NCERT 9.4.3

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Question: Find the solution for the differential equation $\frac{dy}{dx} + y = 1$

I. Mathematical Approach

$$\frac{dy}{dx} = 1 - y$$

On rearranging the terms,

$$\frac{dy}{1-y} = dx$$
$$\int \frac{dy}{1-y} = \int dx$$
$$-\log|y-1| + c_1 = x + c_2$$

On simplification

$$-\log|y-1| = x + c$$
$$|y-1| = \pm e^{-x}$$
$$y = 1 \pm e^{-x}$$

For the numerical approach we are assuming that the function passes through origin
Thus is the function y is,

$$y = 1 - e^{-x}$$

II. Laplace Transform

Take the Laplace transform of both sides,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = \mathcal{L}\{1 - y\}$$

Using the Laplace transform properties,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = sY(s) - y(0), \quad \mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{y(t)\} = Y(s)$$

On rearranging,

$$\begin{aligned} sY(s) - y(0) &= \frac{1}{s} - Y(s) \\ sY(s) + Y(s) &= \frac{1}{s} + y(0) \\ Y(s)(s+1) &= \frac{1}{s} + y(0) \\ Y(s) &= \frac{1}{s(s+1)} + \frac{y(0)}{s+1} \end{aligned}$$

Taking the inverse laplace transform

$$\begin{aligned} \mathcal{L}^{-1}\left(Y(s)\right) &= y(x) \\ \mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) &= 1 - e^{-x} \\ \mathcal{L}^{-1}\left(\frac{y(0)}{s+1}\right) &= y(0)\, e^{-x} \\ y(x) &= 1 - e^{-x} + y(0)\, e^{-x} \end{aligned}$$

As $y(0) = 0$ The solution is

$$y(x) = 1 - e^{-x}$$

III. Z Transform

$$\begin{aligned} \frac{dy}{dx} &= \frac{y_{n+1} - y_n}{h} \\ y_{n+1} &= y_n + \frac{dy}{dx}h \\ y_{n+1} &= y_n + (1 - y_n)\, h \end{aligned}$$

Applying Z Transform on both sides

$$\begin{aligned} \mathcal{Z}\{y(k+1)\} &= z.Y(z) - z.y(0) \\ \mathcal{Z}\{y(k)\} &= Y(z) \\ \mathcal{Z}\{1 - y(k)\} &= \frac{1}{1 - z^{-1}} - Y(z) \\ zY(z) - zy(0) &= Y(z) + h\left(\frac{1}{1 - z^{-1}} - Y(z)\right) \end{aligned}$$

Solving for $Y(z)$,

$$Y(z) = \frac{zy(0) + h}{z - 1 + h} = \frac{h}{z - 1 + h} \qquad (y(0) = 0)$$

Applying Z Inverse,

$$y(x) = h(1-h)^x u[x]$$

IV. Numerical Approach

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{h\to 0} \frac{y_{n+1} - y_n}{h}$$

If h is sufficiently small,

$$\frac{dy}{dx} = \frac{y_{n+1} - y_n}{h}$$
$$y_{n+1} = y_n + \frac{dy}{dx} h$$
$$y_{n+1} = y_n + (1 - y_n) h$$

Thus the value of y_{n+1} can be predicted if we know the value of y_n
For this question

1) The interval $[0, 5]$ is divided into 51 equal parts each of width 0.1units
2) On starting from a known point (x, y) the value for $y(x + 0.1)$ is calculated. This procedure is repeated until the value of x reaches 5.
3) Knowing all the y values for the equally spaced x values in interval, the solution plot can be plotted.

The plot is given below