INTRODUCTION TO HOMOLOGICAL ALGEBRA

ROBERT CARDONA, MASSY KHOSHBIN, AND SIAVASH MORTEZAVI

0. MATH 697 Homework Zero.Two

AM Exercise 2.1: Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$ if m and n are coprime.

Proof. Choose $a \otimes b \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$. Since m and n are coprime, there exist $s, t \in \mathbb{Z}$ such that $ms + nt = 1$ Observe that

$$a = a \cdot 1 = a(ms + nt) = ams + ant \equiv ant \pmod{m}.$$

Now observe that

$$a \otimes b = atn \otimes b = a \otimes nb = at \otimes 0 = 0.$$

We have shown that any simple tensor is zero, so any finite linear combination of simple tensors is zero. Conclude $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) = 0$.

□

AM Exercise 2.2: Let R be a ring, I an ideal of R, M an R-module. Show that $(R/I) \otimes_R M$ is isomorphic to M/IM.

Proof. Define $\varphi : R/I \times M \to M/IM$ by $\varphi(r + I, m) = rm + IM$, which we shall henceforth write as $\varphi(\overline{r}, m) = \overline{rm}$. Let $(\overline{r}, m) = (\overline{s}, m)$. Then $\overline{r} = \overline{s} \implies r \in \overline{s} \implies r = s + i$, some $i \in I$. Then $\varphi(\overline{r}, m) = \overline{rm} = \overline{(s+i)m} = \overline{sm + im} = \overline{sm} + \overline{im} = \overline{sm} + \overline{0} = \overline{sm} = \varphi(\overline{s}, m)$. Thus φ is well-defined.

Observe $\varphi(\overline{r} + \overline{s}, m) = \varphi(\overline{r+s}, m) = \overline{(r+s)m} = \overline{rm + sm} = \overline{rm} + \overline{sm} = \varphi(\overline{r}, m) + \varphi(\overline{s}, m)$. Similarly, $\varphi(\overline{r}, m + n) = \varphi(\overline{r}, m) + \varphi(\overline{r}, n)$. Lastly, $\varphi(\overline{rs}, m) = \overline{(rs)m} = \overline{r(sm)} = \varphi(\overline{r}, sm)$. Thus φ is R-biadditive (In fact, φ is R-bilinear).

Now we are guaranteed a unique R-homomorphism $\phi : R/I \otimes_R M \to M/IM$ given by $\phi(\overline{r} \otimes m) = \overline{rm}$. Notice if we define $f : M/IM \to R/I \otimes_R M$ via $f(\overline{m}) = \overline{1} \otimes m$ then f is a $\mathbb{Z}$-homomorphism which makes $f \circ \phi$ and $\phi \circ f$ the identity map in $R/I \otimes_R M$ and M/IM, respectively. So ϕ has a two-sided inverse, hence a bijective function, and accordingly is an isomorphism when considered as an R-map. □

DF §10.5 Proposition 24: (*The Short Five Lemma*) Let α, β, γ be homomorphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) If α and γ are injective then so is β.

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that $b = 0$. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know $g(b) = 0$ so $b \in \ker g$ but since we are in an exact sequence we have $\operatorname{im} f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with $f(a) = b$. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so $a = 0$. So $0 = f(a) = b$. □

(2) If α and γ are surjective then so is β.

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that $g(b) = c$. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness $\operatorname{im} f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$. □

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

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Proof. Follows from (1) and (2). □

R Exercise 2.14: Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of module maps. Prove that $gf = 0$ if and only if $\operatorname{im} f \subseteq \ker g$. Give an example of such a sequence that is not exact.

