Handbook of Mathematics

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0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	8	10
9	1001	9	11
10	1010	A	12
11	1011	B	13
12	1100	C	14
13	1101	D	15
14	1110	E	16
15	1111	F	17

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$

Polar coordinates
$x = r\cos\varphi$
$y = r\sin\varphi$
$\varphi \in (-\pi, \pi]$
$\det J = r$

Cylindrical coordinates
$x = r_{xy}\cos\varphi$
$y = r_{xy}\sin\varphi$
$z = z$
$\det J = r_{xy}$

Spherical coordinates
$x = r\sin\theta\,\cos\varphi$
$y = r\sin\theta\,\sin\varphi$
$z = r\cos\theta$
$\varphi \in (-\pi, \pi],\ \theta \in [0, \pi]$
$\det J = r^2\sin\theta$

$\theta = \beta - \pi/2$
$\beta \in [-\pi/2, \pi/2]$
$\cos\theta = \sin\beta$
$\sin\theta = \cos\beta$

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1 Basics and foundations

1.1 Complex numbers

1.1.1 Operations

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{z_2\overline{z}_2} = \frac{z_1\overline{z}_2}{|z_2|^2}, \tag{1.1}$$
$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}. \tag{1.2}$$

1.1.2 Absolut value

For all $z_1, z_2 \in \mathbb{C}$:

$$|z_1 z_2| = |z_1|\,|z_2|, \tag{1.3}$$
$$z_2 \neq 0 \implies \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{1.4}$$
$$z\,\overline{z} = |z|^2. \tag{1.5}$$

1.1.3 Conjugation

For all $z_1, z_2 \in \mathbb{C}$:

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2, \qquad \overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2, \tag{1.6}$$
$$\overline{z_1 z_2} = \overline{z}_1\,\overline{z}_2, \qquad z_2 \neq 0 \implies \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}, \tag{1.7}$$
$$\overline{\overline{z}} = z, \qquad |\overline{z}| = |z|, \qquad z\,\overline{z} = |z|^2, \tag{1.8}$$
$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \operatorname{Im}(z) = \frac{z - \overline{z}}{2\mathrm{i}}, \tag{1.9}$$
$$\overline{\cos(z)} = \cos(\overline{z}), \qquad \overline{\sin(z)} = \sin(\overline{z}), \tag{1.10}$$
$$\overline{\exp(z)} = \exp(\overline{z}). \tag{1.11}$$

1.2 Logic

1.2.1 Propositional logic

1.2.1.1 Boolean algebra

Laws of distributivity:

$$A \lor (B \land C) = (A \lor B) \land (A \lor C), \tag{1.12}$$
$$A \land (B \lor C) = (A \land B) \lor (A \land C). \tag{1.13}$$

1.2.1.2 Functions in two arguments

There are 16 boolean functions in two arguments.

AB	value
00	a
01	b
10	c
11	d

No.	dcba	Function	Name
0	0000	0	Contradiction
1	0001	$\overline{A \lor B}$	NOR
2	0010	$\overline{B \Rightarrow A}$	
3	0011	$\overline{A}$	
4	0100	$\overline{A \Rightarrow B}$	
5	0101	$\overline{B}$	
6	0110	$A \oplus B$	Contravalence
7	0111	$\overline{A \land B}$	NAND
8	1000	$A \land B$	Conjunction
9	1001	$A \Leftrightarrow B$	Equivalence
10	1010	B	Projection
11	1011	$A \Rightarrow B$	Implication
12	1100	A	Projection
13	1101	$B \Rightarrow A$	Implication
14	1110	$A \lor B$	Disjunction
15	1111	1	Tautology

1.2.2 Predicate logic

1.2.2.1 Basic laws

Negation (De Morgan's laws):

$$\overline{\forall x[P(x)]} \iff \exists x[\overline{P(x)}], \tag{1.14}$$
$$\overline{\exists x[P(x)]} \iff \forall x[\overline{P(x)}]. \tag{1.15}$$

Generalized laws of distributivity:

$$P \lor \forall x[Q(x)] \iff \forall x[P \lor Q(x)], \tag{1.16}$$
$$P \land \exists x[Q(x)] \iff \exists x[P \land Q(x)]. \tag{1.17}$$

