

# INTRODUCTION TO CATEGORY THEORY

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ABSTRACT. Category Theory: Remain calm and carry on when all the mathematics you've ever known and loved gets abstracted away into dots and arrows.

## 0. MATH 697 HOMEWORK ZERO

**Exercise 0.1.** (DF §10.2 Theorem 4): Prove Theorem 4 (Isomorphism Theorems):

- (1) (*The First Isomorphism Theorem for Modules*) Let  $M, N$  be  $R$ -modules and let  $\varphi : M \rightarrow N$  be an  $R$ -modules homomorphism. Then  $\ker \varphi$  is a submodule of  $M$  and  $M/\ker \varphi \cong \varphi(M)$ .

*Proof.* Let  $M, N$  be  $R$ -modules and let  $\varphi : M \rightarrow N$  be an  $R$ -modules homomorphism. Then by definition  $\varphi(x + y) = \varphi(x) + \varphi(y)$  and  $\varphi(rx) = r\varphi(x)$  for all  $x, y \in M, r \in R$ . We want to show that  $\ker \varphi = \{m \in M : \varphi(m) = 0\}$  is a submodule. Observe that since  $M$  is a module then  $M$  is an abelian group by definition so there exists  $0 \in M$  such that  $m + 0 = m$  for all  $m \in M$ . In particular  $\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0)$  implying  $\varphi(0) = 0$ . Conclude that  $0 \in \ker \varphi \neq \emptyset$ . Now let  $r \in R, x, y \in \ker \varphi$ . Observe that  $\varphi(x + ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r \cdot 0 = 0 + 0 = 0$ . Hence  $x + ry \in \ker \varphi$ . Conclude by the submodule criterion that  $\ker \varphi$  is in fact a submodule.

Now define  $\Phi : M/\ker \varphi \rightarrow \varphi(M)$  by  $\Phi(m + \ker \varphi) = \varphi(m)$ . We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose  $m + \ker \varphi = m' + \ker \varphi$  it follows by property of cosets that  $m - m' \in \ker \varphi$ , in particular  $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$  and hence  $\varphi(m) = \varphi(m')$ . But since  $\varphi(m) = \Phi(m + \ker \varphi)$  and  $\varphi(m') = \Phi(m' + \ker \varphi)$  we have  $\Phi(m + \ker \varphi) = \Phi(m' + \ker \varphi)$ . Conclude that  $\Phi$  is in fact well-defined.

Suppose that  $\Phi(m + \ker \varphi) = \Phi(m' + \ker \varphi)$ . Then it follows that  $\varphi(m) = \varphi(m')$  and so  $\varphi(m - m') = 0$  and so  $m - m' \in \ker \varphi$ . By property of cosets it follows that  $m + \ker \varphi = m' + \ker \varphi$  and hence  $\Phi$  is injective.

Let  $n \in \varphi(M)$ . Then by definition of image of  $\varphi$  there exists  $m \in M$  such that  $n = \varphi(m)$ . It is immediate that  $m + \ker \varphi \in M/\ker \varphi$  and we can conclude that  $\Phi$  is surjective.

Now we must show that  $\Phi$  is an  $R$ -module homomorphism. Let  $x, y \in M/\ker \varphi$  where  $x = m + \ker \varphi$  and  $y = m' + \ker \varphi$  for some  $m, m' \in M$  and let  $r \in R$ . Observe that

$$\begin{aligned} \Phi(x + y) &= \Phi(m + m' + \ker \varphi) \\ &= \varphi(m + m') \\ &= \varphi(m) + \varphi(m') \\ &= \Phi(m + \ker \varphi) + \Phi(m' + \ker \varphi) \\ &= \Phi(x) + \Phi(y) \end{aligned}$$

and

$$\begin{aligned} \Phi(rx) &= \Phi(r(m + \ker \varphi)) \\ &= \Phi(rm + \ker \varphi) \\ &= \varphi(rm) \\ &= r\varphi(m) \\ &= r\Phi(m + \ker \varphi) \\ &= r\Phi(x) \end{aligned}$$

Hence we have shown that  $\Phi$  is a well-defined bijective homomorphism and thus we can conclude by definition of  $R$ -module isomorphism that  $M/\ker \varphi \cong \varphi(M)$ .  $\square$

- (2) (*The Second Isomorphism Theorem*) Let  $A, B$  be submodules of the  $R$ -module  $M$ . Then  $(A + B)/B \cong A/(A \cap B)$ .

