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ME 524

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FEA of 2D Steady-State Heat Diffusion with Biquadratic Elements

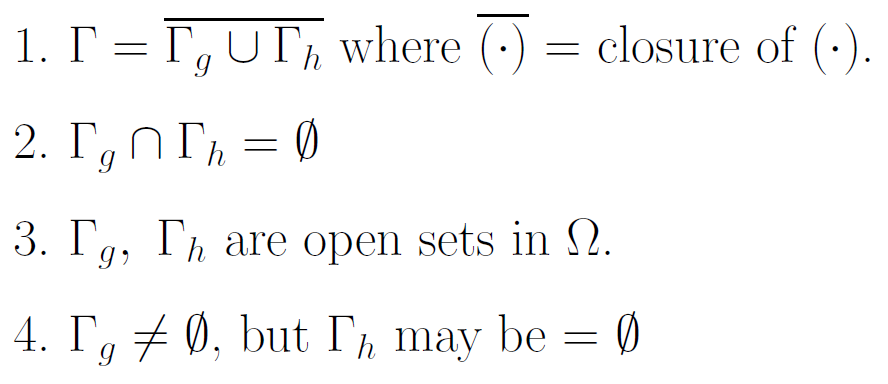
For steady-state (time-independent) heat diffusion the equations are

*qi,i = f* in Ω

*u = g* on Γ*g*

*-qini = h* on Γ*h*

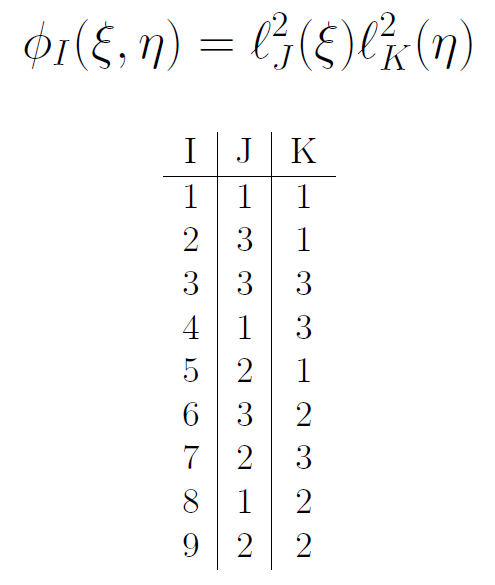
where *qi = -κiju,j*. First start by analyzing *elemstiff.m* where D should be replaced by a diagonal matrix assuming that our material is isotropic. Also since temperature is a scalar field we set the degrees of freedom to 1 and match matrix sizes throughout the program, including changing **BJ** to have 2 rows representing the gradient of the shape functions in the 2 spatial dimensions. Then in *applybcs.m* we have to make sure that with the total boundary being Γ, the boundary conditions Γ*g* and Γ*h* satisfy



are satisfied. In this program *f* is assumed to be 0, and *h* is assumed to be 0 on Γ*h* where Γ*h*= Γ-Γ*g*. Finally in *heat2d.m* since we cannot assume that boundary conditions are homogeneous, if row *m* of the forcing vector lies on the Dirichlet boundary, we do not set row *m* of **bigk** to 0 but we still set column *m* of **bigk** to 0 and **bigk**(*m*,*m*) to 1.

The 9-node biquadratic elements consists of 9 shape functions that satisfies the Kronecker delta property for the local shape function number and the local node number. This is formed by taking the product of the 1D 3-node quadratic element in ξ and η. This still satisfies C1 element continuity, C0 global continuity, and linear precision so we can expect convergence of the resulting approximation. The reason why we may want to use biquadratic elements is that we can generally achieve higher accuracy than just the bilinear elements.

For the 9-node biquadratic elements start by analyzing at the original *elemstiff.m* file. Based on Lecture 13, there would be 9 nodes instead of 4 that would follow the ordering

