INTRODUCTION TO HOMOLOGICAL ALGEBRA

ROBERT CARDONA, MASSY KHOSHBIN, AND SIAVASH MORTEZAVI

0. MATH 697 Homework Zero.Two

AM Exercise 2.1: Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if m and n are coprime.

Proof. Choose $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. Since m and n are coprime, there exist $s, t \in \mathbb{Z}$ such that $ms + nt = 1$ Observe that

$$a = a \cdot 1 = a(ms + nt) = ams + ant \equiv ant \pmod{m}.$$

Now observe that

$$a \otimes b = atn \otimes b = a \otimes nb = at \otimes 0 = 0.$$

We have shown that any simple tensor is zero, so any finite linear combination of simple tensors is zero. Conclude $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$. □

AM Exercise 2.2: Let R be a ring, I an ideal of R, M an R-module. Show that $(R/I) \otimes_R M$ is isomorphic to M/IM.

Proof. Define $\varphi : R/I \times M \to M/IM$ by $\varphi(r + I, m) = rm + IM$, which we shall henceforth write as $\varphi(\overline{r}, m) = \overline{rm}$. Let $(\overline{r}, m) = (\overline{s}, m)$. Then $\overline{r} = \overline{s} \implies r \in \overline{s} \implies r = s + i$, some $i \in I$. Then $\varphi(\overline{r}, m) = \overline{rm} = \overline{(s+i)m} = \overline{sm + im} = \overline{sm} + \overline{im} = \overline{sm} + \overline{0} = \overline{sm} = \varphi(\overline{s}, m)$. Thus φ is well-defined.

Observe $\varphi(\overline{r} + \overline{s}, m) = \varphi(\overline{r+s}, m) = \overline{(r+s)m} = \overline{rm + sm} = \overline{rm} + \overline{sm} = \varphi(\overline{r}, m) + \varphi(\overline{s}, m)$. Similarly, $\varphi(\overline{r}, m + n) = \varphi(\overline{r}, m) + \varphi(\overline{r}, n)$. Lastly, $\varphi(\overline{rs}, m) = \overline{(rs)m} = \overline{r(sm)} = \varphi(\overline{r}, sm)$. Thus φ is R-biadditive (In fact, φ is R-bilinear).

Now we are guaranteed a unique R-homomorphism $\phi : R/I \otimes_R M \to M/IM$ given by $\phi(\overline{r} \otimes m) = \overline{rm}$. Notice if we define $f : M/IM \to R/I \otimes_R M$ via $f(\overline{m}) = \overline{1} \otimes m$ then f is a $\mathbb{Z}$-homomorphism which makes $f \circ \phi$ and $\phi \circ f$ the identity map in $R/I \otimes_R M$ and M/IM, respectively. So ϕ has a two-sided inverse, hence a bijective function, and accordingly is an isomorphism when considered as an R-map. □

DF §10.5 Proposition 24: (*The Short Five Lemma*) Let α, β, γ be homomorphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) If α and γ are injective then so is β.

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that $b = 0$. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know $g(b) = 0$ so $b \in \ker g$ but since we are in an exact sequence we have $\operatorname{im} f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with $f(a) = b$. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so $a = 0$. So $0 = f(a) = b$. □

(2) If α and γ are surjective then so is β.

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that $g(b) = c$. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness $\operatorname{im} f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$. □

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

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Proof. Follows from (1) and (2). □

R Exercise 2.14: Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of module maps. Prove that $gf = 0$ if and only if $\operatorname{im} f \subseteq \ker g$. Give an example of such a sequence that is not exact.

Proof. Suppose $gf = 0$, that is, $f(g(a)) = 0$ for all $a \in A$. Let $b \in \operatorname{im} f$ then by definition there exists $a \in A$ such that $f(a) = b$. But we know by hypothesis that $0 = g(f(a)) = g(b)$ so $b \in \ker g$. Conclude that $\operatorname{im} f \subseteq \ker g$. Conversely, suppose that $\operatorname{im} f \subseteq \ker g$. Let $a \in A$ and observe that $f(a) \in \operatorname{im} f$. By hypothesis $f(a) \in \ker g$ so $g(f(a)) = 0$. Since a was arbitrary conclude $gf = 0$. □

Consider the sequence of module maps $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$ where $f(\overline{x}) = 2\overline{x}$ and $g(\overline{y}) = \overline{y}$. Observe that $\operatorname{im} f = \mathbb{Z}/3\mathbb{Z}$ and $\ker g = \{0\}$. Since $\operatorname{im} f \neq \ker g$ it follows that this is **not** an exact sequence.

