INTRODUCTION TO HOMOLOGICAL ALGEBRA

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1. MATH 697 Notes

A **category** C consists of three ingredients: a class $\text{obj}(C)$ of **objects**, a set of **morphisms** $\text{Hom}(A, B)$ for every ordered pair (A, B) of objects, and **composition** $\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$, denoted by $(f, g) \mapsto gf$. for every ordered triple A, B, C of objects. These ingredients are subject to the following axioms:

(1) The Hom sets are pairwise disjoint; that is, each $f \in \text{Hom}(A, B)$ has a unique **domain** A and a unique **target** B.
(2) For each object A, there is an **identity morphism** $1_A \in \text{Hom}(A, A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$.
(3) Composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(fg) = (hg)f$.

Examples of categories: **Sets**, **Groups**, **Top**, **Ab**, **Rings**, **ComRings**.

A **left** R**-module**, where R is a ring, is an additive abelian group M having a *scalar multiplication* $R \times M \to M$, denoted by $(r, m) \mapsto rm$, such that, for all $m, m' \in M$ and $r, r' \in R$. We often write ${}_RM$ to denote M being a left R-module.

(1) $r(m + m') = rm + rm'$
(2) $(r + r')m = rm + r'm$
(3) $(rr')m = r(r'm)$
(4) $1m = m$.

If M and N are left R-modules, then an R-**homomorphism** (or an R-**map**) is a function $f : M \to N$ such that, for all $m, m' \in M$ and $r \in R$,

(1) $f(m + m') = f(m) + f(m')$
(2) $f(rm) = r(fm)$

An R-**isomorphism** is a bijective R-homomorphism.

A **right** R**-module**, where R is a ring, is an additive abelian group M having a *scalar multiplication* $M \times R \to M$, denoted by $(m, r) \mapsto mr$, such that, for all $m, m' \in M$ and $r, r' \in R$. We often write M_R to denote M being a right R-module.

(1) $(m + m')r = mr + m'r$
(2) $m(r + r') = mr + mr'$
(3) $m(rr') = (mr)r'$
(4) $m = m1$

The category ${}_R\mathbf{Mod}$ of all **left** R**-modules** (where R is a ring) has as its objects all left R-modules, has as its morphisms all R-homomorphisms, and as its composition the usual composition of functions. We denote the sets $\text{Hom}(A, B)$ in ${}_R\mathbf{Mod}$ by $\text{Hom}_R(A, B)$.

The category $\mathbf{Mod}_R$ of all **right** R**-modules** (where R is a ring) has as its objects all **right** R**-modules**, as its morphisms all R-homomorphisms, and as its composition the usual composition. We denote the sets $\text{Hom}(A, B)$ in $\mathbf{Mod}_R$ by $\text{Hom}_R(A, B)$.

A category S is a **subcategory** of a category C if

(1) $\text{obj}(S) \subseteq \text{obj}(C)$

R Proposition 2.18:

(1) A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective.
(2) A sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.
(3) A sequence $0 \to A \xrightarrow{h} B \to 0$ is exact if and only if h is an isomorphism.

DF §10.5 Proposition 24: (*The Short Five Lemma*) Let α, β, γ be homomorphisms of short exact sequences:

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$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

(1) If α and γ are injective then so is β.

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that $b = 0$. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know $g(b) = 0$ so $b \in \ker g$ but since we are in an exact sequence we have $\operatorname{im} f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with $f(a) = b$. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so $a = 0$. So $0 = f(a) = b$. □

(2) If α and γ are surjective then so is β.

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that $g(b) = c$. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness $\operatorname{im} f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$. □

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

Proof. Follows from (1) and (2). □

R Proposition 2.72: (*Five Lemma*) Consider the commutative diagram with exact rows.

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "\alpha"] & A_2 \arrow[r, "g"] \arrow[d, "\beta"] & A_3 \arrow[r, "h"] \arrow[d, "\gamma"] & A_4 \arrow[r, "k"] \arrow[d, "\delta"] & A_5 \arrow[d, "\varepsilon"] \\ B_1 \arrow[r, "f'"] & B_2 \arrow[r, "g'"] & B_3 \arrow[r, "h'"] & B_4 \arrow[r, "k'"] & B_5 \end{tikzcd}$$

(1) If β and δ are surjective and ε is injective, then γ is surjective.
(2) If β and δ are injective and α is surjective, then γ is injective.
(3) If α, β, δ and ε are isomorphisms, then γ is an isomorphism.

R Exercise 2.14: Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of module maps. Prove that $gf = 0$ if and only if $\operatorname{im} f \subseteq \ker g$. Give an example of such a sequence that is not exact.

Proof. Suppose $gf = 0$, that is, $f(g(a)) = 0$ for all $a \in A$. Let $b \in \operatorname{im} f$ then by definition there exists $a \in A$ such that $f(a) = b$. But we know by hypothesis that $0 = g(f(a)) = g(b)$ so $b \in \ker g$. Conclude that $\operatorname{im} f \subseteq \ker g$. Conversely, suppose that $\operatorname{im} f \subseteq \ker g$. Let $a \in A$ and observe that $f(a) \in \operatorname{im} f$. By hypothesis $f(a) \in \ker g$ so $g(f(a)) = 0$. Since a was arbitrary conclude $gf = 0$. □

Consider the sequence of module maps $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$ where $f(\overline{x}) = 2\overline{x}$ and $g(\overline{y}) = \overline{y}$. Observe that $\operatorname{im} f = \mathbb{Z}/3\mathbb{Z}$ and $\ker g = \{0\}$. Since $\operatorname{im} f \neq \ker g$ it follows that this is **not** an exact sequence.

R Exercise 2.15:

(1) Prove that $f : M \to N$ is surjective if and only if $\operatorname{coker} f = \{0\}$.

Proof. Suppose $f : M \to N$ is surjective then for $n \in N$, there exists $m \in M$ such that $f(m) = n$. By definition $\operatorname{coker} f = M/\operatorname{im} f = M/M = 0$. Conversely, suppose that $\operatorname{coker} f = 0$, i.e., $M/\operatorname{im} f = 0$ implying that if $m + \operatorname{im} f \in M/\operatorname{im} f$ then $m + \operatorname{im} f = 0$ or equivalently $m \in \operatorname{im} f$. Since m is arbitrary, conclude $M = \operatorname{im} f$ and hence f is surjective by definition. □

(2) If $f : M \to N$ is a map, prove that there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Proof. Define $h : \ker f \to M$ by $h(m) = m$, that is, map each element to itself. It follows immediately that $\operatorname{im} h = \ker f$. Define $g : N \to \operatorname{coker} f = N/\operatorname{im} f$ by $g(n) = n + \operatorname{im} f$, that is, the canonical/projection mapping. Observe that $\ker g = \operatorname{im} f$. Conclude that the following sequence is in fact exact:

$$0 \xrightarrow{\text{identity}} \ker f \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} \operatorname{coker} f \xrightarrow{\text{zero}} 0. \quad \square$$

R Exercise 2.16:

(1) If $0 \to M \to 0$ is an exact sequence, prove that $M = \{0\}$.

Proof. Consider $0 \xrightarrow{f} M \xrightarrow{g} 0$. Since f is surjective then for $m \in M$ there exists $x \in 0$ such that $f(x) = m$ but x must be 0 so $m = 0$. ☐

(2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is an exact sequence, prove that f is surjective if and only if h is injective.

Proof. Suppose f is surjective. Then $\operatorname{im} f = B = \ker g$ but this immediately implies that $\operatorname{im} g = 0 = \ker h$ so h is injective by definitoin. Conversely, suppose h is injective. Then $\ker h = 0 = \operatorname{im} g$ which immediately implies $\ker g = B = \operatorname{im} f$. Conclude by definition f is surjective. ☐

(3) Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$ be exact. If α and δ are isomorphisms, prove that $C = \{0\}$.

Proof. Observe that, by previous exercise, β is surjective and γ is injective so we have $\operatorname{im} \beta = C$ and $\ker \gamma = 0$. Result follows by exactness: $C = \operatorname{im} \beta = \ker \gamma = 0$. Conclude $C = \{0\}$. ☐

R Exercise 2.17: If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ is exact, prove that there is an exact sequence $0 \to \operatorname{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \ker k \to 0$, where $\alpha : b + \operatorname{im} f \mapsto g(b)$ and $\beta : c \mapsto h(c)$.

