INTRODUCTION TO HOMOLOGICAL ALGEBRA

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1. MATH 697 Notes

R Proposition 2.18:

(1) A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective.
(2) A sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.
(3) A sequence $0 \to A \xrightarrow{h} B \to 0$ is exact if and only if h is an isomorphism.

DF §10.5 Proposition 24: (*The Short Five Lemma*) Let α, β, γ be homomorphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) If α and γ are injective then so is β.

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that $b = 0$. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know $g(b) = 0$ so $b \in \ker g$ but since we are in an exact sequence we have $\operatorname{im} f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with $f(a) = b$. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so $a = 0$. So $0 = f(a) = b$. $\square$

(2) If α and γ are surjective then so is β.

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that $g(b) = c$. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness $\operatorname{im} f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$. $\square$

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

Proof. Follows from (1) and (2). $\square$

R Proposition 2.72: (*Five Lemma*) Consider the commutative diagram with exact rows.

$$\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "\alpha"] & A_2 \arrow[r, "g"] \arrow[d, "\beta"] & A_3 \arrow[r, "h"] \arrow[d, "\gamma"] & A_4 \arrow[r, "k"] \arrow[d, "\delta"] & A_5 \arrow[d, "\varepsilon"] \\
B_1 \arrow[r, "f'"] & B_2 \arrow[r, "g'"] & B_3 \arrow[r, "h'"] & B_4 \arrow[r, "k'"] & B_5
\end{tikzcd}$$

(1) If β and δ are surjective and ε is injective, then γ is surjective.
(2) If β and δ are injective and α is surjective, then γ is injective.
(3) If α, β, δ and ε are isomorphisms, then γ is an isomorphism.

R Exercise 2.14: Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of module maps. Prove that $gf = 0$ if and only if $\operatorname{im} f \subseteq \ker g$. Give an example of such a sequence that is not exact.

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Proof. Suppose $gf = 0$, that is, $f(g(a)) = 0$ for all $a \in A$. Let $b \in \operatorname{im} f$ then by definition there exists $a \in A$ such that $f(a) = b$. But we know by hypothesis that $0 = g(f(a)) = g(b)$ so $b \in \ker g$. Conclude that $\operatorname{im} f \subseteq \ker g$. Conversely, suppose that $\operatorname{im} f \subseteq \ker g$. Let $a \in A$ and observe that $f(a) \in \operatorname{im} f$. By hypothesis $f(a) \in \ker g$ so $g(f(a)) = 0$. Since a was arbitrary conclude $gf = 0$. $\square$

Consider the sequence of module maps $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$ where $f(\overline{x}) = 2\overline{x}$ and $g(\overline{y}) = \overline{y}$. Observe that $\operatorname{im} f = \mathbb{Z}/3\mathbb{Z}$ and $\ker g = \{0\}$. Since $\operatorname{im} f \neq \ker g$ it follows that this is **not** an exact sequence.

R Exercise 2.15:

(1) Prove that $f : M \to N$ is surjective if and only if $\operatorname{coker} f = \{0\}$.

Proof. Suppose $f : M \to N$ is surjective then for $n \in N$, there exists $m \in M$ such that $f(m) = n$. By definition $\operatorname{coker} f = M/\operatorname{im} f = M/M = 0$. Conversely, suppose that $\operatorname{coker} f = 0$, i.e., $M/\operatorname{im} f = 0$ implying that if $m + \operatorname{im} f \in M/\operatorname{im} f$ then $m + \operatorname{im} f = 0$ or equivalently $m \in \operatorname{im} f$. Since m is arbitrary, conclude $M = \operatorname{im} f$ and hence f is surjective by definition. $\square$

(2) If $f : M \to N$ is a map, prove that there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Proof. Define $h : \ker f \to M$ by $h(m) = m$, that is, map each element to itself. It follows immediately that $\operatorname{im} h = \ker f$. Define $g : N \to \operatorname{coker} f = N/\operatorname{im} f$ by $g(n) = n + \operatorname{im} f$, that is, the canonical/projection mapping. Observe that $\ker g = \operatorname{im} f$. Conclude that the following sequence is in fact exact:

$$0 \xrightarrow{\text{identity}} \ker f \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} \operatorname{coker} f \xrightarrow{\text{zero}} . \quad \square$$

R Exercise 2.16:

(1) If $0 \to M \to 0$ is an exact sequence, prove that $M = \{0\}$.

Proof. Consider $0 \xrightarrow{f} M \xrightarrow{g} 0$. Since f is surjective then for $m \in M$ there exists $x \in 0$ such that $f(x) = m$ but x must be 0 so $m = 0$. $\square$

(2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective if and only if h is injective.

Proof. Suppose f is surjective. Then $\operatorname{im} f = B = \ker g$ but this immediately implies that $\operatorname{im} g = 0 = \ker h$ so h is injective by definitoin. Conversely, suppose h is injective. Then $\ker h = 0 = \operatorname{im} g$ which immediately implies $\ker g = B = \operatorname{im} f$. Conclude by definition f is surjective. $\square$

(3) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ be exact. If α and δ are isomorphisms, prove that $C = \{0\}$.

Proof. Observe that, by previous exercise, β is surjective and γ is injective so we have $\operatorname{im} \beta = C$ and $\ker \gamma = 0$. Result follows by exactness: $C = \operatorname{im} \beta = \ker \gamma = 0$. Conclude $C = \{0\}$. $\square$

R Exercise 2.17: If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that there is an exact sequence $0 \to \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \to 0$, where $\alpha : b + \operatorname{im} f \mapsto g(b)$ and $\beta : c \mapsto h(c)$.

Proof. Observe that $\ker \beta = \{c \in C : \beta(c) = h(c) = 0\} = \ker h = \operatorname{im} g$ by exactness and since α can run through any $b \in B$ conclude $\operatorname{im} \alpha = \operatorname{im} g = \ker \beta$ $\square$

AM Proposition 2.10: (*Snake Lemma*) Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{g} & M & \xrightarrow{h} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow{g'} & N & \xrightarrow{h'} & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \to 0$$

in which $\overline{u}$, $\overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

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