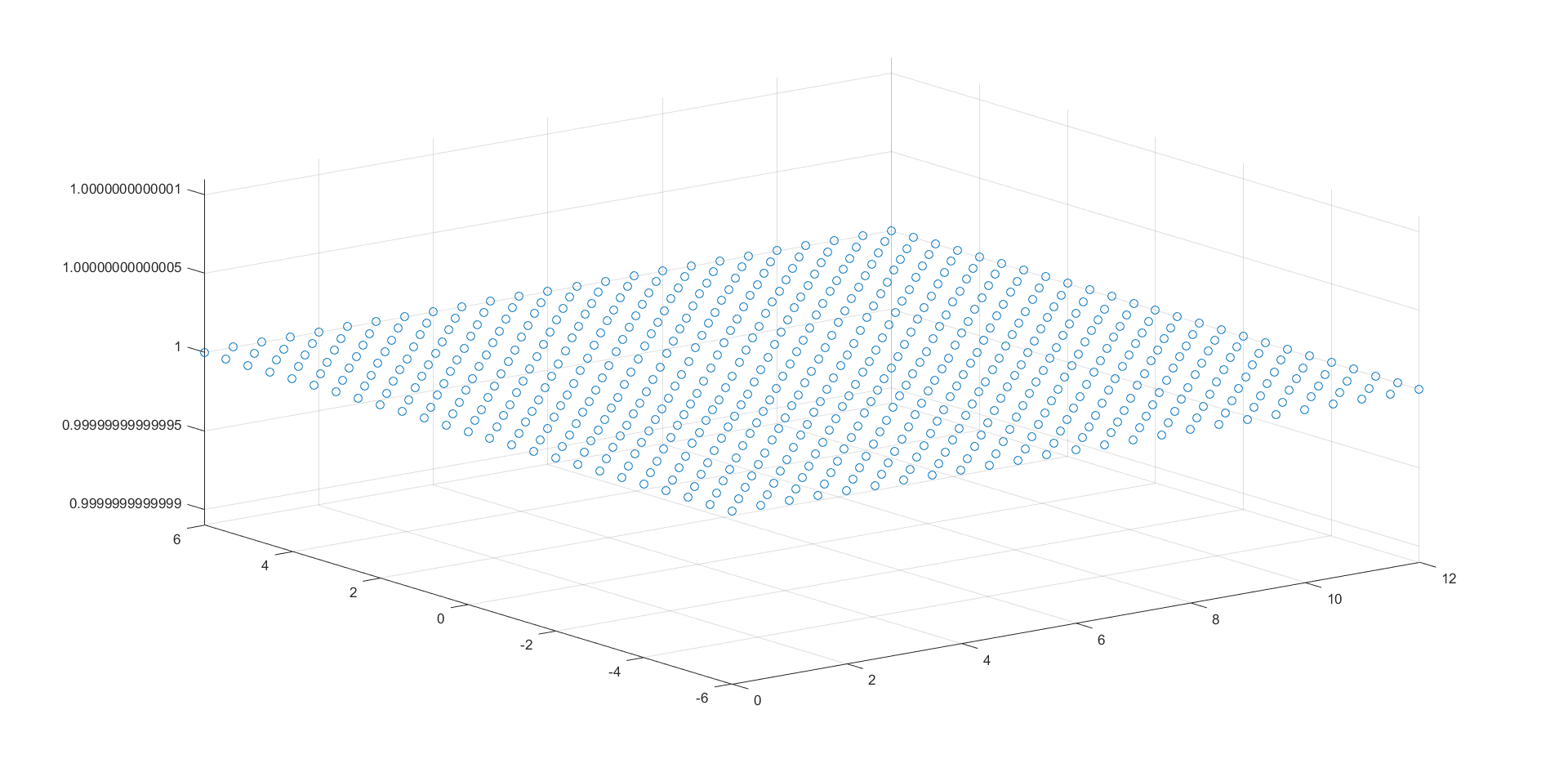


so begin by writing down the 9 shape functions for each node of the element in terms of ξ and η. Then replace **NJpsi** with the row vector of derivatives of the shape functions with respect to ξ in order and do the same for **NJeta** for derivatives with respect to η. Now these become 1x9 matrices so the next step is to match the dimensions of other matrix quantities. **ke** is initialized to be 18x18 and **BJ** to be 2x18 since there are 9 stacked sets of 2x2 **B** matrices. **BJ** would then have to load in the proper values from the 2x9 matrix **NJdxy**. We can still use 2-point Gaussian quadrature since it gives us forth-order accuracy. Finally, the new coordinates of the element nodes would have to be loaded from **node** so that **xe** and **ye** have the correct values, so next we move on to *mesh2d.m*, where **node** is generated.

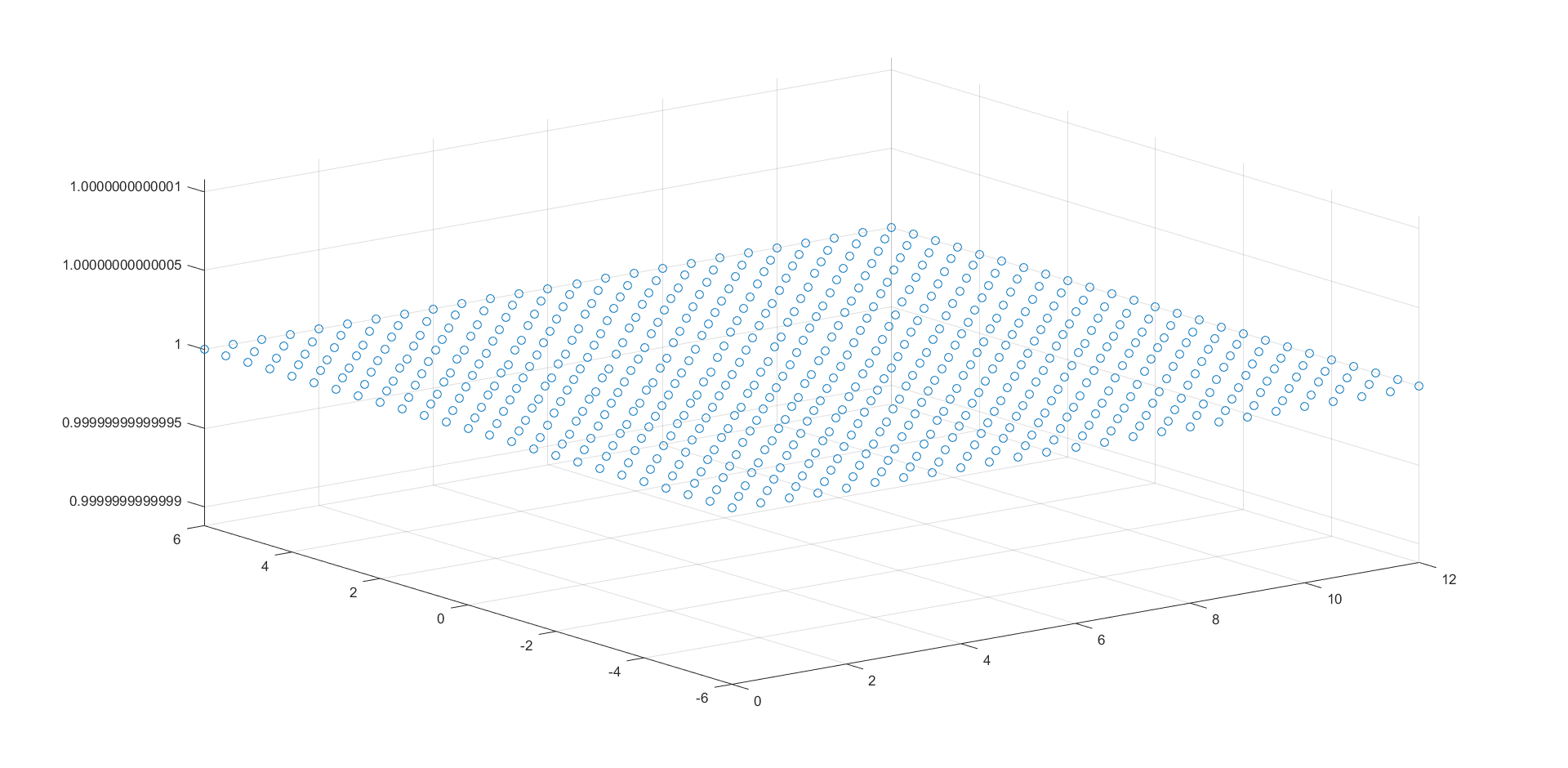
Since we are now using biquadratic elements, now *n* elements in a dimension corresponds to having *2\*n+1* nodes so we update *numnod* and other occurrences of looping over nodes and interpolating nodal positions to reflect this. **nodet** will now have 9 rows with a numbering format similar to the original bilinear nodes. Lastly, *nlink* in the main *heat2d.m* is set to 9.

