Implementation of special functions

Contents

1 Elliptic integrals and related

1.1 Complete elliptic integral of the first kind

The arithmetic geometric mean $\operatorname{agm}(x,y)$ is defined and calculated as the limit of the iteration:

$$\begin{bmatrix} a_0 \\ g_0 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a_{n+1} \\ g_{n+1} \end{bmatrix} := \begin{bmatrix} \frac{1}{2}(a_n + g_n) \\ \sqrt{a_n g_n} \end{bmatrix}. \tag{1}$$

The iteration can be stopped if a_n and g_n are sufficiently close to each other. If this condition fails for some reason, to have a more stable algorithm, a maximum number n_{max} of iterations should be specified. Numerical experiments show that $n_{\mathrm{max}} = 14$ is enough for 64-bit floating point arithmetic with

$$(x,y) \in [10^{-307}, 10^{308}] \times [10^{-307}, 10^{308}]. \tag{2}$$

The complete elliptic integral of the first kind is defined as

$$K(m) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{3}$$

It is calculated by the arithmetic geometric mean:

$$K(m) = \frac{\pi}{2\operatorname{agm}(1, \sqrt{1-m})}. \tag{4}$$

The domain of $K(m)$ is $m < 1$, but (3) allows more generally

$$m \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}. \tag{5}$$

The relation between the arithmetic geometric mean and $K(m)$ holds even for complex numbers, but one has to take care of the branch cut of the square root.

2 Polynomials and related

2.1 Associated Legendre functions

The associated Legendre functions $P_n^m(x)$ are solutions of the general Legendre equation

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_n^m(x)-2x\frac{\mathrm{d}}{\mathrm{d}x}P_n^m(x)+\Big[(n+1)n-\frac{m^2}{1-x^2}\Big]P_n^m(x)=0. \tag{6}$$

In case of $m=n$ one has the recurrence

$$P_0^0(x)=1,\quad P_n^n(x)=-(2n-1)\sqrt{1-x^2}P_{n-1}^{n-1}(x), \tag{7}$$

which has the solution

$$P_n^n(x)=(-1)^n(2n-1)!!(1-x^2)^{n/2}. \tag{8}$$

By

$$(2n-1)!!=(-2)^n\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} \tag{9}$$

we obtain

$$P_n^n(x)=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)}(2\sqrt{1-x^2})^n. \tag{10}$$

In case of $m=n-1$ one has

$$P_n^{n-1}(x)=(2n-1)xP_{n-1}^{n-1}(x). \tag{11}$$

Now we use the recurrence

$$(n-m)P_n^m(x)=(2n-1)xP_{n-1}^m(x)-(n-1+m)P_{n-2}^m(x) \tag{12}$$

to get $n\ge m$ down to m. The recurrence will be converted into a bottom up iteration like in the calculation of the Fibonacci sequence. We can remove quadratic complexity by this trick.

This leads us to the following algorithm:

function $P_n^m(x)$
    if $n=m$
        return $\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)}(2\sqrt{1-x^2})^n$
    else if $n-1=m$
        return $(2n-1)xP_m^m(x)$
    else
        let mut $a:=P_m^m(x)$
        let mut $b:=P_{m+1}^m(x)$
        for k in $[m+2..n]$
            let $h:=\frac{(2k-1)xb-(k-1+m)a}{k-m}$
            $a:=b;\ \ b:=h$
        end
        return b
    end
end

3 Gamma function and related

3.1 Gamma function

The easiest way to compute an approximate value of the gamma function is Stirling's approximation

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x = \sqrt{2\pi}\,x^{x+1/2}\mathrm{e}^{-x}. \tag{13}$$

This approximation is also an asymptotic formula. That means, the relative error gets smaller as $x \to \infty$. We can profit from this property if me make x larger by the functional equation

$$\Gamma(x+1) = x\,\Gamma(x). \tag{14}$$

Performing the functional equation one time yields again an easy formula, because in this case a simplification is possible:

$$\Gamma(x) \approx \sqrt{2\pi}\,x^{x-1/2}\mathrm{e}^{-x}. \tag{15}$$

Formula (15) is more precise than (13). Iteration of this technique yields

$$\Gamma(x) \approx \sqrt{2\pi}\,\frac{(x+n)^{x+n-1/2}}{\mathrm{e}^{x+n}}\prod_{k=0}^{n-1}\frac{1}{x+k}. \tag{16}$$

We can obtain more and more precise values, but convergence as $n \to \infty$ is very slow.

More precise than (15) is

$$\Gamma(x) \approx \sqrt{2\pi}\,x^{x-1/2}\exp\left(-x+\frac{1}{12x}\right). \tag{17}$$

More precise than (17) is

$$\Gamma(x) \approx \sqrt{\frac{2\pi}{x}}\left(\frac{1}{e}\left(x+\frac{1}{12x-\frac{1}{10x}}\right)\right)^x, \tag{18}$$

found by G. Nemes. More formulas are discussed in [1].

The standard algorithm for 64-bit floating point numbers is Lanczos approximation, see [2].

References

[1] Peter Luschny: *Approximation Formulas for the Factorial Function*.

[2] Glendon Pugh (2004): *An analysis of the Lanczos Gamma approximation*.