Home Work 1
Com S 435/535

When asked to calculate probabilities of events, your solutions must be mathematically rigorous and you must write all the steps of your derivation. You will not receive full credit for merely stating the probability and/or providing an intuitive or high level argument as why your solution is correct.

1. For each of the following experiments, write the sample space and the probability function. (40 Points)

   (a) Toss a biased coin (probability of head 2/3) 4 times.

   (b) Roll a fair dice, if the outcome is an even number, then toss a fair coin. Otherwise throw another roll another fair dice.

   (c) Toss a balance dice twice. Output the following 3-tuple $\langle a_1, a_2, a_1 + a_2 \rangle$: Where a_1 is the outcome of the first dice, and a_2 is the outcome of the second dice.

   (d) Let $S = \{1, 2, \cdots, n\}$. Uniformly at random pick x from $\{1, 2, \cdots, n\}$. Uniformly at random pick y from $\{1, 2, \cdots, n\} - \{x\}$. Output $\langle x, y \rangle$.

2. Consider the experiment from 1d. Let m be a number between 1 and n. Calculate the probability that one of the numbers picked is m. (20 Points)

3. Suppose that you have n white balls, numbered 1 thru n. Consider an experiment that repeats the following process L times: Uniformly at random pick an integer x from $\{1, 2, \cdots, n\}$. If $x \neq n$, then color the balls numbered x and $x + 1$ black. If x equals n, then color the balls numbered x and 1 black. (60 Points)

   - Calculate the probability that Ball number 1 is black.

   - Use union bound, to obtain an upper bound on the probability that there exists a ball that is not colored black.

4. Let $U = \{1, 2, \cdots, n\}$. We pick a random subset of S of U as follows. For each $i \in U$, place it in S with probability 2/3. Suppose that we randomly picked two sets S_1 and then S_2. Calculate the exact probability that $S_1 \subseteq S_2$. (40 Points)

5. You have two coins, a fair coin and a biased coin (Probability of H is 2/3). Consider the following experiment. Throw the fair coin. If outcome is H, then $b_1 = 1$ else $b_1 = 0$. Throw the biased coin, if the outcome is H, then $b_2 = 1$, else $b_2 = 0$. Output the following three bits: $b_1, b_2, b_1 \oplus b_2$.

(a) E_1: First bit is 1

(b) E_2: Second bit is 0

(c) E_3: Third bit is 1

Answer the following questions.   (30 Points)

(a) What are probabilities of E_1, E_2, and E_3.

(b) Are E_1 and E_2 independent? Prove your answer.

(c) Are E_1 and E_3 independent? Prove your answer.

Guidelines:

- You are allowed to discuss with your classmates, while doing your homework. However, I strongly suggest that you think about the problems on your own before discussing.

- Definition of *classmates*: Students who are taking CS 435/535 in Fall 18.

- However, You should write the final solutions alone, without consulting your classmates. Your writing should demonstrate that you understand the proofs completely. If we suspect that you wrote the proofs without understanding, we may ask you to explain the proofs in person. In such scenarios, failure to explain proofs will be taken as evidence of *academic dishonesty*.

- For each problem, you should acknowledge the students with whom you discussed. This will not affect your grade. Failure to acknowledge is considered *academic dishonesty*, and it will affect your grade.

- Any student found guilty of academic dishonesty will receive "F" in the. course.

- When proofs are required, make them both clear and rigorous. Do not hand wave. Even when proofs are not required, you should justify your answers and explain your work.

- Late homeworks are not accepted.