R Exercise 2.15:

(1) Prove that $f : M \to N$ is surjective if and only if $\operatorname{coker} f = \{0\}$.

Proof. Suppose $f : M \to N$ is surjective then for $n \in N$, there exists $m \in M$ such that $f(m) = n$. By definition $\operatorname{coker} f = M/\operatorname{im} f = M/M = 0$. Conversely, suppose that $\operatorname{coker} f = 0$, i.e., $M/\operatorname{im} f = 0$ implying that if $m + \operatorname{im} f \in M/\operatorname{im} f$ then $m + \operatorname{im} f = 0$ or equivalently $m \in \operatorname{im} f$. Since m is arbitrary, conclude $M = \operatorname{im} f$ and hence f is surjective by definition. □

(2) If $f : M \to N$ is a map, prove that there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Proof. Define $h : \ker f \to M$ by $h(m) = m$, that is, map each element to itself. It follows immediately that $\operatorname{im} h = \ker f$. Define $g : N \to \operatorname{coker} f = N/\operatorname{im} f$ by $g(n) = n + \operatorname{im} f$, that is, the canonical/projection mapping. Observe that $\ker g = \operatorname{im} f$. Conclude that the following sequence is in fact exact:

$$0 \xrightarrow{\text{identity}} \ker f \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} \operatorname{coker} f \xrightarrow{\text{zero}} 0. \quad \square$$

R Exercise 2.16:

(1) If $0 \to M \to 0$ is an exact sequence, prove that $M = \{0\}$.

Proof. Consider $0 \xrightarrow{f} M \xrightarrow{g} 0$. Since f is surjective then for $m \in M$ there exists $x \in 0$ such that $f(x) = m$ but x must be 0 so $m = 0$. □

(2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective if and only if h is injective.

Proof. Suppose f is surjective. Then $\operatorname{im} f = B = \ker g$ but this immediately implies that $\operatorname{im} g = 0 = \ker h$ so h is injective by definitoin. Conversely, suppose h is injective. Then $\ker h = 0 = \operatorname{im} g$ which immediately implies $\ker g = B = \operatorname{im} f$. Conclude by definition f is surjective. □

(3) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ be exact. If α and δ are isomorphisms, prove that $C = \{0\}$.

Proof. Observe that, by previous exercise, β is surjective and γ is injective so we have $\operatorname{im} \beta = C$ and $\ker \gamma = 0$. Result follows by exactness: $C = \operatorname{im} \beta = \ker \gamma = 0$. Conclude $C = \{0\}$. □

R Exercise 2.17: If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that there is an exact sequence $0 \to \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \to 0$, where $\alpha : b + \operatorname{im} f \mapsto g(b)$ and $\beta : c \mapsto h(c)$.

Proof. Observe that $\ker \beta = \{c \in C : \beta(c) = h(c) = 0\} = \ker h = \operatorname{im} g$ by exactness and since α can run through any $b \in B$ conclude $\operatorname{im} \alpha = \operatorname{im} g = \ker \beta$ □

R Exercise 2.27: Let V and W be finite-dimensional vector spaces over a field F and let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be bases of V and W, respectively. Let $S : V \to V$ be a linear transformation having matrix $A = [a_{ij}]$, and let $T : W \to W$ be a linear transformation having matrix $B = [b_{kl}]$. Show that the matrix of $S \otimes T : V \otimes W \to V \otimes W$, with respect to a suitable listing of the vectors $v_i \otimes w_j$, is the $nm \times nm$ matrix K, which we write in block form:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}$$

