and any constant $\lambda > 0$, there exists profile θ where no m-unit IC diffusion auction with no-reward achieves $(1/m + \lambda)$-weakly efficient.*

Proof. Consider the graph as shown in Fig. 3 which depicts a situation with $2m$ buyers. The valuations and connections of the buyers are defined as follows:

- $i_1, i_2, i_3, \dots, i_{m-1}$ have valuation $n > 1$,
- $j_1, j_2, \dots, j_{m-1}$ have valuation $n^2 - \tau$ for a small $\tau > 0$,
- i_m has valuation n^2, and
- i_{m+1} has valuation n^3.

To guarantee IC, a mechanism must allocate $m-1$ items to buyers $i_1, i_2, \dots, i_{m-1}$ and the last item to i_m. Thus the social welfare of any IC mechanism is at most $n^2 + (m-1)n$. For such a mechanism, $\mathsf{SW}_{\text{wopt}}$ is $mn^2 - (m-1)\tau$. For sufficiently large n,

$$\begin{aligned}
&\frac{n^2 + (m-1)n}{mn^2 - (m-1)\tau} \\
=&\frac{1}{m} + \frac{(m-1)\tau/m + (m-1)n}{mn^2 - (m-1)\tau} \\
\leq&\frac{1}{m} + \frac{1}{mn^2 - m} + \frac{1}{n - (m-1)\tau/mn} \\
<&\frac{1}{m} + \lambda.
\end{aligned}$$

□

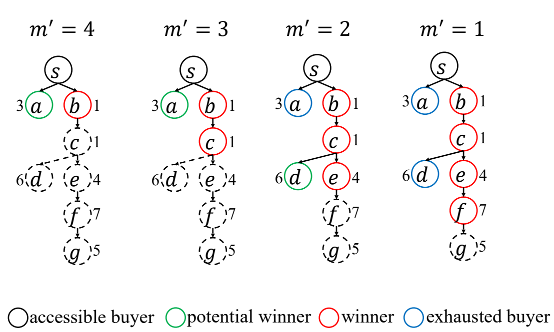

Figure 4: Detailed run-through of MUDAN on the network shown in Figure 1

E The MUDAR mechanism

We discuss the properties ensured by the MUDAR mechanism.

Lemma 10. *The MUDAR mechanism satisfies IR, ND, and NW.*

Proof. **IR:** By our payment rule, for winner $w \in W_{\mathsf{A}}$, $p_w = v'_{m+1}$ and thus w's utility is $u_w = v_w - p_w = v_w - v'_{m+1}$; for winner $w \in W_{\mathsf{R}}$, the utility $u_w = -p_w = v'_w - v'_{m+1}$. If she reports the true valuation, then $v_w = v'_w \geq v'_{m+1}$, which ensures $u_w \geq 0$. For any other agent $i \in A \setminus W$, $u_i = 0$.

NW: When $|V_{\theta'} - 1| \leq m$, the mechanism puts all buyers into set W_{A} and $\sum_{i \in V_{\theta'} \setminus \{s\}} \pi_i(\theta') = |V_{\theta'}| - 1$. When $|V_{\theta'} - 1| > m$, the mechanism puts the m buyers with the top-m valuations in W_{A} and $\sum_{i \in V_{\theta'} \setminus \{s\}} \pi_i(\theta') = m$.

ND: List all winners in the set W as $w_1, w_2, \ldots, w_{|W|}$ in the order as they are added in W. Below we separately discuss w_k for $k \leq m$ and for $k > m$:

1. $1 \leq k \leq m$: The tentative payment $\hat{p}_{w_k} \geq 0$. If $w_k \in W_{\mathsf{R}}$, the payment $p_{w_k} = \hat{p}_{w_k} - v_{w_k} \geq 0 - v_{w_k}$; if $w_k \in W_{\mathsf{A}}$, $p_{w_k} \geq 0$.
2. $m < k \leq |W|$: The tentative payment $\hat{p}_{w_k}$ is the $(m+1)$th highest valuation in A at iteration k. Let ψ_k be the $(m+1)$th highest valuation in the set W at iteration k. Since $W \subseteq A$, $\hat{p}_{w_k} \geq \psi_k$. If $w_k \in W_{\mathsf{R}}$, the payment is $p_{w_k} = \hat{p}_{w_k} - v_{w_k} \geq \psi_k - v_{w_k}$; if agent $w_k \in W_{\mathsf{A}}$, the payment is $p_{w_k} \geq \psi_k$.

