

INTRODUCTION TO CATEGORY THEORY

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0. MATH 697 HOMEWORK ZERO.ONE

§10.2 Theorem 4: (Isomorphism Theorems):

- (1) (*The First Isomorphism Theorem for Modules*) Let M, N be R -modules and let $\varphi : M \rightarrow N$ be an R -modules homomorphism. Then $\ker \varphi$ is a submodule of M and $M/\ker \varphi \cong \varphi(M)$.

Proof. φ is, in particular, a group homomorphism from M to N . By First Isomorphism Theorem for groups, $\ker \varphi \trianglelefteq M$ and \exists group isomorphism $\phi : M/\ker \varphi \rightarrow \varphi(M)$ satisfying $\phi(\overline{m}) = \varphi(m)$. Since φ is an R -module homomorphism, for $r \in R$, have $\phi(r\overline{m}) = \phi(\overline{rm}) = \varphi(rm) = r\varphi(m) = r\phi(\overline{m})$. Thus ϕ is an R -module isomorphism. \square

- (2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R -module M . Then $(A + B)/B \cong A/(A \cap B)$.

Proof. Define $\varphi : A \rightarrow (A + B)/B$ by $\varphi(a) = a + B$. By the Second Isomorphism Theorem for groups, φ is a group homomorphism. Let $r \in R$, then

$$\begin{aligned}\varphi(ra) &= ra + B \\ &= r(a + B) \\ &= r\varphi(a)\end{aligned}$$

and so φ is an R -module homomorphism by definition. Observe that $\ker \varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$. Now let $x \in (A + B)/B$ then $x = a + b + B$ for some $a \in A, b \in B$. But observe that $a + b + B = a + B$ by absorption. So φ is immediately surjective. In particular we have $\varphi(A) = (A + B)/B$. By the First Isomorphism Theorem for Modules, $A/\ker \varphi = A/(A \cap B) \cong (A + B)/B = \varphi(A)$. \square

- (3) (*The Third Isomorphism Theorem*) Let M be an R -module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

Proof. Define $\varphi : M/A \rightarrow M/B$ by $\varphi(m + A) = m + B$. By the Third Isomorphism Theorem for groups, φ is a group homomorphism. Let $r \in R$, then

$$\begin{aligned}\varphi(r(m + A)) &= \varphi(rm + A) \\ &= rm + B \\ &= r(m + A) \\ &= r\varphi(m + A)\end{aligned}$$

and thus φ is an R -module homomorphism.

Observe that $\ker \varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A : \varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$. Let $m + B \in M/B$. Clearly $\varphi(m + A) = m + B$ and hence φ is surjective. Now by the First Isomorphism Theorem for Modules we have $(M/A)/\ker \varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$. \square

- (4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R -module M . There is a bijection between the submodules of M which contain N and the submodules of M/N . The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Proof. Let N be a submodule of M . Define $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$, $T = \{L : L \text{ is a submodule of } M/N\}$. Define $\varphi : S \rightarrow T$ by $\varphi(K) = K/N$. We want to show that this mapping is bijective.

Let $K_1, K_2 \in S$ and suppose that $\varphi(K_1) = \varphi(K_2)$. Then $K_1/N = K_2/N$. We want to show that $K_1 = K_2$. Let $x \in K_1$, then $x + N \in K_1/N = K_2/N$, in particular there exists $y \in K_2$ such that $x + N = y + N$. By property of cosets it follows that $x - y \in N$. But since $N \subseteq K_2$ by construction $x - y \in K_2$. Since K_2 is a submodule of M , it is closed under addition and so $(x - y) + y = x \in K_2$. Conclude that $K_1 \subseteq K_2$. By symmetric argument $K_2 \subseteq K_1$ and hence $K_1 = K_2$. Thus by definition φ is injective.

Let L be a submodule of M/N . Consider the natural projection map $\pi : M \rightarrow M/N$ defined by $\pi(m) = m + N$. We want to show that there exists $K \in S$ such that $\varphi(K) = L$. To do this we will show that $\pi^{-1}(L)$ is a submodule of M and that $N \subseteq \pi^{-1}(L)$. Recall that $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$. Observe that $0 \in \pi^{-1}(L)$ since $\pi(0) = 0$ and hence $\pi^{-1}(L) \neq \emptyset$. Let $x, y \in \pi^{-1}(L)$ and $r \in R$. Observe that $\pi(x + ry) = \pi(x) + r\pi(y)$. Since $\pi(y) \in L$ by definition and L is a submodule of M/N , it follows that since scalar multiplication is closed $r\pi(y) \in L$. Thus it follows that $\pi(x) + r\pi(y) \in L$ and hence $x + ry \in \pi^{-1}(L)$. Thus $\pi^{-1}(L)$ is a submodule. Now let $n \in N$ and observe that $\pi(n) = n + N = 0 + N \in L$ so by definition it follows that $n \in \pi^{-1}(L)$. Conclude that $N \subseteq \pi^{-1}(L)$ and hence φ is surjective.

Conclude φ is bijective and result follows. \square

§10.2 #1: Use the submodule criterion to show that the kernels and images of R -module homomorphisms are submodules.

Proof. Let M, N be R -modules and $\varphi : M \rightarrow N$ an R -module homomorphism. Recall that $\ker \varphi = \{m \in M : \varphi(m) = 0\}$ and $\text{im } \varphi = \{n \in N : \text{there exists } m \in M \text{ with } \varphi(m) = n\}$.