Proof. Since $v_1, \ldots, v_m$ is a basis for V and $w_1, \ldots, w_n$ is a basis for W, then $\{v_i \otimes w_j | 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $V \otimes W$. Note that the action of $S \otimes T$ on $V \otimes W$ is induced by S and T on the basis vectors of V and W, respectively:

$(S \otimes T)(v_i \otimes w_j) = S(v_i) \otimes T(w_j)$, which has the matrix representation $A(v_i) \otimes B(w_j)$. Writing $A = [a_{ij}]$ and $B = [b_{kl}]$ we have

$$A(v_i) = \sum_{t=1}^m a_{it}v_t \text{ and } B(w_j) = \sum_{s=1}^n b_{js}w_s.$$

Therefore

$$(A \otimes B)(v_i \otimes w_j) = \sum_{t=1}^m a_{it}v_t \otimes \sum_{s=1}^n b_{js}w_s.$$

By expanding on the basis of V only, we can represent

$$(A \otimes B)(v_i \otimes w_j) = a_{i1}v_1 \otimes B(w_j) + a_{i2}v_2 \otimes B(w_j) + \cdots + a_{im}v_m \otimes B(w_j)$$

in block form as the i'th "row" in $A \otimes B$ times the j'th column basis vector of $A \otimes B$:

$$\begin{bmatrix} a_{i1}B & a_{i2}B & \cdots & a_{im}B \end{bmatrix} \begin{bmatrix} v_1 \otimes w_j \\ v_2 \otimes w_j \\ \vdots \\ v_m \otimes w_j \end{bmatrix}.$$

With $1 \leq i \leq m$ and B being an $n \times n$ matrix, we have each "row" is an $n \times mn$ matrix. Since there are m such "rows", we have that $A \otimes B$ is the $mn \times mn$ matrix as given above. $\square$

R Exercise 2.28: Let R be a domain with $Q = \mathrm{Frac}(R)$, its field of fractions. If A is an R-module, prove that every element of $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$ (i.e. every element is a simple tensor).

Proof. Let $\sum_1^n q_i \otimes a_i \in Q \otimes_R A$. We can write $\sum_1^n q_i \otimes a_i = \sum_1^n \frac{r_i}{s_i} \otimes a_i$ for $r_i, s_i \in R, s_i \neq 0$. Write $s = s_1 s_2 \cdots s_n$ and $\widehat{s_i} = \frac{s}{s_i}$. Then $\sum_1^n \frac{r_i}{s_i} \otimes a_i = \sum_1^n (1 \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n (\frac{\widehat{s_i}}{\widehat{s_i}} \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n \frac{\widehat{s_i} r_i}{s} \otimes a_i = \sum_1^n (\frac{1}{s}) \widehat{s_i} r_i \otimes a_i = \sum_1^n \frac{1}{s} \otimes (\widehat{s_i} r_i) a_i = \frac{1}{s} \otimes (\sum_1^n \widehat{s_i} r_i a_i)$. $\square$

R Exercise 2.29:**(i)** Let p be a prime, and let p, q be relatively prime. Prove that if A is a p-primary group and $a \in A$, then there exists $x \in A$ with $qx = a$.

(ii) If D is a finite cyclic group of order m, prove that D/nD is a cyclic group of order $d = (m, n)$.

(iii) Let m and n be positive integers, and let $d = (m, n)$. Prove that there is an isomorphism of abelian groups

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d.$$

(iv) Let G and H be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \text{ and } H = B_1 \oplus \cdots \oplus B_m,$$

where A_i and B_j are cyclic groups. Compute $G \otimes_{\mathbb{Z}} H$ explicitely.

Proof. (i) $a \in A$ so $p^k a = 0$, some $k \in \mathbb{Z}^+$. Since p is prime, $(q, p) = 1 \implies (q, p^k) = 1$. So there exist $m, n \in \mathbb{Z}$ such that $qm + np^k = 1$. Now $a = 1 \cdot a = (qm + np^k)a = qma + np^k a = q(ma) + n(p^k a) = qx + 0 = qx$. Observe $p^k x = p^k(ma) = m(p^k a) = 0$ so $x \in A$.