Generalized laws of idempotence:

$$\begin{aligned}\exists x{\in}M\,[P] &\iff (M \neq \{\}) \land P\\ &\iff \begin{cases} P & \text{if } M \neq \{\},\\ 0 & \text{if } M = \{\}.\end{cases}\end{aligned} \tag{1.18}$$
$$\begin{aligned}\forall x{\in}M\,[P] &\iff (M = \{\}) \lor P\\ &\iff \begin{cases} P & \text{if } M \neq \{\},\\ 1 & \text{if } M = \{\}.\end{cases}\end{aligned} \tag{1.19}$$

Equivalences:

$$\forall x \forall y[P(x,y)] \iff \forall y \forall x[P(x,y)], \tag{1.20}$$
$$\exists x \exists y[P(x,y)] \iff \exists y \exists x[P(x,y)], \tag{1.21}$$
$$\forall x[P(x) \land Q(x)] \iff \forall x[P(x)] \land \forall x[Q(x)], \tag{1.22}$$
$$\exists x[P(x)] \lor Q(x)] \iff \forall x[P(x)] \lor \forall x[Q(x)], \tag{1.23}$$
$$\forall x[P(x) \Rightarrow Q] \iff \exists x[P(x)] \Rightarrow Q, \tag{1.24}$$
$$\forall x[P \Rightarrow Q(x)] \iff P \Rightarrow \forall x[Q(x)], \tag{1.25}$$
$$\exists x[P(x) \Rightarrow Q(x)] \iff \forall x[P(x)] \Rightarrow \exists x[Q(x)]. \tag{1.26}$$

Implications:

$$\exists x \forall y[P(x,y)] \implies \forall y \exists x[P(x,y)], \tag{1.27}$$
$$\forall x[P(x)] \lor \forall x[Q(x)] \implies \forall x[P(x) \lor Q(x)], \tag{1.28}$$
$$\exists x[P(x) \land Q(x)] \implies \exists x[P(x)] \land \exists x[Q(x)], \tag{1.29}$$
$$\forall x[P(x) \Rightarrow Q(x)] \implies (\forall x[P(x)] \Rightarrow \forall x[Q(x)]), \tag{1.30}$$
$$\forall x[P(x) \Leftrightarrow Q(x)] \implies (\forall x[P(x)] \Leftrightarrow \forall x[Q(x)]). \tag{1.31}$$

Table 1.1: Operations

Name	**Operation**	**Polar form**	**Cartesian form**
Identity	z	$= r\mathrm{e}^{\mathrm{i}\varphi}$	$= a + b\mathrm{i}$
Addition	$z_1 + z_2$		$= (a_1 + a_2) + (b_1 + b_2)\mathrm{i}$
Subtraction	$z_1 - z_2$		$= (a_1 - a_2) + (b_1 - b_2)\mathrm{i}$
Multiplication	$z_1 z_2$	$= r_1 r_2 \mathrm{e}^{\mathrm{i}(\varphi_1 + \varphi_2)}$	$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\mathrm{i}$
Division	$\dfrac{z_1}{z_2}$	$= \dfrac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\varphi_1 - \varphi_2)}$	$= \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\mathrm{i}$
Reciprocal	$\dfrac{1}{z}$	$= \dfrac{1}{r}\mathrm{e}^{-\mathrm{i}\varphi}$	$= \dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2}\mathrm{i}$
Real part	$\mathrm{Re}(z)$	$= \cos\varphi$	$= a$
Imaginary part	$\mathrm{Im}(z)$	$= \sin\varphi$	$= b$
Conjugation	$\overline{z}$	$= r\mathrm{e}^{-\varphi\mathrm{i}}$	$= a - b\mathrm{i}$
Absolut value	$\lvert z \rvert$	$= r$	$= \sqrt{a^2 + b^2}$
Argument	$\arg(z)$	$= \varphi$	$= s(b)\arccos\left(\dfrac{a}{r}\right)$

$$s(b) := \begin{cases} +1 & \text{if } b \ge 0, \\ -1 & \text{if } b < 0 \end{cases}$$