*Proof.* Define  $\varphi : A \rightarrow (A+B)/B$  by  $\varphi(a) = a + B$ . This mapping is clearly well-defined. We want to show that  $\varphi$  is a homomorphism. Let  $r \in R$ ,  $a, a' \in A$  and observe that

$$\begin{aligned}\varphi(a + a') &= a + a' + B \\ &= a + B + a' + B \\ &= \varphi(a) + \varphi(a')\end{aligned}$$

and

$$\begin{aligned}\varphi(ra) &= ra + B \\ &= r(a + B) \\ &= r\varphi(a)\end{aligned}$$

and so  $\varphi$  is an  $R$ -module homomorphism by definition. Observe that  $\ker \varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$ . Now let  $x \in (A+B)/B$  then  $x = a + b + B$  for some  $a \in A$ ,  $b \in B$ . But observe that  $a + b + B = a + B$  by absorption. So  $\varphi$  is immediately surjective. In particular we have  $\varphi(A) = (A+B)/B$ . Conclude by the First Isomorphism Theorem for Modules that  $A/\ker \varphi = A/(A \cap B) \cong (A+B)/B = \varphi(A)$ .  $\square$

- (3) (*The Third Isomorphism Theorem*) Let  $M$  be an  $R$ -module, and let  $A$  and  $B$  be submodules of  $M$  with  $A \subseteq B$ . Then  $(M/A)/(B/A) \cong M/B$ .

*Proof.* Define  $\varphi M/A \rightarrow M/B$  by  $\varphi(m + A) = m + B$ . We need to show  $\varphi$  is well-defined. Suppose  $m + A = m' + A$  then  $m - m' \in A \subseteq B$  by property of cosets. It also follows that  $m + B = m' + B$ . Hence  $\varphi(m + A) = m + B = m' + B = \varphi(m' + A)$  and hence  $\varphi$  is well-defined.

Now we must show  $\varphi$  is an  $R$ -module homomorphism. Let  $m, m' \in M$  and  $r \in R$ . Observe that

$$\begin{aligned}\varphi((m + A) + (m' + A)) &= \varphi(m + m' + A) \\ &= m + m' + B \\ &= (m + B) + (m' + B) \\ &= \varphi(m + A) + \varphi(m' + A)\end{aligned}$$

and

$$\begin{aligned}\varphi(r(m + A)) &= \varphi(rm + A) \\ &= rm + B \\ &= r(m + B) \\ &= r\varphi(m + A)\end{aligned}$$

and hence we can conclude by definition that  $\varphi$  is an  $R$ -module homomorphism.

Observe that  $\ker \varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A : \varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$ . Let  $m + B \in M/B$ . Clearly  $\varphi(m + A) = m + B$  and hence  $\varphi$  is surjective. Now by the First Isomorphism Theorem for Modules we have  $(M/A)/\ker \varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$ .  $\square$

- (4) (*The Fourth or Lattice Isomorphism Theorem*) Let  $N$  be a submodule of the  $R$ -module  $M$ . There is a bijection between the submodules of  $M$  which contain  $N$  and the submodules of  $M/N$ . The correspondence is given by  $A \leftrightarrow A/N$ , for all  $A \supseteq N$ . The correspondence commutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of  $M/N$  and the lattice of submodules of  $M$  which contain  $N$ ).

*Proof.* Let  $N$  be a submodule of  $M$ . Define  $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$ ,  $T = \{L : L \text{ is a submodule of } M/N\}$ . Define  $\varphi : S \rightarrow T$  by  $\varphi(K) = K/N$ . We want to show that this mapping is bijective.

