

# INTRODUCTION TO CATEGORY THEORY

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ABSTRACT. Module Theory.

## 0. MATH 697 HOMEWORK ZERO

**Exercise 0.1.** (DF §10.2 Theorem 4): Prove Theorem 4 (Isomorphism Theorems):

- (1) (*The First Isomorphism Theorem for Modules*) Let  $M, N$  be  $R$ -modules and let  $\varphi : M \rightarrow N$  be an  $R$ -modules homomorphism. Then  $\ker \varphi$  is a submodule of  $M$  and  $M/\ker \varphi \cong \varphi(M)$ .

*Proof.* Let  $M, N$  be  $R$ -modules and let  $\varphi : M \rightarrow N$  be an  $R$ -modules homomorphism. Then by definition  $\varphi(x + y) = \varphi(x) + \varphi(y)$  and  $\varphi(rx) = r\varphi(x)$  for all  $x, y \in M, r \in R$ . We want to show that  $\ker \varphi = \{m \in M : \varphi(m) = 0\}$  is a submodule. Observe that since  $M$  is a module then  $M$  is an abelian group by definition so there exists  $0 \in M$  such that  $m + 0 = m$  for all  $m \in M$ . In particular  $\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0)$  implying  $\varphi(0) = 0$ . Conclude that  $0 \in \ker \varphi \neq \emptyset$ . Now let  $r \in R, x, y \in \ker \varphi$ . Observe that  $\varphi(x + ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r \cdot 0 = 0 + 0 = 0$ . Hence  $x + ry \in \ker \varphi$ . Conclude by the submodule criterion that  $\ker \varphi$  is in fact a submodule.

Now define  $\Phi : M/\ker \varphi \rightarrow \varphi(M)$  by  $\Phi(m + \ker \varphi) = \varphi(m)$ . We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose  $m + \ker \varphi = m' + \ker \varphi$  it follows by property of cosets that  $m - m' \in \ker \varphi$ , in particular  $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$  and hence  $\varphi(m) = \varphi(m')$ . But since  $\varphi(m) = \Phi(m + \ker \varphi)$  and  $\varphi(m') = \Phi(m' + \ker \varphi)$  we have  $\Phi(m + \ker \varphi) = \Phi(m' + \ker \varphi)$ . Conclude that  $\Phi$  is in fact well-defined.

Suppose that  $\Phi(m + \ker \varphi) = \Phi(m' + \ker \varphi)$ . Then it follows that  $\varphi(m) = \varphi(m')$  and so  $\varphi(m - m') = 0$  and so  $m - m' \in \ker \varphi$ . By property of cosets it follows that  $m + \ker \varphi = m' + \ker \varphi$  and hence  $\Phi$  is injective.

Let  $n \in \varphi(M)$ . Then by definition of image of  $\varphi$  there exists  $m \in M$  such that  $n = \varphi(m)$ . It is immediate that  $m + \ker \varphi \in M/\ker \varphi$  and we can conclude that  $\Phi$  is surjective.

Now we must show that  $\Phi$  is an  $R$ -module homomorphism. Let  $x, y \in M/\ker \varphi$  where  $x = m + \ker \varphi$  and  $y = m' + \ker \varphi$  for some  $m, m' \in M$  and let  $r \in R$ . Observe that

$$\begin{aligned} \Phi(x + y) &= \Phi(m + m' + \ker \varphi) \\ &= \varphi(m + m') \\ &= \varphi(m) + \varphi(m') \\ &= \Phi(m + \ker \varphi) + \Phi(m' + \ker \varphi) \\ &= \Phi(x) + \Phi(y) \end{aligned}$$

and

$$\begin{aligned} \Phi(rx) &= \Phi(r(m + \ker \varphi)) \\ &= \Phi(rm + \ker \varphi) \\ &= \varphi(rm) \\ &= r\varphi(m) \\ &= r\Phi(m + \ker \varphi) \\ &= r\Phi(x) \end{aligned}$$

Hence we have shown that  $\Phi$  is a well-defined bijective homomorphism and thus we can conclude by definition of  $R$ -module isomorphism that  $M/\ker \varphi \cong \varphi(M)$ .  $\square$

- (2) (*The Second Isomorphism Theorem*) Let  $A, B$  be submodules of the  $R$ -module  $M$ . Then  $(A + B)/B \cong A/(A \cap B)$ .

