INTRODUCTION TO HOMOLOGICAL ALGEBRA

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0. MATH 697 Homework Zero.Two

AM 2.1: Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if m and n are coprime.

Proof. Choose $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. Since m and n are coprime, there exist $s, t \in \mathbb{Z}$ such that $ms + nt = 1$ Observe that

$$a = a \cdot 1 = a(ms + nt) = ams + ant \equiv ant \pmod{m}.$$

Now observe that

$$a \otimes b = atn \otimes b = a \otimes nb = at \otimes 0 = 0.$$

We have shown that any simple tensor is zero, so any finite linear combination of simple tensors is zero. Conclude $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$. □

AM 2.2: Let R be a ring, I an ideal of R, M an R-module. Show that $(R/I) \otimes_R M$ is isomorphic to M/IM.

Proof. Define $\varphi : R/I \times M \to M/IM$ by $\varphi(r + I, m) = rm + IM$, which we shall henceforth write as $\varphi(\overline{r}, m) = \overline{rm}$. Let $(\overline{r}, m) = (\overline{s}, m)$. Then $\overline{r} = \overline{s} \implies r \in \overline{s} \implies r = s + i$, some $i \in I$. Then $\varphi(\overline{r}, m) = \overline{rm} = \overline{(s + i)m} = \overline{sm + im} = \overline{sm} + \overline{im} = \overline{sm} + \overline{0} = \overline{sm} = \varphi(\overline{s}, m)$. Thus φ is well-defined.

Observe $\varphi(\overline{r} + \overline{s}, m) = \varphi(\overline{r + s}, m) = \overline{(r + s)m} = \overline{rm + sm} = \overline{rm} + \overline{sm} = \varphi(\overline{r}, m) + \varphi(\overline{s}, m)$. Similarly, $\varphi(\overline{r}, m + n) = \varphi(\overline{r}, m) + \varphi(\overline{r}, n)$. Lastly, $\varphi(\overline{rs}, m) = \overline{(rs)m} = \overline{r(sm)} = \varphi(\overline{r}, sm)$. Thus φ is R-biadditive (In fact, φ is R-bilinear).

Now we are guaranteed a unique R-homomorphism $\phi : R/I \otimes_R M \to M/IM$ given by $\phi(\overline{r} \otimes m) = \overline{rm}$. Notice if we define $f : M/IM \to R/I \otimes_R M$ via $f(\overline{m}) = \overline{1} \otimes m$ then f is a $\mathbb{Z}$-homomorphism which makes $f \circ \phi$ and $\phi \circ f$ the identity map in $R/I \otimes_R M$ and M/IM, respectively. So ϕ has a two-sided inverse, hence a bijective function, and accordingly is an isomorphism when considered as an R-map. □

R 2.28: Let R be a domain with $Q = \mathrm{Frac}(R)$, its field of fractions. If A is an R-module, prove that every element of $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$ (i.e. every element is a simple tensor).

Proof. Let $\sum_1^n q_i \otimes a_i \in Q \otimes_R A$. We can write $\sum_1^n q_i \otimes a_i = \sum_1^n \frac{r_i}{s_i} \otimes a_i$ for $r_i, s_i \in R, s_i \neq 0$. Write $s = s_1 s_2 \cdots s_n$ and $\widehat{s_i} = \frac{s}{s_i}$. Then $\sum_1^n \frac{r_i}{s_i} \otimes a_i = \sum_1^n (1 \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n (\frac{\widehat{s_i}}{\widehat{s_i}} \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n \frac{\widehat{s_i} r_i}{s} \otimes a_i = \sum_1^n (\frac{1}{s}) \widehat{s_i} r_i \otimes a_i = \sum_1^n \frac{1}{s} \otimes (\widehat{s_i} r_i) a_i = \frac{1}{s} \otimes (\sum_1^n \widehat{s_i} r_i a_i)$. □

R 2.29:**(i)** Let p be a prime, and let p, q be relatively prime. Prove that is A is a p-primary group and $a \in A$, then there exists $x \in A$ with $qx = a$.

(ii) If D is a finite cyclic group of order m, prove that D/nD is a cyclic group of order $d = (m, n)$.