(ii) D is cyclic, so D/nD is cyclic. If we write $D =< a >$, then $nD =< na >$. This is because for any $nb \in nD$, we can write $b = ka$, some $k \in \mathbb{Z}^+$, since a generates D. Now $nb = n(ka) = k(na)$, and we have that na generates nD.

Claim $|na| = \frac{m}{d}$. Observe $\frac{m}{d}(na) = \frac{n}{d}(ma) = \frac{n}{d} \cdot 0 = 0$, which implies $|na|$ divides $\frac{m}{d}$. On the other hand, if $k(na) = 0$, then $(kn)a = 0 \implies m | kn \implies \frac{m}{d} | k\frac{n}{d}$. But $\frac{m}{d}$ and $\frac{n}{d}$ are relatively prime by construction, forcing $\frac{m}{d} | k$. In particular, we have $\frac{m}{d}$ divides $|na|$. Thus $|na| = d$. Now $|nD| = | < na > | = |na| = \frac{m}{d}$. Lagrange's theorem gives us $|\frac{D}{nD}| = \frac{|D|}{|nD|} = \frac{m}{\frac{m}{d}} = d$.

(iii) By Proposition 2.68, we have that $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_m / n\mathbb{Z}_m$. But by part (ii), $\mathbb{Z}_m / n\mathbb{Z}_m$ is a cyclic group of order $d = (m, n)$ so $\mathbb{Z}_m / n\mathbb{Z}_m \cong \mathbb{Z}_d$.

(iv) There are three cases: If both A_i and B_j are finite, then by part (iii) we have $A_i \otimes_{\mathbb{Z}} B_j \cong \mathbb{Z}_{s_i} \otimes \mathbb{Z}_{t_j} \cong \mathbb{Z}_{(s_i, t_j)}$, where $|A_i| = s_i$ and $|B_j| = t_i$. If both A_i and B_j are infinite, then $A_i \otimes_{\mathbb{Z}} B_j \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ by Proposition 2.58. If (wlog) A_i is finite and B_j is infinite, then $A_i \otimes_{\mathbb{Z}} B_j \cong Z_{s_i} \otimes \mathbb{Z} \cong \mathbb{Z}_{s_i}$ by Proposition 2.58. Thus we have

$$G \otimes_{\mathbb{Z}} H = \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j = \sum_{i,j: A_i \text{ and } B_j \text{ finite}} \mathbb{Z}_{(s_i, t_i)} + \sum_{i,j: A_i \text{ finite and } B_j \text{ infinite}} \mathbb{Z}_{s_i} + \sum_{i,j: A_i \text{ or } B_j \text{ infinite}} \mathbb{Z}.$$

$\square$

R Exercise 2.32: (3×3 *Lemma*) Consider the following commutative diagram in ${}_R\mathbf{Mod}$ having exact columns.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "\alpha_1"] \arrow[d, "f'"] & A \arrow[r, "\alpha_2"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "\beta_1"] \arrow[d, "g'"] & B \arrow[r, "\beta_2"] \arrow[d, "g"] & B'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & C' \arrow[r, "\gamma_1"] \arrow[d] & C \arrow[r, "\gamma_2"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

Proof. Suppose that the bottom two rows are exact. We must show four things:

(1) α_1 is injective: Suppose $a' \in A'$ with $\alpha_1(a') = 0$. Observe that by commutativity $f(\alpha_1(a')) = \beta_1(f'(a')) = 0$ so since β_1 is injective, by exactness of the second row, $f'(a') = 0$. But since the first column is exact by hypothesis, we have $a' = 0$. Conclude that α_1 is in fact injective.

(2) $\operatorname{im}\alpha_1 \subseteq \ker\alpha_2$: Choose $a \in \operatorname{im}\alpha_1$ then by definition there exists $a' \in A'$ such that $\alpha_1(a') = a$. Observe that $f(a) = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. Moreover $\beta_2(f(a)) = f''(\alpha_2(a))$ again by commutativity. Hence we have
$$\beta_2(f(a)) = \beta_2(\beta_1(f'(a'))) = 0 = f''(\alpha_2(a)).$$
Since f'' is injective by exactness of the third column, $\alpha_2(a) = 0$ and so $a \in \ker\alpha_2$. Conclude that $\operatorname{im}\alpha_2 \subseteq \ker\alpha_2$ as desired.