Table 1.2: Boolean algebra

Disjunction	**Conjunction**	
$A \lor A \Leftrightarrow A$	$A \land A \Leftrightarrow A$	Laws of idempotence
$A \lor 0 \Leftrightarrow A$	$A \land 1 \Leftrightarrow A$	Laws of neutrality
$A \lor 1 \Leftrightarrow 1$	$A \land 0 = 0$	Laws of annihilation
$A \lor \overline{A} \Leftrightarrow 1$	$A \land \overline{A} \Leftrightarrow 0$	Laws of complementation
$A \lor B \Leftrightarrow B \lor A$	$A \land B \Leftrightarrow B \land A$	Laws of commutativity
$(A \lor B) \lor C \Leftrightarrow A \lor (B \lor C)$	$(A \land B) \land C \Leftrightarrow A \land (B \land C)$	Laws of associativity
$\overline{A \lor B} \Leftrightarrow \overline{A} \land \overline{B}$	$\overline{A \land B} \Leftrightarrow \overline{A} \lor \overline{B}$	De Morgan's laws
$A \lor (A \land B) \Leftrightarrow A$	$A \land (A \lor B) \Leftrightarrow A$	Laws of absorption

1.2.2.2 Finite sets

Let $M = \{x_1, \ldots, x_n\}$. One has:

$$\forall x{\in}M\,[P(x)] \iff P(x_1) \land \ldots \land P(x_n), \tag{1.32}$$
$$\exists x{\in}M\,[P(x)] \iff P(x_1) \lor \ldots \lor P(x_n). \tag{1.33}$$

1.2.2.3 Restricted quantification

$$\begin{aligned}\forall x{\in}M\,[P(x)] &:\iff \forall x[x \notin M \lor P(x)] \\ &\iff \forall x[x \in M \Rightarrow P(x)],\end{aligned} \tag{1.34}$$
$$\exists x{\in}M\,[P(x)] :\iff \exists x[x \in M \land P(x)], \tag{1.35}$$
$$\forall x{\in}M\backslash N\,[P(x)] \iff \forall x[x \notin N \Rightarrow P(x)]. \tag{1.36}$$

1.2.2.4 Product sets as domains of discourse

$$\forall (x,y)\,[P(x,y)] \iff \forall x \forall y[P(x,y)], \tag{1.37}$$
$$\exists (x,y)\,[P(x,y)] \iff \exists x \exists y[P(x,y)]. \tag{1.38}$$

By analogy:

$$\forall (x,y,z) \iff \forall x \forall y \forall z, \tag{1.39}$$
$$\exists (x,y,z) \iff \exists x \exists y \exists z \tag{1.40}$$

etc.

1.2.2.5 Alternative representation

Let $P\colon G \to \{0,1\}$ and $M \subseteq G$. Let $P(M)$ be the image of M under P. One has

$$\begin{aligned}\forall x{\in}M\,[P(x)] &\iff P(M) = \{1\} \\ &\iff M \subseteq \{x{\in}G \mid P(x)\}\end{aligned} \tag{1.41}$$

and

$$\begin{aligned}\exists x{\in}M\,[P(x)] &\iff \{1\} \subseteq P(M) \\ &\iff M \cap \{x{\in}G \mid P(x)\} \ne \{\}.\end{aligned} \tag{1.42}$$

1.2.2.6 Uniqueness

Quantifier of unique existence:

$$\begin{aligned}&\exists! x\,[P(x)] \\ &:\iff \exists x\,[P(x) \land \forall y\,[P(y) \Rightarrow x = y]] \\ &\iff \exists x\,[P(x)] \land \forall x \forall y[P(x) \land P(y) \Rightarrow x = y].\end{aligned} \tag{1.43}$$

1.3 Set theory

1.3.1 Definitions

Subset relation:

$$A \subseteq B \;:\iff\; \forall x\,[x \in A \implies x \in B]. \tag{1.44}$$

Equality:

$$A = B \;:\iff\; \forall x\,[x \in A \iff x \in B]. \tag{1.45}$$

Union:

$$A \cup B := \{x \mid x \in A \lor x \in B\}. \tag{1.46}$$

Intersection:

$$A \cap B := \{x \mid x \in A \land x \in B\}. \tag{1.47}$$

Difference set:

$$A \setminus B := \{x \mid x \in A \land x \notin B\}. \tag{1.48}$$

Symmetric difference:

$$A \triangle B := \{x \mid x \in A \oplus x \in B\}. \tag{1.49}$$

1.3.2 Boolean algebra

Laws of distributivity:

$$M \cup (A \cap B) = (M \cup A) \cap (M \cup B), \tag{1.50}$$
$$M \cap (A \cup B) = (M \cap A) \cup (M \cap B). \tag{1.51}$$

1.3.3 Subset relation

Decomposition of equality:

$$A = B \iff A \subseteq B \land B \subseteq A. \tag{1.52}$$

Pharaphrasing of subset relations:

$$\begin{aligned} A \subseteq B &\iff A \cap B = A \\ &\iff A \cup B = B \\ &\iff A \setminus B = \{\}. \end{aligned} \tag{1.53}$$