**Example problems**

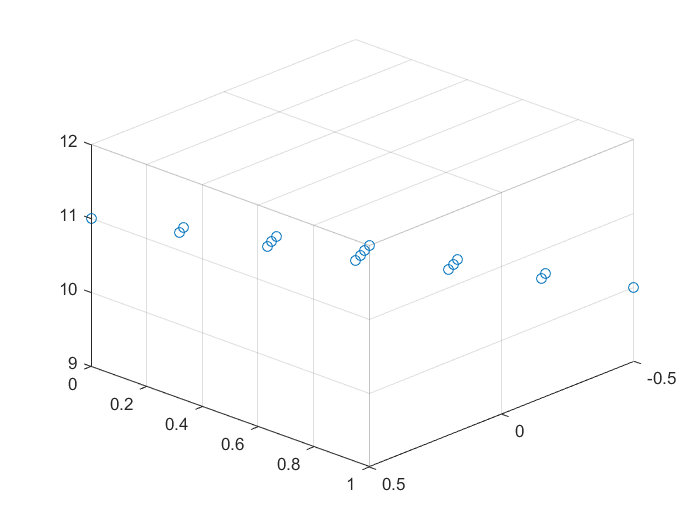
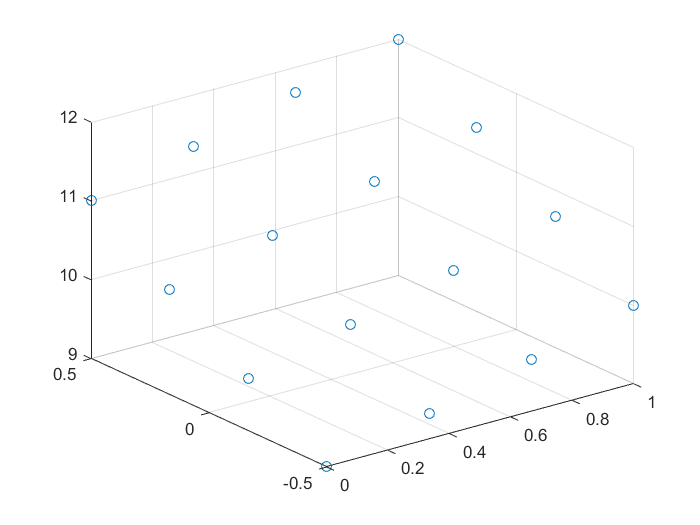
Length=12, Height=12, 25 nodes in x and y directions, u=1 on Γ*g* = All 4 edges, Γ*h* = ∅, Bilinear elements:



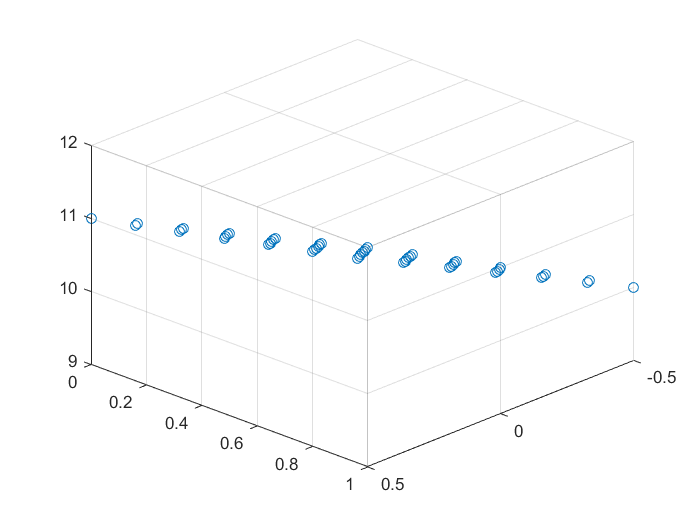
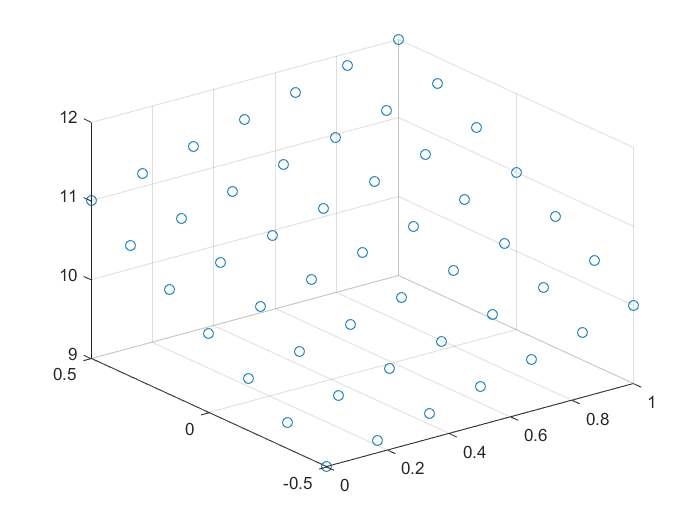
Length=12, Height=12, 25 nodes in x and y directions, u=1 on Γ*g* = All 4 edges, Γ*h* = ∅, Biquadratic elements:

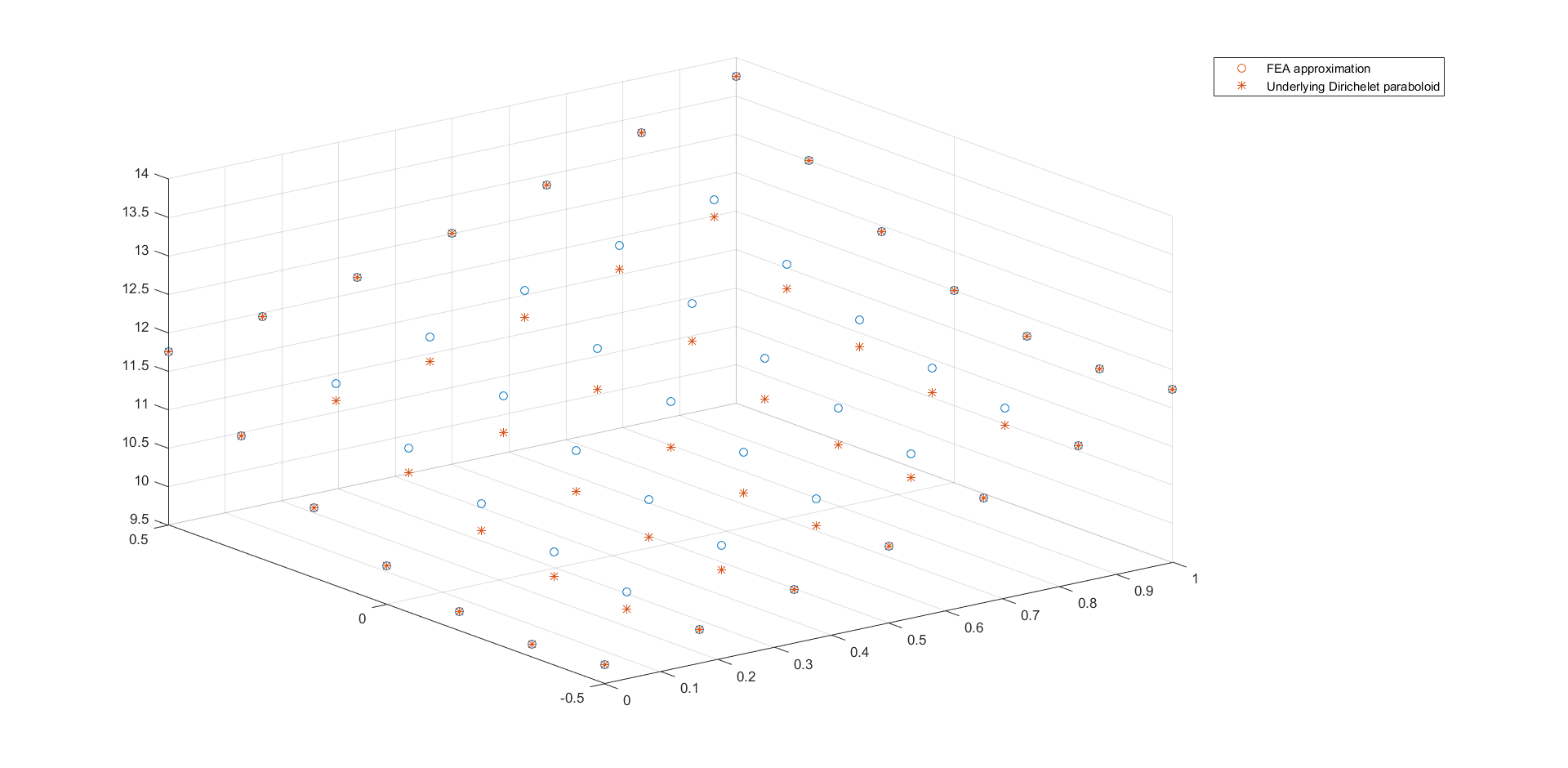
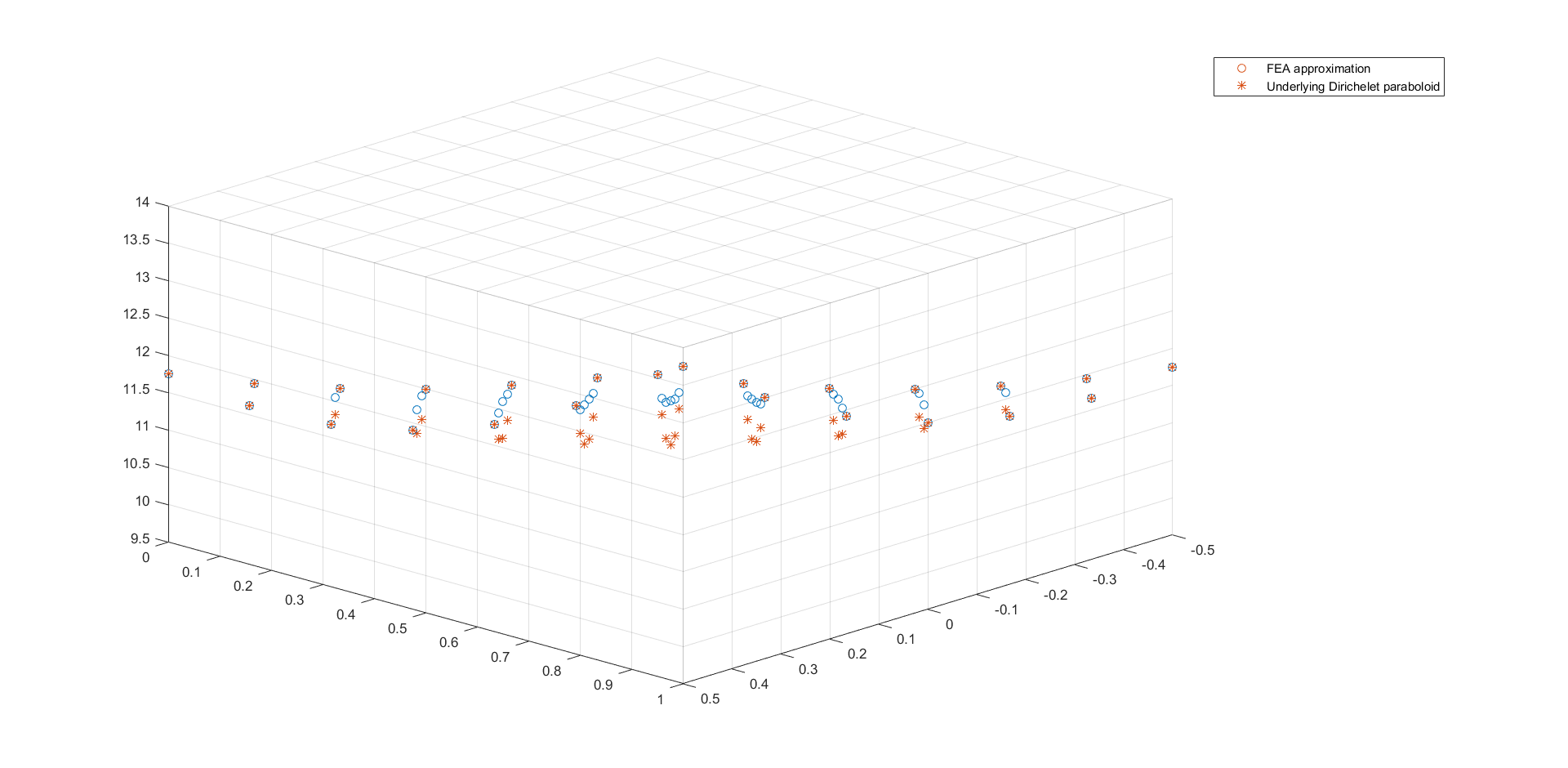


Length=1, Height=1, 3 nodes in x and y directions (patch test), u= 10+x+2y on Γ*g* = All 4 edges, Γ*h* = ∅, Bilinear elements:

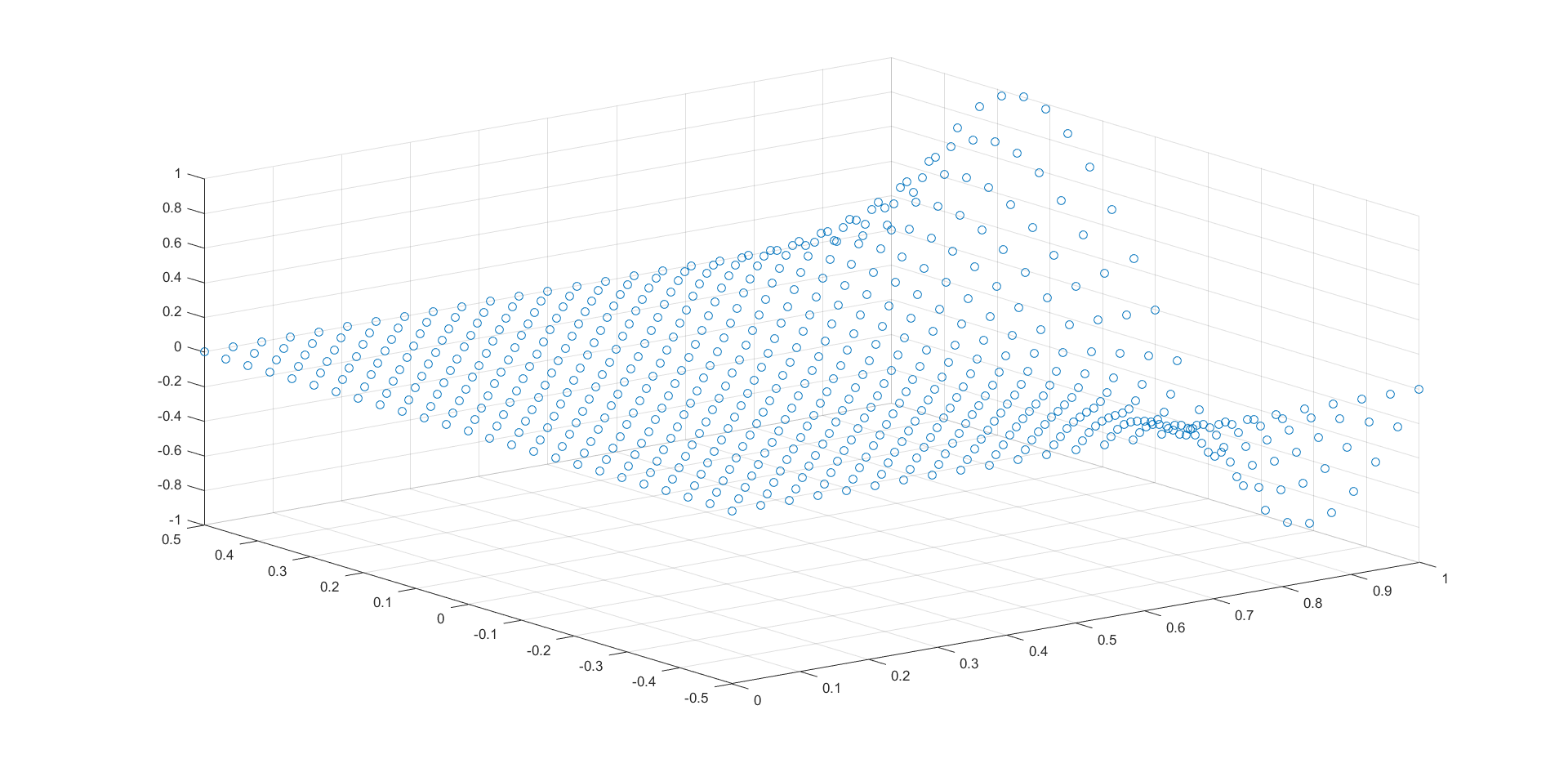


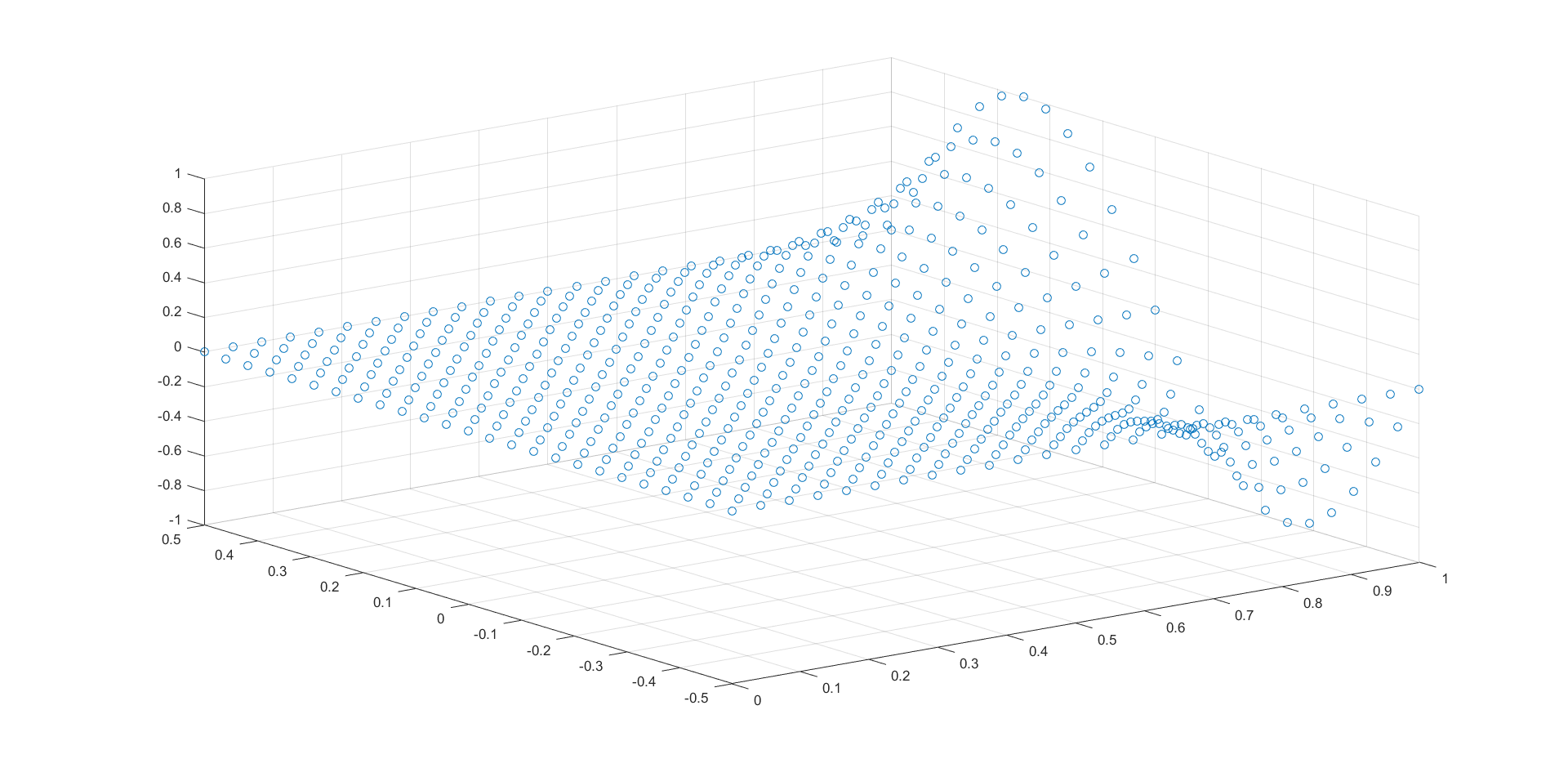
Length=1, Height=1, 3 nodes in x and y directions (patch test), u= 10+x+2y on Γ*g* = All 4 edges, Γ*h* = ∅, Biquadratic elements:



Length=1, Height=1, 3 nodes in x and y directions (patch test), u= 10+x+x2+2y-3y2 on Γ*g* = All 4 edges, Γ*h* = ∅, Biquadratic elements:

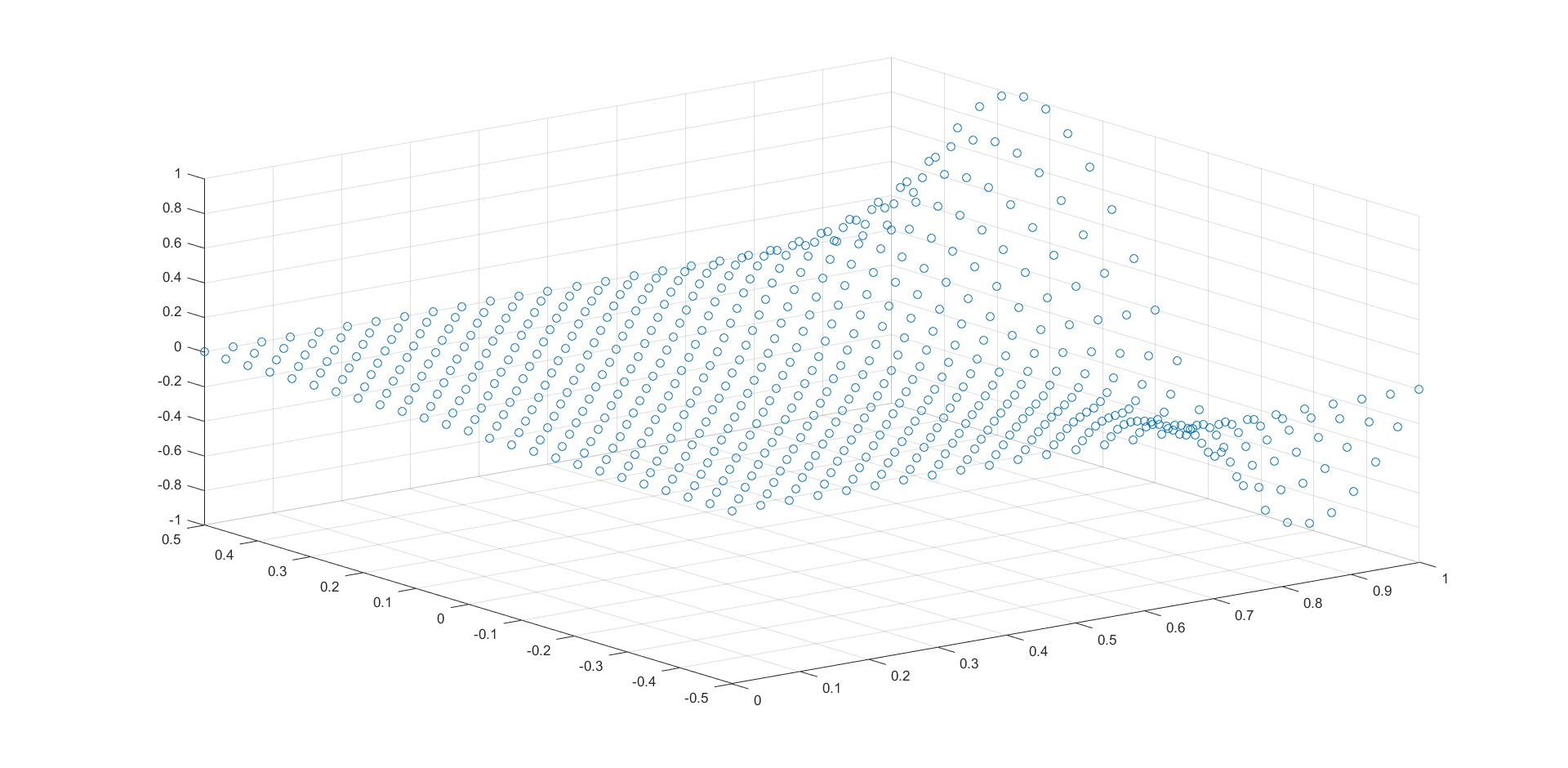
Length=1, Height=1, 25 nodes in x and y directions, u=0 on x=0, y=-0.5, y=0.5 and u=sin(2πy/1), Γ*h* = ∅, Analytical solution:



Length=1, Height=1, 25 nodes in x and y directions, u=0 on x=0, y=-0.5, y=0.5 and u=sin(2πy/1), Γ*h* = ∅, Bilinear elements:

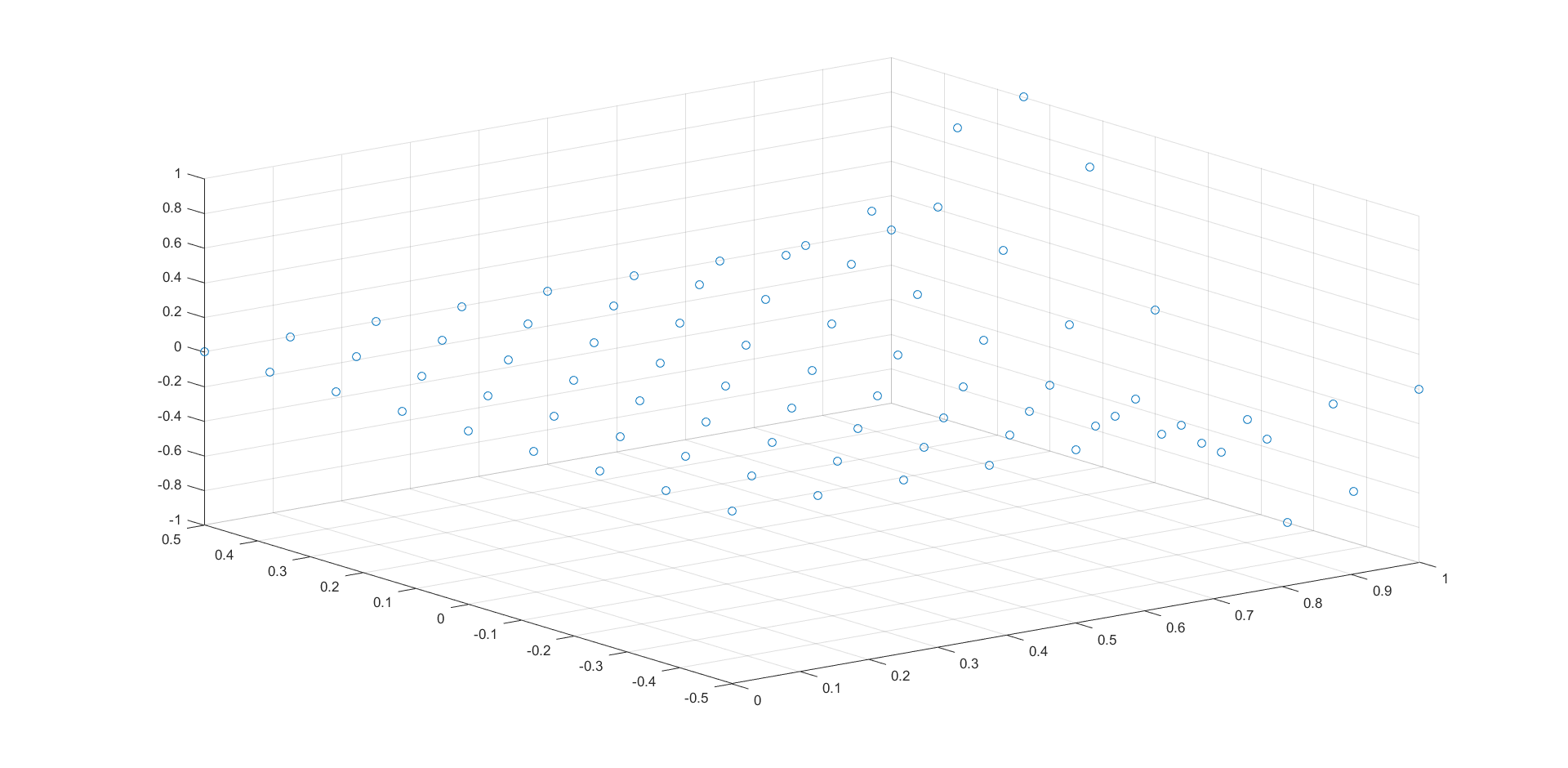
L2-norm of error over each node = 0.194

Length=1, Height=1, 25 nodes in x and y directions, u=0 on x=0, y=-0.5, y=0.5 and u=sin(2πy/1), Γ*h* = ∅, Biquadratic elements:



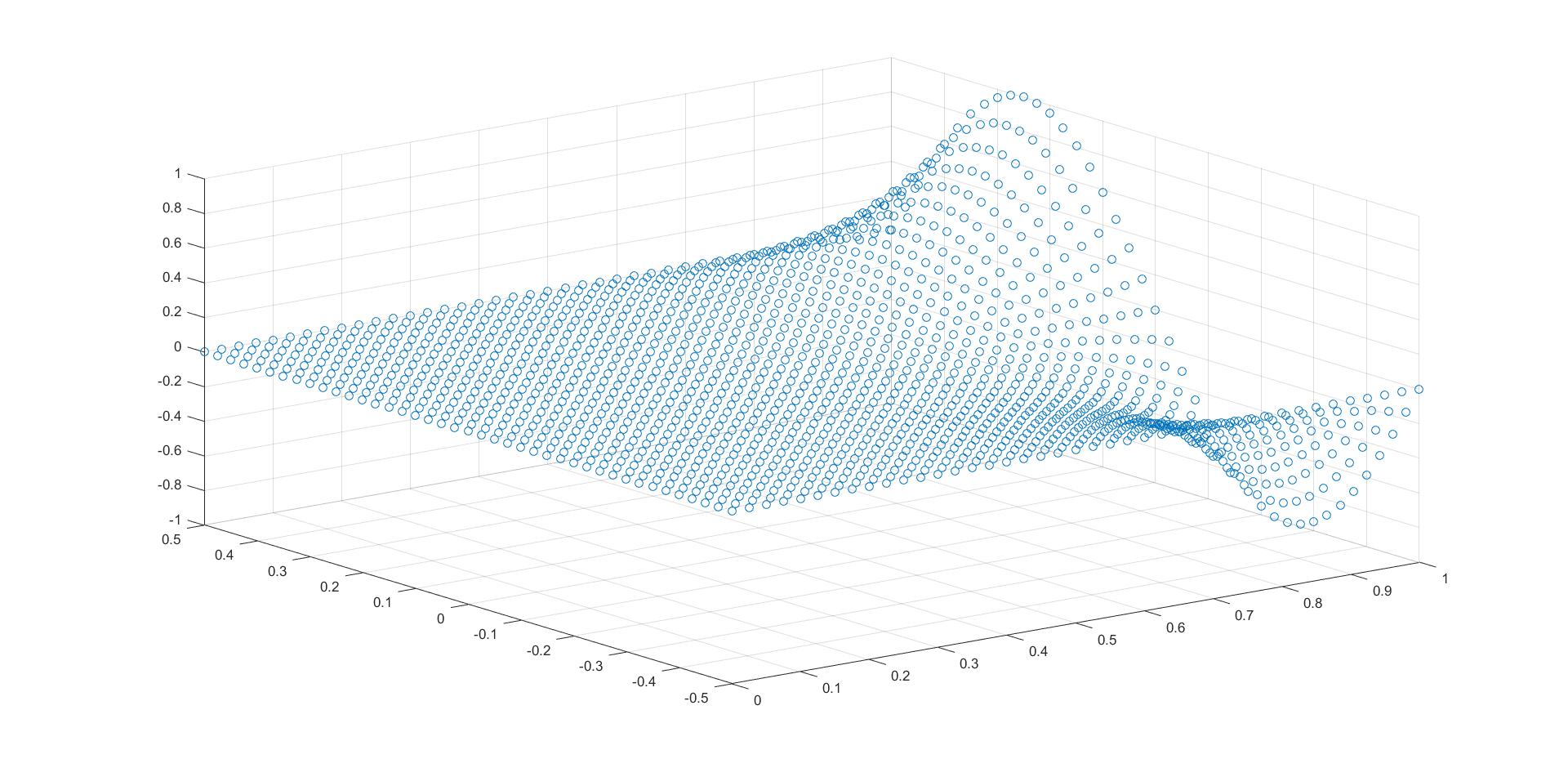
L2-norm of error over each node = 6.9115\*10-4