Observe that $\varphi(0) = 0$ so $0 \in \ker \varphi \neq \emptyset$. Let $m, m' \in M$, $r \in R$. Now $\varphi(m + rm') = \varphi(m) + r\varphi(m') = 0 + r \cdot 0 = 0 + 0 = 0$. So $m + rm' \in \ker \varphi$. Thus by the submodule criterion $\ker \varphi$ is a submodule.

Observe that $\varphi(0) = 0 \in N$ so $0 \in \text{im } \varphi \neq \emptyset$. Let $n, n' \in N$, $r \in R$. Then there exists $m, m' \in M$ such that $\varphi(m) = n$ and $\varphi(m') = n'$. Now consider $n + rn'$. $\varphi(m + rm') = \varphi(m) + r\varphi(m') = n + rn'$ so $n + rn' \in \text{im } \varphi$. Conclude that $\text{im } \varphi$ is a submodule. \square

§10.2 #2: Show that the relation “is R -module isomorphic to” is an equivalence relation on any set of R -modules.

Proof. Let X be a set of R -modules.

- Let $M \in X$. Observe that M is isomorphic to M trivially. So relation is reflexive.
- Let $M, N \in X$. Suppose M is isomorphic to N then by definition there exists $\varphi : M \rightarrow N$ that is bijective. Immediately we have $\varphi^{-1} : N \rightarrow M$ which is also bijective so N is isomorphic to M . By definition the relation is symmetric.
- Let $L, M, N \in X$. Suppose L is isomorphic to M , then by definition there exists $\varphi : L \rightarrow M$ a bijective R -module homomorphism. Suppose M is isomorphic to N , then there exists $\Phi : M \rightarrow N$ a bijective R -module homomorphism. Observe that $\varphi \circ \Phi : L \rightarrow N$ is again a bijective R -module homomorphism by property of composition of mappings. Hence by definition L is isomorphic to N .

Conclude that the relation “is R -module isomorphic to” is an equivalence relation on any set of R -modules. \square

§10.2 #3: Give an explicit example of a map from one R -module to another which is a group homomorphism but not an R -module homomorphism.

Solution. Consider the Quaternions $\mathbb{H} = R$; they form a commutative group under addition and a noncommutative group under multiplication. Hence \mathbb{H} is a noncommutative ring with unity. In particular \mathbb{H} is an R -module over itself. Define $\varphi : \mathbb{H} \rightarrow \mathbb{H}$ by $\varphi(h) = ih$. This is a group homomorphism since $\varphi(h + h') = i(h + h') = ih + ih' = \varphi(h) + \varphi(h')$. But note that $\varphi(j \cdot 1) = \varphi(j) = ij = k \neq -k = ji = j(i \cdot 1) = j\varphi(1)$. Conclude that φ is not an R -module homomorphism since the definition is not satisfied. \blacktriangleleft

§10.2 #4: Let A be a \mathbb{Z} -module, let a be any element of A and let n be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$ given by $\varphi(\bar{k}) = ka$ is a well-defined \mathbb{Z} -module homomorphism if and only if $na = 0$. Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A : na = 0\}$ (So A_n is the annihilator in A of the ideal (n) of \mathbb{Z}).

Proof. Suppose that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$ given by $\varphi(\bar{k}) = ka$ is a well-defined \mathbb{Z} -module homomorphism. Then by definition if $\bar{m} = \bar{k}$ then $\varphi(\bar{m}) = \varphi(\bar{k})$ or equivalently $ma = ka$. Moreover $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ra) = r\varphi(a)$ for all $a, b \in A$ and $r \in \mathbb{Z}$. Observe that $\bar{0} = \bar{k}$ so by hypothesis $\varphi(\bar{0}) = \varphi(\bar{n})$ but observe that $\varphi(\bar{0}) = 0 \cdot a = 0$ and $\varphi(\bar{n}) = na$. Hence by equality $na = 0$. Conversely suppose that $na = 0$. We want to show that $\varphi : \mathbb{Z}/n\mathbb{Z} \rightarrow A$ defined by $\varphi(\bar{k}) = ka$ is a well-defined R -module homomorphism. Say $\bar{k} = \bar{m}$ then by property of cosets $k - m \in \mathbb{Z}/n\mathbb{Z}$ and so by definition $n \mid k - m$ and hence there exists $t \in \mathbb{Z}$ such that $k - m = nt$. Observe that

$$\begin{aligned} k - m &= nt \\ (k - m)a &= nta \\ ka - ma &= (na)t \\ ka - ma &= 0 \\ ka &= ma \end{aligned}$$

Thus we have $\varphi(\bar{k}) = \varphi(\bar{m})$ and we can conclude that φ is a well-defined R -module homomorphism.

Now we want to show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n = \{a \in A : na = 0\}$. *Note:* We are making the assumption that we want to show this in an isomorphism of R -modules as exercise does not specify group, ring or module isomorphism. Define $\Phi : A_n \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ by $\Phi(a) = \varphi_a$. We will show that this is an R -module homomorphism, then that is a bijection.

Let $a, a' \in A_n$, $r \in \mathbb{Z}$. Observe that

$$\begin{aligned}\Phi(a + a')(\bar{k}) &= \varphi_{a+a'}(\bar{k}) \\ &= (a + a')k \\ &= ak + a'k \\ &= \varphi_a(\bar{k}) + \varphi_{a'}(\bar{k}) \\ &= \Phi(a)(\bar{k}) + \Phi(a')(\bar{k})\end{aligned}$$

So $\Phi(a + a') = \Phi(a) + \Phi(a')$ by definition. Moreover

$$\begin{aligned}\Phi(ra)(\bar{k}) &= \varphi_{ra}(\bar{k}) \\ &= rak \\ &= r\varphi_a(\bar{k}) \\ &= r\Phi(a)(\bar{k})\end{aligned}$$

Hence $\Phi(ra) = r\Phi(a)$ by definition. Conclude that Φ is an R -module homomorphism.

