INTRODUCTION TO HOMOLOGICAL ALGEBRA

ROBERT CARDONA, MASSY KHOSHBIN, AND SIAVASH MORTEZAVI

1. MATH 697 Notes

R Proposition 2.18:

(1) A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective.
(2) A sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.
(3) A sequence $0 \to A \xrightarrow{h} B \to 0$ is exact if and only if h is an isomorphism.

DF §10.5 Proposition 24: (*The Short Five Lemma*) Let α, β, γ be homomorphisms of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) If α and γ are injective then so is β.

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that $b = 0$. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know $g(b) = 0$ so $b \in \ker g$ but since we are in an exact sequence we have $\operatorname{im} f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with $f(a) = b$. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so $a = 0$. So $0 = f(a) = b$. $\square$

(2) If α and γ are surjective then so is β.

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that $g(b) = c$. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness $\operatorname{im} f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$. $\square$

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

Proof. Follows from (1) and (2). $\square$

R Proposition 2.72: (*Five Lemma*) Consider the commutative diagram with exact rows.

$$\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "\alpha"] & A_2 \arrow[r, "g"] \arrow[d, "\beta"] & A_3 \arrow[r, "h"] \arrow[d, "\gamma"] & A_4 \arrow[r, "k"] \arrow[d, "\delta"] & A_5 \arrow[d, "\epsilon"] \\
B_1 \arrow[r, "f'"] & B_2 \arrow[r, "g'"] & B_3 \arrow[r, "h'"] & B_4 \arrow[r, "k'"] & B_5
\end{tikzcd}$$

(1) If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.
(2) If h_2 and h_4 are injective and h_1 is surjective, then h_3 is injective.
(3) If h_1, h_2, h_4 and h_5 are isomorphisms, then h_3 is an isomorphism.

AM Proposition 2.10: (*Snake Lemma*) Let

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "g"] \arrow[d, "f'"] & M \arrow[r, "h"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "g'"] & N \arrow[r, "h'"] & N'' \arrow[r] & 0
\end{tikzcd}$$

Date: August 3, 2013.

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \to 0$$

in which $\overline{u}$, $\overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

Department of Mathematics, California State University Long Beach
E-mail address: mrrobertcardona@gmail.com and massy255@gmail.com and siavash.mortezavi@gmail.com