NCERT 9.4.3

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Question: Find the solution for the differential equation $\frac{dy}{dx} + y = 1$

I. Mathematical Approach

$$\frac{dy}{dx} = 1 - y$$

On rearranging the terms,

$$\begin{aligned} \frac{dy}{1-y} &= dx \\ \int \frac{dy}{1-y} &= \int dx \\ -\log|y-1| + c_1 &= x + c_2 \end{aligned}$$

On simplification

$$\begin{aligned} -\log|y-1| &= x + c \\ |y-1| &= \pm e^{-x} \\ y &= 1 \pm e^{-x} \end{aligned}$$

For the numerical approach we are assuming that the function passes through origin
Thus is the function y is,

$$y = 1 - e^{-x}$$

II. Laplace Transform

Take the Laplace transform of both sides,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = \mathcal{L}\{1 - y\}$$

Using the Laplace transform properties,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = sY(s) - y(0), \quad \mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{y(t)\} = Y(s)$$

On rearranging,

$$
\begin{aligned}
sY(s) - y(0) &= \frac{1}{s} - Y(s) \\
sY(s) + Y(s) &= \frac{1}{s} + y(0) \\
Y(s)(s+1) &= \frac{1}{s} + y(0) \\
Y(s) &= \frac{1}{s(s+1)} + \frac{y(0)}{s+1}
\end{aligned}
$$

Taking the inverse laplace transform

$$
\begin{aligned}
\mathcal{L}^{-1}\left(Y\left(s\right)\right) &= y\left(x\right) \\
\mathcal{L}^{-1}\left(\frac{1}{s\left(s+1\right)}\right) &= 1 - e^{-x} \\
\mathcal{L}^{-1}\left(\frac{y\left(0\right)}{s+1}\right) &= y\left(0\right)e^{-x} \\
y\left(x\right) &= 1 - e^{-x} + y\left(0\right)e^{-x}
\end{aligned}
$$

As $y\left(0\right) = 0$ The solution is

$$
y\left(x\right) = 1 - e^{-x}
$$

III. Numerical Approach

$$
\frac{dy}{dx} = \lim_{h \to 0} \frac{y\left(x+h\right) - y\left(x\right)}{h} = \lim_{h \to 0} \frac{y_{n+1} - y_n}{h}
$$

If h is sufficiently small,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y_{n+1} - y_n}{h} \\
y_{n+1} &= y_n + \frac{dy}{dx}h \\
y_{n+1} &= y_n + \left(1 - y_n\right)h
\end{aligned}
$$

Thus the value of y_{n+1} can be predicted if we know the value of y_n
For this question

1) The interval $[0, 5]$ is divided into 51 equal parts each of width 0.1units
2) On starting from a known point (x, y) the value for $y\left(x + 0.1\right)$ is calculated. This procedure is repeated until the value of x reaches 5.
3) Knowing all the y values for the equally spaced x values in interval, the solution plot can be plotted.

The plot is given below