Recall that $\ker \Phi = \{a \in A_n : \Phi(a) = 0\}$ and observe that

$$\begin{aligned}\ker \Phi &= \{a \in A_n : \Phi(a) = 0\} \\ &= \{a \in A_n : \varphi_a(\bar{k}) = 0 \text{ for all } k \in \mathbb{Z}/n\mathbb{Z}\} \\ &= \{0\}\end{aligned}$$

So we conclude that $\ker \Phi = \{0\}$ and hence Φ is injective.

Let $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$. Define $a = \varphi(\bar{1})$ and hence $na = n\varphi(\bar{1}) = \varphi(n\bar{1}) = \varphi(\bar{n}) = \varphi(\bar{0}) = 0$ and hence $a \in A_n$. Observe that for all $k \in \mathbb{Z}/n\mathbb{Z}$ it follows that $\varphi(\bar{k}) = \varphi(\bar{k} \cdot \bar{1}) = \varphi(k\bar{1}) = k\varphi(\bar{1}) = ka = \varphi_a(\bar{k})$. Thus by definition $\varphi = \varphi_a$. So for any $\varphi \in \text{Hom}(\mathbb{Z}/n\mathbb{Z}, A_n)$ we can find $a \in A_n$ such that $\Phi(a) = \varphi_a = \varphi$. Conclude by definition that Φ is surjective. \square

§10.2 #5: Exhibit all \mathbb{Z} -module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$.

Proof. By previous exercise we know $\text{Hom}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/21\mathbb{Z}) \cong (\mathbb{Z}/21\mathbb{Z})_{30}$ where $(\mathbb{Z}/21\mathbb{Z})_{30} = \{a \in \mathbb{Z}/21\mathbb{Z} : 30a = 0\} = A_{30} = \{0, 7, 14\}$ and it has three elements. Hence we know that there are three homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$. The three homomorphisms are the ones defined by the trivial homomorphism, $\varphi_7(\bar{x}) = 7x$, $\varphi_{14}(\bar{x}) = 14x$. \square

§10.2 #6: Prove that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

Proof. By previous exercise we know $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})_n = \{a \in \mathbb{Z}/m\mathbb{Z} : na \equiv 0 \pmod{m}\}$. It will suffice to show that $(\mathbb{Z}/m\mathbb{Z})_n \cong \mathbb{Z}/(n, m)\mathbb{Z}$. Let $d = \gcd(n, m)$, so by definition there exist a, b relatively prime, such that $n = ad$ and $m = bd$. Observe that $b \in (\mathbb{Z}/m\mathbb{Z})_n$ since

$$\begin{aligned}nb &\equiv (ad)b \pmod{m} \\ &\equiv a(db) \pmod{m} \\ &\equiv am \pmod{m} \\ &\equiv 0 \pmod{m}\end{aligned}$$

Define $\varphi : \mathbb{Z} \rightarrow (\mathbb{Z}/m\mathbb{Z})_n$ by $\varphi(z) = zb \pmod{m}$. We must first show that this is a \mathbb{Z} -module homomorphism. Let $z, y \in \mathbb{Z}$, $r \in \mathbb{Z}$ and observe that

$$\begin{aligned}\varphi(z + y) &\equiv (z + y)b \pmod{m} \\ &\equiv (zb + yb) \pmod{m} \\ &\equiv zb \pmod{m} + yb \pmod{m} \\ &\equiv \varphi(z) + \varphi(y)\end{aligned}$$

and

$$\begin{aligned}\varphi(rz) &\equiv (rz)b \pmod{m} \\ &\equiv r(zb) \pmod{m} \\ &\equiv r\varphi(z)\end{aligned}$$

We must now show that φ is surjective. Choose $t \in (\mathbb{Z}/m\mathbb{Z})_n$, by definition $nt \equiv 0 \pmod{m}$ and hence $m \mid nt$ or equivalently $bd \mid adt$ or equivalently $b \mid at$. Since $\gcd(a, b) = 1$, it must follow that $b \mid t$ so there exists $s \in \mathbb{Z}$ such that $t = sb$. Hence $\varphi(s) = sb \pmod{m} = t \pmod{m}$. Thus φ is surjective.

We will now show that $\ker \varphi = d\mathbb{Z}$. Observe that $\varphi(d) = db \pmod{m} \equiv m \pmod{m} \equiv 0 \pmod{m}$. So $d \in \ker \varphi$ and immediately $d\mathbb{Z} \subseteq \ker \varphi$. Now let $s \in \ker \varphi$. Then by definition $\varphi(s) = sb \pmod{m} \equiv 0 \pmod{m}$ so, by definition, $m \mid sb$ or

equivalently $bd \mid sb$ or equivalently $d \mid s$ so $s \in d\mathbb{Z}$. Hence $\ker \varphi \subset d\mathbb{Z}$. Conclude that $\ker \varphi = d\mathbb{Z}$.

