INTRODUCTION TO CATEGORY THEORY

CARDONA, ROBERT *MRROBERTCARDONA@GMAIL*, KHOSHBIN, MASSY *MASSY255@GMAIL.COM*,
AND MORTEZAVI, SIAVASH *SIAVASH.MORTEZAVI@GMAIL.COM*

Abstract. Module Theory.

1. MATH 697 Homework One

Exercise 1.1. Prove Theorem 4 (Isomorphism Theorems):

(1) (*The First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then $\ker\varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proof. Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then by definition $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$, $r \in R$. We want to show that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ is a submodule. Observe that since M is a module then M is an abelian group by definition so there exists $0 \in M$ such that $m+0 = m$ for all $m \in M$. In particular $\varphi(0) = \varphi(0+0) = \varphi(0)+\varphi(0)$ implying $\varphi(0) = 0$. Conclude that $0 \in \ker\varphi \neq \emptyset$. Now let $r \in R$, $x, y \in \ker\varphi$. Observe that $\varphi(x+ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r\cdot 0 = 0 + 0 = 0$. Hence $x + ry \in \ker\varphi$. Conclude by the submodule criterion that $\ker\varphi$ is in fact a submodule.

Now define $\Phi : M/\ker\varphi \to \varphi(M)$ by $\Phi(m + \ker\varphi) = \varphi(m)$. We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose $m + \ker\varphi = m' + \ker\varphi$ it follows by property of cosets that $m - m' \in \ker\varphi$, in particular $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$ and hence $\varphi(m) = \varphi(m')$. But since $\varphi(m) = \Phi(m + \ker\varphi)$ and $\varphi(m') = \Phi(m' + \ker\varphi)$ we have $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Conclude that Φ is in fact well-defined.

Suppose that $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Then it follows that $\varphi(m) = \varphi(m')$ and so $\varphi(m - m') = 0$ and so $m - m' \in \ker\varphi$. By property of cosets it follows that $m + \ker\varphi = m' + \ker\varphi$ and hence Φ is injective.

Let $n \in \varphi(M)$. Then by definition of image of φ there exists $m \in M$ such that $n = \varphi(m)$. It is immediate that $m + \ker\varphi \in M/\ker\varphi$ and we can conclude that Φ is surjective.

Now we must show that Φ is an R-module homomorphism. Let $x, y \in M/\ker\varphi$ where $x = m + \ker\varphi$ and $y = m' + \ker\varphi$ for some $m, m' \in M$ and let $r \in R$. Observe that

$$\begin{aligned}
\Phi(x+y) &= \Phi(m + m' + \ker\varphi) \\
&= \varphi(m + m') \\
&= \varphi(m) + \varphi(m') \\
&= \Phi(m + \ker\varphi) + \Phi(m' + \ker\varphi) \\
&= \Phi(x) + \Phi(y)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(rx) &= \Phi(r(m + \ker\varphi)) \\
&= \Phi(rm + \ker\varphi) \\
&= \varphi(rm) \\
&= r\varphi(m) \\
&= r\Phi(m + \ker\varphi) \\
&= r\Phi(x)
\end{aligned}$$

Hence we have shown that Φ is a well-defined bijective homomorphism and thus we can conclude by definition of R-module isomorphism that $M/\ker\varphi \cong \varphi(M)$. □

(2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R-module M. Then $(A+B)/B \cong A/(A\cap B)$.

(3) (*The Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

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(4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence cummutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Department of Mathematics, California State University Long Beach