Length=1, Height=1, 9 nodes in x and y directions, u=0 on x=0, y=-0.5, y=0.5 and u=sin(2πy/1), Γ*h* = ∅, Biquadratic elements:



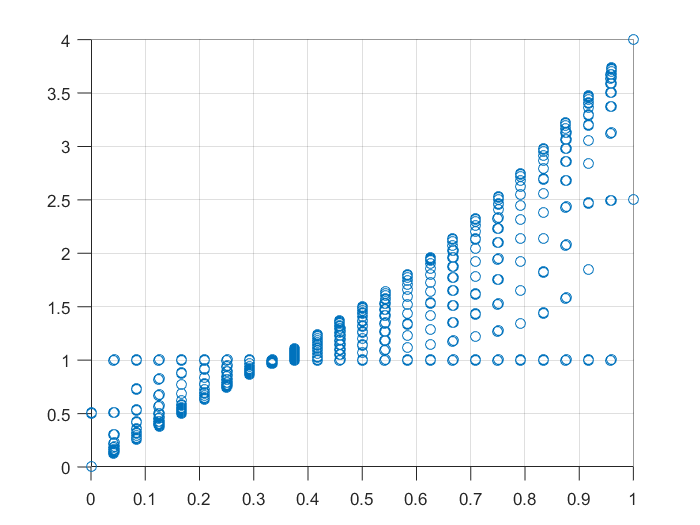
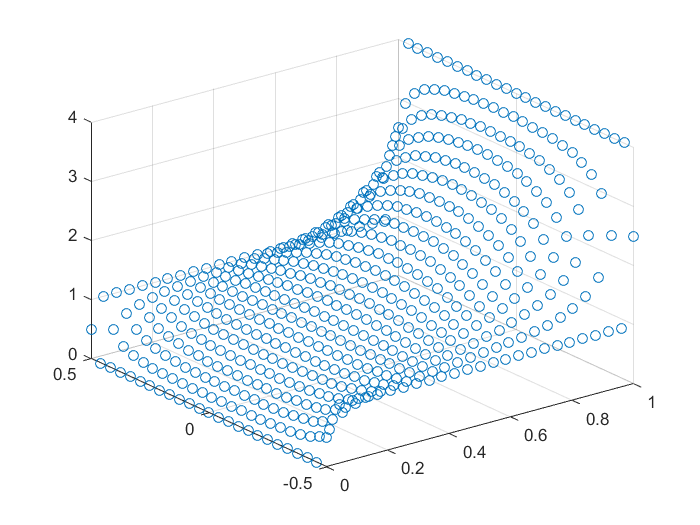
L2-norm of error over each node = 0.0168

Length=1, Height=1, 41 nodes in x and y directions, u=0 on x=0, y=-0.5, y=0.5 and u=sin(2πy/1), Γ*h* = ∅, Biquadratic elements:

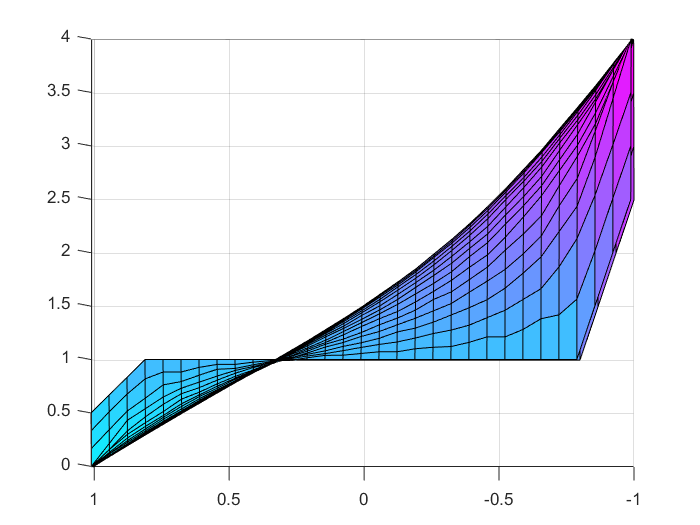
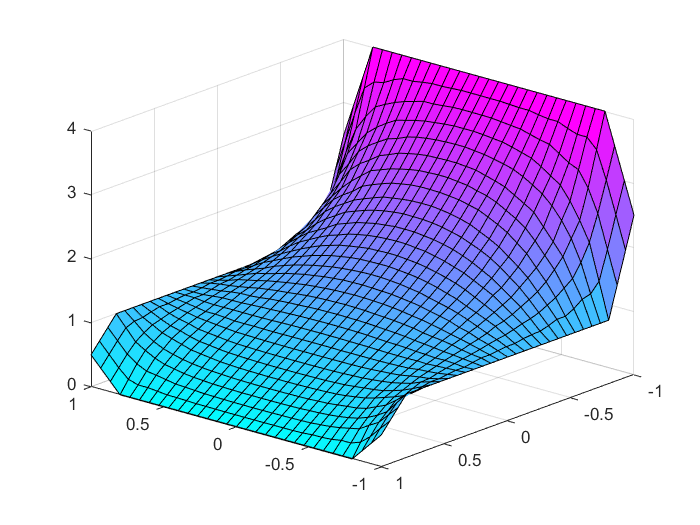


L2-norm of error over each node = 1.5087\*10-4

Length=1, Height=1, 25 nodes in x and y directions, u=0 on x=0, u=1 on y=-height/2 and height/2, u=4 on x=length (with interpolated corners), Γ*h* = ∅, Biquadratic elements:



Length=1, Height=1, u=0 on x=0, u=1 on y=-height/2 and height/2, u=4 on x=length (with interpolated corners), Γ*h* = ∅, MATLAB Parabolic PDE solver (based on <http://www.mathworks.com/help/pde/examples/inhomogeneous-heat-equation-on-a-square-domain.html>):



Length=20, Height=10, 49 nodes in x and 25 nodes in y directions, u=10-y2/20 on x=20 and u=1+sin(x) on y=-5, *-qini*=0 on x=0 and y=5, Biquadratic elements:

