NCERT 10.3.3.3.6

EE24BTECH11032- John Bobby

Question:
Five years from now on, the age of Jacob will be 3 times that of his son. Five years ago, Jacob's age was 7 times that of his son. What are their present ages?
Solution:
Given information can be interpreted as,

$$x - 3y = 10 \tag{1}$$
$$x - 7y = -30 \tag{2}$$

Simplifying and using matrix notation,

$$\begin{pmatrix} 1 & -3 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -30 \end{pmatrix} \tag{3}$$

The matrix A can be decomposed into:

$$A = L \cdot U, \tag{4}$$

where:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{5}$$
$$U = \begin{pmatrix} 1 & -3 \\ 0 & -4 \end{pmatrix}. \tag{6}$$

Factorization of LU:
Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:
1. Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
2. For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{7}$$

3. For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{8}$$

The system $A\mathbf{x} = \mathbf{b}$ is transformed into $L \cdot U \cdot \mathbf{x} = \mathbf{b}$. Let $\mathbf{y}$ satisfy $L\mathbf{y} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 10 \\ -30 \end{pmatrix}. \tag{9}$$

Using forward substitution:

$$y_1 = 10 \tag{10}$$
$$y_1 + y_2 = -30 \tag{11}$$
$$y_2 = -40 \tag{12}$$

Thus:

$$\mathbf{y} = \begin{pmatrix} 10 \\ -40 \end{pmatrix}. \tag{13}$$

Next, solve $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 1 & -3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -40 \end{pmatrix}. \tag{14}$$

Using back substitution:

$$-4y = -40 \tag{15}$$

$$y = 10 \tag{16}$$

$$x - 3y = 10 \tag{17}$$

$$x = 40 \tag{18}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 10 \end{pmatrix} \tag{19}$$

is the solution of the given system of equations.

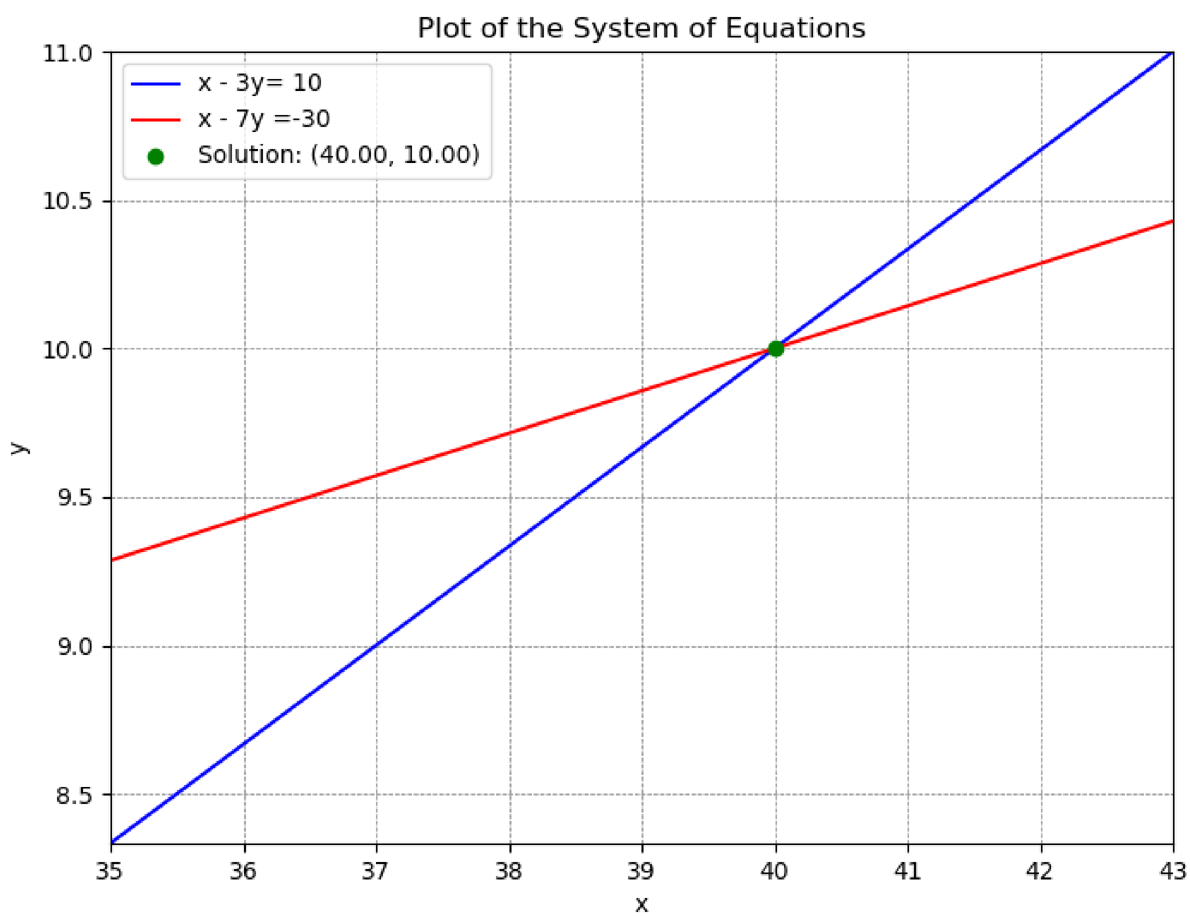

Fig. 1. Solution to set of linear equations