Summarising the above, the revenue is

$$\mathsf{RV}(\theta') = \sum_{k=1}^{|W|} p_{w_k}$$

which equals to

$$\begin{aligned}
&\sum_{k=1}^{m} \{\hat{p}_{w_k} - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=1}^{m} \{\hat{p}_{w_k} \mid w_k \in W_{\mathsf{A}}\} + \\
&\sum_{k=m+1}^{|W|} \{\hat{p}_{w_k} - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=m+1}^{|W|} \{\hat{p}_{w_k} \mid w_k \in W_{\mathsf{A}}\} \\
\geq &\sum_{k=1}^{m} \{0 - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + 0 + \\
&\sum_{k=m+1}^{|W|} \{\psi_k - v_{w_k} \mid w_k \in W_{\mathsf{R}}\} + \sum_{k=m+1}^{|W|} \{\psi_k \mid w_k \in W_{\mathsf{A}}\} \\
= &\sum_{k=m+1}^{|W|} \psi_k - \sum_{w_k \in W_{\mathsf{R}}} v_{w_k} = 0
\end{aligned}$$

The last equation holds because ψ_k, $m \leq k \leq |W|$, coincide with the valuations of $w_k \in W_{\mathsf{R}}$, $1 \leq k \leq |W|$. □

Theorem 13. *MUDAR terminates within time $O(n^2 + |E|)$, satisfies μ-IC, IR, ND, NW, and efficiency, where μ is the mth highest valuation among all buyers.*

Proof. **Complexity.** The time complexity of the algorithm can be proved in the same way as for Theorem 8. We also maintain a sorted list to store the explored nodes whose first m elements are those in P, and label a node if it is added in W. In this implementation, the algorithm runs in $O(n^2 + |E|)$.

Efficiency. For the efficiency condition, note that when the termination condition is met, all buyers are explored. Moreover, W_{A} contains those buyers in A with the top-m reported valuations. By Lemma 12, they are also the buyers with the top-m true valuations. This means that the mechanism will allocate items optimally, thus ensuring efficiency. The rest of the proof follows from Lemma 10 and Lemma 12. □

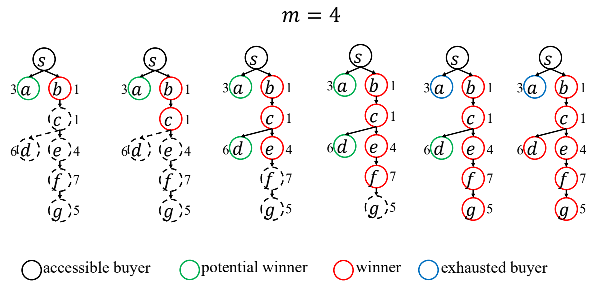

Figure 5: Detailed run-through of MUDAR on the network shown in Figure 1

F Multi-demand multi-unit auction

A detailed description of the multi-demand model. In the multi-demand multi-unit diffusion auction, each buyer $i \in B$ may demand more than one item and attaches a valuation $v_{i,j}$ to the jth item she gets, where $1 \leq j \leq m$. The valuation for all items is denoted by a vector $\vec{v_i} \coloneqq (v_{i,1}, \ldots, v_{i,m}) \in \mathbb{R}^m_{\geq 0}$; we call $\vec{v_i}$ the *valuation vector* of the buyer i. Each additional unit often brings less additional utility than that from the previous unit, which is known as the *law of diminishing marginal utility* in micro-economics [2]. Therefore, we assume that the buyers have *diminishing valuation* towards the items: $v_{i,j} \geq v_{i,j+1}$ for $j = 1, \ldots, m-1$. For simplicity, we omit 0 in this vector. In this setting, we denote the *(multi-demand) profile* of a buyer i as $\eta_i \coloneqq (\vec{v_i}, t_i)$ where $\vec{v_i}$ is the valuation vector and $t_i \subseteq B$ is the set of neighbours of i. The global profile is $\eta' \coloneqq (\eta'_1, \ldots, \eta'_n)$ that corresponds to profile graph $G_{\eta'}$. Let H denote the set of all possible (multi-demand) global profiles. A *mechanism* $\mathcal{M}$ in this setting consists of *allocation rule* $\pi\colon H \to \{0,1\}^{n\times m}$ and *payment rule* $p\colon H \to \mathbb{R}^{n\times m}$. Here each $\pi_i(\eta')$, $p_i(\eta')$ are m-dimensional vectors; We write them as $\pi_i(\eta') = (\pi_{i,1}(\eta'), \ldots, \pi_{i,m}(\eta'))$ and $p_i(\eta') = (p_{i,1}(\eta'), \ldots, p_{i,m}(\eta'))$, respectively.

