INTRODUCTION TO CATEGORY THEORY

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Abstract. Category Theory: Remain calm and carry on when all the mathematics you've ever known and loved gets abstracted away into dots and arrows.

0. MATH 697 Homework Zero

Exercise 0.1. (DF §10.2 Theorem 4): Prove Theorem 4 (Isomorphism Theorems):

(1) (*The First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then $\ker\varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proof. Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then by definition $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$, $r \in R$. We want to show that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ is a submodule. Observe that since M is a module then M is an abelian group by definition so there exists $0 \in M$ such that $m+0 = m$ for all $m \in M$. In particular $\varphi(0) = \varphi(0+0) = \varphi(0)+\varphi(0)$ implying $\varphi(0) = 0$. Conclude that $0 \in \ker\varphi \neq \emptyset$. Now let $r \in R$, $x, y \in \ker\varphi$. Observe that $\varphi(x + ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r \cdot 0 = 0 + 0 = 0$. Hence $x + ry \in \ker\varphi$. Conclude by the submodule criterion that $\ker\varphi$ is in fact a submodule.

Now define $\Phi : M/\ker\varphi \to \varphi(M)$ by $\Phi(m + \ker\varphi) = \varphi(m)$. We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose $m + \ker\varphi = m' + \ker\varphi$ it follows by property of cosets that $m - m' \in \ker\varphi$, in particular $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$ and hence $\varphi(m) = \varphi(m')$. But since $\varphi(m) = \Phi(m + \ker\varphi)$ and $\varphi(m') = \Phi(m' + \ker\varphi)$ we have $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Conclude that Φ is in fact well-defined.

Suppose that $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Then it follows that $\varphi(m) = \varphi(m')$ and so $\varphi(m - m') = 0$ and so $m - m' \in \ker\varphi$. By property of cosets it follows that $m + \ker\varphi = m' + \ker\varphi$ and hence Φ is injective.

Let $n \in \varphi(M)$. Then by definition of image of φ there exists $m \in M$ such that $n = \varphi(m)$. It is immediate that $m + \ker\varphi \in M/\ker\varphi$ and we can conclude that Φ is surjective.

Now we must show that Φ is an R-module homomorphism. Let $x, y \in M/\ker\varphi$ where $x = m + \ker\varphi$ and $y = m' + \ker\varphi$ for some $m, m' \in M$ and let $r \in R$. Observe that

$$\begin{aligned}\Phi(x+y) &= \Phi(m + m' + \ker\varphi) \\ &= \varphi(m + m') \\ &= \varphi(m) + \varphi(m') \\ &= \Phi(m + \ker\varphi) + \Phi(m' + \ker\varphi) \\ &= \Phi(x) + \Phi(y)\end{aligned}$$

and

$$\begin{aligned}\Phi(rx) &= \Phi(r(m + \ker\varphi)) \\ &= \Phi(rm + \ker\varphi) \\ &= \varphi(rm) \\ &= r\varphi(m) \\ &= r\Phi(m + \ker\varphi) \\ &= r\Phi(x)\end{aligned}$$

Hence we have shown that Φ is a well-defined bijective homomorphism and thus we can conclude by definition of R-module isomorphism that $M/\ker\varphi \cong \varphi(M)$. $\square$

(2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R-module M. Then $(A + B)/B \cong A/(A \cap B)$.

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Proof. Define $\varphi : A \to (A+B)/B$ by $\varphi(a) = a + B$. This mapping is clearly well-defined. We want to show that φ is a homomorphism. Let $r \in R$, $a, a' \in A$ and observe that

$$\begin{aligned}\varphi(a + a') &= a + a' + B \\ &= a + B + a' + B \\ &= \varphi(a) + \varphi(a')\end{aligned}$$

and

$$\begin{aligned}\varphi(ra) &= ra + B \\ &= r(a + B) \\ &= r\varphi(a)\end{aligned}$$

and so φ is an R-module homomorphism by definition. Observe that $\ker\varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$. Now let $x \in (A+B)/B$ then $x = a + b + B$ for some $a \in A$, $b \in B$. But observe that $a + b + B = a + B$ by absorbption. So φ is immediately surjective. In particular we have $\varphi(A) = (A+B)/B$. Conclude by the First Isomorphism Theorem for Modules that $A/\ker\varphi = A/(A \cap B) \cong (A+B)/B = \varphi(A)$. □

(3) (*The Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

Proof. Define $\varphi M/A \to M/B$ by $\varphi(m + A) = m + B$. We need to show φ is well-defined. Suppose $m + A = m' + A$ then $m - m' \in A \subseteq B$ by property of cosets. It also follows that $m + B = m' + B$. Hence $\varphi(m + A) = m + B = m' + B = \varphi(m' + A)$ and hence φ is well-defined.

Now we must show φ is an R-module homomorphism. Let $m, m' \in M$ and $r \in R$. Observe that

$$\begin{aligned}\varphi((m + A) + (m' + a)) &= \varphi(m + m' + A) \\ &= m + m' + B \\ &= (m + B) + (m' + B) \\ &= \varphi(m + A) + \varphi(m' + A)\end{aligned}$$

and

$$\begin{aligned}\varphi(r(m + A)) &= \varphi(rm + A) \\ &= rm + B \\ &= r(m + A) \\ &= r\varphi(m + A)\end{aligned}$$

and hence we can conclude by definition that φ is an R-module homomorphism.

Observe that $\ker\varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A"\varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$. Let $m + B \in M/B$. Clearly $\varphi(m + A) = m + B$ and hence φ is surjective. Now by the First Isomorphism Theorem for Modules we have $(M/A)/\ker\varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$. □

(4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence cummutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Proof. Let N be a submodule of M. Define $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$, $T = \{L : L \text{ is a submodule of } M/N\}$. Define $\varphi : S \to T$ by $\varphi(K) = K/N$. We want to show that this mapping is bijective.

Let $K_1, K_2 \in S$ and suppose that $\varphi(K_1) = \varphi(K_2)$. Then $K_1/N = K_2/N$. We want to show that $K_1 = K_2$. Let $x \in K_1$, then $x + N \in K_1/N = K_2/N$, in particular there exists $y \in K_2$ such that $x + N = y + N$. By property of cosets it follows that $x - y \in N$. But since $N \subseteq K_2$ by construction $x - y \in K_2$. Since K_2 is a submodule of M, it is closed under addition and so $(x - y) + y = x \in K_2$. Conclude that $K_1 \subseteq K_2$. By symmetric argument $K_2 \subseteq K_1$ and hence $K_1 = K_2$. Thus by definition φ is injective.

Let L be a submodule of M/N. Consider the natural projection map $\pi : M \to M/N$ defined by $\pi(m) = m + N$. We want to show that there exists $K \in S$ such that $\varphi(K) = L$. To do this we will show that $\pi^{-1}(L)$ is a submodule of M and that $N \subseteq \pi^{-1}(L)$. Recall that $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$. Observe that $0 \in \pi^{-1}(L)$ since $\pi(0) = 0$ and hence $\pi^{-1}(L) \neq \emptyset$. Let $x, y \in \pi^{-1}(L)$ and $r \in R$. Observe that $\pi(x + ry) = \pi(x) + r\pi(y)$. Since $\pi(y) \in L$ by definition and L is a submodule of M/N, it follows that since scalar multiplication is closed $r\pi(y) \in L$. Thus it follows that $\pi(x) + r\pi(y) \in L$ and hence $x + ry \in \pi^{-1}(L)$. Thus by the submodule criterion we can conclude that $\pi^{-1}(L)$ is in fact a submodule. Now let $n \in N$ and observe that $\pi(n) = n + N = 0 + N \in L$ so by definition it follows that $n \in \pi^{-1}(L)$. Conclude that $N \subseteq \pi^{-1}(L)$ and hence φ is surjective.

Conclude φ is bijective and result follows. $\square$

Exercise 0.2. (DF §10.2 Exercise 1): Use the submodule criterion to show that the kernels and images of R-module homomorphisms are submodules.

Proof. Let M, N be R-modules and $\varphi : M \to N$ an R-module homomorphism. Recall that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ and $\operatorname{im}\varphi = \{n \in N : \text{there exists } m \in M \text{ with } \varphi(m) = n\}$.

Observe that $\varphi(0) = 0$ so $0 \in \ker\varphi \neq \emptyset$. Let $m, m' \in M$, $r \in R$. Now $\varphi(m + rm') = \varphi(m) + r\varphi(m') = 0 + r \cdot 0 = 0 + 0 = 0$. So $m + rm' \in \ker\varphi$. Thus by the submodule criterion $\ker\varphi$ is a submodule.

Observe that $\varphi(0) = 0 \in N$ so $0 \in \operatorname{im}\varphi \neq \emptyset$. Let $n, n' \in N$, $r \in R$. Then there exists $m, m' \in M$ such that $\varphi(m) = n$ and $\varphi(m') = n'$. Now consider $n + rn'$. $\varphi(m + rm') = \varphi(m) + r\varphi(m') = n + rn'$ so $n + rn' \in \operatorname{im}\varphi$. Conclude by submodule criterion that $\operatorname{im}\varphi$ is in fact a submodule. $\square$

Exercise 0.3. (DF §10.2 Exercise 2): Show that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules.

Proof. Let X be a set of R-modules.

- Let $M \in X$. Observe that M is isomorphic to M trivially. So relation is reflexive.
- Let $M, N \in X$. Suppose M is isomorphic to N then by definition there exists $\varphi : M \to N$ that is bijective. Immediately we have $\varphi^{-1} : N \to M$ which is also bijective so N is isomorphic to M. By definition the relation is symmetric.
- Let $L, M, N \in X$. Suppose L is isomorphic to M, then by definition there exists $\varphi : L \to M$ a bijective R-module homomorphism. Suppose M is isomorphic to N, then there exists $\Phi : M \to N$ a bijective R-module homomorphism. Observe that $\varphi \circ \Phi : L \to N$ is again a bijective R-module homomorphism by property of composition of mappings. Hence by definition L is isomorphic to N.

Conclude that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules. $\square$

Exercise 0.4. (DF §10.2 Exercise 3): Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

Solution. Consider the Quaternions $\mathbb{H} = R$; they form a commutative group under addition and a noncommutative group under multiplication. Hence $\mathbb{H}$ is a noncommutative ring with unity. In particular $\mathbb{H}$ is an R-module over itself. Define $\varphi : \mathbb{H} \to \mathbb{H}$ by $\varphi(h) = ih$. This is a group homomorphism since $\varphi(h + h') = i(h + h') = ih + ih' = \varphi(h) + \varphi(h')$. But note that $\varphi(j \cdot 1) = \varphi(j) = ij = k \neq -k = ji = j(i \cdot 1) = j\varphi(1)$. Conclude that φ is not an R-module homomorphism since the definition is not satisfied. ◀

Exercise 0.5. (DF §10.2 Exercise 4): Let A be and $\mathbb{Z}$-module, let a be any element of A and let n be a positive integer. Prove that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A : na = 0\}$ (So A_n is the annihilator in A of the ideal (n) of $\mathbb{Z}$).

Proof. Suppose that the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\varphi(\overline{k}) = ka$ is a well-defined $\mathbb{Z}$-module homomorphism. Then by definition if $\overline{m} = \overline{k}$ then $\varphi(\overline{m}) = \varphi(\overline{k})$ or equivalently $ma = ka$. Moreover $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ra) = r\varphi(a)$ for all $a, b \in A$ and $r \in \mathbb{Z}$. Observe that $\overline{0} = \overline{k}$ so by hypothesis $\varphi(\overline{0}) = \varphi(\overline{k})$ but observe that $\varphi(\overline{0}) = 0 \cdot a = 0$ and $\varphi(\overline{k}) = ka$. Hence by equality $ka = 0$. Conversely suppose that $na = 0$. We want to show that $\varphi : \mathbb{Z}/n\mathbb{Z} \to A$ defined by $\varphi(\overline{k}) = ka$ is a well-defined R-module homomorphism. Say $\overline{k} = \overline{m}$ then by property of cosets $k - m \in \mathbb{Z}/n\mathbb{Z}$ and so $n \cong k - m$. By definition $n \mid k - m$ and hence there exists $t \in \mathbb{Z}$ such that $k - m = nt$. Observe that

$$\begin{aligned} k - m &= nt \\ (k - m)a &= nta \\ ka - ma &= (na)t \\ ka - ma &= 0 \\ ka &= ma \end{aligned}$$

Thus we have $\varphi(\overline{k}) = \varphi(\overline{m})$ and we can conclude that φ is in fact a well-defined R-module homomorphism.

Now we want to show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n = \{a \in A : na = 0\}$. *Note*: We are making the assumption that we want to show this in an isomorphism of R-modules as exercise does not specifiy group, ring or module isomorphism. Define $\Phi : A_n \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ by $\Phi(a) = \varphi_a$. We will show that this is in fact an R-module homomorphism, then that is a bijection.

Let $a, a' \in A_n$, $r \in \mathbb{Z}$. Observe that

$$\begin{aligned} \Phi(a + a')(\overline{k}) &= \varphi_{a+a'}(\overline{k}) \\ &= (a + a')k \\ &= ak + a'k \\ &= \varphi_a(\overline{k}) + \varphi_{a'}(\overline{k}) \end{aligned}$$

$$= \Phi(a)(\overline{k}) + \Phi(a')(\overline{k})$$

So $\Phi(a + a') = \Phi(a) + \Phi(a')$ by definition. Moreover

$$\begin{aligned}\Phi(ra)(\overline{k}) &= \varphi_{ra}(\overline{k}) \\ &= rak \\ &= r\varphi_a(\overline{k}) \\ &= r\Phi(a)(\overline{k})\end{aligned}$$

Hence $\Phi(ra) = r\Phi(a)$ by definition. Conclude by definition that Φ is in fact an R-module homomorphism.

Recall that $\ker \Phi = \{a \in A_n : \Phi(a) = 0\}$ and observe that

$$\begin{aligned}\ker \varphi &= \{a \in A_n : \Phi(a) = 0\} \\ &= \{a \in A_n : \varphi_a(\overline{k}) = 0 \text{ for all k } \in \mathbb{Z}/n\mathbb{Z}\} \\ &= \{0\}\end{aligned}$$

So we conclude that $\ker \Phi = \{0\}$ and hence Φ is injective.

Let $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A_n)$. Define $a = \varphi(\overline{1})$ and hence $na = n\varphi(\overline{1}) = \varphi(n\overline{1}) = \varphi(\overline{n}) = \varphi(\overline{0}) = 0$ and hence $a \in A_n$. Observe that for all $\overline{k} \in \mathbb{Z}/n\mathbb{Z}$ it follows that $\varphi(\overline{k}) = \varphi(\overline{k} \cdot \overline{1}) = \varphi(k\overline{1}) = k\varphi(\overline{1}) = ka = \varphi_a(\overline{k})$. Thus by definition $\varphi = \varphi_a$. So for any $\varphi \in \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A_n)$ we can find $a \in A_n$ such that $\Phi(a) = \varphi_a = \varphi$. Conclude by definition that Φ is surjective.

Now applying the First Isomorphism Theorem for R-modules we can conclude that $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A_n) \cong A_n$. □

Exercise 0.6. (DF §10.2 Exercise 5):

Exercise 0.7. (DF §10.2 Exercise 6):

Exercise 0.8. (DF §10.2 Exercise 7): Let z be a fixed element of the center of R. Prove that the map $m \to zm$ is an R-module homomorphism from M to itself. Show that for a commutative ring R the map from R to $\mathrm{End}_R\, M$ given by $r \to rI$ is a ring homomorphism (where I is the identity endomorphism).)

Proof. Recall that the center of R is $\{z \in R : zr = rz \text{ for all } r \in R\}$. Let z be in the center of R then by definition $zr = rz$ for all $r \in R$. Define $\varphi : M \to M$ by $\varphi(m) = zm$. We claim that this is an R-module homomorphism. Let $m, m' \in M$, $r \in R$. Observe that $\varphi(m + m') = z(m + m')zm + zm' = \varphi(m) + \varphi(m')$ and $\varphi(rm) = zrm = rzm = r\varphi(m)$. Conclude by definition that φ is in fact an R-module homomorphism.

Note: We are making the assumption that "I being the identity endomorphism" means *multiplicative identity.* Now let R be a commutative ring and define $\Phi : R \to \mathrm{End}_R(M)$ by $\Phi(r) = rI$ where $I : M \to M$, defined by $I(m) = m$, is the identity endomorphism. We want to show that Φ is a ring homomorphism. Let $r, s \in R$ and observe that for all $m \in M$

$$\begin{aligned}\Phi(r + s)(m) &= (r + s)I(m) \\ &= (r + s)m \\ &= rm + rs \\ &= rI(m) + sI(m) \\ &= \Phi(r)(m) + \Phi(s)(m)\end{aligned}$$

So by definition $\Phi(r + s) = \Phi(r) + \Phi(s)$. Moreover,

$$\begin{aligned}\Phi(rs) &= rsI(m) \\ &= rsI(m)I(m) \\ &= rI(m) \cdot sI(m) \\ &= \Phi(r)(m)\Phi(s)(m)\end{aligned}$$

And hence $\Phi(rs) = \Phi(r)\Phi(s)$ and we can conclude by definition that Φ is a ring homomorphism. □

Exercise 0.9. (DF §10.2 Exercise 8): Let $\varphi : M \to N$ be an R-module homomorphism. Prove that $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$.

Proof. Recall that $\mathrm{Tor}(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Now it follows that $\varphi(\mathrm{Tor}(M)) = \{n \in N : n = \varphi(m)$ for some $m \in \mathrm{Tor}(M)\}$. Let $n \in \varphi(\mathrm{Tor}(M))$ then $n = \varphi(m)$ for some $m \in \mathrm{Tor}(M)$ by definition. Since $m \in \mathrm{Tor}(M)$ there exists $0 \neq r \in R$ such that $rm = 0$. Hence $rn = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$. Conclude that $n \in \mathrm{Tor}(N)$ and hence $\varphi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$. □

Exercise 0.10. (DF §10.2 Exercise 9): Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

Proof. Define $\Phi : \text{Hom}_R(R, M) \to M$ by $\Phi(\varphi) = \varphi(1)$. We must first show that this is in fact an R-module homomorphism. Observe that for all $\varphi, \xi \in \text{Hom}_R(R, M)$ and all $r \in R$ it follows that

$$\begin{aligned}\Phi(\varphi+\xi) &== (\varphi+\xi)(1)\\ &= \varphi(1) + \varphi(\xi)\\ &= \Phi(\varphi) + \Phi(\xi)\end{aligned}$$

and also by Proposition 2 we have

$$\begin{aligned}\Phi(r\varphi) &= (r\varphi)(1)\\ &= r\varphi(1)\\ &= r\Phi(\varphi)\end{aligned}$$

Hence we can now conclude that Φ is an R-module homomorphism.

We must now show that Φ is injective. Suppose that $\Phi(\varphi) = \Phi(\xi)$ then by definition $\varphi(1) = \xi(1)$ or equivalently $\varphi(1) - \xi(1) = 0$ and hence $(\varphi - \xi)(1) = 0$. But since $\varphi - \xi \in \text{Hom}_R(R, M)$ it is an R-module homomorphism so $(\varphi - \xi)(x) = (\varphi - \xi)(x \cdot 1) = x(\varphi - \xi)(1) = x \cdot 0 = 0$ for all $x \in R$. Conclude that $\varphi(x) = \xi(x)$ for all $x \in R$ and hence by definition $\varphi = \xi$. Conclude that Φ is inective.

We must now show that Φ is surjective. Let $m \in M$ be arbitrary. We want to show that there exists $\varphi \in \text{Hom}_R(R, M)$ such that $\Phi(\varphi) = m$. Let us define $\varphi : R \to M$ by $\varphi(x) = xm$. We need to show that $\varphi \in \text{Hom}_R(R, M)$. Observe that $\varphi(x + y) = (x + y)m = xm + ym = \varphi(x) + \varphi(y)$ for all $x, y \in R$ and $\varphi(rx) = rxm = r\varphi(x)$ for all $x \in R$, $r \in R$. Hence we have shown that φ is in fact an R-module homomorphism. Now observe that $\Phi(\varphi) = \varphi(1) = 1 \cdot m = m$. Conclude by definition that Φ is surjective.

We have shown that Φ is a bijective R-module homomorphism. Conclude that $\text{Hom}_R(R, M) \cong M$. □

Exercise 0.11. (DF §10.2 Exercise 10): Let R be a commutative ring. Prove that $\text{Hom}_R(R, R)$ and R are isomorphic as rings.

Proof. Define $\Phi : \text{Hom}_R(R, R) \to R$ by $\Phi(\varphi) = \varphi(1)$. We will first show that Φ is a ring homomorphism. Observe that for all $\varphi, \xi \in \text{Hom}_R(R, R)$ and all $r \in R$ by Proposition 2 we have

$$\begin{aligned}\Phi(\varphi+\xi) &= (\varphi+\xi)(1)\\ &= \varphi(1) + \xi(1)\\ &= \Phi(\varphi) + \Phi(\xi)\end{aligned}$$

and also by property of commutativity

$$\begin{aligned}\Phi(\varphi \circ \xi) &= (\varphi \circ \xi)(1)\\ &= \varphi(\xi(1))\\ &= \varphi(\xi(1) \cdot 1)\\ &= \xi(1)\varphi(1)\\ &= \varphi(1)\xi(1)\\ &= \Phi(\varphi)\Phi(\xi)\end{aligned}$$

Hence we can conclude that Φ is in fact a ring homomorphism.

We must now show that Φ is injective. Suppose that $\Phi(\varphi) = \Phi(\xi)$ then by definition $\varphi(1) = \xi(1)$ or equivalently $\varphi(1) - \xi(1) = 0$ and hence $(\varphi - \xi)(1) = 0$. But since $\varphi - \xi \in \text{Hom}_R(R, R)$ it is an R-module homomorphism so $(\varphi - \xi)(x) = (\varphi - \xi)(x \cdot 1) = x(\varphi - \xi)(1) = x \cdot 0 = 0$ for all $x \in R$. Conclude that $\varphi(x) = \xi(x)$ for all $x \in R$ and hence by definition $\varphi = \xi$. Conclude that Φ is inective.

We must now show that Φ is surjective. Let $r \in R$ be arbitrary. We want to show that there exists $\varphi \in \text{Hom}_R(R, R)$ such that $\Phi(\varphi) = r$. Let us define $\varphi : R \to R$ by $\varphi(x) = xr$. We need to show that $\varphi \in \text{Hom}_R(R, R)$. Observe that $\varphi(x + y) = (x + y)r = xr + yr = \varphi(x) + \varphi(y)$ for all $x, y \in R$ and $\varphi(sx) = sxr = s\varphi(x)$ for all $s \in R$ and all $x \in R$. Hence we have shown that φ is in fact an R-module homomorphism. Now observe that $\Phi(\varphi) = \varphi(1) = 1 \cdot r = r$. Conclude by definition that Φ is surjective.

We have shown that Φ is a bijective ring homomorphism. Conclude that $\text{Hom}_R(R, R) \cong R$. □

Exercise 0.12. (DF §10.2 Exercise 11): Let $A_1, A_2, \ldots, A_n$ be R-modules and let B_i be submodules of A_i for each $i = 1, 2, \ldots, n$. Prove that

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

Proof. Define $\varphi : A_1 \times A_2 \times \cdots \times A_n \to (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$ by $\varphi(a_1, a_2, \dots, a_n) = (a_1+B_1, a_2+B_2, \dots, a_n+B_n)$. We want to show that this is an R-module homomorphism and that $\ker\varphi = B_1 \times B_2 \times \cdots \times B_n$ and that the mapping is surjective. Then the first isomorphism theorem for modules yields the result.

Let $x, y \in A_1 \times A_2 \times \cdots \times A_n$ where $x = (a_1, a_2, \dots, a_n)$ and $y = (a_1', a_2', \dots, a_n')$. Observe that

$$\begin{aligned}
\varphi(x+y) &= \varphi((a_1, a_2, \dots, a_n) + (a_1', a_2', \dots, a_n')) \\
&= \varphi(a_1 + a_1', a_2 + a_2', \dots, a_n + a_n') \\
&= (a_1 + a_1' + B_1, a_2 + a_2' + B_2, \dots, a_n + a_n' + B_n) \\
&= (a_1 + B_1 + a_1' + B_1, a_2 + B_2 + a_2' + B_2, \dots, a_n + B_n + a_n' + B_n) \\
&= (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) + (a_1' + B_1, a_2' + B_2, \dots, a_n' + B_n) \\
&= \varphi(a_1, a_2, \dots, a_n) + \varphi(a_1', a_2', \dots, a_n') \\
&= \varphi(x) + \varphi(y)
\end{aligned}$$

and for $r \in R$

$$\begin{aligned}
\varphi(rx) &= \varphi(r(a_1, a_2, \dots, a_n)) \\
&= \varphi(ra_1, ra_2, \dots, ra_n) \\
&= (ra_1 + B_1, ra_2 + B_2, \dots, ra_n + B_n) \\
&= (r(a_1 + B_1), r(a_2 + B_2), \dots, r(a_n + B_n)) \\
&= r(a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) \\
&= r\varphi(a_1, a_2, \dots, a_n) \\
&= r\varphi(x)
\end{aligned}$$

Thus by definition we can conclude that φ is an R-moduel homomorphism.

Now we want to show that $\ker\varphi = B_1 \times B_2 \times \cdots \times B_n$. Observe that

$$\begin{aligned}
\ker\varphi &= \{x \in A_1 \times A_2 \times \cdots \times A_n : \varphi(x) = 0\} \\
&= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : \varphi(a_1, a_2, \dots, a_n) = 0\} \\
&= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) = (0, 0, \dots, 0)\} \\
&= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : a_1 \in B_1, a_2 \in B_2, \dots, a_n \in B_n\}
\end{aligned}$$

Hence $\ker\varphi \subseteq B_1 \times B_2 \times \cdots \times B_n$ and trivially $B_1 \times B_2 \times \cdots \times B_n \subseteq \ker\varphi$ by construction of φ. Conclude that $\ker\varphi = B_1 \times B_2 \times \cdots \times B_n$.

The mapping is trivially surjective. Applying first Isomorphism theorem yields the result. □

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