(3) $\ker\alpha_2 \subseteq \operatorname{im}\alpha_1$: Let $a \in \ker\alpha_2$ then by definition $\alpha_2(a) = 0$ and so $f''(\alpha(a)) = \beta_2(f(a)) = 0$. So $f(a) \in \ker\beta_2 = \operatorname{im}\beta_1$ by exactness of the second column, so there exists $b' \in B'$ such that $\beta_1(b') = f(a)$. Now
$$\gamma_1(g'(b)) = g(\beta_1(b')) = g(f(a)) = 0$$
by commutativity and exactness so there exists $a' \in A'$ such that $f'(a') = b'$. Now by commutativity
$$f(\alpha_1(a')) = \beta_1(f'(a')) = \beta_1(b') = f(a)$$
and so $f(a - \alpha_1(a')) = 0$ since f is a homomorphism. f is injective since the second column is exact so it follows that $\alpha_1(a') = a$ and so $a \in \operatorname{im}\alpha_1$. Conclude that $\ker\alpha_2 \subseteq \operatorname{im}\alpha_1$ is in fact true.

(4) α_2 is surjective: Choose $a'' \in A''$. Consider $f''(a'') \in B''$. Since the second row is exact β_2 is surjective so there exists $b \in B$ such that $\beta_2(b) = f''(a'')$. By commutativity $g''(\beta_1(b)) = \gamma_2(g(b))$ but by exactness of the third column $g''(\beta_2(b)) = g''(f''(a'')) = 0$. So $\gamma_2(g(b)) = 0$ which implies that $g(b) \in \ker\gamma_2 = \operatorname{im}\gamma_1$ by exactness of the third row. So there exists $c' \in C'$ such that $\gamma_1(c') = g(b)$. Since the first column is exact g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. So $\gamma_1(c') = \gamma_1(g'(b')) = g(\beta_1(b'))$. Observe that
$$g(b - \beta_1(b')) = g(b) - g(\beta_1(b')) = 0$$
since $g(b) = \gamma_1(c)$ and $g(\beta_1(b')) = \gamma_1(c)$ also. So $b - \beta_1(b') \in \ker g = \operatorname{im} f$ by exactness of the second column. So there exists $a \in A$ such that $f(a) = b - \beta_2(b')$. Now since $f''(\alpha(a)) = \beta_2(f(a)) = \beta_2(b - \beta_1(b'))$ we get
$$f''(\alpha_2(a)) = \beta_2(b) - \beta_2(\beta_1(b')) = \beta_2(b) = f''(a'').$$
In particular we get $f''(\alpha_2(a) - a'') = 0$ but f'' is injective by exactness of the third column so $\alpha_2(a) = a''$. Conclude that α_2 is in fact surjective.

We have shown that if the bottom two rows are exact, then the top row is exact.

Suppose that the top two rows are exact. We must show four things:

(1) γ_1 is injective: Suppose $c' \in C'$ with $\gamma_1(c') = 0$. Since the first column is exact, g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. Now by commutativity
$$0 = \gamma_1(c') = \gamma_1(g'(b')) = g(\beta_1(b')).$$
So $\beta_1(b') \in \ker g = \operatorname{im} f$ since the second row is exact. So there exists $a \in A$ such that $f(a) = \beta_1(b')$. But by commutativity
$$f''(\alpha_2(a)) = \beta_2(f(a)) = \beta_2(\beta_1(b')) = 0$$
so $\alpha_2(a) = 0$. Since f'' is injective, because the third column is exact. Now $a \in \ker\alpha_2 = \operatorname{im}\alpha_1$ by exactness of the first row so there exists $a' \in A'$ such that $\alpha_2(a') = a$. Now
$$f(a) = f(\alpha_1(a')) = \beta_1(f'(a')) = \beta_1(b')$$

So $\beta_1(f'(a') - b') = 0$. Since the second row is exact, β_1 is injective so $f'(a') - b = 0$. Hence $f'(a') = b'$. Taking g' of both sides we get

$$0 = g'(f'(a')) = g'(b') = c'.$$

Hence γ_1 is injective.

