INTRODUCTION TO HOMOLOGICAL ALGEBRA

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0. MATH 697 Homework Zero.Two

AM 2.1: Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if m and n are coprime.

Proof. Observe that since m and n are relatively prime, there exists $s, t \in \mathbb{Z}$ such that $ms + nt = 1$. Now let $a \otimes b \in (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$. Observe that

$$\begin{aligned}
a \otimes b &= a \cdot 1 \otimes b \cdot 1 \\
&= a(ms + nt) \otimes b(ms + nt) \\
&= (ams + ant) \otimes (bms + bnt) \\
&= ams \otimes (bms + bnt) + ant \otimes (bms + bnt) \\
&= ams \otimes bms + ams \otimes bnt + ant \otimes bms + ant \otimes bnt \\
&= 0 \otimes bms + ams \otimes 0 + ant \otimes bms + ant \otimes 0 \\
&= ant \otimes bms \\
&= atn \otimes bms \\
&= at \otimes nbms \\
&= at \otimes 0 \\
&= 0
\end{aligned}$$

Hence we have shown that any simple tensor is zero, so any finite linear combination of simple tensors is zero.

Conclude that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ holds. ☐

AM 2.2: Let R be a ring, I an ideal, M an R-module. Show that $(R/I) \otimes_I M$ is isomorphic to M/IM. [Tensor the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ with M.]

Proof. Define $\varphi : R/I \otimes_R M \to M/IM$ by $\varphi(r + I \otimes m) = rm + IM$. We must first show this mapping is well-defined. Suppose $r_1 + I \otimes m_1 = r_2 + I \otimes m_2$.

Question: Can we assume $m_1 = m_2$? If we can then the well-definedness follows immediately. But I am unsure. Recall that the simple tensor is just a coset itself, i.e., $r + I \otimes m = (r + I, m) + (R/I) \otimes_R M$. ☐

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