Exercice 4

2.

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\begin{aligned}
D(n) &= 2D\left(\frac{n}{5}\right) + n && \langle \text{ Définition de } D(n), n \in \{3i | i \in \mathbb{N}^*\} \rangle \\
&= 2\left(2D\left(\frac{n}{5^2}\right) + \frac{n}{5}\right) + n && \left\langle \text{ Substitution } D\left(\frac{n}{5}\right) = 2D\left(\frac{n}{5^2}\right) + \frac{n}{5} \right\rangle \\
&= 2^2 D\left(\frac{n}{5^2}\right) + \frac{2n}{5} + n && \langle \text{ arithmétique } \rangle \\
&= 2^2\left(2D\left(\frac{n}{5^3}\right) + \frac{2}{5^2}\right) + \frac{2n}{5} + n && \left\langle \text{ Substitution } D\left(\frac{n}{5^2}\right) = 2D\left(\frac{n}{5^3}\right) + \frac{n}{5^2} \right\rangle \\
&= 2^3 D\left(\frac{n}{5^3}\right) + \frac{2^2 n}{5^2} + \frac{2n}{5} + n && \langle \text{ arithmétique } \rangle \\
&= 2^3 D\left(\frac{n}{5^3}\right) + n\left(\frac{2}{5}\right)^2 + n\left(\frac{2}{5}\right)^1 + n\left(\frac{2}{5}\right)^0 && \langle \text{ arithmétique } \rangle \\
&\vdots && \left\langle \forall i = 1, 2, 3, \ldots, \text{ On divise par 5 jusqu'à } \frac{n}{5^i} = 1. \right\rangle \\
&\vdots && \left\langle i \geq 1 \iff \frac{n}{5^i} \geq 1 \iff n \geq 5^i \text{ et, } \log_5 n = \log_5 5^i = i \right\rangle \\
&= 2^i D\left(\frac{n}{5^i}\right) + n\left(\frac{2}{5}\right)^{i-1} + \cdots + n\left(\frac{2}{5}\right)^1 + n\left(\frac{2}{5}\right)^0 && \langle \text{ Donc, } i = 1, \ldots, \log_5 n. \rangle \\
&= 2^{\log_5(n)} D(1) + n\left(\frac{2}{5}\right)^{i-1} + \cdots + n\left(\frac{2}{5}\right)^1 + n\left(\frac{2}{5}\right)^0 && \langle \text{ avec } [i := \log_5 n] \rangle \\
&= 2^{\log_5(n)} D(1) + n\sum_{j=0}^{i-1}\left(\frac{2}{5}\right)^j && \langle \text{ notation sigma } \rangle \\
&= 2^{\log_5(n)} D(1) + n\left(\frac{1 - \left(\frac{2}{5}\right)^i}{1 - \frac{2}{5}}\right) && \langle \text{ Théorème 2.4.7 } \rangle \\
&= 2^{\log_5(n)} D(1) + n\left(\frac{1 - \left(\frac{2}{5}\right)^i}{\frac{3}{5}}\right) && \langle \text{ arithmétique } \rangle \\
&= 2^{\log_5(n)} D(1) + \frac{5}{3}n\left(1 - \left(\frac{2}{5}\right)^{\log_5(n)}\right) && \langle \text{ arithmétique } \rangle \\
&= D(1)n^{\log_5(2)} + \frac{5}{3}n\left(1 - \left(\frac{n^{\log_5(2)}}{n}\right)\right) && \left\langle a^{\log_b(x)} = x^{\log_b(a)} \text{ dans notre cas } 2^{\log_5(n)} = n^{\log_5(2)} \right\rangle \\
&= 7n^{\log_5(2)} + \frac{5}{3}n\left(\frac{n - \left(n^{\log_5(2)}\right)}{n}\right) && \langle \text{ arithmétique et substitution de la condition initiale } \rangle \\
&= 7n^{\log_5(2)} + \frac{5}{3}\left(n - n^{\log_5(2)}\right) && \langle \text{ arithmétique } \rangle \\
&= \frac{16}{3}n^{\log_5(2)} + \frac{5}{3}n && \langle \text{ arithmétique } \rangle
\end{aligned}
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3.