(iii) Let m and n be positive integers, and let $d = (m, n)$. Prove that there is an isomorphism of abelian groups

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d.$$

(iv) Let G and H be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \text{ and } H = B_1 \oplus \cdots \oplus B_m,$$

where A_i and B_j are cyclic groups. Compute $G \otimes_{\mathbb{Z}} H$ explicitely.

Proof. (i) $a \in A$ so $p^k a = 0$, some $k \in \mathbb{Z}^+$. Since p is prime, $(q, p) = 1 \implies (q, p^k) = 1$. So there exist $m, n \in \mathbb{Z}$ such that $qm + np^k = 1$. Now $a = 1 \cdot a = (qm + np^k)a = qma + np^k a = q(ma) + n(p^k a) = qx + 0 = qx$. Observe $p^k x = p^k(ma) = m(p^k a) = 0$ so $x \in A$.

(ii) D is cyclic, so D/nD is cyclic. If we write $D =< a >$, then $nD =< na >$. This is because for any $nb \in nD$, we can write $b = ka$, some $k \in \mathbb{Z}^+$, since a generates D. Now $nb = n(ka) = k(na)$, and we have that na generates nD.

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Claim $|na| = \frac{m}{d}$. Observe $\frac{m}{d}(na) = \frac{n}{d}(ma) = \frac{n}{d} \cdot 0 = 0$, which implies $|na|$ divides $\frac{m}{d}$. On the other hand, if $k(na) = 0$, then $(kn)a = 0 \implies m|kn \implies \frac{m}{d}|k\frac{n}{d}$. But $\frac{m}{d}$ and $\frac{n}{d}$ are relatively prime by construction, forcing $\frac{m}{d}|k$. In particular, we have $\frac{m}{d}$ divides $|na|$. Thus $|na| = d$. Now $|nD| = |<na>| = |na| = \frac{m}{d}$. Lagrange's theorem gives us $|\frac{D}{nD}| = \frac{|D|}{|nD|} = \frac{m}{\frac{m}{d}} = d$.

(iii) By Proposition 2.68, we have that $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_m/n\mathbb{Z}_m$. But by part (ii), $\mathbb{Z}_m/n\mathbb{Z}_m$ is a cyclic group of order $d = (m, n)$ so $\mathbb{Z}_m/n\mathbb{Z}_m \cong \mathbb{Z}_d$.

(iv)

□

R 2.32: Consider the following commutative diagram in ${}_R\textbf{Mod}$ having exact columns.

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

Proof. α_1 is injective: Let $a' \in \ker \alpha_1$. Then $\alpha_1(a') = 0$. So $f(\alpha_1(a')) = 0$. Now $0 = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. The injectivity of β_1 implies $f'(a') = 0$ and the injectivity of f' gives us $a' = 0$. Thus $\ker \alpha_1 = 0$ and α_1 is injective.

α_2 is surjective:

im $\alpha_1 \subseteq \ker \alpha_2$: Let $a \in$ im α_1. Then there exists $a' \in A'$ with $a = \alpha_1(a')$. Observe $f(a) = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. Thus $\beta_2(f(a)) = \beta_2(\beta_1(f'(a'))) = 0$ by exactness. Now $0 = \beta_2(f(a)) = f''(\alpha_2(a))$ by commutativity. The injectivity of f'' gives us $\alpha_2(a) = 0$. Hence $a \in \ker \alpha_2$.

$\ker \alpha_2 \subseteq$ im α_1: Let $a \in \ker \alpha_2$. Then $\alpha_2(a) = 0$. So $f''(\alpha_2(a)) = 0$. By commutativity, $\beta_2(f(a)) = 0$. Now $f(a) \in \ker \beta_2 =$ im β_1, so there exists $b' \in B'$ such that $f(a) = \beta_1(b')$. Now $g(f(a)) = 0$ by exactness, so $g(\beta_1(b')) = 0$. By commutativity, $\gamma_1(g'(b')) = 0$. Since γ_1 is injective, $g'(b') = 0$. Now $b' \in \ker g' =$ im f' so there exists $a' \in A'$ such that $b' = f'(a')$. Thus $f(a) = \beta_1(b') = \beta_1(f'(a'))$. By commutativity, $\beta_1(f'(a')) = f(\alpha_1(a'))$. So $f(a) = f(\alpha_1(a'))$. Since f is injective, we have $a = \alpha_1(a')$, and therefore $a \in$ im α_1.

□

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