Proof. Suppose $gf = 0$, that is, $f(g(a)) = 0$ for all $a \in A$. Let $b \in \operatorname{im} f$ then by definition there exists $a \in A$ such that $f(a) = b$. But we know by hypothesis that $0 = g(f(a)) = g(b)$ so $b \in \ker g$. Conclude that $\operatorname{im} f \subseteq \ker g$. Conversely, suppose that $\operatorname{im} f \subseteq \ker g$. Let $a \in A$ and observe that $f(a) \in \operatorname{im} f$. By hypothesis $f(a) \in \ker g$ so $g(f(a)) = 0$. Since a was arbitrary conclude $gf = 0$. □

Consider the sequence of module maps $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$ where $f(\overline{x}) = 2\overline{x}$ and $g(\overline{y}) = \overline{y}$. Observe that $\operatorname{im} f = \mathbb{Z}/3\mathbb{Z}$ and $\ker g = \{0\}$. Since $\operatorname{im} f \neq \ker g$ it follows that this is **not** an exact sequence.

R Exercise 2.15:

(1) Prove that $f : M \to N$ is surjective if and only if $\operatorname{coker} f = \{0\}$.

Proof. Suppose $f : M \to N$ is surjective then for $n \in N$, there exists $m \in M$ such that $f(m) = n$. By definition $\operatorname{coker} f = M/\operatorname{im} f = M/M = 0$. Conversely, suppose that $\operatorname{coker} f = 0$, i.e., $M/\operatorname{im} f = 0$ implying that if $m + \operatorname{im} f \in M/\operatorname{im} f$ then $m + \operatorname{im} f = 0$ or equivalently $m \in \operatorname{im} f$. Since m is arbitrary, conclude $M = \operatorname{im} f$ and hence f is surjective by definition. □

(2) If $f : M \to N$ is a map, prove that there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Proof. Define $h : \ker f \to M$ by $h(m) = m$, that is, map each element to itself. It follows immediately that $\operatorname{im} h = \ker f$. Define $g : N \to \operatorname{coker} f = N/\operatorname{im} f$ by $g(n) = n + \operatorname{im} f$, that is, the canonical/projection mapping. Observe that $\ker g = \operatorname{im} f$. Conclude that the following sequence is in fact exact:

$$0 \xrightarrow{\text{identity}} \ker f \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} \operatorname{coker} f \xrightarrow{\text{zero}} 0. \quad \square$$

R Exercise 2.16:

(1) If $0 \to M \to 0$ is an exact sequence, prove that $M = \{0\}$.

Proof. Consider $0 \xrightarrow{f} M \xrightarrow{g} 0$. Since f is surjective then for $m \in M$ there exists $x \in 0$ such that $f(x) = m$ but x must be 0 so $m = 0$. □

(2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective if and only if h is injective.

Proof. Suppose f is surjective. Then $\operatorname{im} f = B = \ker g$ but this immediately implies that $\operatorname{im} g = 0 = \ker h$ so h is injective by definitoin. Conversely, suppose h is injective. Then $\ker h = 0 = \operatorname{im} g$ which immediately implies $\ker g = B = \operatorname{im} f$. Conclude by definition f is surjective. □

(3) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ be exact. If α and δ are isomorphisms, prove that $C = \{0\}$.

Proof. Observe that, by previous exercise, β is surjective and γ is injective so we have $\operatorname{im} \beta = C$ and $\ker \gamma = 0$. Result follows by exactness: $C = \operatorname{im} \beta = \ker \gamma = 0$. Conclude $C = \{0\}$. □

R Exercise 2.17: If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that there is an exact sequence $0 \to \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \to 0$, where $\alpha : b + \operatorname{im} f \mapsto g(b)$ and $\beta : c \mapsto h(c)$.

Proof. Observe that $\ker \beta = \{c \in C : \beta(c) = h(c) = 0\} = \ker h = \operatorname{im} g$ by exactness and since α can run through any $b \in B$ conclude $\operatorname{im} \alpha = \operatorname{im} g = \ker \beta$ □

R Exercise 2.28: Let R be a domain with $Q = \operatorname{Frac}(R)$, its field of fractions. If A is an R-module, prove that every element of $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$ (i.e. every element is a simple tensor).