We formally define properties of a mechanism below:

- The *utility* $u_i(\eta')$ of the buyer i is defined as $\sum_{j=1}^m v_{i,j}\pi_{i,j} - p_{i,j}$.
- The *social welfare* $\mathsf{SW}(\eta')$ of the mechanism $\mathcal{M}$ is the sum of the utilities of all the agents, i.e., $\sum_{i=1}^n \sum_{j=1}^m v_{i,m}\pi_{i,j}$.
- The *optimal social welfare* SW_{opt} is the sum of the top-m valuations among $v_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m$.
- The *revenue* $\mathsf{RV}(\eta')$ is the sum of the payment of all buyers, i.e., $\sum_{i=1}^n \sum_{j=1}^m p_{i,j}$.

Consider a mechanism $\widetilde{\mathcal{M}}$ that is applied to $\widetilde{B}$ given the constructed profile θ, and the corresponding mechanism $\mathcal{M}$ that is applied to B given the profile η. The following lemma is straightforward from the construction:

Lemma 17. *The following statements hold:*

1. *For each buyer $i \in B$, the utility of i received from $\mathcal{M}$ equals the sum of utilities of $i_1, \ldots, i_m$ received from $\widetilde{\mathcal{M}}$, i.e., $u_i(\eta') = \sum_{j=1}^m u_{i_j}(\theta')$.*
2. *The social welfare $\mathsf{SW}(\eta')$ of $\mathcal{M}$ equals the social welfare $\mathsf{SW}(\theta')$ of $\widetilde{\mathcal{M}}$.*
3. *The optimal social welfare on B equals the optimal social welfare on $\widetilde{B}$.*
4. *The revenue $\mathsf{RV}(\eta')$ of $\mathcal{M}$ equals the revenue $\mathsf{RV}(\theta')$ of $\widetilde{\mathcal{M}}$.*
5. *The number of items allocated by $\mathcal{M}$ equals the number of items allocated by $\widetilde{\mathcal{M}}$.*

□

In the next definition, let $\eta'_{-i} \coloneqq (\eta'_1, \ldots, \eta'_{i-1}, \eta'_{i+1}, \ldots, \eta'_n)$ denote the profiles of all buyers but i.

Definition 18. *Let $\mathcal{M}$ be a mechanism.*

1. *$\mathcal{M}$ is* incentive compatible *(IC) if for any buyer reporting truthfully is a dominant strategy: for all $i \in B$, all global profiles η' and η'', we have $u_i(\eta_i, \eta'_{-i}) \geq u_i(\eta''_i, \eta''_{-i})$* [4].

[4] As t''_i may be different from t'_i, some agents j who are reachable in $G_{\eta'}$ may become unreachable had we replace η'_i with η''_i. η''_{-i} is obtained from η'_{-i} by replacing η'_j with the silent profile for all such agents j.

2. $\mathcal{M}$ *is* individually rational *(IR) if any buyer by reporting truthfully receives non-negative utility, i.e., for all* $i \in B$, $u_i(\eta) \geq 0$.

3. $\mathcal{M}$ *is* non-deficit *(ND) if for any global profile* η'*, the revenue is non-negative, i.e.,* $\mathsf{RV}(\eta') \geq 0$.

4. $\mathcal{M}$ *is* non-wasteful *(NW) if all items are allocated to buyers (up to the number of reachable buyers), i.e., for any global profile* η', $\sum_{i \in V_{\eta'} \setminus \{s\}} \sum_{j=1}^{m} \pi_{i,j}(\eta') = \min\{m, m(|V_{\eta'}| - 1)\}$.

5. $\mathcal{M}$ *is* efficient *if it achieves optimal social welfare, i.e., for any* η', $\mathsf{SW}(\eta') = \mathsf{SW}_{\mathrm{opt}}$.

The next lemma follows direction from Lemma 17 so we omit the proof.

Lemma 19. *The following statements holds:*

1. *If* $\widetilde{\mathcal{M}}$ *is IR, then so is* $\mathcal{M}$.
2. *If* $\widetilde{\mathcal{M}}$ *is ND, then so is* $\mathcal{M}$.
3. *If* $\widetilde{\mathcal{M}}$ *is NW, then so is* $\mathcal{M}$.
4. *If* $\widetilde{\mathcal{M}}$ *is efficient, then so is* $\mathcal{M}$.

$\square$

The MUDAN-m mechanism. We first prove the following lemma.

Lemma 20. *The MUDAN-m mechanism is IC.*

Proof. We first prove that no buyer in B can benefit from misreporting her valuation vector. To run $\mathcal{M}$, we first execute the MUDAN mechanism $\widetilde{\mathcal{M}}$ given the constructed profile θ' on $\widetilde{B}$. Now consider the execution of $\widetilde{\mathcal{M}}$.

If i_1 is selected as a winner, then $\widetilde{\mathcal{M}}$ will proceed to select more i_j as winners until the valuation is so low that i_j is not added in P. That is because after i_j is chosen as winner, the only buyer in $\widetilde{B}$ that is added to A is i_{j+1}. By definition of the priority on B, if i_{j+1} is added to P, then i_{j+1} will be the next node with the highest priority and is thus chosen as the winner.

Consider a given iteration of $\widetilde{\mathcal{M}}$. By the argument above, if a buyer $i_j \in P$ is not chosen as the winner, then $j = 1$. Moreover, if i_j is exhausted, then i_ℓ for all $\ell > j$ are exhausted also due to the diminishing valuation. This means that for any buyer $i \in B$, either $i_1 \in P$, or $\{i_1, \ldots, i_m\} \cap P = \varnothing$. This observation is crucial for the proof of truthfulness.

Now suppose in the given iteration, w_j is chosen as the winner. Suppose $w \in B$ misreports her valuation vector so that $\vec{v_w'} \neq \vec{v_w}$. We show that this strategy will not give w extra utility at this iteration:

(a) Suppose $v_{w,\ell} \neq v'_{w,\ell}$ where $\ell > j$. At this iteration, w_ℓ would *not* have been added in A. Therefore for w, misreporting $v_{w,\ell}$ does not affect the tentative payment $\hat{p}(w_j)$ for her jth item.

(b) Suppose $v_{w,k} \neq v'_{w,k}$ where $k < j$. Note that w_k must have been a winner in an earlier iteration. So the valuation of w_k is not taken into consideration when the algorithm sets the tentative payment $\hat{p}(w_j)$.

(c) Now suppose w misreports the valuation of $v_{w,j}$ in such a way that $v'_{w,j} > v_{w,j} > p_{\hat{w_j}}$ or $v_{w,j} > v'_{w,j} > p_{\hat{w_j}}$. Here, $p_{\hat{w_j}}$ is the tentative payment of w_j assuming $\tilde{\mathcal{M}}$ sets w_j as the winner. As this does not change w_j's priority, w is still chosen as the winner in this iteration and is allocated her jth item with payment $p_{\hat{w_j}}$. This item will add w a utility of $u_{w_j}((v'_{w_j}, r_{w_j}), \theta_{-w_j}) = u_{w_j}((v_{w_j}, r_{w_j}), \theta_{-w_j})$.

(d) On the other hand, if $v'_{w,j} < p_{\hat{w_j}} < v_{w,j}$, then w loses the jth item as all w_ℓ where $\ell \geq j$ are exhausted, giving her no extra utility.

Summarising (a)-(d) above, in the given iteration, misreporting any element in the valuation vector $(v_{w,1}, \ldots, v_{w,m})$ will not give w extra utility.

Next, consider buyer $i \neq w$ where $i_1 \in P$. We prove that i cannot benefit from misreporting her valuation using a similar argument as above:

(a) Consider $j > 1$. The node i_j would not have been added in A. Therefore for i, misreporting $v_{i,j}$ cannot lead to a higher priority nor a less payment.

(b) Suppose i misreports her valuation such that $v'_{i_1} < v'_{m'+1} < v_{i_1}$. Recall that we used $v'_{m'+1}$ in MUDAN to denote the $m' + 1$th highest valuation in this iteration, where m' has the same meaning as defined in Sec. 4. Then i_1 would not be a potential winner, but rather be exhausted. This means that no item would be allocated to i and her utility would be 0.

(c) Suppose i misreports her valuation such that $v'_{i_1} > v_{i_1} > v'_{m'+1}$ or $v_{i_1} > v'_{i_1} > v'_{m'+1}$. Then i_1 would still be a potential winner in this iteration, giving her no extra utility.

Summarising (a)-(c) above, in the given iteration, misreporting any (combination of) elements in the valuation vector $(v_{i,1}, \dots, v_{i,m})$ will not give i extra utility.

Last, consider buyer $i \neq w$ where $\{i_1, \dots, i_m\} \cap P = \varnothing$. We prove that i cannot benefit from misreporting her valuation again using a similar argument:

(a) Consider $j > 1$. The node i_j would not have been added in A. Therefore for i, misreporting v_{i_j} cannot lead to a higher priority nor a less payment.

(b) Suppose i misreports her valuation such that $v_{i_1} < v'_{m'} \leq v'_{i_1}$. Then we have: (i) If i_1 has the highest priority, then her utility of this item is $u_{i_1}((v'_{i_1}, r_{i_1}), \theta_{-i_1}) = v_{i_1} - v'_{m'+1} < 0 = u_{i_1}((v_{i_1}, r_{i_1}), \theta_{-i_1})$. (ii) Otherwise, her utility remains 0.

(c) Suppose i misreports her valuation such that $v_{i_1} < v'_{i_1} < v'_{m'}$ or $v'_{i_1} < v_{i_1} < v'_{m'}$. Her utility remains 0 and all i_j where $j \geq 1$ are exhausted.

