INTRODUCTION TO CATEGORY THEORY

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Abstract. Module Theory.

0. MATH 697 Homework Zero

Exercise 0.1. (DF §10.2 Theorem 4): Prove Theorem 4 (Isomorphism Theorems):

(1) (*The First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then $\ker\varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proof. Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then by definition $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$, $r \in R$. We want to show that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ is a submodule. Observe that since M is a module then M is an abelian group by definition so there exists $0 \in M$ such that $m+0 = m$ for all $m \in M$. In particular $\varphi(0) = \varphi(0+0) = \varphi(0)+\varphi(0)$ implying $\varphi(0) = 0$. Conclude that $0 \in \ker\varphi \neq \emptyset$. Now let $r \in R$, $x, y \in \ker\varphi$. Observe that $\varphi(x+ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r\cdot 0 = 0 + 0 = 0$. Hence $x + ry \in \ker\varphi$. Conclude by the submodule criterion that $\ker\varphi$ is in fact a submodule.

Now define $\Phi : M/\ker\varphi \to \varphi(M)$ by $\Phi(m + \ker\varphi) = \varphi(m)$. We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose $m + \ker\varphi = m' + \ker\varphi$ it follows by property of cosets that $m - m' \in \ker\varphi$, in particular $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$ and hence $\varphi(m) = \varphi(m')$. But since $\varphi(m) = \Phi(m + \ker\varphi)$ and $\varphi(m') = \Phi(m' + \ker\varphi)$ we have $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Conclude that Φ is in fact well-defined.

Suppose that $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Then it follows that $\varphi(m) = \varphi(m')$ and so $\varphi(m - m') = 0$ and so $m - m' \in \ker\varphi$. By property of cosets it follows that $m + \ker\varphi = m' + \ker\varphi$ and hence Φ is injective.

Let $n \in \varphi(M)$. Then by definition of image of φ there exists $m \in M$ such that $n = \varphi(m)$. It is immediate that $m + \ker\varphi \in M/\ker\varphi$ and we can conclude that Φ is surjective.

Now we must show that Φ is an R-module homomorphism. Let $x, y \in M/\ker\varphi$ where $x = m + \ker\varphi$ and $y = m' + \ker\varphi$ for some $m, m' \in M$ and let $r \in R$. Observe that

$$\begin{aligned}
\Phi(x+y) &= \Phi(m + m' + \ker\varphi) \\
&= \varphi(m + m') \\
&= \varphi(m) + \varphi(m') \\
&= \Phi(m + \ker\varphi) + \Phi(m' + \ker\varphi) \\
&= \Phi(x) + \Phi(y)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(rx) &= \Phi(r(m + \ker\varphi)) \\
&= \Phi(rm + \ker\varphi) \\
&= \varphi(rm) \\
&= r\varphi(m) \\
&= r\Phi(m + \ker\varphi) \\
&= r\Phi(x)
\end{aligned}$$

Hence we have shown that Φ is a well-defined bijective homomorphism and thus we can conclude by definition of R-module isomorphism that $M/\ker\varphi \cong \varphi(M)$. ☐

(2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R-module M. Then $(A+B)/B \cong A/(A\cap B)$.

Proof. Define $\varphi : A \to (A+B)/B$ by $\varphi(a) = a + B$. This mapping is clearly well-defined. We want to show that φ is a homomorphism. Let $r \in R$, $a, a' \in A$ and observe that

$$\varphi(a + a') = a + a' + B$$

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$$= a + B + a' + B$$
$$= \varphi(a) + \varphi(a')$$

and

$$\begin{aligned}\varphi(ra) &= ra + B \\ &= r(a+B) \\ &= r\varphi(a)\end{aligned}$$

and so φ is an R-module homomorphism by definition. Observe that $\ker\varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$. Now let $x \in (A+B)/B$ then $x = a + b + B$ for some $a \in A$, $b \in B$. But observe that $a + b + B = a + B$ by absorbption. So φ is immediately surjective. In particular we have $\varphi(A) = (A+B)/B$. Conclude by the First Isomorphism Theorem for Modules that $A/\ker\varphi = A/(A \cap B) \cong (A+B)/B = \varphi(A)$. □

(3) (*The Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

Proof. Define $\varphi M/A \to M/B$ by $\varphi(m + A) = m + B$. We need to show φ is well-defined. Suppose $m + A = m' + A$ then $m - m' \in A \subseteq B$ by property of cosets. It also follows that $m + B = m' + B$. Hence $\varphi(m + A) = m + B = m' + B = \varphi(m' + A)$ and hence φ is well-defined.

