NCERT 9.4.3

EE24BTECH11032- John Bobby

Question: Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$,$x = 4$,$y = 0$,$y = 4$ into 3 equal parts.
Solution:

I. Theoretical Method

The variables used in $y^2 = 4x$ are given below

Variable	Description	values
V	Quadratic form of the matrix	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
u	Linear coefficient vector	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
f	constant term	0

The variables used in $x^2 = 4y$ are given below

Variable	Description	values
V	Quadratic form of the matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
u	Linear coefficient vector	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
f	constant term	0

The point of intersection of the line with the parabolas is

$$x_i = h + k_i m \tag{1}$$

where, k_i is a constant and is calculated as follows:

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh + u)\right]^2 - g(h)\left(m^\top V m\right)}\right). \tag{2}$$

Substituting the input parameters into k_i,
For the line $x = 0$, $\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{m} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
For the function $y^2 = 4x$ we get $k = 0$ Thus $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
For the function $x^2 = 4y$ we get $k = 0$ Thus $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Similarly, on repeating the same process for the line $x = 4$ to each of the functions, we get the intersection points as $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Area between $y^2 = 4x$ and $x^2 = 4y$ $=\int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4}dx = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$
Area between $x^2 = 4y$ and $x = 0$ and $x = 4$$=\int_0^4 2\sqrt{x}dx = \frac{16}{3}$
Area between $y^2 = 4x$ and $y = 0$ and $y = 4$ $=\int_0^4 \frac{y^2}{4}dy = \frac{16}{3}$
We can see that the 3 areas are equal.

II. Trapezoidal Method

Using trapezoidal rule

$$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

Applying trapezoid rule for all values of x between 0 and 2π
where h is the step size

$$A = \frac{1}{2}h\left(y(x_1)+y(x_0)\right)+\frac{1}{2}h\left(y(x_2)+y(x_1)\right)+\cdots+\frac{1}{2}h\left(y(x_n)+y(x_{n-1})\right)$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A(x_n+h) = A(x_n) + \frac{1}{2}h\left(y(x_n+h)+y(x_n)\right) \tag{3}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1}+y_n\right) \tag{4}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n+hy'_n)+y_n\right) \tag{5}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n+hy'_n\right) \tag{6}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{7}$$

$$x_{n+1} = x_n + h \tag{8}$$

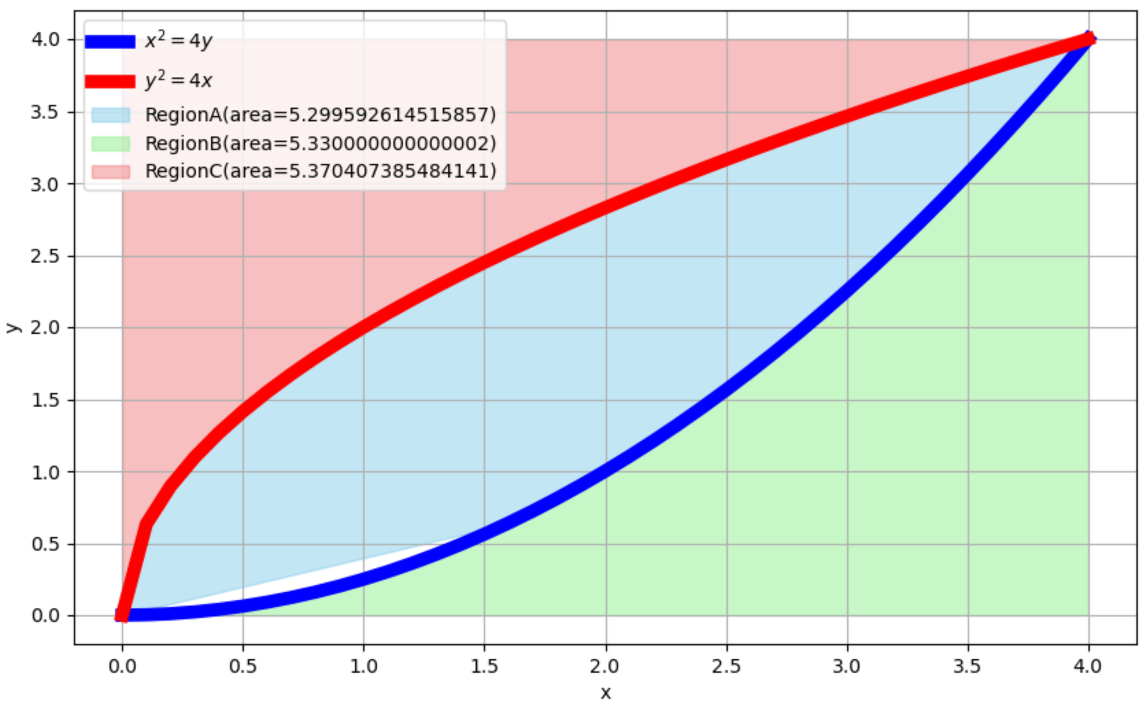
For $y^2 = 4x$, $y_n = 2\sqrt{x_n}$ and $y'_n = \frac{1}{\sqrt{x_n}}$
For $x^2 = 4y$, $y_n = \frac{x^2}{4}$ and $y'_n = \frac{x}{2}$
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{9}$$

$$A_{n+1} = A_n + h(\cos x_n) + \frac{1}{2}h^2(-\sin x_n) \tag{10}$$

$$x_{n+1} = x_n + h \tag{11}$$

Iterating till we reach $x_n = 4$ will return the required area for each function.


$x^2 = 4y$
$y^2 = 4x$
RegionA(area=5.299592614515857)
RegionB(area=5.330000000000002)
RegionC(area=5.370407385484141)
x
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0