Summarising (a)-(c) above, in the given iteration, misreporting any element in the valuation vector $(v_{i,1}, \dots, v_{i,m})$ will not give i extra utility. This we proved the fact for all iterations, no buyer in B has incentive to misreport their valuation vector.

It remains to prove that no buyer in B can benefit from misreporting her neighbour set. Our argument is the following: For each agent $i_j \in \widetilde{B}$, her priority cannot increase when she hide any of her neighbours. And her neighbours is only added in A either when i_m is chosen as a winner or when i_m is exhausted so that her neighbours cannot influence her allocation. The rest of the proof is the same as in the proof of Lemma 4. □

For analysing the social welfare of MUDAN-m, we make the following definition:

- Suppose $w^*_j \in \widetilde{B}$ is the winner chosen by the MUDAN mechanism $\hat{\mathcal{M}}$ in the last iteration. We say that w^* is *critical* for a buyer i if all paths from s to i in $G_{\eta'}$ pass through w^*.
- Let $B^* \subseteq B$ denote the set of buyers for whom w^* is not critical.
- The *weakly-optimal social welfare* $\mathsf{SW}_{\text{wopt}}$ denote the sum of the top-m valuations among $\{v_{i,j}\}_{i \in B^*, 1 \leq j \leq m}$.
- A mechanism $\mathcal{M}$ is ϵ*-weakly efficient* if for any global profile η', we have $\mathsf{SW}(\eta') \geq \epsilon \mathsf{SW}_{\text{wopt}}$.

Theorem 14. *MUDAN-m is IC, IR, ND, NW, and $1/m$-weakly efficient.*

Proof. IC follows from Lemma 20, IR, ND, and NW follow from Lemma 19 and Theorem 8. $1/m$-weak efficiency follows directly from the fact that $\widetilde{\mathcal{M}}$ selects the buyer i_1 who has the highest valuation v_{i_1} among the explored buyers. □

The MUDAR-m mechanism. For analysing the truthfulness of MUDAR-m, we make the following definition:

For a positive value $\mu > 0$, a mechanism $\mathcal{M}$ is *μ-bounded incentive compatible* (μ-IC) if for any buyer $i \in B$, for any $1 \leq j \leq m$, i cannot benefit from either reporting a lower valuation for her jth item, or reporting a valuation that is higher than μ, i.e., for any $i \in B$, for any $1 \leq j \leq m$, for any global profiles η' and η'' such that $v''_{i,j} \geq \mu \geq v_{i,j}$ or $v''_{i,j} < v_{i,j}$,

$$u_i(\eta_i, \eta'_{-i}) \geq u_i(\eta''_i, \eta''_{-i}).[5]$$

We first show that MUDAR-m satisfies the same truthfulness condition as MUDAR.

[5] Similar to the def. of IC in Def. 2, η''_{-i} is obtained from η'_{-i}.

Lemma 21. *The MUDAR-m mechanism satisfies μ-IC where μ is the mth highest valuation among $v_{i,j}$ for all $1 \le i \le n$, $1 \le j \le m$.*

Proof. Firstly we prove that no agent $i \in B$ can benefit from misreporting her valuation vector in such a way that some of its elements $v'_{i,j}$ are lower than the true valuation $v_{i,j}$, or higher than μ. In other words, suppose $v'_{i,j} < v_{i,j}$ or $v'_{i,j} > \mu$ for some $1 \le j \le m$, then the utility that i receives is strictly less than as if she reports $\vec{v_i}$ truthfully.

Just like in the proof of Lemma 20, in the following we analyse the execution of the $\widetilde{\mathcal{M}}$ mechanism over buyers $\widetilde{B}$. Consider a given iteration of $\widetilde{\mathcal{M}}$.

Let w_j be the chosen winner by $\widetilde{\mathcal{M}}$. Suppose w_j is selected as an allocation winner. Suppose $w \in B$ misreports her valuation vector so that $\vec{v'_w} \neq \vec{v_w}$. We show that this strategy will not give w extra utility at this iteration:

(a) Suppose $v_{w,\ell} \neq v'_{w,\ell}$ where $\ell > j$. In this iteration, w_l would not have been added in A. Therefore for w, misreporting $v_{w,\ell}$ does affect the tentative payment $\hat{p}(w_j)$.

(b) Suppose $v_{w,k} \neq v'_{w,k}$ where $k < j$. Note that w_k must have been an allocation winner in an earlier iteration. We have $v'_{w,k} > v'_{w,j} > \hat{p}(w_j)$, so that $v'_{w,k}$ cannot be tentative payment of w_j. Therefore for w, misreporting $v_{w,\ell}$ does affect the tentative payment $\hat{p}(w_j)$.