Now we must show φ is an R-module homomorphism. Let $m, m' \in M$ and $r \in R$. Observe that

$$\begin{aligned}\varphi((m + A) + (m' + a)) &= \varphi(m + m' + A) \\ &= m + m' + B \\ &= (m + B) + (m' + B) \\ &= \varphi(m + A) + \varphi(m' + A)\end{aligned}$$

and

$$\begin{aligned}\varphi(r(m + A)) &= \varphi(rm + A) \\ &= rm + B \\ &= r(m + A) \\ &= r\varphi(m + A)\end{aligned}$$

and hence we can conclude by definition that φ is an R-module homomorphism.

Observe that $\ker\varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A"\varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$. Let $m + B \in M/B$. Clearly $\varphi(m + A) = m + B$ and hence φ is surjective. Now by the First Isomorphism Theorem for Modules we have $(M/A)/\ker\varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$. □

(4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence cummutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Proof. Let N be a submodule of M. Define $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$, $T = \{L : L \text{ is a submodule of } M/N\}$. Define $\varphi : S \to T$ by $\varphi(K) = K/N$. We want to show that this mapping is bijective.

Let $K_1, K_2 \in S$ and suppose that $\varphi(K_1) = \varphi(K_2)$. Then $K_1/N = K_2/N$. We want to show that $K_1 = K_2$. Let $x \in K_1$, then $x + N \in K_1/N = K_2/N$, in particular there exists $y \in K_2$ such that $x + N = y + N$. By property of cosets it follows that $x - y \in N$. But since $N \subseteq K_2$ by construction $x - y \in K_2$. Since K_2 is a submodule of M, it is closed under addition and so $(x - y) + y = x \in K_2$. Conclude that $K_1 \subseteq K_2$. By symmetric argument $K_2 \subseteq K_1$ and hence $K_1 = K_2$. Thus by definition φ is injective.

Let L be a submodule of M/N. Consider the natural projection map $\pi : M \to M/N$ defined by $\pi(m) = m + N$. We want to show that there exists $K \in S$ such that $\varphi(K) = L$. To do this we will show that $\pi^{-1}(L)$ is a submodule of M and that $N \subseteq \pi^{-1}(L)$. Recall that $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$. Observe that $0 \in \pi^{-1}(L)$ since $\pi(0) = 0$ and hence $\pi^{-1}(L) \neq \emptyset$. Let $x, y \in \pi^{-1}(L)$ and $r \in R$. Observe that $\pi(x + ry) = \pi(x) + r\pi(y)$. Since $\pi(y) \in L$ by definition and L is a submodule of M/N, it follows that since scalar multiplication is closed $r\pi(y) \in L$. Thus it follows that $\pi(x) + r\pi(y) \in L$ and hence $x + ry \in \pi^{-1}(L)$. Thus by the submodule criterion we can conclude that $\pi^{-1}(L)$ is in fact a submodule. Now let $n \in N$ and observe that $\pi(n) = n + N = 0 + N \in L$ so by definition it follows that $n \in \pi^{-1}(L)$. Conclude that $N \subseteq \pi^{-1}(L)$ and hence φ is surjective.

Conclude φ is bijective and result follows. □

Exercise 0.2. (DF §10.2 Exercise 1): Use the submodule criterion to show that the kernels and images of R-module homomorphisms are submodules.

Proof. Let M, N be R-modules and $\varphi : M \to N$ an R-module homomorphism. Recall that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ and $\operatorname{im}\varphi = \{n \in N : \text{there exists } m \in M \text{ with } \varphi(m) = n\}$.

Observe that $\varphi(0) = 0$ so $0 \in \ker\varphi \neq \emptyset$. Let $m, m' \in M$, $r \in R$. Now $\varphi(m + rm') = \varphi(m) + r\varphi(m') = 0 + r \cdot 0 = 0 + 0 = 0$. So $m + rm' \in \ker\varphi$. Thus by the submodule criterion $\ker\varphi$ is a submodule.

Observe that $\varphi(0) = 0 \in N$ so $0 \in \operatorname{im}\varphi \neq \emptyset$. Let $n, n' \in N$, $r \in R$. Then there exists $m, m' \in M$ such that $\varphi(m) = n$ and $\varphi(m') = n'$. Now consider $n + rn'$. $\varphi(m + rm') = \varphi(m) + r\varphi(m') = n + rn'$ so $n + rn' \in \operatorname{im}\varphi$. Conclude by submodule criterion that $\operatorname{im}\varphi$ is in fact a submodule. □

Exercise 0.3. (DF §10.2 Exercise 2): Show that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules.

Proof. Let X be a set of R-modules.