*Proof.* Define  $\varphi : A \rightarrow (A + B)/B$  by  $\varphi(a) = a + B$ . This mapping is clearly well-defined. We want to show that  $\varphi$  is a homomorphism. Let  $r \in R, a, a' \in A$  and observe that

$$\varphi(a + a') = a + a' + B$$

$$\begin{aligned}
&= a + B + a' + B \\
&= \varphi(a) + \varphi(a')
\end{aligned}$$

and

$$\begin{aligned}
\varphi(ra) &= ra + B \\
&= r(a + B) \\
&= r\varphi(a)
\end{aligned}$$

and so  $\varphi$  is an  $R$ -module homomorphism by definition. Observe that  $\ker \varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$ . Now let  $x \in (A + B)/B$  then  $x = a + b + B$  for some  $a \in A, b \in B$ . But observe that  $a + b + B = a + B$  by absorption. So  $\varphi$  is immediately surjective. In particular we have  $\varphi(A) = (A + B)/B$ . Conclude by the First Isomorphism Theorem for Modules that  $A/\ker \varphi = A/(A \cap B) \cong (A + B)/B = \varphi(A)$ .  $\square$

- (3) (*The Third Isomorphism Theorem*) Let  $M$  be an  $R$ -module, and let  $A$  and  $B$  be submodules of  $M$  with  $A \subseteq B$ . Then  $(M/A)/(B/A) \cong M/B$ .

*Proof.* Define  $\varphi M/A \rightarrow M/B$  by  $\varphi(m + A) = m + B$ . We need to show  $\varphi$  is well-defined. Suppose  $m + A = m' + A$  then  $m - m' \in A \subseteq B$  by property of cosets. It also follows that  $m + B = m' + B$ . Hence  $\varphi(m + A) = m + B = m' + B = \varphi(m' + A)$  and hence  $\varphi$  is well-defined.

Now we must show  $\varphi$  is an  $R$ -module homomorphism. Let  $m, m' \in M$  and  $r \in R$ . Observe that

$$\begin{aligned}
\varphi((m + A) + (m' + A)) &= \varphi(m + m' + A) \\
&= m + m' + B \\
&= (m + B) + (m' + B) \\
&= \varphi(m + A) + \varphi(m' + A)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(r(m + A)) &= \varphi(rm + A) \\
&= rm + B \\
&= r(m + B) \\
&= r\varphi(m + A)
\end{aligned}$$

and hence we can conclude by definition that  $\varphi$  is an  $R$ -module homomorphism.

Observe that  $\ker \varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A : \varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$ . Let  $m + B \in M/B$ . Clearly  $\varphi(m + A) = m + B$  and hence  $\varphi$  is surjective. Now by the First Isomorphism Theorem for Modules we have  $(M/A)/\ker \varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$ .  $\square$

- (4) (*The Fourth or Lattice Isomorphism Theorem*) Let  $N$  be a submodule of the  $R$ -module  $M$ . There is a bijection between the submodules of  $M$  which contain  $N$  and the submodules of  $M/N$ . The correspondence is given by  $A \leftrightarrow A/N$ , for all  $A \supseteq N$ . The correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of  $M/N$  and the lattice of submodules of  $M$  which contain  $N$ ).

*Proof.* Let  $N$  be a submodule of  $M$ . Define  $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$ ,  $T = \{L : L \text{ is a submodule of } M/N\}$ . Define  $\varphi : S \rightarrow T$  by  $\varphi(K) = K/N$ . We want to show that this mapping is bijective.

Let  $K_1, K_2 \in S$  and suppose that  $\varphi(K_1) = \varphi(K_2)$ . Then  $K_1/N = K_2/N$ . We want to show that  $K_1 = K_2$ . Let  $x \in K_1$ , then  $x + N \in K_1/N = K_2/N$ , in particular there exists  $y \in K_2$  such that  $x + N = y + N$ . By property of cosets it follows that  $x - y \in N$ . But since  $N \subseteq K_2$  by construction  $x - y \in K_2$ . Since  $K_2$  is a submodule of  $M$ , it is closed under addition and so  $(x - y) + y = x \in K_2$ . Conclude that  $K_1 \subseteq K_2$ . By symmetric argument  $K_2 \subseteq K_1$  and hence  $K_1 = K_2$ . Thus by definition  $\varphi$  is injective.

