

1)

a) induction on upper bound of summation

Basis: when $n = 1$, $\sum_{i=1}^n i^3 = 1$, $(\sum_{i=1}^n i)^2 = 1$ **Inductive Hypothesis:** $\forall n \geq 1$, $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$ **Inductive Step:**

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n+1} i \right)^2 \\
(n+1)^3 + \sum_{i=1}^n i^3 &= (n+1 + \sum_{i=1}^n i)(n+1 + \sum_{i=1}^n i) \\
(n+1)^3 + \left(\sum_{i=1}^n i \right)^2 &= (n+1)^2 + 2(n+1) \sum_{i=1}^n i + \left(\sum_{i=1}^n i \right)^2 \\
(n+1)^3 &= (n+1)^2 + 2(n+1) \left(\frac{n(n+1)}{2} \right) \\
(n+1)(n+1)^2 &= (n+1)^2 + n(n+1)^2 \\
n+1 &= 1+n
\end{aligned}$$

proving $\sum_{i=1}^n i^3 = (\sum_{i=1}^n i)^2$, $\forall n \geq 1$ through mathematical inductionb) induction on the term n **Basis:** when $n = 4$, $2^n = 16$, $n! = 24$ **Inductive Hypothesis:** $\forall n \geq 4$, $2^n < n!$ **Inductive Step:**

$$\begin{aligned}
2^{n+1} &< (n+1)! \\
2 * 2^n &< n! * (n+1) \\
2 * 2^n &< n! * (n+1)
\end{aligned}$$

since 2^n is always less than $n!$ by the Inductive Hypothesis the equation only depends on the following

$$\begin{aligned}
2 &\leq (n+1) \\
1 &\leq n
\end{aligned}$$

since n was defined as being greater than equal to 4, n will always be greater than 1 so this proves the claim by induction

2)

a)

Claim: For every Full b-tree T , $n(T) \geq h(T)$.

Base Case: The b-tree with only one node has $n(T) = 1$ and $h(T) = 1$ so the claim holds.

Inductive Hypothesis: assume X, Y are two B-Trees such that $n(X) \geq h(X)$, $n(Y) \geq h(Y)$, $h(X) \geq h(Y)$

Inductive Step: by the definition of a full b-tree we can create a new full b-tree Z by adding a new node, and making X the right child of this node and Y the left child of this node. this would make $n(Z) = 1 + n(X) + n(Y)$, $h(Z) = 1 + h(X)$, and since $n(X) \geq h(X)$, $n(Y) \geq h(Y)$ this would make $n(Z) \geq h(Z)$ this proves the claim by structural induction.

b)

Claim: For every Full Binary Tree T , $i(T) \geq h(T) - 1$

Base Case: consider the B-Tree T with one node $i(T) = 0$, $h(T) = 1$, $i(T) = h(T) - 1$

Inductive Hypothesis: assume X, Y are two B-Trees such that $i(X) \geq h(X) - 1$, $i(Y) \geq h(Y) - 1$, $h(X) \geq h(Y)$

Inductive Step: a new full b-tree Z can be constructed by adding a new node and make X the right child of this node, and Y the left child of this node. $h(Z) = h(X) + 1$, $i(Z) = i(X) + i(Y) + 1$, since $i(X) \geq h(X) - 1$ $i(X) \geq h(X) - 1$

this proves the claim by structural induction

c)

Claim: For every Full Binary Tree T , $\ell(T) = (n(T) + 1)/2$

Base Case: consider the B-Tree T with one node $\ell(T) = 1$, $\frac{n(T)+1}{2} = 1$

Inductive Hypothesis: assume X, Y are two B-Trees such that $\ell(X) = (n(X) + 1)/2$, $\ell(Y) = (n(Y) + 1)/2$, $n(X) \geq n(Y)$

Inductive Step: