Implementation of special functions

Contents

1 Elliptic integrals and related

1.1 Complete elliptic integral of the first kind

The arithmetic geometric mean $\operatorname{agm}(x, y)$ is defined and calculated as the limit of the iteration:

$$\begin{bmatrix} a_0 \\ g_0 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a_{n+1} \\ g_{n+1} \end{bmatrix} := \begin{bmatrix} \frac{1}{2}(a_n + g_n) \\ \sqrt{a_n g_n} \end{bmatrix}. \tag{1}$$

The iteration can be stopped if a_n and g_n are sufficiently close to each other. If this condition fails for some reason, to have a more stable algorithm, a maximum number n_{max} of iterations should be specified. Numerical experiments show that $n_{\mathrm{max}} = 14$ is enough for 64-bit floating point arithmetic with

$$(x, y) \in [10^{-307}, 10^{308}] \times [10^{-307}, 10^{308}]. \tag{2}$$

The complete elliptic integral of the first kind is defined as

$$K(m) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{3}$$

It is calculated by the arithmetic geometric mean:

$$K(m) = \frac{\pi}{2\operatorname{agm}(1, \sqrt{1-m})}. \tag{4}$$

The domain of $K(m)$ is $m < 1$, but (3) allows more generally

$$m \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}. \tag{5}$$

The relation between the arithmetic geometric mean and $K(m)$ holds even for complex numbers, but one has to take care of the branch cut of the square root.