By the First Isomorphism Theorem for Modules we have $\mathbb{Z}/\ker \varphi \cong (\mathbb{Z}/m\mathbb{Z})_n$. Result follows by equality

$$\mathbb{Z}/\ker \varphi = \mathbb{Z}/d\mathbb{Z} = \mathbb{Z}/(n, m)\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z})_n \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}). \quad \square$$

§10.2 #7: Let z be a fixed element of the center of R . Prove that the map $m \rightarrow zm$ is an R -module homomorphism from M to itself. Show that for a commutative ring R the map from R to $\text{End}_R M$ given by $r \rightarrow rI$ is a ring homomorphism (where I is the identity endomorphism).

Proof. Let z be in the center of R . Define $\varphi : M \rightarrow M$ by $\varphi(m) = zm$. Let $m, m' \in M$, $r \in R$. Observe that $\varphi(m + m') = z(m + m') = zm + zm' = \varphi(m) + \varphi(m')$ and $\varphi(rm) = zrm = rzm = r\varphi(m)$. Thus φ is an R -module homomorphism.

Now let R be a commutative ring and define $\Phi : R \rightarrow \text{End}_R(M)$ by $\Phi(r) = rI$ where $I : M \rightarrow M$, defined by $I(m) = m$, is the identity endomorphism. We want to show that Φ is a ring homomorphism. Let $r, s \in R$ and observe that for all $m \in M$

$$\begin{aligned} \Phi(r + s)(m) &= (r + s)I(m) \\ &= (r + s)m \\ &= rm + sm \\ &= rI(m) + sI(m) \\ &= \Phi(r)(m) + \Phi(s)(m) \end{aligned}$$

So by definition $\Phi(r + s) = \Phi(r) + \Phi(s)$. Moreover,

$$\begin{aligned} \Phi(rs) &= rsI(m) \\ &= rsI(m)I(m) \\ &= rI(m) \cdot sI(m) \\ &= \Phi(r)(m)\Phi(s)(m) \end{aligned}$$

And hence $\Phi(rs) = \Phi(r)\Phi(s)$ and we can conclude by definition that Φ is a ring homomorphism. \square

Exercise. §10.2 #8: Let $\varphi : M \rightarrow N$ be an R -module homomorphism. Prove that $\varphi(\text{Tor}(M)) \subseteq \text{Tor}(N)$.

Proof. Recall that $\text{Tor}(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Now it follows that $\varphi(\text{Tor}(M)) = \{n \in N : n = \varphi(m) \text{ for some } m \in \text{Tor}(M)\}$. Let $n \in \varphi(\text{Tor}(M))$ then $n = \varphi(m)$ for some $m \in \text{Tor}(M)$ by definition. Since $m \in \text{Tor}(M)$ there exists $0 \neq r \in R$ such that $rm = 0$. Hence $rn = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$. Conclude that $n \in \text{Tor}(N)$ and hence $\varphi(\text{Tor}(M)) \subseteq \text{Tor}(N)$. \square

§10.2 #9: Let R be a commutative ring. Prove that $\text{Hom}_R(R, M)$ and M are isomorphic as left R -modules.

Proof. Define $\Phi : \text{Hom}_R(R, M) \rightarrow M$ by $\Phi(\varphi) = \varphi(1)$. We must first show that this is an R -module homomorphism. Observe that for all $\varphi, \xi \in \text{Hom}_R(R, M)$ and all $r \in R$ it follows that

$$\begin{aligned} \Phi(\varphi + \xi) &= (\varphi + \xi)(1) \\ &= \varphi(1) + \xi(1) \\ &= \Phi(\varphi) + \Phi(\xi) \end{aligned}$$

and also by Proposition 2 we have

$$\begin{aligned} \Phi(r\varphi) &= (r\varphi)(1) \\ &= r\varphi(1) \\ &= r\Phi(\varphi). \end{aligned}$$

We must now show that Φ is injective. Suppose that $\Phi(\varphi) = \Phi(\xi)$. Then by definition $\varphi(1) = \xi(1)$ or equivalently $\varphi(1) - \xi(1) = 0$ and hence $(\varphi - \xi)(1) = 0$. But since $\varphi - \xi \in \text{Hom}_R(R, M)$ it is an R -module homomorphism so $(\varphi - \xi)(x) = (\varphi - \xi)(x \cdot 1) = x(\varphi - \xi)(1) = x \cdot 0 = 0$ for all $x \in R$. Conclude that $\varphi(x) = \xi(x)$ for all $x \in R$ and hence by definition $\varphi = \xi$. Hence Φ is injective.

We now show that Φ is surjective. Let $m \in M$ be arbitrary. We want to show that there exists $\varphi \in \text{Hom}_R(R, M)$ such that $\Phi(\varphi) = m$. Define $\varphi : R \rightarrow M$ by $\varphi(x) = xm$. We need to show that $\varphi \in \text{Hom}_R(R, M)$. Observe that $\varphi(x + y) = (x + y)m = xm + ym = \varphi(x) + \varphi(y)$ for all $x, y \in R$ and $\varphi(rx) = rxm = r\varphi(x)$ for all $x \in R$, $r \in R$. Hence we have shown that φ is an R -module homomorphism. Now observe that $\Phi(\varphi) = \varphi(1) = 1 \cdot m = m$. Conclude by definition that Φ is surjective.

Whence Φ is bijective and $\text{Hom}_R(R, M) \cong M$. \square

§10.2 #10: Let R be a commutative ring. Prove that $\text{Hom}_R(R, R)$ and R are isomorphic as rings.

