1)

a) induction on upper bound of summation

Basis: when $n = 1, \sum_{i=1}^{n} i^3 = 1, (\sum_{i=1}^{n} i)^2 = 1$
Inductive Hypothesis: $\forall n \geq 1, \sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$
Inductive Step:

$$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n+1} i\right)^2$$
$$(n+1)^3 + \sum_{i=1}^{n} i^3 = (n+1+\sum_{i=1}^{n} i)(n+1+\sum_{i=1}^{n} i)$$
$$(n+1)^3 + \left(\sum_{i=1}^{n} i\right)^2 = (n+1)^2 + 2(n+1)\sum_{i=1}^{n} i + \left(\sum_{i=1}^{n} i\right)^2$$
$$(n+1)^3 = (n+1)^2 + 2(n+1)(\frac{n(n+1)}{2})$$
$$(n+1)(n+1)^2 = (n+1)^2 + n(n+1)^2$$
$$n+1 = 1+n$$

proving $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2, \forall n \geq 1$ through mathematical induction

b) induction on the term n

Basis: when $n = 4, 2^n = 16, n! = 24$
Inductive Hypothesis: $\forall n \geq 4, 2^n < n!$
Inductive Step:

$$2^{n+1} < (n+1)!$$
$$2 * 2^n < n! * (n+1)$$
$$2 * 2^n < n! * (n+1)$$

since 2^n is always less than $n!$ by the Inductive Hypothesis the equation only depends on the following

$$2 \leq (n+1)$$
$$1 \leq n$$

since n was defined as being greater than equal to 4, n will always be greater than 1 so this proves the claim by induction

2)

a)

Claim: For every Full b-tree $T, n(T) \geq h(T)$.
Base Case: The b-tree with only one node has $n(T) = 1$ and $h(T) = 1$ so the claim holds.
Inductive Hypothesis: assume X, Y are two B-Trees such that $n(X) \geq h(X), n(X) \geq n(Y), h(X) \geq h(Y)$
Inductive Step: by the definition of a full b-tree we can create a new full b-tree Z by adding a new node, and making X the right child of this node and Y the left child of this node. this would make $n(Z) = 1+n(X)+n(Y), h(Z) = 1+h(X)$, and since $n(x) \geq h(X), n(Y) \geq h(X)$ this would make $n(Z) > h(Z)$ this proves the claim by structural induction.

b)

Claim: For every Full Binary Tree $T, i(T) \geq h(T) - 1$
Base Case: consider the B-Tree T with one node $i(T) = 0, h(T) = 1, i(T) = h(T) - 1$
Inductive Hypothesis: assume X, Y are two B-Trees such that $i(X) \geq h(X) - 1, i(Y) \geq h(Y) - 1, h(X) \geq h(Y)$
Inductive Step: a new full b-tree Z can be constructed by adding a new node and make X the right child of this node, and Y the left child of this node. $h(Z) = h(X) + 1, i(Z) = i(X) + i(Y) + 1$, since $i(X) \geq h(X) - 1$ $i(X) \geq H(X) - 1$
this proves the claim by structural induction

c)

Claim: For every Full Binary Tree $T, \ell(T) = (n(T) + 1)/2$
Base Case: consider the B-Tree T with one node $\ell(T) = 1, \frac{n(T)+1}{2} = 1$
Inductive Hypothesis: assume X, Y are two B-Trees such that $\ell(X) = (n(X) + 1)/2, \ell(Y) = (n(Y) + 1)/2$
Inductive Step: a b-tree Z can be constructed with one new node that has X as the right child and Y as the left child.

$$\begin{aligned}
\ell(Z) &= \frac{n(Z)+1}{2} \\
\ell(X) + \ell(Y) &= \frac{n(X)+n(Y)+2}{2} \\
\ell(X) + \ell(Y) &= \frac{n(X)+1}{2} + \frac{n(Y)+1}{2} \\
\ell(X) + \ell(Y) &= \ell(X) + \ell(Y)
\end{aligned}$$

proving the claim by structural induction