- Let $M \in X$. Observe that M is isomorphic to M trivially. So relation is reflexive.
- Let $M, N \in X$. Suppose M is isomorphic to N then by definition there exists $\varphi : M \to N$ that is bijective. Immediately we have $\varphi^{-1} : N \to M$ which is also bijective so N is isomorphic to M. By definition the relation is symmetric.
- Let $L, M, N \in X$. Suppose L is isomorphic to M, then by definition there exists $\varphi : L \to M$ a bijective R-module homomorphism. Suppose M is isomorphic to N, then there exists $\Phi : M \to N$ a bijective R-module homomorphism. Observe that $\varphi \circ \Phi : L \to N$ is again a bijective R-module homomorphism by property of composition of mappings. Hence by definition L is isomorphic to N.

Conclude that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules. □

Exercise 0.4. (DF §10.2 Exercise 3): Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

Solution. Consider the Quaternions $\mathbb{H} = R$; they form a commutative group under addition and a noncommutative group under multiplication. Hence $\mathbb{H}$ is a noncommutative ring with unity. In particular $\mathbb{H}$ is an R-module over itself. Define $\varphi : \mathbb{H} \to \mathbb{H}$ by $\varphi(h) = ih$. This is a group homomorphism since $\varphi(h + h') = i(h + h') = ih + ih' = \varphi(h) + \varphi(h')$. But note that $\varphi(j \cdot 1) = \varphi(j) = ij = k \neq -k = ji = j(i \cdot 1) = j\varphi(1)$. Conclude that φ is not an R-module homomorphism since the definition is not satisfied. ◀

Exercise 0.5. (DF §10.2 Exercise 4): Let A be and $\mathbb{Z}$-module, let a be any element of A and let n be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A : na = 0\}$ (So A_n is the annihilator in A of the ideal (n) of $\mathbb{Z}$).

Proof. Suppose that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism. Then by definition if $\overline{m} = \overline{k}$ then $\varphi(\overline{m}) = \varphi(\overline{k})$ or equivalently $ma = ka$. Moreover $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ra) = r\varphi(a)$ for all $a, b \in A$ and $r \in \mathbb{Z}$. Observe that $\overline{0} = \overline{k}$ so by hypothesis $\varphi(\overline{0}) = \varphi(\overline{k})$ but observe that $\varphi(\overline{0}) = 0 \cdot a = 0$ and $\varphi(\overline{k}) = ka$. Hence by equality $ka = 0$. Conversely suppose that $na = 0$. We want to show that $\varphi : \mathbb{Z}/n\mathbb{Z} \to A$ defined by $\varphi(\overline{k}) = ka$ is a well-defined R-module homomorphism. Say $\overline{k} = \overline{m}$ then by property of cosets $k - m \in \mathbb{Z}/n\mathbb{Z}$ and so $n \cong k - m$. By definition $n \mid k - m$ and hence there exists $t \in \mathbb{Z}$ such that $k - m = nt$. Observe that

$$\begin{aligned} k - m &= nt \\ (k - m)a &= nta \\ ka - ma &= (na)t \\ ka - ma &= 0 \\ ka &= ma \end{aligned}$$

Thus we have $\varphi(\overline{k}) = \varphi(\overline{m})$ and we can conclude that φ is in fact a well-defined R-module homomorphism.

Now □

Exercise 0.6. (DF §10.2 Exercise 5):

Exercise 0.7. (DF §10.2 Exercise 6):

Exercise 0.8. (DF §10.2 Exercise 7):

Exercise 0.9. (DF §10.2 Exercise 8): Let $\varphi : M \to N$ be an R-module homomorphism. Prove that $\varphi(\operatorname{Tor}(M)) \subseteq \operatorname{Tor}(N)$.

Proof. Recall that $\operatorname{Tor}(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Now it follows that $\varphi(\operatorname{Tor}(M)) = \{n \in N : n = \varphi(m)$ for some $m \in \operatorname{Tor}(M)\}$. Let $n \in \varphi(\operatorname{Tor}(M))$ then $n = \varphi(m)$ for some $m \in \operatorname{Tor}(M)$ by definition. Since $m \in \operatorname{Tor}(M)$ there exists $0 \neq r \in R$ such that $rm = 0$. Hence $rn = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$. Conclude that $n \in \operatorname{Tor}(N)$ and hence $\varphi(\operatorname{Tor}(M)) \subseteq \operatorname{Tor}(N)$. □

Exercise 0.10. (DF §10.2 Exercise 9):

Exercise 0.11. (DF §10.2 Exercise 10):

Exercise 0.12. (DF §10.2 Exercise 11):

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