(c) Now suppose w misreports the valuation of $v_{w,j}$ in such a way that $v'_{w_j} > v_{w_j} > \hat{p}(w_j)$ or $v_{w_j} > v'_{w_j} > \mu$, where $\hat{p}(w_j)$ is the tentative payment of w_j. As this does not change w_j's priority, w is still chosen as the winner in this iteration and is allocated her jth item and with payment $\hat{p}(w_j)$. This item will add w a utility of $u_{w_j}((v'_{w_j}, r_{w_j}), \theta_{-w_j}) = u_{w,j}((v'_{w,j}, r_{w_j}), \theta_{-w_j})$.

(d) Suppose w misreports her valuation such that $v_{w_j} > \mu > v'_{w_j} > \hat{p}(w_j)$, then w_j becomes a reward winner in this iteration. This iteration gives w a lower extra utility of $u_{w_j}((v'_{w_j}, r_{w_j}), \theta_{-w_j}) = v'_{i,j} - \hat{p}(w_j) \le v_{i,j} - \hat{p}(w_j) = u_{w_j}((v'_{w,j}, r_{w_j}), \theta_{-w_j})$.

(e) On the other hand, if w misreports her valuation such that $v'_{w_j} < \hat{p}(w_j) < v_{w,j}$, then w loses her jth item as all w_ℓ where $\ell > j$ are exhausted, giving her no extra utility.

Summarising (a)-(e) above, in the given iteration, misreporting any element in the valuation vector $(v_{w,1}, \dots, v_{w,m})$ will not give w extra utility.

Next, consider buyer w_j is selected as a reward winner. We prove that w cannot benefit from misreporting any element of her valuation vector such that $v'_{w,j} < v_{w,j}$ or $v'_{w,j} > \mu$:

(a) Suppose$v_{w,\ell} \neq v'_{w,\ell}$ where $\ell > j$. In this iteration, w_l would not have been added in A. Therefore for w, misreporting $v_{w,\ell}$ does affect the tentative payment $\hat{p}(w_j)$.

(b) Suppose $v_{w,k} \neq v'_{w,k}$ where $k < j$. Note that w_k must have been a winner in an earlier iteration. We have $v'_{w,k} > v'_{w,j} > \hat{p}(w_j)$, so that $v'_{w,k}$ cannot be tentative payment of w_j. Therefore for w, misreporting $v_{w,\ell}$ does affect the tentative payment $\hat{p}(w_j)$.

(c) Now suppose w misreports the valuation of $v_{w,j}$ in such a way that $v'_{w_j} > \mu > v_{w_j} > \hat{p}(w_j)$, then w_j is selected as an allocation winner. This iteration gives w an extra utility of $v_{w,j} - \hat{p}(w_j)$. This equals to the utility of truthful reporting.

(d) On the other hand, if w misreports her valuation such that $v'_{w_j} < \hat{p}(w_j) < v_{w_j}$, then w loses her reward of the j's item as all w_ℓ where $\ell > j$ are exhausted, giving her no extra utility.

Summarising (a)-(d) above, in the given iteration, misreporting any element in the valuation vector $(v_{w,1}, \dots, v_{w,m})$ will not give w extra utility. Next, consider buyer $i \neq w$ where $i_1 \in P$. We prove that i cannot benefit from misreporting her valuation using a similar argument as above:

(a) Consider $j > 1$. The node i_j would not have been added in A. Therefore for i, misreporting $v_{i,j}$ cannot lead to a higher priority nor a less payment.

(b) Suppose i misreports her valuation such that $v'_{i_1} < v'_{m'+1} < v_{i_1}$. Then i_1 would not be a potential winner, but rather be exhausted. This means that no item would be allocated to i and her utility would be 0.

(c) Suppose i misreports her valuation such that $v'_{i_1} > v_{i_1} > v'_{m'+1}$ or $v_{i_1} > v'_{i_1} > v'_{m'+1}$. Then i_1 would still be a potential winner in this iteration, giving her no extra utility.

Summarising (a)-(c) above, in the given iteration, misreporting any (combination of) elements in the valuation vector $(v_{i,1}, \ldots, v_{i,m})$ will not give i extra utility.

Last, consider buyer $i \neq w$ where $\{i_1, \ldots, i_m\} \cap P = \varnothing$. We prove that i cannot benefit from misreporting her valuation again using a similar argument:

(a) Consider $j > 1$. The node i_j would not have been added in A. Therefore for i, misreporting v_{i_j} cannot lead to a higher priority nor a less payment.