Proof. Let $\sum_1^n q_i \otimes a_i \in Q \otimes_R A$. We can write $\sum_1^n q_i \otimes a_i = \sum_1^n \frac{r_i}{s_i} \otimes a_i$ for $r_i, s_i \in R, s_i \neq 0$. Write $s = s_1 s_2 \cdots s_n$ and $\widehat{s_i} = \frac{s}{s_i}$. Then $\sum_1^n \frac{r_i}{s_i} \otimes a_i = \sum_1^n (1 \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n (\frac{\widehat{s_i}}{\widehat{s_i}} \cdot \frac{r_i}{s_i}) \otimes a_i = \sum_1^n \frac{\widehat{s_i} r_i}{s} \otimes a_i = \sum_1^n (\frac{1}{s}) \widehat{s_i} r_i \otimes a_i = \sum_1^n \frac{1}{s} \otimes (\widehat{s_i} r_i) a_i = \frac{1}{s} \otimes (\sum_1^n \widehat{s_i} r_i a_i)$. □

R Exercise 2.29:**(i)** Let p be a prime, and let p, q be relatively prime. Prove that if A is a p-primary group and $a \in A$, then there exists $x \in A$ with $qx = a$.

(ii) If D is a finite cyclic group of order m, prove that D/nD is a cyclic group of order $d = (m, n)$.

(iii) Let m and n be positive integers, and let $d = (m, n)$. Prove that there is an isomorphism of abelian groups

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d.$$

(iv) Let G and H be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \text{ and } H = B_1 \oplus \cdots \oplus B_m,$$

where A_i and B_j are cyclic groups. Compute $G \otimes_{\mathbb{Z}} H$ explicitely.

Proof. (i) $a \in A$ so $p^k a = 0$, some $k \in \mathbb{Z}^+$. Since p is prime, $(q,p) = 1 \implies (q, p^k) = 1$. So there exist $m, n \in \mathbb{Z}$ such that $qm + np^k = 1$. Now $a = 1 \cdot a = (qm + np^k)a = qma + np^k a = q(ma) + n(p^k a) = qx + 0 = qx$. Observe $p^k x = p^k(ma) = m(p^k a) = 0$ so $x \in A$.

(ii) D is cyclic, so D/nD is cyclic. If we write $D =< a >$, then $nD =< na >$. This is because for any $nb \in nD$, we can write $b = ka$, some $k \in \mathbb{Z}^+$, since a generates D. Now $nb = n(ka) = k(na)$, and we have that na generates nD.

Claim $|na| = \frac{m}{d}$. Observe $\frac{m}{d}(na) = \frac{n}{d}(ma) = \frac{n}{d} \cdot 0 = 0$, which implies $|na|$ divides $\frac{m}{d}$. On the other hand, if $k(na) = 0$, then $(kn)a = 0 \implies m|kn \implies \frac{m}{d}|k\frac{n}{d}$. But $\frac{m}{d}$ and $\frac{n}{d}$ are relatively prime by construction, forcing $\frac{m}{d}|k$. In particular, we have $\frac{m}{d}$ divides $|na|$. Thus $|na| = d$. Now $|nD| = |< na >| = |na| = \frac{m}{d}$. Lagrange's theorem gives us $|\frac{D}{nD}| = \frac{|D|}{|nD|} = \frac{m}{\frac{m}{d}} = d$.

(iii) By Proposition 2.68, we have that $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_m / n\mathbb{Z}_m$. But by part (ii), $\mathbb{Z}_m / n\mathbb{Z}_m$ is a cyclic group of order $d = (m,n)$ so $\mathbb{Z}_m / n\mathbb{Z}_m \cong \mathbb{Z}_d$.

(iv) If both A_i and B_j are finite, then by part (iii) we have $A_i \otimes_{\mathbb{Z}} B_j \cong \mathbb{Z}_{s_i} \otimes \mathbb{Z}_{t_j} \cong \mathbb{Z}_{(s_i, t_j)}$, where $|A_i| = s_i$ and $|B_j| = t_i$. Otherwise, $A_i \otimes_{\mathbb{Z}} B_j \cong \mathbb{Z}$. Thus we have

$$G \otimes_{\mathbb{Z}} H = \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j = \sum_{i,j: A_i \text{ and } B_j \text{ finite}} \mathbb{Z}_{(s_i, t_i)} + \sum_{i,j: A_i \text{ or } B_j \text{ infinite}} \mathbb{Z}.$$

□

R Exercise 2.32: (3×3 *Lemma*) Consider the following commutative diagram in ${}_R\mathbf{Mod}$ having exact columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "\alpha_1"] \arrow[d, "f'"] & A \arrow[r, "\alpha_2"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "\beta_1"] \arrow[d, "g'"] & B \arrow[r, "\beta_2"] \arrow[d, "g"] & B'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & C' \arrow[r, "\gamma_1"] \arrow[d] & C \arrow[r, "\gamma_2"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

Proof. We show the first part of the statement. The second part follows by a symmetric argument.

