

# INTRODUCTION TO HOMOLOGICAL ALGEBRA

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## 0. MATH 697 HOMEWORK ZERO.TWO

**AM 2.1:** Show that  $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$  if  $m$  and  $n$  are coprime.

*Proof.* Choose  $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ . Since  $m$  and  $n$  are coprime, there exist  $s, t \in \mathbb{Z}$  such that  $ms + nt = 1$ . Observe that

$$a = a \cdot 1 = a(ms + nt) = ams + ant \equiv ant \pmod{m}.$$

Now observe that

$$a \otimes b = atn \otimes b = a \otimes nb = at \otimes 0 = 0.$$

We have shown that any simple tensor is zero, so any finite linear combination of simple tensors is zero. Conclude  $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ . □

**AM 2.2:** Let  $R$  be a ring,  $I$  an ideal of  $R$ ,  $M$  an  $R$ -module. Show that  $(R/I) \otimes_R M$  is isomorphic to  $M/IM$ .

*Proof.* Define  $\varphi : R/I \times M \rightarrow M/IM$  by  $\varphi(r + I, m) = rm + IM$ , which we shall henceforth write as  $\varphi(\bar{r}, m) = \overline{rm}$ . Let  $(\bar{r}, m) = (\bar{s}, m)$ . Then  $\bar{r} = \bar{s} \implies r \in \bar{s} \implies r = s + i$ , some  $i \in I$ . Then  $\varphi(\bar{r}, m) = \overline{rm} = \overline{(s + i)m} = \overline{sm + im} = \overline{sm} + \overline{im} = \overline{sm} + \overline{0} = \overline{sm} = \varphi(\bar{s}, m)$ . Thus  $\varphi$  is well-defined.

Observe  $\varphi(\bar{r} + \bar{s}, m) = \varphi(\overline{r + s}, m) = \overline{(r + s)m} = \overline{rm + sm} = \overline{rm} + \overline{sm} = \varphi(\bar{r}, m) + \varphi(\bar{s}, m)$ . Similarly,  $\varphi(\bar{r}, m + n) = \varphi(\bar{r}, m) + \varphi(\bar{r}, n)$ . Lastly,  $\varphi(\bar{r}\bar{s}, m) = \overline{(rs)m} = \overline{r(sm)} = \varphi(\bar{r}, sm)$ . Thus  $\varphi$  is  $R$ -biadditive (In fact,  $\varphi$  is  $R$ -bilinear).

Now we are guaranteed a unique  $R$ -homomorphism  $\phi : R/I \otimes_R M \rightarrow M/IM$  given by  $\phi(\bar{r} \otimes m) = \overline{rm}$ . Notice if we define  $f : M/IM \rightarrow R/I \otimes_R M$  via  $f(\bar{m}) = \bar{1} \otimes m$  then  $f$  is a  $\mathbb{Z}$ -homomorphism which makes  $f \circ \phi$  and  $\phi \circ f$  the identity map in  $R/I \otimes_R M$  and  $M/IM$ , respectively. So  $\phi$  has a two-sided inverse, hence a bijective function, and accordingly is an isomorphism when considered as an  $R$ -map. □

### R Proposition 2.18:

- (1) A sequence  $0 \rightarrow A \xrightarrow{f} B$  is exact if and only if  $f$  is injective.
- (2) A sequence  $B \xrightarrow{g} C \rightarrow 0$  is exact if and only if  $g$  is surjective.
- (3) A sequence  $0 \rightarrow A \xrightarrow{h} B \rightarrow 0$  is exact if and only if  $h$  is an isomorphism.

**DF §10.5 Proposition 24:** (*The Short Five Lemma*) Let  $\alpha, \beta, \gamma$  be homomorphisms of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

- (1) If  $\alpha$  and  $\gamma$  are injective then so is  $\beta$ .

*Proof.* Let  $b \in B$  such that  $\beta(b) = 0$ . We want to show that  $b = 0$ . Observe that  $g'(\beta(b)) = g'(0) = 0$ . By commutativity we have  $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$ . Since  $\gamma$  is injective we know  $g(b) = 0$  so  $b \in \ker g$  but since we are in an exact sequence we have  $\text{im } f = \ker g$  and hence  $b \in \text{im } f$ . By definition there exists  $a \in A$  with  $f(a) = b$ . Now  $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$ . Since  $f'$  is injective, it follows that  $\alpha(a) = (f')^{-1}(0) = 0$ . Now we have  $a = \alpha^{-1}(0)$  and so  $a = 0$ . So  $0 = f(a) = b$ . □

- (2) If  $\alpha$  and  $\gamma$  are surjective then so is  $\beta$ .

*Proof.* Let  $b' \in B'$  then  $g'(b') \in C'$ . Since  $\gamma$  is surjective there exists  $c \in C$  such that  $\gamma(c) = g'(b')$ . Since this is an exact sequence,  $g$  is surjective so there exists  $b \in B$  such that  $g(b) = c$ . By equality we have  $\gamma(c) = \gamma(g(b)) = g'(b')$ . Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular  $b' - \beta(b) \in \ker g'$  but by exactness  $\text{im } f' = \ker g'$  so there exists  $a' \in A'$  such that  $f'(a') = b' - \beta(b)$ . But since  $\alpha$  is surjective, there exists  $a \in A$  such that  $\alpha(a) = a'$ . Now  $f'(a') = f'(\alpha(a)) = b' - \beta(b)$ . By commutativity  $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$  so  $\beta(f(a)) + \beta(b) = b'$  and we have  $\beta(f(a) + b) = b'$ .  $\square$

- (3) If  $\alpha$  and  $\gamma$  are isomorphisms then so is  $\beta$  (and then the two sequences are isomorphism).

*Proof.* Follows from (1) and (2).  $\square$

**R 2.29:(i)** Let  $p$  be a prime, and let  $p, q$  be relatively prime. Prove that if  $A$  is a  $p$ -primary group and  $a \in A$ , then there exists  $x \in A$  with  $qx = a$ .

- (ii) If  $D$  is a finite cyclic group of order  $m$ , prove that  $D/nD$  is a cyclic group of order  $d = (m, n)$ .

- (iii) Let  $m$  and  $n$  be positive integers, and let  $d = (m, n)$ . Prove that there is an isomorphism of abelian groups

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d.$$

- (iv) Let  $G$  and  $H$  be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \text{ and } H = B_1 \oplus \cdots \oplus B_m,$$

where  $A_i$  and  $B_j$  are cyclic groups. Compute  $G \otimes_{\mathbb{Z}} H$  explicitly.

*Proof.* (i)  $a \in A$  so  $p^k a = 0$ , some  $k \in \mathbb{Z}^+$ . Since  $p$  is prime,  $(q, p) = 1 \implies (q, p^k) = 1$ . So there exist  $m, n \in \mathbb{Z}$  such that  $qm + np^k = 1$ . Now  $a = 1 \cdot a = (qm + np^k)a = qma + np^k a = q(ma) + n(p^k a) = qx + 0 = qx$ . Observe  $p^k x = p^k(ma) = m(p^k a) = 0$  so  $x \in A$ .

(ii)  $D$  is cyclic, so  $D/nD$  is cyclic. If we write  $D = \langle a \rangle$ , then  $nD = \langle na \rangle$ . This is because for any  $nb \in nD$ , we can write  $b = ka$ , some  $k \in \mathbb{Z}^+$ , since  $a$  generates  $D$ . Now  $nb = n(ka) = k(na)$ , and we have that  $na$  generates  $nD$ .

Claim  $|na| = \frac{m}{d}$ . Observe  $\frac{m}{d}(na) = \frac{n}{d}(ma) = \frac{n}{d} \cdot 0 = 0$ , which implies  $|na|$  divides  $\frac{m}{d}$ . On the other hand, if  $k(na) = 0$ , then  $(kn)a = 0 \implies m|kn \implies \frac{m}{d}|k\frac{n}{d}$ . But  $\frac{m}{d}$  and  $\frac{n}{d}$  are relatively prime by construction, forcing  $\frac{m}{d}|k$ . In particular, we have  $\frac{m}{d}$  divides  $|na|$ . Thus  $|na| = d$ . Now  $|nD| = |\langle na \rangle| = |na| = \frac{m}{d}$ . Lagrange's theorem gives us  $|\frac{D}{nD}| = \frac{|D|}{|nD|} = \frac{m}{d} = d$ .

(iii) By Proposition 2.68, we have that  $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_m / n\mathbb{Z}_m$ . But by part (ii),  $\mathbb{Z}_m / n\mathbb{Z}_m$  is a cyclic group of order  $d = (m, n)$  so  $\mathbb{Z}_m / n\mathbb{Z}_m \cong \mathbb{Z}_d$ .

- (iv) In the case where  $A_i$  and  $B_j$  are finite cyclic groups we have that  $G \otimes_{\mathbb{Z}} H = \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j \cong \sum_{i,j} \mathbb{Z}_{s_i} \otimes \mathbb{Z}_{t_j} \cong \sum_{i,j} \mathbb{Z}_{(s_i, t_j)}$

by (iii). For the case where we have some  $A_i$  or  $B_j$  infinite observe that

$$G \otimes_{\mathbb{Z}} H = \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j = \sum_{i,j: A_i \text{ finite and } B_j \text{ finite}} \mathbb{Z}_{(s_i, t_j)} + \sum_{i,j: A_i \text{ or } B_j \text{ infinite}} \mathbb{Z}$$

where  $|A_i| = s_i$  and  $|B_j| = t_j$  where  $A_i, B_j$  finite.  $\square$

**R Exercise 2.14:** Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be a sequence of module maps. Prove that  $gf = 0$  if and only if  $\text{im } f \subseteq \ker g$ . Give an example of such a sequence that is not exact.

*Proof.* Suppose  $gf = 0$ , that is,  $f(g(a)) = 0$  for all  $a \in A$ . Let  $b \in \text{im } f$  then by definition there exists  $a \in A$  such that  $f(a) = b$ . But we know by hypothesis that  $0 = g(f(a)) = g(b)$  so  $b \in \ker g$ . Conclude that  $\text{im } f \subseteq \ker g$ . Conversely, suppose that  $\text{im } f \subseteq \ker g$ . Let  $a \in A$  and observe that  $f(a) \in \text{im } f$ . By hypothesis  $f(a) \in \ker g$  so  $g(f(a)) = 0$ . Since  $a$  was arbitrary conclude  $gf = 0$ .  $\square$

Consider the sequence of module maps  $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$  where  $f(\bar{x}) = 2\bar{x}$  and  $g(\bar{y}) = \bar{y}$ . Observe that  $\text{im } f = \mathbb{Z}/3\mathbb{Z}$  and  $\ker g = \{0\}$ . Since  $\text{im } f \neq \ker g$  it follows that this is **not** an exact sequence.

**R Exercise 2.15:**

- (1) Prove that  $f : M \rightarrow N$  is surjective if and only if  $\text{coker } f = \{0\}$ .

*Proof.* Suppose  $f : M \rightarrow N$  is surjective then for  $n \in N$ , there exists  $m \in M$  such that  $f(m) = n$ . By definition  $\text{coker } f = M / \text{im } f = M / M = 0$ . Conversely, suppose that  $\text{coker } f = 0$ , i.e.,  $M / \text{im } f = 0$  implying that if  $m + \text{im } f \in M / \text{im } f$  then  $m + \text{im } f = 0$  or equivalently  $m \in \text{im } f$ . Since  $m$  is arbitrary, conclude  $M = \text{im } f$  and hence  $f$  is surjective by definition.  $\square$

- (2) If  $f : M \rightarrow N$  is a map, prove that there is an exact sequence

$$0 \rightarrow \ker f \rightarrow M \xrightarrow{f} N \rightarrow \text{coker } f \rightarrow 0.$$

*Proof.* Define  $h : \ker f \rightarrow M$  by  $h(m) = m$ , that is, map each element to itself. It follows immediately that  $\text{im } h = \ker f$ . Define  $g : N \rightarrow \text{coker } f = N/\text{im } f$  by  $g(n) = n + \text{im } f$ , that is, the canonical/projection mapping. Observe that  $\ker g = \text{im } f$ . Conclude that the following sequence is in fact exact:

$$0 \xrightarrow{\text{identity}} \ker f \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} \text{coker } f \xrightarrow{\text{zero}} 0. \quad \square$$

**R Exercise 2.16:**

- (1) If  $0 \rightarrow M \rightarrow 0$  is an exact sequence, prove that  $M = \{0\}$ .

*Proof.* Consider  $0 \xrightarrow{f} M \xrightarrow{g} 0$ . Since  $f$  is surjective then for  $m \in M$  there exists  $x \in 0$  such that  $f(x) = m$  but  $x$  must be  $0$  so  $m = 0$ .  $\square$

- (2) If  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$  is an exact sequence, prove that  $f$  is surjective if and only if  $h$  is injective.

*Proof.* Suppose  $f$  is surjective. Then  $\text{im } f = B = \ker g$  but this immediately implies that  $\text{im } g = 0 = \ker h$  so  $h$  is injective by definition. Conversely, suppose  $h$  is injective. Then  $\ker h = 0 = \text{im } g$  which immediately implies  $\ker g = B = \text{im } f$ . Conclude by definition  $f$  is surjective.  $\square$

- (3) Let  $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$  be exact. If  $\alpha$  and  $\delta$  are isomorphisms, prove that  $C = \{0\}$ .

*Proof.* Observe that, by previous exercise,  $\beta$  is surjective and  $\gamma$  is injective so we have  $\text{im } \beta = C$  and  $\ker \gamma = 0$ . Result follows by exactness:  $C = \text{im } \beta = \ker \gamma = 0$ . Conclude  $C = \{0\}$ .  $\square$

**R Exercise 2.17:** If  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$  is exact, prove that there is an exact sequence  $0 \rightarrow \text{coker } f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \rightarrow 0$ , where  $\alpha : b + \text{im } f \mapsto g(b)$  and  $\beta : c \mapsto h(c)$ .

*Proof.* Observe that  $\ker \beta = \{c \in C : \beta(c) = h(c) = 0\} = \ker h = \text{im } g$  by exactness and since  $\alpha$  can run through any  $b \in B$  conclude  $\text{im } \alpha = \text{im } g = \ker \beta$ .  $\square$

**R Exercise 2.28:** Let  $R$  be a domain with  $Q = \text{Frac}(R)$ , its field of fractions. If  $A$  is an  $R$ -module, prove that every element of  $Q \otimes_R A$  has the form  $q \otimes a$  for  $q \in Q$  and  $a \in A$  (i.e. every element is a simple tensor).

*Proof.* Let  $\sum_1^n q_i \otimes a_i \in Q \otimes_R A$ . We can write  $\sum_1^n q_i \otimes a_i = \sum_1^n \frac{r_i}{s_i} \otimes a_i$  for  $r_i, s_i \in R, s_i \neq 0$ . Write  $s = s_1 s_2 \cdots s_n$  and  $\hat{s}_i = \frac{s}{s_i}$ . Then  $\sum_1^n \frac{r_i}{s_i} \otimes a_i = \sum_1^n (1 \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n (\frac{\hat{s}_i}{s_i} \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n \frac{\hat{s}_i r_i}{s} \otimes a_i = \sum_1^n (\frac{1}{s}) \hat{s}_i r_i \otimes a_i = \sum_1^n \frac{1}{s} \otimes (\hat{s}_i r_i) a_i = \frac{1}{s} \otimes (\sum_1^n \hat{s}_i r_i a_i)$ .  $\square$

**R Exercise 2.32:** ( $3 \times 3$  Lemma) Consider the following commutative diagram in  ${}_R \mathbf{Mod}$  having exact columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \longrightarrow 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' \\
 0 & \longrightarrow & B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \longrightarrow 0 \\
 & & \downarrow g' & & \downarrow g & & \downarrow g'' \\
 0 & \longrightarrow & C' & \xrightarrow{\gamma_1} & C & \xrightarrow{\gamma_2} & C'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

*Proof.*  $\alpha_1$  is injective: Let  $a' \in \ker \alpha_1$ . Then  $\alpha_1(a') = 0$ . So  $f(\alpha_1(a')) = 0$ . Now  $0 = f(\alpha_1(a')) = \beta_1(f'(a'))$  by commutativity. The injectivity of  $\beta_1$  implies  $f'(a') = 0$  and the injectivity of  $f'$  gives us  $a' = 0$ . Thus  $\ker \alpha_1 = 0$  and  $\alpha_1$  is injective.

$\alpha_2$  is surjective:

$\text{im } \alpha_1 \subseteq \ker \alpha_2$ : Let  $a \in \text{im } \alpha_1$ . Then there exists  $a' \in A'$  with  $a = \alpha_1(a')$ . Observe  $f(a) = f(\alpha_1(a')) = \beta_1(f'(a'))$  by commutativity. Thus  $\beta_2(f(a)) = \beta_2(\beta_1(f'(a')))) = 0$  by exactness. Now  $0 = \beta_2(f(a)) = f''(\alpha_2(a))$  by commutativity. The injectivity of  $f''$  gives us  $\alpha_2(a) = 0$ . Hence  $a \in \ker \alpha_2$ .

$\ker \alpha_2 \subseteq \operatorname{im} \alpha_1$ : Let  $a \in \ker \alpha_2$ . Then  $\alpha_2(a) = 0$ . So  $f''(\alpha_2(a)) = 0$ . By commutativity,  $\beta_2(f(a)) = 0$ . Now  $f(a) \in \ker \beta_2 = \operatorname{im} \beta_1$ , so there exists  $b' \in B'$  such that  $f(a) = \beta_1(b')$ . Now  $g(f(a)) = 0$  by exactness, so  $g(\beta_1(b')) = 0$ . By commutativity,  $\gamma_1(g'(b')) = 0$ . Since  $\gamma_1$  is injective,  $g'(b') = 0$ . Now  $b' \in \ker g' = \operatorname{im} f'$  so there exists  $a' \in A'$  such that  $b' = f'(a')$ . Thus  $f(a) = \beta_1(b') = \beta_1(f'(a'))$ . By commutativity,  $\beta_1(f'(a')) = f(\alpha_1(a'))$ . So  $f(a) = f(\alpha_1(a'))$ . Since  $f$  is injective, we have  $a = \alpha_1(a')$ , and therefore  $a \in \operatorname{im} \alpha_1$ .  $\square$

**AM Proposition 2.10:** (*Snake Lemma*) Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{g} & M & \xrightarrow{h} & M'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & N' & \xrightarrow{g'} & N & \xrightarrow{h'} & N'' \longrightarrow 0 \end{array}$$

be a commutative diagram of  $R$ -modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \rightarrow \ker(f') \xrightarrow{\bar{u}} \ker(f) \xrightarrow{\bar{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\bar{u}'} \operatorname{coker}(f) \xrightarrow{\bar{v}'} \operatorname{coker}(f'') \rightarrow 0$$

in which  $\bar{u}, \bar{v}$  are restrictions of  $u, v$ , and  $\bar{u}', \bar{v}'$  are induced by  $u', v'$ .

*Proof.* We consider:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker f' & \xrightarrow{\bar{u}} & \ker f & \xrightarrow{\bar{v}} & \ker f'' \\ & & \downarrow g' & & \downarrow g & & \downarrow g'' \\ 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' \longrightarrow 0 \\ & & \downarrow h' & & \downarrow h' & & \downarrow h' \\ & & \operatorname{coker} f' & \xrightarrow{\bar{u}'} & \operatorname{coker} f & \xrightarrow{\bar{v}'} & \operatorname{coker} f'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

By Rotman Exercise 2.32, we have that  $\operatorname{im} \bar{u} = \ker \bar{v}$  and  $\operatorname{im} \bar{u}' = \ker \bar{v}'$ . It is left for us to first define  $d$  and then show that  $\operatorname{im} \bar{v} = \ker d$  and  $\operatorname{im} d = \ker \bar{u}'$ .  $\square$