Let  $L$  be a submodule of  $M/N$ . Consider the natural projection map  $\pi : M \rightarrow M/N$  defined by  $\pi(m) = m + N$ . We want to show that there exists  $K \in S$  such that  $\varphi(K) = L$ . To do this we will show that  $\pi^{-1}(L)$  is a submodule of  $M$  and that  $N \subseteq \pi^{-1}(L)$ . Recall that  $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$ . Observe that  $0 \in \pi^{-1}(L)$  since  $\pi(0) = 0$  and hence  $\pi^{-1}(L) \neq \emptyset$ . Let  $x, y \in \pi^{-1}(L)$  and  $r \in R$ . Observe that  $\pi(x + ry) = \pi(x) + r\pi(y)$ . Since  $\pi(y) \in L$  by definition and  $L$  is a submodule of  $M/N$ , it follows that since scalar multiplication is closed  $r\pi(y) \in L$ . Thus it follows that  $\pi(x) + r\pi(y) \in L$  and hence  $x + ry \in \pi^{-1}(L)$ . Thus by the submodule criterion we can conclude that  $\pi^{-1}(L)$  is in fact a submodule. Now let  $n \in N$  and observe that  $\pi(n) = n + N = 0 + N \in L$  so by definition it follows that  $n \in \pi^{-1}(L)$ . Conclude that  $N \subseteq \pi^{-1}(L)$  and hence  $\varphi$  is surjective.

Conclude  $\varphi$  is bijective and result follows.  $\square$

**Exercise 0.2.** (DF §10.2 Exercise 1): Use the submodule criterion to show that the kernels and images of  $R$ -module homomorphisms are submodules.

*Proof.* Let  $M, N$  be  $R$ -modules and  $\varphi : M \rightarrow N$  an  $R$ -module homomorphism. Recall that  $\ker \varphi = \{m \in M : \varphi(m) = 0\}$  and  $\operatorname{im} \varphi = \{n \in N : \text{there exists } m \in M \text{ with } \varphi(m) = n\}$ .

Observe that  $\varphi(0) = 0$  so  $0 \in \ker \varphi \neq \emptyset$ . Let  $m, m' \in M$ ,  $r \in R$ . Now  $\varphi(m + rm') = \varphi(m) + r\varphi(m') = 0 + r \cdot 0 = 0 + 0 = 0$ . So  $m + rm' \in \ker \varphi$ . Thus by the submodule criterion  $\ker \varphi$  is a submodule.

Observe that  $\varphi(0) = 0 \in N$  so  $0 \in \operatorname{im} \varphi \neq \emptyset$ . Let  $n, n' \in N$ ,  $r \in R$ . Then there exists  $m, m' \in M$  such that  $\varphi(m) = n$  and  $\varphi(m') = n'$ . Now consider  $n + rn'$ .  $\varphi(m + rm') = \varphi(m) + r\varphi(m') = n + rn'$  so  $n + rn' \in \operatorname{im} \varphi$ . Conclude by submodule criterion that  $\operatorname{im} \varphi$  is in fact a submodule.  $\square$

**Exercise 0.3.** (DF §10.2 Exercise 2): Show that the relation “is  $R$ -module isomorphic to” is an equivalence relation on any set of  $R$ -modules.

*Proof.* Let  $X$  be a set of  $R$ -modules.

- Let  $M \in X$ . Observe that  $M$  is isomorphic to  $M$  trivially. So relation is reflexive.
- Let  $M, N \in X$ . Suppose  $M$  is isomorphic to  $N$  then by definition there exists  $\varphi : M \rightarrow N$  that is bijective. Immediately we have  $\varphi^{-1} : N \rightarrow M$  which is also bijective so  $N$  is isomorphic to  $M$ . By definition the relation is symmetric.
- Let  $L, M, N \in X$ . Suppose  $L$  is isomorphic to  $M$ , then by definition there exists  $\varphi : L \rightarrow M$  a bijective  $R$ -module homomorphism. Suppose  $M$  is isomorphic to  $N$ , then there exists  $\Phi : M \rightarrow N$  a bijective  $R$ -module homomorphism. Observe that  $\varphi \circ \Phi : L \rightarrow N$  is again a bijective  $R$ -module homomorphism by property of composition of mappings. Hence by definition  $L$  is isomorphic to  $N$ .

