# NCERT 9.4.3

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**Question:** Find the solution for the differential equation $\frac{dy}{dx} + y = 1$ using trapezoidal rule.
**Solution:**

## I. Trapezoidal Method

From the question

$$\frac{dy}{dx} = 1 - y \tag{1}$$

Let

$$f(x, y) = 1 - y \tag{2}$$
$$y(0) = 0 \tag{3}$$

From Forward Euler method:

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n) \tag{4}$$

From Backward Euler method:

$$\frac{y_{n+1} - y_n}{h} = f(x_{n+1}, y_{n+1}) \tag{5}$$

On adding both equation (4) and (5), We get the Trapezoidal Method

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] \tag{6}$$
$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right] \tag{7}$$
$$y_{n+1} = y_n + \frac{h}{2}\left[1 - y_n + 1 - y_{n+1}\right] = y_n + \frac{h}{2}\left[2 - y_n - y_{n+1}\right] \tag{8}$$

On rearranging, we get the difference equation

$$y_{n+1} = \frac{2-h}{2+h}y_n + \frac{2h}{2+h} \tag{9}$$
$$x_{n+1} = x_n + h \tag{10}$$

## II. Laplace Transform

Take the Laplace transform of both sides,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = \mathcal{L}\{1 - y\} \tag{11}$$

Using the Laplace transform properties,

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = sY(s) - y(0), \quad \mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{y(t)\} = Y(s) \tag{12}$$

On rearranging,

$$sY(s) - y(0) = \frac{1}{s} - Y(s) \tag{13}$$

$$sY(s) + Y(s) = \frac{1}{s} + y(0) \tag{14}$$

$$Y(s)(s+1) = \frac{1}{s} + y(0) \tag{15}$$

$$Y(s) = \frac{1}{s(s+1)} + \frac{y(0)}{s+1} \tag{16}$$

Taking the inverse laplace transform

$$\mathcal{L}^{-1}\left(Y(s)\right) = y(x) \tag{17}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+1)}\right) = 1 - e^{-x} \tag{18}$$

$$\mathcal{L}^{-1}\left(\frac{y(0)}{s+1}\right) = y(0)\, e^{-x} \tag{19}$$

$$y(x) = 1 - e^{-x} + y(0)\, e^{-x} \tag{20}$$

As $y(0) = 0$ The solution is

$$y(x) = 1 - e^{-x} \tag{21}$$

## III. Bilinear Transform

Let the laplace transform of $f(x,y) = 1 - y$ be $X(s)$

$$X(s) = \mathcal{L}\left(f(x,y)\right) \tag{22}$$

Applying laplace transform on both sides of equation

$$sY(s) = X(s) \tag{23}$$

Let $H(s)$ be defined such that

$$H(s) = \frac{Y(s)}{X(s)} \tag{24}$$

$$H(s) = 1/s \tag{25}$$

Applying bilinear transform which converts s-domain to z-domain

$$s = \frac{2}{h}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{26}$$

$$H(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) \tag{27}$$

$$Y(z) = \frac{h}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right)X(z) \tag{28}$$

$$\tag{29}$$

On rearranging,

$$zY(z) - Y(z) = \frac{h}{2}\left(zX(z) + X(z)\right) \tag{30}$$

Applying Z inverse transform,

$$y_{n+1} - y_n = \frac{h}{2}\left(f\left(x_{n+1}, y_{n+1}\right) + f\left(x_n, y_n\right)\right) \tag{31}$$

$$y_{n+1} - y_n = \frac{h}{2}\left(1 - y_{n+1} + 1 - y_n\right) \tag{32}$$

$$y_{n+1} - y_n = \frac{h}{2}\left(2 - y_{n+1} - y_n\right) \tag{33}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(2 - y_{n+1} - y_n\right) \tag{34}$$

Equation (34) is the same difference equation obtained in equation (8)