Law of contraposition:

$$A \subseteq B = \overline{B} \subseteq \overline{A}. \tag{1.54}$$

1.3.4 Inductive sets

Set theoretical model of the natural numbers:

$$\begin{aligned} &0 := \{\}, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \\ &3 := \{0, 1, 2\}, \quad \text{usw.} \end{aligned} \tag{1.55}$$

Successor function:

$$x' := x \cup \{x\}. \tag{1.56}$$

Proof by induction: For a predicate $A(n)$ with $n \in \mathbb{N}$ one has:

$$\begin{aligned} &A(n_0) \land \forall n \geq n_0 \, [A(n) \Rightarrow A(n+1)] \\ &\implies \forall n \geq n_0 \, [A(n)]. \end{aligned} \tag{1.57}$$

Table 1.3: Boolean algebra

Union	**Intersection**	
$A \cup A = A$	$A \cap A = A$	Laws of idempotence
$A \cup \{\} = A$	$A \cap G = A$	Laws of neutrality
$A \cup G = G$	$A \cap \{\} = \{\}$	Laws of annihilation
$A \cup \overline{A} = G$	$A \cap \overline{A} = \{\}$	Laws of complementation
$A \cup B = B \cup A$	$A \cap B = B \cap A$	Laws of commutativity
$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$	Laws of associativity
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	Laws of absorption

G: Universe

2 Appendix

2.1 Mathematical constants

1. Archimede's constant
   $\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ldots$
2. Euler's number
   $\mathrm{e} = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ldots$
3. Euler-Mascheroni constant
   $\gamma = 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ldots$
4. Golden ratio, $(1+\sqrt{5})/2$
   $\varphi = 1.61803\ 39887\ 49894\ 84820\ 45868\ 34365\ldots$
5. First Feigenbaum constant
   $\delta = 4.66920\ 16091\ 02990\ 67185\ 32038\ 20466\ldots$
6. Second Feigenbaum constant
   $\alpha = 2.50290\ 78750\ 95892\ 82228\ 39028\ 73218\ldots$

2.2 Physical constants

1. Speed of light in vacuum
   $c = 299\,792\,458\ \mathrm{m/s}$
2. Electric constant
   $\varepsilon_0 = 8.854\,187\,817\,620\,39 \times 10^{-12}\ \mathrm{F/m}$
3. Magnetic constant
   $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}$
4. Elementary charge
   $e = 1.602\,176\,6208(98) \times 10^{-19}\ \mathrm{C}$

2.3 Greek alphabet

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	R	ϱ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Y	υ	Upsilon
I	ι	Iota	Φ	φ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

2.4 Fraktur script

A a	$\mathfrak{A}\ \mathfrak{a}$	O o	$\mathfrak{O}\ \mathfrak{o}$
B b	$\mathfrak{B}\ \mathfrak{b}$	P p	$\mathfrak{P}\ \mathfrak{p}$
C c	$\mathfrak{C}\ \mathfrak{c}$	Q q	$\mathfrak{Q}\ \mathfrak{q}$
D d	$\mathfrak{D}\ \mathfrak{d}$	R r	$\mathfrak{R}\ \mathfrak{r}$
E e	$\mathfrak{E}\ \mathfrak{e}$	S s	$\mathfrak{S}\ \mathfrak{s}$
F f	$\mathfrak{F}\ \mathfrak{f}$	T t	$\mathfrak{T}\ \mathfrak{t}$
G g	$\mathfrak{G}\ \mathfrak{g}$	U u	$\mathfrak{U}\ \mathfrak{u}$
H h	$\mathfrak{H}\ \mathfrak{h}$	V v	$\mathfrak{V}\ \mathfrak{v}$
I i	$\mathfrak{I}\ \mathfrak{i}$	W w	$\mathfrak{W}\ \mathfrak{w}$
J j	$\mathfrak{J}\ \mathfrak{j}$	X x	$\mathfrak{X}\ \mathfrak{x}$
K k	$\mathfrak{K}\ \mathfrak{k}$	Y y	$\mathfrak{Y}\ \mathfrak{y}$
L l	$\mathfrak{L}\ \mathfrak{l}$	Z z	$\mathfrak{Z}\ \mathfrak{z}$
M m	$\mathfrak{M}\ \mathfrak{m}$		
N n	$\mathfrak{N}\ \mathfrak{n}$		