(b) Suppose i misreports her valuation such that $v_{i_1} < v'_{m'} \leq v'_{i_1}$; Recall that we used $v_{m'}$ in MUDAN to denote the m'th highest valuation in this iteration, where m' has the same meaning as defined in Sec. 4. Then we have: (i) If i_1 has the highest priority, then her utility of this item is $u_{i_1}((v'_{i_1}, r_{i_1}), \theta_{-i_1}) = v_{i_1} - v'_{m'+1} < 0 = u_{i_1}((v_{i_1}, r_{i_1}), \theta_{-i_1})$. (ii) Otherwise, her utility remains 0.

(c) Suppose i misreports her valuation such that $v_{i_1} < v'_{i_1} < v'_{m'}$ or $v'_{i_1} < v_{i_1} < v'_{m'}$. Her utility remains 0 and all i_j where $j \geq 1$ are exhausted.

Summarising (a)-(c) above, in the given iteration, misreporting any element in the valuation vector $(v_{i,1}, \ldots, v_{i,m})$ will not give i extra utility. This we proved the fact for all iterations, no buyer in B has incentive to misreport their valuation vector.

It remains to prove that no buyer in B can benefit from misreporting her neighbour set. Our argument is the following: For each agent $i_j \in \widetilde{B}$, her priority cannot increase when she hide any of her neighbours. And her neighbours is only added in A either when i_m is chosen as a winner or when i_m is exhausted so that her neighbours cannot influence her allocation. The rest of the proof is the same as in the proof of Lemma 4. $\square$

Theorem 15. *MUDAR-m is μ-IC, IR, ND, NW, and efficient, where μ is the mth highest valuation among $v_{i,j}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.*

Proof. IR, ND, NW, and efficiency follow directly from Lemma 19 and Theorem 13. m-IC follows from Lemma 21. $\square$

G Experiments on priority

Finally, we turn our attention to the priority σ_i. The priority determines the selection of winner at an iteration. It affects the outcome of MUDAN-m[6]. The idea of priorities to buyers has been exploited by several existing diffusion auction mechanisms, and the following three priority orderings have been used: **(1) depth-based selection** [22] **(2) distance-based selection** [6], and **(3) degree-based selection** [17]. Yet their effectiveness has not been analysed. Since all three approaches can be adopted by our mechanisms, we now examine how they affect social welfare in MUDAN. This section of the Appendix does not belong to the main part of the paper. Yet we include it for the completeness of exposition.

Depth-based (or distance-based) selection prioritises the buyers who are farthest away from (closest to) the seller. Thus these methods suffer from the risk of omitting buyers with high valuations but close to (far away from) the seller. Degree-based selection prioritises the buyers who have more neighbours. However, this does not guarantee that large number of buyers are explored. With this intuition, we propose a new traversal strategy, i.e., **(4) New-agent-based selection:** Prioritise agents who are able to bring the highest number of unexplored buyers to A. In this strategy, only the contribution to graph exploration counts and thus we expect it achieves higher social welfare. We describe our experiments below.

Dataset. We use three real-world datasets, including Facebook social network [12], Hamsterster friendships [7], and email-Eu-core network [18]. Facebook network has $4,039$ nodes and $88,234$ edges, Hamsterster friendship network has $1,858$ nodes and $12,534$ edges, while email-Eu-core network has $1,005$ nodes and $25,571$ edges. Table 7 shows the key statistics of these three datasets. For each dataset, we randomly select one node as the seller. As the initial setup, especially the neighbour set of the seller, may effect experiment results, we repeat

dataset	$\lvert V\rvert$	$\lvert E\rvert$	clustering coefficient	diameter
Facebook social network	4039	88234	0.6055	8
Hamsterster friendships	1005	25571	0.3994	7
email-Eu-core network	1858	12534	0.0904	14

Table 7: Dataset statistics

each scenario $n/2$ times, where n is the number of nodes, and calculate the average revenue and social welfare as the result for the scenario.

Valuation. We use three different models to generate the agents' valuation.

1. **Model 1:** All the valuations of buyers are sampled at random i.i.d. Specifically, we assume $v_{i,j} \sim U(0, 200000)$ for $1 \le i \le n, 1 \le j \le m$.

2. **Model 2:** To increase diversity, the highest valuations of buyers are sampled i.i.d. while their subsequent valuations are independently but non-identically distributed. Here, we assume that $v_{i,1} \sim U(0, 200000)$ for $1 \le i \le n$, and $v_{i,j} \sim U(1, v_{i,1})$ for $1 \le i \le n, 1 < j \le m$.