Proof. Observe that $\ker \beta = \{c \in C : \beta(c) = h(c) = 0\} = \ker h = \operatorname{im} g$ by exactness and since α can run through any $b \in B$ conclude $\operatorname{im} \alpha = \operatorname{im} g = \ker \beta$ ☐

R Exercise 2.32: (3×3 *Lemma*) Consider the following commutative diagram in ${}_R\mathbf{Mod}$ having exact columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "\alpha_1"] \arrow[d, "f'"] & A \arrow[r, "\alpha_2"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "\beta_1"] \arrow[d, "g'"] & B \arrow[r, "\beta_2"] \arrow[d, "g"] & B'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & C' \arrow[r, "\gamma_1"] \arrow[d] & C \arrow[r, "\gamma_2"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

Proof. Suppose that the bottom two rows are exact. We must show four things:

(1) α_1 is injective: Suppose $a' \in A'$ with $\alpha_1(a') = 0$. Observe that by commutativity $f(\alpha_1(a')) = \beta_1(f'(a')) = 0$ so since β_1 is injective, by exactness of the second row, $f'(a') = 0$. But since the first column is exact by hypothesis, we have $a' = 0$. Conclude that α_1 is in fact injective.

(2) $\operatorname{im} \alpha_1 \subseteq \ker \alpha_2$: Choose $a \in \operatorname{im} \alpha_1$ then by definition there exists $a' \in A'$ such that $\alpha_1(a') = a$. Observe that $f(a) = f(\alpha_1(a')) = \beta_1(f'(a'))$ by commutativity. Moreover $\beta_2(f(a)) = f''(\alpha_2(a))$ again by commutativity. Hence we have
$$\beta_2(f(a)) = \beta_2(\beta_1(f'(a'))) = 0 = f''(\alpha_2(a)).$$
Since f'' is injective by exactness of the third column, $\alpha_2(a) = 0$ and so $a \in \ker \alpha_2$. Conclude that $\operatorname{im} \alpha_2 \subseteq \ker \alpha_2$ as desired.

(3) $\ker \alpha_2 \subseteq \operatorname{im} \alpha_1$: Let $a \in \ker \alpha_2$ then by definition $\alpha_2(a) = 0$ and so $f''(\alpha(a)) = \beta_2(f(a)) = 0$. So $f(a) \in \ker \beta_2 = \operatorname{im} \beta_1$ by exactness of the second column, so there exists $b' \in B'$ such that $\beta_1(b') = f(a)$. Now
$$\gamma_1(g'(b)) = g(\beta_1(b')) = g(f(a)) = 0$$
by commutativity and exactness so there exists $a' \in A'$ such that $f'(a') = b'$. Now by commutativity
$$f(\alpha_1(a')) = \beta_1(f'(a')) = \beta_1(b') = f(a)$$
and so $f(a - \alpha_1(a')) = 0$ since f is a homomorphism. f is injective since the second column is exact so it follows that $\alpha_1(a') = a$ and so $a \in \operatorname{im} \alpha_1$. Conclude that $\ker \alpha_2 \subseteq \operatorname{im} \alpha_1$ is in fact true.

(4) α_2 is surjective: Choose $a'' \in A''$. Consider $f''(a'') \in B''$. Since the second row is exact β_2 is surjective so there exists $b \in B$ such that $\beta_2(b) = f''(a'')$. By commutativity $g''(\beta_1(b)) = \gamma_2(g(b))$ but by exactness of the third column $g''(\beta_2(b)) = g''(f''(a'')) = 0$. So $\gamma_2(g(b)) = 0$ which implies that $g(b) \in \ker\gamma_2 = \operatorname{im}\gamma_1$ by exactness of the third row. So there exists $c' \in C'$ such that $\gamma_1(c') = g(b)$. Since the first column is exact g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. So $\gamma_1(c') = \gamma_1(g'(b')) = g(\beta_1(b'))$. Observe that
$$g(b - \beta_1(b')) = g(b) - g(\beta_1(b')) = 0$$
since $g(b) = \gamma_1(c)$ and $g(\beta_1(b')) = \gamma_1(c)$ also. So $b - \beta_1(b') \in \ker g = \operatorname{im} f$ by exactness of the second column. So there exists $a \in A$ such that $f(a) = b - \beta_2(b')$. Now since $f''(\alpha(a)) = \beta_2(f(a)) = \beta_2(b - \beta_1(b'))$ we get
$$f''(\alpha_2(a)) = \beta_2(b) - \beta_2(\beta_1(b')) = \beta_2(b) = f''(a'').$$
In particular we get $f''(\alpha_2(a) - a'') = 0$ but f'' is injective by exactness of the third column so $\alpha_2(a) = a''$. Conclude that α_2 is in fact surjective.

We have shown that if the bottom two rows are exact, then the top row is exact.