Conclude that the relation “is  $R$ -module isomorphic to” is an equivalence relation on any set of  $R$ -modules.  $\square$

**Exercise 0.4.** (DF §10.2 Exercise 3): Give an explicit example of a map from one  $R$ -module to another which is a group homomorphism but not an  $R$ -module homomorphism.

*Solution.* Consider the Quaternions  $\mathbb{H} = R$ ; they form a commutative group under addition and a noncommutative group under multiplication. Hence  $\mathbb{H}$  is a noncommutative ring with unity. In particular  $\mathbb{H}$  is an  $R$ -module over itself. Define  $\varphi : \mathbb{H} \rightarrow \mathbb{H}$  by  $\varphi(h) = ih$ . This is a group homomorphism since  $\varphi(h + h') = i(h + h') = ih + ih' = \varphi(h) + \varphi(h')$ . But note that  $\varphi(j \cdot 1) = \varphi(j) = ij = k \neq -k = ji = j(i \cdot 1) = j\varphi(1)$ . Conclude that  $\varphi$  is not an  $R$ -module homomorphism since the definition is not satisfied.  $\blacktriangleleft$

**Exercise 0.5.** (DF §10.2 Exercise 4): Let  $A$  be a  $\mathbb{Z}$ -module, let  $a$  be any element of  $A$  and let  $n$  be a positive integer. Prove that the map  $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  given by  $\varphi(\bar{k}) = ka$  is a well-defined  $\mathbb{Z}$ -module homomorphism if and only if  $na = 0$ . Prove that  $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$ , where  $A_n = \{a \in A : na = 0\}$  (So  $A_n$  is the annihilator in  $A$  of the ideal  $(n)$  of  $\mathbb{Z}$ ).

*Proof.* Suppose that the map  $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  given by  $\varphi(\bar{k}) = ka$  is a well-defined  $\mathbb{Z}$ -module homomorphism. Then by definition if  $\bar{m} = \bar{k}$  then  $\varphi(\bar{m}) = \varphi(\bar{k})$  or equivalently  $ma = ka$ . Moreover  $\varphi(a + b) = \varphi(a) + \varphi(b)$  and  $\varphi(ra) = r\varphi(a)$  for all  $a, b \in A$  and  $r \in \mathbb{Z}$ . Observe that  $\bar{0} = \bar{k}$  so by hypothesis  $\varphi(\bar{0}) = \varphi(\bar{k})$  but observe that  $\varphi(\bar{0}) = 0 \cdot a = 0$  and  $\varphi(\bar{k}) = ka$ . Hence by equality  $ka = 0$ . Conversely suppose that  $na = 0$ . We want to show that  $\varphi : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  defined by  $\varphi(\bar{k}) = ka$  is a well-defined  $R$ -module homomorphism. Say  $\bar{k} = \bar{m}$  then by property of cosets  $k - m \in n\mathbb{Z}$  and so  $n \mid k - m$ . By definition  $n \mid k - m$  and hence there exists  $t \in \mathbb{Z}$  such that  $k - m = nt$ . Observe that

$$\begin{aligned} k - m &= nt \\ (k - m)a &= nta \\ ka - ma &= (na)t \\ ka - ma &= 0 \\ ka &= ma \end{aligned}$$

Thus we have  $\varphi(\bar{k}) = \varphi(\bar{m})$  and we can conclude that  $\varphi$  is in fact a well-defined  $R$ -module homomorphism.

Now  $\square$

**Exercise 0.6.** (DF §10.2 Exercise 5):

**Exercise 0.7.** (DF §10.2 Exercise 6):

**Exercise 0.8.** (DF §10.2 Exercise 7):

**Exercise 0.9.** (DF §10.2 Exercise 8): Let  $\varphi : M \rightarrow N$  be an  $R$ -module homomorphism. Prove that  $\varphi(\operatorname{Tor}(M)) \subseteq \operatorname{Tor}(N)$ .