Proof. Define $\Phi : \text{Hom}_R(R, R) \rightarrow R$ by $\Phi(\varphi) = \varphi(1)$. By the previous exercise, all that remains to show is $\Phi(\varphi \circ \xi) = \Phi(\varphi)\Phi(\xi)$:

$$\begin{aligned}\Phi(\varphi \circ \xi) &= (\varphi \circ \xi)(1) \\ &= \varphi(\xi(1)) \\ &= \varphi(\xi(1) \cdot 1) \\ &= \xi(1)\varphi(1) \\ &= \varphi(1)\xi(1) \\ &= \Phi(\varphi)\Phi(\xi).\end{aligned}$$

□

§10.2 #11: Let A_1, A_2, \dots, A_n be R -modules and let B_i be submodules of A_i for each $i = 1, 2, \dots, n$. Prove that

$$(A_1 \times A_2 \times \cdots \times A_n) / (B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

Proof. Define $\varphi : A_1 \times A_2 \times \cdots \times A_n \rightarrow (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$ by $\varphi(a_1, a_2, \dots, a_n) = (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n)$.

Let $x, y \in A_1 \times A_2 \times \cdots \times A_n$ where $x = (a_1, a_2, \dots, a_n)$ and $y = (a'_1, a'_2, \dots, a'_n)$. Observe that

$$\begin{aligned}\varphi(x + y) &= \varphi((a_1, a_2, \dots, a_n) + (a'_1, a'_2, \dots, a'_n)) \\ &= \varphi(a_1 + a'_1, a_2 + a'_2, \dots, a_n + a'_n) \\ &= (a_1 + a'_1 + B_1, a_2 + a'_2 + B_2, \dots, a_n + a'_n + B_n) \\ &= (a_1 + B_1 + a'_1 + B_1, a_2 + B_2 + a'_2 + B_2, \dots, a_n + B_n + a'_n + B_n) \\ &= (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) + (a'_1 + B_1, a'_2 + B_2, \dots, a'_n + B_n) \\ &= \varphi(a_1, a_2, \dots, a_n) + \varphi(a'_1, a'_2, \dots, a'_n) \\ &= \varphi(x) + \varphi(y)\end{aligned}$$

and for $r \in R$

$$\begin{aligned}\varphi(rx) &= \varphi(r(a_1, a_2, \dots, a_n)) \\ &= \varphi(ra_1, ra_2, \dots, ra_n) \\ &= (ra_1 + B_1, ra_2 + B_2, \dots, ra_n + B_n) \\ &= (r(a_1 + B_1), r(a_2 + B_2), \dots, r(a_n + B_n)) \\ &= r(a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) \\ &= r\varphi(a_1, a_2, \dots, a_n) \\ &= r\varphi(x)\end{aligned}$$

Thus φ is an R -module homomorphism.

Now we want to show that $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$. Observe that

$$\begin{aligned}\ker \varphi &= \{x \in A_1 \times A_2 \times \cdots \times A_n : \varphi(x) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : \varphi(a_1, a_2, \dots, a_n) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) = (0, 0, \dots, 0)\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : a_1 \in B_1, a_2 \in B_2, \dots, a_n \in B_n\}\end{aligned}$$

Hence $\ker \varphi \subseteq B_1 \times B_2 \times \cdots \times B_n$ and trivially $B_1 \times B_2 \times \cdots \times B_n \subseteq \ker \varphi$ by construction of φ . Conclude that $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$.

The mapping is trivially surjective. Applying first Isomorphism theorem yields the result. □

§10.3 #3: Show that the $F[x]$ -modules in Exercises 18 and 19 of Section 1 are both cyclic.

Proof. For Problem 18: $F = \mathbb{R}$, $V = \mathbb{R}^2$, $T : V \rightarrow V$ is defined by $T(x, y) = (y, -x)$. Let $(a, b) \in \mathbb{R}^2$ be arbitrary. Observe that $(ax + b)(0, 1) = aT(0, 1) + b(0, 1) = a(1, 0) + b(0, 1) = (a, b)$. Hence it follows by definition that $V = \mathbb{R}[x](0, 1)$. Moreover it can also be written as $V = \mathbb{R}[x](1, 0)$ with $p(x) = a - bx$, so the representation is not unique.

For Problem 19: $F = \mathbb{R}$, $V = \mathbb{R}^2$, $T : V \rightarrow V$ is defined by $T(x, y) = (0, y)$. Let $(a, b) \in \mathbb{R}^2$ be arbitrary. Observe that $(a + (b - a)x)(1, 1) = (a, a) + (b - a)T(1, 1) = (a, a) + (0, b - a) = (a, b)$. Hence it follows by definition that $V = \mathbb{R}[x](1, 1)$. □

§10.3 #4: An R -module M is called a *torsion* module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$, where r may depend on m (i.e., $M = \text{Tor}(M)$ in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion \mathbb{Z} -module. Give an example of an infinite abelian group that is a torsion \mathbb{Z} -module.

Proof. Let M be a finite abelian group. Let $m \in M$. We want to show that there exists $0 \neq r \in R = \mathbb{Z}$ such that $rm = 0$. Consider $1m, 2m, 3m, 4m, \dots$. These are not all distinct, because if they were we would have infinitely many, a contradiction. So we are assured $km = lm$ for some $k, l \in \mathbb{Z}$ nonzero with $k \neq l$. It follows that $km - lm = 0$ and hence by property of modules, $(k - l)m = 0$. Finally observe that $k - l \neq 0$ so $m \in \text{Tor}(M)$. Conclude that $M = \text{Tor}(M)$.

