

Implementation of special functions

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1 Elliptic integrals and related

1.1 Complete elliptic integral of the first kind

The arithmetic geometric mean $\text{agm}(x, y)$ is defined and calculated as the limit of the iteration:

$$\begin{bmatrix} a_0 \\ g_0 \end{bmatrix} := \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a_{n+1} \\ g_{n+1} \end{bmatrix} := \begin{bmatrix} \frac{1}{2}(a_n + g_n) \\ \sqrt{a_n g_n} \end{bmatrix}. \quad (1)$$

The iteration can be stopped if a_n and g_n are sufficiently close to each other. If this condition fails for some reason, to have a more stable algorithm, a maximum number n_{\max} of iterations should be specified. Numerical experiments show that $n_{\max} = 14$ is enough for 64-bit floating point arithmetic with

$$(x, y) \in [10^{-307}, 10^{308}] \times [10^{-307}, 10^{308}]. \quad (2)$$

The complete elliptic integral of the first kind is defined as

$$K(m) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}. \quad (3)$$

It is calculated by the arithmetic geometric mean:

$$K(m) = \frac{\pi}{2 \text{agm}(1, \sqrt{1-m})}. \quad (4)$$

The domain of $K(m)$ is $m < 1$, but (3) allows more generally

$$m \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}. \quad (5)$$

The relation between the arithmetic geometric mean and $K(m)$ holds even for complex numbers, but one has to take care of the branch cut of the square root.

2 Polynomials and related

2.1 Associated Legendre functions

The associated Legendre functions $P_n^m(x)$ are solutions of the general Legendre equation

$$(1-x^2)\frac{d^2}{dx^2}P_n^m(x) - 2x\frac{d}{dx}P_n^m(x) + \left[(n+1)n - \frac{m^2}{1-x^2}\right]P_n^m(x) = 0. \quad (6)$$

In case of $m = n$ one has the recurrence

$$P_0^n(x) = 1, \quad P_n^n(x) = -(2n-1)\sqrt{1-x^2}P_{n-1}^{n-1}(x), \quad (7)$$

which has the solution

$$P_n^n(x) = (-1)^n(2n-1)!!(1-x^2)^{n/2}. \quad (8)$$

By

$$(2n-1)!! = (-2)^n \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} \quad (9)$$

we obtain

$$P_n^n(x) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} (2\sqrt{1-x^2})^n. \quad (10)$$

In case of $m = n-1$ one has

$$P_n^{n-1}(x) = (2n-1)xP_{n-1}^{n-1}(x). \quad (11)$$

Now we use the recurrence

$$(n-m)P_n^m(x) = (2n-1)xP_{n-1}^m(x) - (n-1+m)P_{n-2}^m(x) \quad (12)$$

to get $n \geq m$ down to m . The recurrence will be converted into a bottom up iteration like in the calculation of the Fibonacci sequence. We can remove quadratic complexity by this trick.

This leads us to the following algorithm:

```
function  $P_n^m(x)$ 
  if  $n = m$ 
    return  $\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} (2\sqrt{1-x^2})^n$ 
  else if  $n-1 = m$ 
    return  $(2n-1)xP_m^m(x)$ 
  else
    let mut  $a := P_m^m(x)$ 
    let mut  $b := P_{m+1}^m(x)$ 
    for  $k$  in  $[m+2..n]$ 
      let  $h := \frac{(2k-1)xb - (k-1+m)a}{k-m}$ 
       $a := b$ ;  $b := h$ 
    end
    return  $b$ 
end
end
```