Home Work 1
Com S 311
Due: Sep 7, 4PM

Late Submission Due: Sep 8, 4PM (25% penalty)

This HW is based on the expected background knowledge necessary for Com S 311.
Outcomes. Be comfortable with proof techniques and use them to prove program correctness.
There are 5 problems and each problem is worth 50 points.

1. Use mathematical induction to prove the following statements

   (a)
   $$\forall n \geq 1, \sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

   (b)
   $$\forall n \geq 4, 2^n < n!$$

   Your proof must use mathematical induction; otherwise you will receive zero credit.

2. Refer to the definition of Full Binary Tree from the notes. For a Full Binary Tree T, we use $n(T), h(T), i(T)$ and $\ell(T)$ to refer to number of nodes, height, number of internal nodes (non-leaf nodes) and number of leaves respectively. Note that the height of a tree with single node is 1 (not zero). Using structural induction, prove the following:

   (a) For every Full Binary Tree T, $n(T) \geq h(T)$.
   (b) For every Full Binary Tree T, $i(T) \geq h(T) - 1$
   (c) For every Full Binary Tree T, $\ell(T) = (n(T) + 1)/2$.

   Your proof must use structural induction; otherwise you will receive zero credit.

3. Let a be an array of size n indexed by $0, 1, \cdots n-1$. Consider the following code (inner loop of Bubble Sort)

```
for i in the range [0, n-2]
    if (a[i] > a[i+1])
         swap (a[i], a[i+1]); //swap the values of a[i] and a[i+1]
```

   Show the following using induction :" At the start of ith iteration of the loop, a[i] is the maximum among $a[0], a[1], \cdots a[i]$.

4. Let a be an array of size n indexed by $0, 1, \cdots n-1$. Now consider Bubble sort pseudo code.

```
for j in the range [0, n-1]
   for i in the range [0, n-j-2]
      if a[i] > a[i+1]
          swap(a[i], a[i+1])
```

Use mathematical induction to show: At the start of jth iteration of outer loop the following conditions hold:

- $a[n-j] \leq a[n-j+1] \leq a[n-1]$
- $a[n-j], a[n-j+1], \cdots a[n-1]$ are the j largest elements of the array.

5. A student proved the following statement using mathematical induction:

"The population of every city in US is the same"

The students proof proceeds as follows.

The above statement is equivalent to: "For every n, if S is any set of n cities in US, then the population of all cities from S is the same".

Base Case. Size of S is 1. There is only one city in S. Thus population of every city in S is the same.

Inductive Hypothesis. Let S be a set of m cities and assume that population of every city in S is the same.

Induction Step. We will prove that if S is a set of $m+1$ cities, then the population of every city in S is the same. Let $S = \{c_1, c_2, \cdots, c_m, c_{m+1}\}$. Consider the following two subsets of S:

$$S_1 = \{c_1, \cdots, c_m\},$$
$$S_2 = \{c_2, \cdots, c_{m+1}\}$$

Note that both S_1 and S_2 are of size m. Thus by induction hypothesis:

$$population(c_1) = polulation(c_2) = \cdots = population(c_m),$$

$$population(c_2) = polulation(c_2) = \cdots = population(c_{m+1}).$$

Since c_2 appears in both sets, we have

$$population(c_1) = polulation(c_2) = \cdots = population(c_m) = population(c_{m+1}).$$

Thus for every set of $m+1$ cities, their population is the same. By induction principle, every city in US has the same population.

Of course, the above statement is wrong. Identify the problem in the above proof. To receive credit, you must explain why the above proof is correct—identify **the exact place** where the proof fails.

GUIDE LINES:

- Graded home works will be returned in the recitations. So please write your recitation number, time and TA name.
- It is important to know whether your really know! For each problem, if you write the statement "I do not know how to solve this problem" (and nothing else), you will receive 20% credit for that problem. If you do write a solution, then your grade could be anywhere between 0% to 100%. To receive this 20% credit, you must explicitly state that you do not know how to solve the problem.
- You must work on the homework problems on your own. You should write the final solutions alone, without consulting any one. Your writing should demonstrate that you understand the proofs completely.
- When proofs are required, you should make them both clear and rigorous. Do not hand waive.
- If you hand writing is not legible, then your homework will not be graded.
- Any concerns about grading should be made within one week of returning the homework.
- There is physical dropbox in the first Floor of Atanasoff (labeled 311), place your solutions in the box before the due date.