As for the example. Let $n \in \mathbb{Z}$ be greater than 1. Consider $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \dots$ and observe that this is an infinite abelian group. This can be seen as a \mathbb{Z} -module. Let $(a_1, a_2, \dots) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \dots$ be arbitrary. Observe that $n(a_1, a_2, \dots) = (na_1, na_2, \dots) = (0, 0, \dots) = \mathbf{0}$. \square

§10.3 #5: Let R be an integral domain. Prove that every finitely generated torsion R -module has a nonzero annihilator i.e., there is a nonzero element $r \in R$ such that $rm = 0$ for all $m \in M$ – here r does not depend on m (the annihilator of a module was defined in Exercise 9 of Section 1). Give an example of a torsion R -module whose annihilator is the zero ideal.

Proof. Let M be a finitely generated torsion R -module where R is an integral domain. By definition $M = Rm_1 + \dots + Rm_n$ for some $m_1, m_2, \dots, m_n \in M$. Let $m \in M$. Since M is finitely generated there exist $r_1, r_2, \dots, r_n \in R$ such that $m = r_1m_1 + \dots + r_nm_n$. Since M is torsion, there exists $0 \neq \bar{r}_i \in R$ such that $\bar{r}_im_i = 0$ for $i = 1, 2, \dots, n$. Define $r = \bar{r}_1\bar{r}_2 \dots \bar{r}_n$. Note that R is an integral domain, so that $r \neq 0$. Now observe that

$$\begin{aligned} rm &= r(r_1m_1 + \dots + r_nm_n) \\ &= (r_1\bar{r}_2 \dots \bar{r}_n)\bar{r}_1m_1 + \dots + (\bar{r}_1 \dots \bar{r}_{n-1}r_n)\bar{r}_nm_n \\ &= (r_1\bar{r}_2 \dots \bar{r}_n) \cdot 0 + \dots + (\bar{r}_1 \dots \bar{r}_{n-1}r_n) \cdot 0 \\ &= 0 \end{aligned}$$

hence it follows that $0 \neq r \in \text{Ann}(m)$ and thus $\text{Ann}(m) \neq 0$.

As for the example: recall that \mathbb{Q} is not finitely generated, and hence $\mathbb{Q}/\mathbb{Z} = M$ is also not finitely generated. Observe that M is torsion since for any rational, non-integer number x , multiplication by its denominator, which is an integer, yields an integer, which in this case would be 0 in M . Suppose by way of contradiction that there is a nonzero annihilator, say $0 \neq a \in R = \mathbb{Z}$. Choose $b \in \mathbb{Z}$ such that $b \nmid a$. Now $a \cdot 1/b = 0$ by property of being annihilator so $a/b = k$ is an integer. But then $a = bk$ and hence $b \mid a$, a contradiction. So there are no nonzero annihilators. \square

§10.3 #9: An R -module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M . Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as its generator. Determine all the irreducible \mathbb{Z} -modules.

Proof. Suppose M is irreducible. By definition $M \neq 0$. Let $0 \neq m \in M$. Note that $Rm \subseteq M$ is nonzero since $1m = m \in Rm$. Since M is irreducible and we have already shown $Rm \neq 0$, conclude that $Rm = M$ for any $0 \neq m \in M$.

Conversely suppose that $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Let N be a submodule of M , so $0 \subseteq N \subseteq M$. If $N = 0$ we are done, so suppose $N \neq 0$, then there exists $n \in N$ such that $n \neq 0$. But $n \in M$ and so $M = Rn$. Since $Rn \subseteq N$ immediately we have just shown $M \subseteq N$. Since both $N \subseteq M$ and $M \subseteq N$ we have $M = N$.

Let M be an irreducible \mathbb{Z} -module then M is cyclic as an abelian group and moreover M has finite order. Since it has only two subgroups, the trivial one and the whole group itself, we can conclude that the irreducible \mathbb{Z} modules are of the form $\mathbb{Z}/p\mathbb{Z}$ where p is prime. \square

§10.3 #10: Assume R is commutative. Show that an R -module M is irreducible if and only if M is isomorphic (as an R -module) to R/I where I is a maximal ideal of R . [By the previous exercise, if M is irreducible there is a natural map $R \rightarrow M$ defined by $r \mapsto rm$, where m is any fixed nonzero element of M .]

Proof. Let M be an R -module. Suppose M is irreducible. Then by definition $M \neq 0$ and the only submodules of M are 0 and M . Define $\varphi_m : R \rightarrow M$ by $\varphi(r) = rm$ where $0 \neq m$ is a fixed element of M . We want to show that this is an R -module homomorphism. Let $x, y \in R$ and $r \in R$ and observe that $\varphi(x+y) = (x+y)m = xm + ym = \varphi(x) + \varphi(y)$ and $\varphi(rx) = rxm = r\varphi(x)$. This mapping is surjective since M is a cyclic module with any nonzero element as its generator. So by the First Isomorphism Theorem for Modules, it follows that $R/\ker \varphi_m \cong M$. $\ker \varphi_m$ is a submodule by a previous exercise and is trivially an ideal of R . It will suffice to show that $\ker \varphi_m$ is a maximal ideal for the result to follow. Let $0 \neq \bar{r} \in R/\ker \varphi_m$ where $\bar{r} = r + \ker \varphi_m$. It follows that $\varphi_m(r) = rm \neq 0$. We want to show that \bar{r} has a multiplicative inverse. Since M is irreducible and $rm \neq 0$, we have $M = R(rm)$ by previous exercise. In particular $m = s(rm)$ for some $s \in R$. Then by equality $1 \cdot m - (sr)m = 0$ resulting in $(1 - sr) \in \ker \varphi_m$. By property of cosets, we have $1 + \ker \varphi_m = sr + \ker \varphi_m = (s + \ker \varphi_m)(r + \ker \varphi_m)$. We have just shown that $1 \in \bar{r}\bar{s}$ giving us \bar{s} as the multiplicative inverse of \bar{r} , an arbitrary nonzero element (by commutativity of R , it is both a left and right inverse). Thus $R/\ker \varphi_m$ is a field and so $\ker \varphi_m$ is a maximal ideal.

Conversely suppose $M \cong R/I$ where I is a maximal ideal of R (as an R -module homomorphism). Then by definition there exists $\varphi : M \rightarrow R/I$ such that $\varphi(m+n) = \varphi(m) + \varphi(n)$ and $\varphi(rm) = r\varphi(m)$ for all $m, n \in M$, $r \in R$. Observe that $M \neq 0$ since $M \cong R/I$ and I is maximal, by definition $I \neq R$ so R/I cannot be trivially 0 so M cannot be trivially 0. Suppose N is a submodule of M , then $0 \subseteq N \subseteq M$. Suppose, by way of contradiction, that $0 \neq N \neq M$. Note that $\varphi(N)$ is a submodule of

R/I and trivially $\varphi(N)$ is an ideal of R/I which is not 0 and not R/I , a contradiction to $0 \neq N \neq M$ since R/I is a field, the only ideals of R/I are 0 and R/I . So either $0 = N$ or $N = M$. Conclude by definition that M is irreducible. \square

§10.3 #15: An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R , prove that $M = eM \oplus (1 - e)M$. [Recall Exercise 14 in Section 1.]

Proof. Suppose r is a central idempotent in R . We want to show that $M = eM \oplus (1 - e)M$, that is, any $m \in M$ can be written *uniquely* of the form $em_1 + (1 - e)m_2$ for some $m_1, m_2 \in M$. Let $m \in M$ and observe that $m = em + (m - em)$ so $M \subseteq eM + (1 - e)M$ and $eM + (1 - e)M \subseteq M$ trivially by closure of modules. Now we need to show the uniqueness. Suppose $m \in eM \cap (1 - e)M$ then $m = em_1 = (1 - e)m_2$ for some $m_1, m_2 \in M$. But then $em_1 = m_2 - em_2$ or equivalently $e(m_1 + m_2) = m_2$. Multiplying both sides by e , we get $e^2(m_1 + m_2) = em_2$ and since $e^2 = e$ we have $em_1 + em_2 = em_2$, making $em_1 = 0$. Hence $m = 0$. Conclude that $eM \cap (1 - e)M = 0$. \square

§10.3 #16: For any ideal I of R let IM be the submodule defined in Exercise 5 of Section 1. Let A_1, \dots, A_k be any ideals in the ring R . Prove that the map

$$M \rightarrow M/A_1M \times \cdots \times M/A_kM \text{ defined by } m \mapsto (m + A_1M, \dots, m + A_kM)$$

is an R -module homomorphism with kernel $A_1M \cap A_2M \cap \cdots \cap A_kM$.

Proof. We first must show that $\varphi : M \rightarrow M/A_1M \times \cdots \times M/A_kM$ defined by $\varphi(m) = (m + A_1M, \dots, m + A_kM)$ is an R -module homomorphism. Let $m_1, m_2 \in M, r \in R$. Observe that

$$\begin{aligned} \varphi(m_1 + m_2) &= (m_1 + m_2 + A_1M, \dots, m_1 + m_2 + A_kM) \\ &= ((m_1 + A_1M) + (m_2 + A_1M), \dots, (m_1 + A_kM) + (m_2 + A_kM)) \\ &= (m_1 + A_1M, \dots, m_1 + A_kM) + (m_2 + A_1M, \dots, m_2 + A_kM) \\ &= \varphi(m_1) + \varphi(m_2) \end{aligned}$$

and

$$\begin{aligned} \varphi(rm_1) &= (rm_1 + A_1M, \dots, rm_1 + A_kM) \\ &= (r(m_1 + A_1M), \dots, r(m_1 + A_kM)) \\ &= r(m_1 + A_1M, \dots, m_1 + A_kM) \\ &= r\varphi(m_1) \end{aligned}$$

Observe

$$\begin{aligned} \ker \varphi &= \{m \in M : \varphi(m) = 0\} \\ &= \{m \in M : (m + A_1M, \dots, m + A_kM) = (0, \dots, 0)\} \\ &= \{m \in M : m \in A_1M, \dots, m \in A_kM\} \\ &= A_1M \cap \cdots \cap A_kM \end{aligned}$$

\square

§10.3 #22: Let R be a Principal Ideal Domain, let M be a torsion R -module (cf. Exercise 4) and let p be a prime in R (do not assume M is finitely generated, hence it need not have a nonzero annihilator – cf. Exercise 5). The *p -primary component* of M is the set of all elements of M that are annihilated by some positive power of p .

(1) Prove that the p -primary component is a submodule. [See Exercise 13 in Section 1.]

Proof. Let A be the p -primary component, i.e., $A = \{m \in M : p^i m = 0 \text{ for some } i \in \mathbb{N}\}$. Observe that $pm = 0$ for $m = 0$ so $A \neq \emptyset$. Let $m, n \in A, r \in R$ a PID. Since $m \in A$ there exists $i \in \mathbb{N}$ such that $p^i m = 0$. Since $n \in A$ there exists $j \in \mathbb{N}$ such that $p^j n = 0$. Choose $l = \max\{i, j\}$ and observe that

$$\begin{aligned} p^l(m + rn) &= p^l m + rp^l n \\ &= p^{l-i}(p^i m) + rp^{l-j}(p^j n) \\ &= p^{l-i} \cdot 0 + rp^{l-j} \cdot 0 \\ &= 0 \end{aligned}$$

Conclude that $m + rn \in A$ and thus A is a submodule. \square

(2) Prove that this definition of p -primary component agrees with the one given in Exercise 18 when M has a nonzero annihilator.

Proof. Suppose $\text{Ann}(M) \neq 0$, then there exists $0 \neq a \in \text{Ann}(M)$ such that $\text{Ann}(M) = (a)$ since R is a Principal Ideal Domain. Since every PID is a UFD and primes are irreducibles in here, we can decompose a , say $a = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. We want to show that $M_{p_i} = \{m \in M : p_i^j m = 0 \text{ for some } j \in \mathbb{N}\}$ is equal to $M_i = \{m \in M : p_i^{\alpha_i} m = 0\}$.

Let $m \in M_i$, then by definition we know that $p_i^{\alpha_i} m = 0$ and immediately it follows that $m \in M_{p_i}$. Conclude that $M_i \subseteq M_{p_i}$.

Conversely suppose that $m \in M_{p_i}$ then by definition there exists $j \in \mathbb{N}$ such that $p_i^j m = 0$. Consider (a, p_i^j) and observe that since R is a PID, it must follow that $(a, p_i^j) = (b)$ for some $b \in R$. But also note that $(p_i^j) \subseteq (a, p_i^j) = (b)$, so by property of ideals, we have $b \mid p_i^j$ which leads us to conclude by property of primes that $b = p_i^t$ for some $t \leq j$. But we also know that $(a) \subseteq (p_i^t)$ so it follows that

$$\begin{aligned} p_i^t &\mid p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n} \\ p_i^{t-\alpha_i} p_i^{\alpha_i} &\mid p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n} \\ p_i^{t-\alpha_i} &\mid p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n} \end{aligned}$$

So by definition $p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n} = p_i^{t-\alpha_i} s$ for some $s \in R$. It must follow that $t - \alpha_i = 0$, or equivalently $t = \alpha_i$, because if it does not, we have a contradiction, since p_i is not in $\{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n\}$. So since $(p_i^j) \subseteq (a, p_i^j) = (p_i^{\alpha_i})$ we have $p_i^{\alpha_i} = p_i^j \cdot k$ for some $k \in R$. So $p_i^{\alpha_i} m = p_i^j \cdot k \cdot m = k \cdot (p_i^j \cdot m) = k \cdot 0 = 0$. Conclude that $m \in M_i$ and hence $M_{p_i} \subseteq M_i$.

Finally, by double inclusion, we can conclude that $M_i = M_{p_i}$. □

- (3) Prove that M is the (possibly infinite) direct sum of its p -primary components, as p runs over all primes of R .

Proof. Denote $P \subseteq R$ as the set of primes in R . Let $m \in M$. Since M is a torsion R -module, there exist $r \in R$ such that $rm = 0$. R is a PID, so can write r into its unique prime factorization, say $r = \prod_{i=1}^n p_i^{\alpha_i}$. Define $q_j = \prod_{i \neq j} p_i^{\alpha_i}$. Then $(q_1, q_2, \dots, q_n) = R$. Thus we can write $1 = \sum_{i=1}^n a_i q_i$ for some $a_i \in R$. We have $0 = rm = (p_i^{\alpha_i} q_i) m = p_i^{\alpha_i} (q_i m)$. Thus $q_i m \in M_{p_i}$, so that $a_i q_i m \in M_{p_i}$. But $m = 1 \cdot m = (\sum_{i=1}^n a_i q_i) \cdot m = \sum_{i=1}^n a_i q_i m \in \sum_{i=1}^n M_{p_i}$. Thus $m \in \sum_{p \in P} M_p$ and we have $M = \sum_{p \in P} M_p$.

Claim that this sum is a direct sum. First, fix a prime $p \in P$ and denote $Q = P \setminus \{p\}$. Suppose $m \in M_p \cap (\sum_Q M_q)$. Write $m = m_p = \sum_Q m_q$, where $m_p \in M_p$ and $m_q \in M_q$ for each $q \in Q$. By definition, we know there exist $k \geq 0$ such that $p^k m = 0$. Thus $(p^k) \subseteq \text{Ann}_R(m)$. We also know there exist $e_q \geq 0$ such that $q^{e_q} m_q = 0$, for each $q \in Q$. Claim that $\prod_Q q^{e_q} \in \text{Ann}_R(m)$. To see this, observe that $\prod_Q q^{e_q} m = [\prod_Q q^{e_q}] [\sum_Q m_q] = \sum_Q [(\prod_Q q^{e_q}) m_q] = \sum_Q 0 = 0$. Now $(\prod_Q q^{e_q}) \subseteq \text{Ann}_R(m)$, implying $(p^k, \prod_Q q^{e_q}) \subseteq \text{Ann}_R(m)$. But p^k and $\prod_Q q^{e_q}$ are relatively prime by construction, so $(p^k, \prod_Q q^{e_q}) = R$, forcing $\text{Ann}_R(m) = R$. In particular, we have $m = 1 \cdot m = 0$. Thus $M_p \cap (\sum_Q M_q) = 0$ and we have that m is written uniquely in $\sum_{p \in P} M_p$. I.e., $M = \bigoplus_{p \in P} M_p$. □