3. **Model 3:** The valuation of the buyers are affected by their neighbours, in particular, the *homophily principle* asserts that agents who are tightly connected tend to exhibit similar preferences [13]. To capture possible dependence among closely-tied buyers, we deploy the DeGroot model, an established model of social influence [5], to generate the highest valuations $v_{i,1}$ for $1 \le i \le n$. DeGroot model [5] assumes that each agent's valuation for the next iteration is derived from a weighted average of her own valuation and those of her neighbours in the network. The weight is assigned by the agent, and it represents her confidence in her own valuation or her friendship with others. The other valuations of a certain buyer $v_{i,j} \sim U(1, v_{i,1})$ for $1 \le i \le n, 1 < j \le m$.

Benchmark. The only currently known IC multi-unit diffusion auction mechanism, LDM-Tree [11], is chosen as our benchmark. *Random selection* is used as a benchmark for traversal strategies. It randomly allocates a buyer from the potential winner set. For comparison, we also calculate the *optimal* social welfare by taking the sum of top-m valuations; this will be achieved by MUDAR-m. We evaluate all four implementations of MUDAN-m as well as these benchmarks in terms of *social welfare* SW and *revenue* RV as defined in Sec. 2.

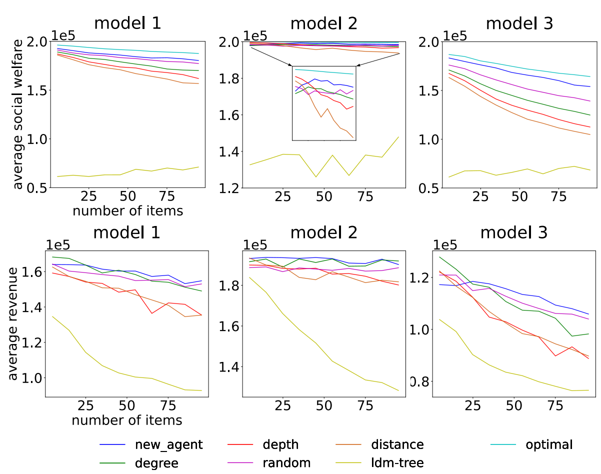

Figure 6: Social welfare and revenue of different traversal strategies in three models for Facebook network

Results. Figs. 6, 7 & 8 show average SW and RV per item. As shown, MUDAN-m significantly outperforms LDM-Tree. In particular, new-agent selection in general outperforms the other traversal schemes. When the number of items is small, new-agent selection is slightly less than that of other strategies, but grows faster as the number of items increases. Further, MUDAN-m with new-agent selection loses by at most 9% from the optimal social welfare. Across the valuation models, MUDAN-m with new-agent selection performs better in general.

[6] As MUDAR-m ensures to explore the entire graph $G_{\theta'}$, all buyers are explored by MUDAR-m. In this sense, the priority does not affect the social welfare. We therefore focus on MUDAN-m in this section.

This is particularly visible in model 3, which is consistent with our expectation: When the valuations are dependant, buyers with similar valuations form communities, new-agent-based selection is more advantageous as it could more easily jump out from lower-valuation communities. We may draw consistent conclusions from the results for all three datasets showing robustness of our mechanism.

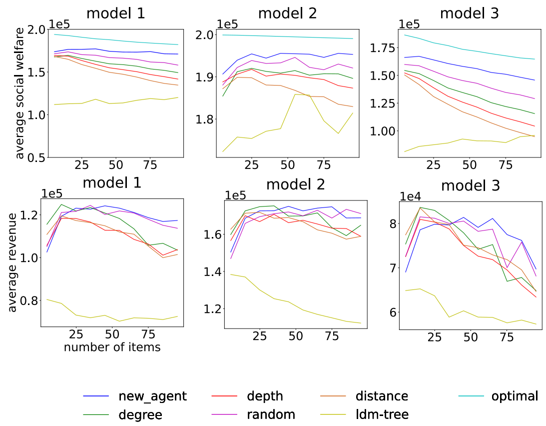

Figure 7: Social welfare and revenue of different traversal strategies in three models for Hamsterster network

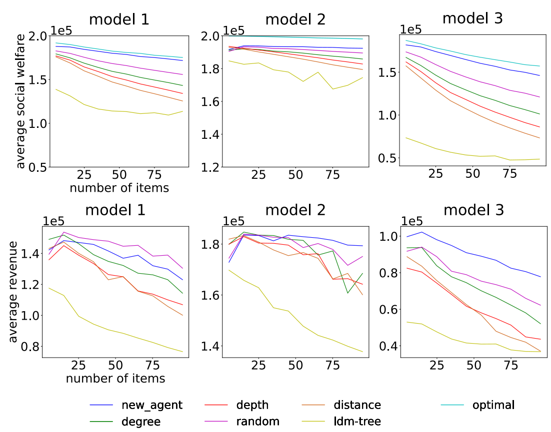

Figure 8: Social welfare and revenue of different traversal strategies in three models for Email network