*Proof.* Recall that  $\operatorname{Tor}(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$ . Now it follows that  $\varphi(\operatorname{Tor}(M)) = \{n \in N : n = \varphi(m) \text{ for some } m \in \operatorname{Tor}(M)\}$ . Let  $n \in \varphi(\operatorname{Tor}(M))$  then  $n = \varphi(m)$  for some  $m \in \operatorname{Tor}(M)$  by definition. Since  $m \in \operatorname{Tor}(M)$  there exists  $0 \neq r \in R$  such that  $rm = 0$ . Hence  $rn = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$ . Conclude that  $n \in \operatorname{Tor}(N)$  and hence  $\varphi(\operatorname{Tor}(M)) \subseteq \operatorname{Tor}(N)$ .  $\square$

**Exercise 0.10.** (DF §10.2 Exercise 9):

**Exercise 0.11.** (DF §10.2 Exercise 10):

**Exercise 0.12.** (DF §10.2 Exercise 11): Let  $A_1, A_2, \dots, A_n$  be  $R$ -modules and let  $B_i$  be submodules of  $A_i$  for each  $i = 1, 2, \dots, n$ . Prove that

$$(A_1 \times A_2 \times \cdots \times A_n) / (B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

*Proof.* Define  $\varphi : A_1 \times A_2 \times \cdots \times A_n \rightarrow (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$  by  $\varphi(a_1, a_2, \dots, a_n) = (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n)$ . We want to show that this is an  $R$ -module homomorphism and that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$  and that the mapping is surjective. Then the first isomorphism theorem for modules yields the result.

Let  $x, y \in A_1 \times A_2 \times \cdots \times A_n$  where  $x = (a_1, a_2, \dots, a_n)$  and  $y = (a'_1, a'_2, \dots, a'_n)$ . Observe that

$$\begin{aligned} \varphi(x + y) &= \varphi((a_1, a_2, \dots, a_n) + (a'_1, a'_2, \dots, a'_n)) \\ &= \varphi(a_1 + a'_1, a_2 + a'_2, \dots, a_n + a'_n) \\ &= (a_1 + a'_1 + B_1, a_2 + a'_2 + B_2, \dots, a_n + a'_n + B_n) \\ &= (a_1 + B_1 + a'_1 + B_1, a_2 + B_2 + a'_2 + B_2, \dots, a_n + B_n + a'_n + B_n) \\ &= (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) + (a'_1 + B_1, a'_2 + B_2, \dots, a'_n + B_n) \\ &= \varphi(a_1, a_2, \dots, a_n) + \varphi(a'_1, a'_2, \dots, a'_n) \\ &= \varphi(x) + \varphi(y) \end{aligned}$$

and for  $r \in R$

$$\begin{aligned} \varphi(rx) &= \varphi(r(a_1, a_2, \dots, a_n)) \\ &= \varphi(ra_1, ra_2, \dots, ra_n) \\ &= (ra_1 + B_1, ra_2 + B_2, \dots, ra_n + B_n) \\ &= (r(a_1 + B_1), r(a_2 + B_2), \dots, r(a_n + B_n)) \\ &= r(a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) \\ &= r\varphi(a_1, a_2, \dots, a_n) \\ &= r\varphi(x) \end{aligned}$$

Thus by definition we can conclude that  $\varphi$  is an  $R$ -module homomorphism.

Now we want to show that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$ . Observe that

$$\begin{aligned} \ker \varphi &= \{x \in A_1 \times A_2 \times \cdots \times A_n : \varphi(x) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : \varphi(a_1, a_2, \dots, a_n) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) = (0, 0, \dots, 0)\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : a_1 \in B_1, a_2 \in B_2, \dots, a_n \in B_n\} \end{aligned}$$

Hence  $\ker \varphi \subseteq B_1 \times B_2 \times \cdots \times B_n$  and trivially  $B_1 \times B_2 \times \cdots \times B_n \subseteq \ker \varphi$  by construction of  $\varphi$ . Conclude that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$ .

The mapping is trivially surjective. Applying first Isomorphism theorem yields the result.  $\square$