NCERT 10.4.EX1

EE24BTECH11032- John Bobby

Alex and Sam together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they have is 124. Find how many marbles they had at the beginning. Solution:

Let the number of marbles Alex have at the beginning= x
The number of marbles Sam have = $45 - x$
After losing 5 marbles,
Number of marbles alex has $x - 5$
Sam now has $45 - x - 5 = 50 - x$
Product of marbles = 124
$(x - 5)(50 - x) = 124$
On rearranging,
$x^2 - 45x + 324 = 0$

I. Quadratic Formula

Consider an equation,

$$ax^2 + bx + c = 0 \tag{1}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{2}$$

$$x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \tag{3}$$

$$\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right) = 0 \tag{4}$$

$$\left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a} \tag{5}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

which is the quadratic formula.
For the quadratic equation $x^2 - 45x + 324 = 0$,
$a = 1, b = -45, c = 324$
On solving we get the roots to be $x = 9$ and $x = 36$

II. Eigen Values-Method

Consider the equation, (2). It can be rearranged as

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \tag{7}$$

$$\lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} = 0 \tag{8}$$

$$-\lambda\left(-\lambda - \frac{b}{a}\right) - (-1)\frac{c}{a} = 0 \tag{9}$$

This can be considered equivalent to the determinant of the matrix,

$$\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & \frac{-b}{a} - \lambda \end{pmatrix} \tag{10}$$

Clearly, it can be seen that the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{-c}{a} & \frac{-b}{a} \end{pmatrix} \tag{11}$$

are the roots of the required quadratic equation. This matrix, (11) is called the **Companion matrix** (**C**). For the given question,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \tag{12}$$

QR ALGORITHM : Eigenvalues of the companion matrix can be found using QR algorithm. Using the Gram-Schmidt orthogonalization, the matrix **C** can be factorized into

$$\mathbf{C} = \mathbf{QR} \tag{13}$$

where,

$$\mathbf{Q} = Orthonormalmatrix \tag{14}$$

$$\mathbf{R} = Uppertriangularmatrix \tag{15}$$

This process can be continues as

$$\mathbf{C_k} = \mathbf{Q_k R_k} \tag{16}$$

$$\mathbf{C_{k+1}} = \mathbf{R_k Q_k} \tag{17}$$

As $k \to \infty$, the diagonal elements of $\mathbf{Q_k}$ converge to the eigenvalues of the matrix. It can be seen that eigenvalues are 1 and 2. **Newton-Raphson method** :
We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{18}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 45x_n + 324}{2x_n - 45} \tag{19}$$

Iterating and updating the value of x_n, we can obtain the roots of the quadratic equation.
The roots found using this method taking the initial guesses as 5 and 25 are 9 and 36 respectively.

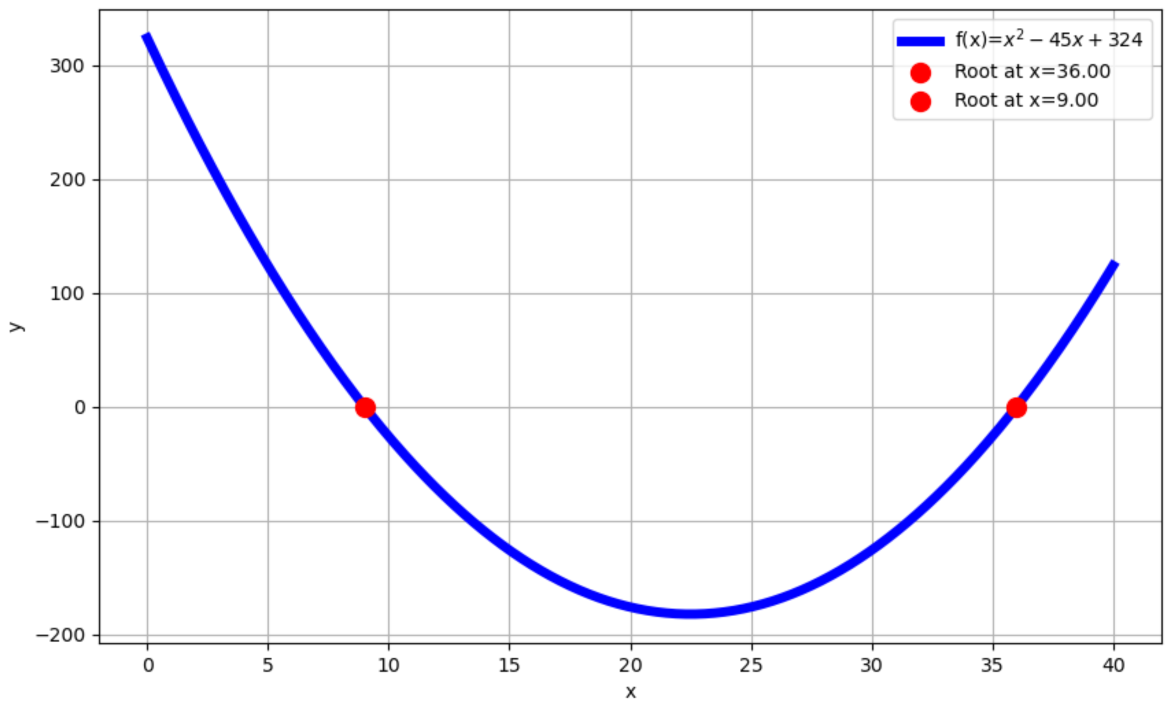

Fig. 1. Plot using Newton-Raphson method