(2) $\operatorname{im}\gamma_1 \subseteq \ker\gamma_2$: Let $c \in \operatorname{im}\gamma_1$ then by definition there exists $c' \in C'$ such that $\gamma_1(c') = c$. Since the first column is exact g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. Observe that

$$g(\beta_1(b')) = \gamma_1(g'(b')) = \gamma_1(c') = c$$

by commutativity. Again by commutativity we observe

$$\gamma_2(c) = \gamma_1(g(\beta_1(b'))) = g''(\beta_1(\beta_1(b'))) = g''(0) = 0$$

and so $c \in \ker\gamma_2$. Conclude that $\operatorname{im}\gamma_1 \subseteq \ker\gamma_2$.

(3) $\ker\gamma_2 \subseteq \operatorname{im}\gamma_1$: Choose $c \in \ker\gamma_2$ then by definition $\gamma_2(c) = 0$. Since the second column is exact, g is surjective. So there exists $b \in B$ such that $g(b) = c$. By commutativity

$$g''(\beta_2(b)) = \gamma_2(g(b)) = \gamma_2(c) = 0.$$

So $\beta_2(b) \in \ker g'' = \operatorname{im} f''$ by exactness of the third column so there exists $a'' \in A''$ such that $f''(a'') = \beta_2(b)$. Since the first row is exact α_2 is surjective so there exists $a \in A$ such that $\alpha_2(a) = a''$. Now

$$\beta_2(f(a)) = f''(\alpha_2(a)) = f''(a'') = \beta_2(b)$$

so $\beta_2(f(a) - b) = 0$ which implies that $f(a) - b \in \ker\beta_2 = \operatorname{im}\beta_1$ by exactness of the second row. So there exists $b' \in B'$ such that $\beta_1(b') = f(a) - b'$. But now

$$\gamma_1(g'(b')) = g(\beta_1(b')) = g(f(a) - b) = g(f(a)) = g(b) = -c$$

yielding $\gamma_1(-g'(b')) = c$. So $c \in \operatorname{im}\gamma_1$ and we can conclude that $\ker\gamma_2 \subseteq \operatorname{im}\gamma_1$.

(4) γ_2 is surjective: Choose $c'' \in C''$. Since the third column is exact g'' is surjective so there exists $b'' \in B''$ such that $g''(b'') = c''$. But since the second row is exact β_2 is surjective so there exists $b \in B$ such that $\beta_2(b) = b''$. Now by commutativity

$$c'' = g''(b'') = g''(\beta_2(b)) = \gamma_2(g(b)).$$

Conclude that γ_2 is in fact surjective.

We have shown that if the top two rows are exact, then the bottom row is exact. □

AM Proposition 2.10: (*Snake Lemma*) Let

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "g"] \arrow[d, "f'"] & M \arrow[r, "h"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "g'"] & N \arrow[r, "h'"] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists an exact sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \to 0$$

in which $\overline{u}$, $\overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

Proof. We consider:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f' \arrow[r, "\overline{u}"] \arrow[d, "g'"] & \ker f \arrow[r, "\overline{v}"] \arrow[d, "g"] & \ker f'' \arrow[d, "g''"] \arrow[dddll, "d"'] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d, "h'"] & N \arrow[r, "v'"] \arrow[d, "h'"] & N'' \arrow[r] \arrow[d, "h'"] & 0 \\
 & \operatorname{coker} f' \arrow[r, "\overline{u}'"] \arrow[d] & \operatorname{coker} f \arrow[r, "\overline{v}'"] \arrow[d] & \operatorname{coker} f'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Rotman Exercise 2.32, we have that $\operatorname{im} \overline{u} = \ker \overline{v}$ and $\operatorname{im} \overline{u}' = \ker \overline{v}'$. It is left for us to first define d and then show that $\operatorname{im} \overline{v} = \ker d$ and $\operatorname{im} d = \ker \overline{u}'$. $\square$

Department of Mathematics, California State University Long Beach
E-mail address: mrrobertcardona@gmail.com and massy255@gmail.com and siavash.mortezavi@gmail.com