Suppose that the top two rows are exact. We must show four things:

(1) γ_1 is injective: Suppose $c' \in C'$ with $\gamma_1(c') = 0$. Since the first column is exact, g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. Now by commutativity
$$0 = \gamma_1(c') = \gamma_1(g'(b')) = g(\beta_1(b')).$$
So $\beta_1(b') \in \ker g = \operatorname{im} f$ since the second row is exact. So there exists $a \in A$ such that $f(a) = \beta_1(b')$. But by commutativity
$$f''(\alpha_2(a)) = \beta_2(f(a)) = \beta_2(\beta_1(b')) = 0$$
so $\alpha_2(a) = 0$. Since f'' is injective, because the third column is exact. Now $a \in \ker\alpha_2 = \operatorname{im}\alpha_1$ by exactness of the first row so there exists $a' \in A'$ such that $\alpha_2(a') = a$. Now
$$f(a) = f(\alpha_1(a')) = \beta_1(f'(a')) = \beta_1(b')$$
So $\beta_1(f'(a') - b') = 0$. Since the second row is exact, β_1 is injective so $f'(a') - b = 0$. Hence $f'(a') = b'$. Taking g' of both sides we get
$$0 = g'(f'(a')) = g'(b') = c'.$$
Hence γ_1 is injective.

(2) $\operatorname{im}\gamma_1 \subseteq \ker\gamma_2$: Let $c \in \operatorname{im}\gamma_1$ then by definition there exists $c' \in C'$ such that $\gamma_1(c') = c$. Since the first column is exact g' is surjective so there exists $b' \in B'$ such that $g'(b') = c'$. Observe that
$$g(\beta_1(b')) = \gamma_1(g'(b')) = \gamma_1(c') = c$$
by commutativity. Again by commutativity we observe
$$\gamma_2(c) = \gamma_1(g(\beta_1(b'))) = g''(\beta_1(\beta_1(b'))) = g''(0) = 0$$
and so $c \in \ker\gamma_2$. Conclude that $\operatorname{im}\gamma_1 \subseteq \ker\gamma_2$.

(3) $\ker\gamma_2 \subseteq \operatorname{im}\gamma_1$: Choose $c \in \ker\gamma_2$ then by definition $\gamma_2(c) = 0$. Since the second column is exact, g is surjective. So there exists $b \in B$ such that $g(b) = c$. By commutativity
$$g''(\beta_2(b)) = \gamma_2(g(b)) = \gamma_2(c) = 0.$$
So $\beta_2(b) \in \ker g'' = \operatorname{im} f''$ by exactness of the third column so there exists $a'' \in A''$ such that $f''(a'') = \beta_2(b)$. Since the first row is exact α_2 is surjective so there exists $a \in A$ such that $\alpha_2(a) = a''$. Now
$$\beta_2(f(a)) = f''(\alpha_2(a)) = f''(a'') = \beta_2(b)$$
so $\beta_2(f(a) - b) = 0$ which implies that $f(a) - b \in \ker\beta_2 = \operatorname{im}\beta_1$ by exactness of the second row. So there exists $b' \in B'$ such that $\beta_1(b') = f(a) - b'$. But now
$$\gamma_1(g'(b')) = g(\beta_1(b')) = g(f(a) - b) = g(f(a)) = g(b) = -c$$
yielding $\gamma_1(-g'(b')) = c$. So $c \in \operatorname{im}\gamma_1$ and we can conclude that $\ker\gamma_2 \subseteq \operatorname{im}\gamma_1$.

(4) γ_2 is surjective: Choose $c'' \in C''$. Since the third column is exact g'' is surjective so there exists $b'' \in B''$ such that $g''(b'') = c''$. But since the second row is exact β_2 is surjective so there exists $b \in B$ such that $\beta_2(b) = b''$. Now by commutativity
$$c'' = g''(b'') = g''(\beta_2(b)) = \gamma_2(g(b)).$$
Conclude that γ_2 is in fact surjective.

We have shown that if the top two rows are exact, then the bottom row is exact. □

AM Proposition 2.10: (*Snake Lemma*) Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{g} & M & \xrightarrow{h} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow{g'} & N & \xrightarrow{h'} & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \to 0$$

in which $\overline{u}$, $\overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

Proof. We consider:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f' \arrow[r, "\overline{u}"] \arrow[d, "g'"] & \ker f \arrow[r, "\overline{v}"] \arrow[d, "g"] & \ker f'' \arrow[d, "g''"] \arrow[dddll] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d, "h'"] & N \arrow[r, "v'"] \arrow[d, "h'"] & N'' \arrow[r] \arrow[d, "h'"] & 0 \\
 & \operatorname{coker} f' \arrow[r, "\overline{u}'"] \arrow[d] & \operatorname{coker} f \arrow[r, "\overline{v}'"] \arrow[d] & \operatorname{coker} f'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

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