α_1 is injective: Let $a' \in \ker \alpha_1$. Then $\alpha_1(a') = 0$. So $f(\alpha_1(a')) = 0$. Now $0 = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. The injectivity of β_1 implies $f'(a') = 0$ and the injectivity of f' gives us $a' = 0$. Thus $\ker \alpha_1 = 0$ and α_1 is injective.

α_2 is surjective:

$\text{im } \alpha_1 \subseteq \ker \alpha_2$: Let $a \in \text{im } \alpha_1$. Then there exists $a' \in A'$ with $a = \alpha_1(a')$. Observe $f(a) = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. Thus $\beta_2(f(a)) = \beta_2(\beta_1(f'(a'))) = 0$ by exactness. Now $0 = \beta_2(f(a)) = f''(\alpha_2(a))$ by commutativity. The injectivity of f'' gives us $\alpha_2(a) = 0$. Hence $a \in \ker \alpha_2$.

$\ker \alpha_2 \subseteq \text{im } \alpha_1$: Let $a \in \ker \alpha_2$. Then $\alpha_2(a) = 0$. So $f''(\alpha_2(a)) = 0$. By commutativity, $\beta_2(f(a)) = 0$. Now $f(a) \in \ker \beta_2 = \text{im } \beta_1$, so there exists $b' \in B'$ such that $f(a) = \beta_1(b')$. Now $g(f(a)) = 0$ by exactness, so $g(\beta_1(b')) = 0$. By commutativity, $\gamma_1(g'(b')) = 0$. Since γ_1 is injective, $g'(b') = 0$. Now $b' \in \ker g' = \text{im } f'$ so there exists $a' \in A'$ such that $b' = f'(a')$. Thus $f(a) = \beta_1(b') = \beta_1(f'(a'))$. By commutativity, $\beta_1(f'(a')) = f(\alpha_1(a'))$. So $f(a) = f(\alpha_1(a'))$. Since f is injective, we have

$a = \alpha_1(a')$, and therefore $a \in \text{im } \alpha_1$.

□

AM Proposition 2.10: (*Snake Lemma*) Let

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "g"] \arrow[d, "f'"] & M \arrow[r, "h"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "g'"] & N \arrow[r, "h'"] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \text{coker}(f') \xrightarrow{\overline{u}'} \text{coker}(f) \xrightarrow{\overline{v}'} \text{coker}(f'') \to 0$$

in which $\overline{u}$, $\overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

Proof. We consider:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f' \arrow[r, "\overline{u}"] \arrow[d, "g'"] & \ker f \arrow[r, "\overline{v}"] \arrow[d, "g"] & \ker f'' \arrow[d, "g''"] \arrow[dddll, "d"'] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d, "h'"] & N \arrow[r, "v'"] \arrow[d, "h'"] & N'' \arrow[r] \arrow[d, "h'"] & 0 \\
 & \text{coker } f' \arrow[r, "\overline{u}'"] \arrow[d] & \text{coker } f \arrow[r, "\overline{v}'"] \arrow[d] & \text{coker } f'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Rotman Exercise 2.32, we have that $\text{im}\,\overline{u} = \ker \overline{v}$ and $\text{im}\,\overline{u}' = \ker \overline{v}'$. It is left for us to first define d and then show that $\text{im}\,\overline{v} = \ker d$ and $\text{im}\, d = \ker \overline{u}'$. □

Department of Mathematics, California State University Long Beach
E-mail address: mrrobertcardona@gmail.com and massy255@gmail.com and siavash.mortezavi@gmail.com