INTRODUCTION TO CATEGORY THEORY

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ABSTRACT. Module Theory.

1. MATH 697 HOMEWORK ONE

Exercise 1.1. Prove Theorem 4 (Isomorphism Theorems):

(1) (*The First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then $\ker\varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proof. Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then by definition $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$, $r \in R$. We want to show that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ is a submodule. Observe that since M is a submodule then M is an abelian group so there exists $0 \in M$ such that $m + 0 = m$ for all $m \in M$. In particular $\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$ implying $\varphi(0) = 0$. Conclude that $0 \in \ker\varphi \neq \emptyset$. Now let $r \in R$, $x, y \in \ker\varphi$. Observe that $\varphi(x + ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r \cdot 0 = 0 + 0 = 0$. Hence $x + ry \in \ker\varphi$. Conclude by the submodule criterion that $\ker\varphi$ is in fact a submodule.

Now □

(2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R-module M. Then $(A+B)/B \cong A/(A \cap B)$.

(3) (*The Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

(4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence cummutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

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