Let  $K_1, K_2 \in S$  and suppose that  $\varphi(K_1) = \varphi(K_2)$ . Then  $K_1/N = K_2/N$ . We want to show that  $K_1 = K_2$ . Let  $x \in K_1$ , then  $x + N \in K_1/N = K_2/N$ , in particular there exists  $y \in K_2$  such that  $x + N = y + N$ . By property of cosets it follows that  $x - y \in N$ . But since  $N \subseteq K_2$  by construction  $x - y \in K_2$ . Since  $K_2$  is a submodule of  $M$ , it is closed under addition and so  $(x - y) + y = x \in K_2$ . Conclude that  $K_1 \subseteq K_2$ . By symmetric argument  $K_2 \subseteq K_1$  and hence  $K_1 = K_2$ . Thus by definition  $\varphi$  is injective.

Let  $L$  be a submodule of  $M/N$ . Consider the natural projection map  $\pi : M \rightarrow M/N$  defined by  $\pi(m) = m + N$ . We want to show that there exists  $K \in S$  such that  $\varphi(K) = L$ . To do this we will show that  $\pi^{-1}(L)$  is a submodule of  $M$  and that  $N \subseteq \pi^{-1}(L)$ . Recall that  $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$ . Observe that  $0 \in \pi^{-1}(L)$  since  $\pi(0) = 0$  and hence  $\pi^{-1}(L) \neq \emptyset$ . Let  $x, y \in \pi^{-1}(L)$  and  $r \in R$ . Observe that  $\pi(x + ry) = \pi(x) + r\pi(y)$ . Since  $\pi(y) \in L$  by definition and  $L$  is a submodule of  $M/N$ , it follows that since scalar multiplication is closed  $r\pi(y) \in L$ . Thus it follows that  $\pi(x) + r\pi(y) \in L$  and hence  $x + ry \in \pi^{-1}(L)$ . Thus by the submodule criterion we can conclude that  $\pi^{-1}(L)$  is in fact a submodule. Now let  $n \in N$  and observe that  $\pi(n) = n + N = 0 + N \in L$  so by definition it follows that  $n \in \pi^{-1}(L)$ . Conclude that  $N \subseteq \pi^{-1}(L)$  and hence  $\varphi$  is surjective.

Conclude  $\varphi$  is bijective and result follows.  $\square$

**Exercise 0.2.** (DF §10.2 Exercise 1): Use the submodule criterion to show that the kernels and images of  $R$ -module homomorphisms are submodules.

*Proof.* Let  $M, N$  be  $R$ -modules and  $\varphi : M \rightarrow N$  an  $R$ -module homomorphism. Recall that  $\ker \varphi = \{m \in M : \varphi(m) = 0\}$  and  $\text{im } \varphi = \{n \in N : \text{there exists } m \in M \text{ with } \varphi(m) = n\}$ .

Observe that  $\varphi(0) = 0$  so  $0 \in \ker \varphi \neq \emptyset$ . Let  $m, m' \in M, r \in R$ . Now  $\varphi(m + rm') = \varphi(m) + r\varphi(m') = 0 + r \cdot 0 = 0 + 0 = 0$ . So  $m + rm' \in \ker \varphi$ . Thus by the submodule criterion  $\ker \varphi$  is a submodule.

Observe that  $\varphi(0) = 0 \in N$  so  $0 \in \text{im } \varphi \neq \emptyset$ . Let  $n, n' \in N, r \in R$ . Then there exists  $m, m' \in M$  such that  $\varphi(m) = n$  and  $\varphi(m') = n'$ . Now consider  $n + rn'$ .  $\varphi(m + rm') = \varphi(m) + r\varphi(m') = n + rn'$  so  $n + rn' \in \text{im } \varphi$ . Conclude by submodule criterion that  $\text{im } \varphi$  is in fact a submodule.  $\square$

**Exercise 0.3.** (DF §10.2 Exercise 2): Show that the relation “is  $R$ -module isomorphic to” is an equivalence relation on any set of  $R$ -modules.

*Proof.* Let  $X$  be a set of  $R$ -modules.

- Let  $M \in X$ . Observe that  $M$  is isomorphic to  $M$  trivially. So relation is reflexive.
- Let  $M, N \in X$ . Suppose  $M$  is isomorphic to  $N$  then by definition there exists  $\varphi : M \rightarrow N$  that is bijective. Immediately we have  $\varphi^{-1} : N \rightarrow M$  which is also bijective so  $N$  is isomorphic to  $M$ . By definition the relation is symmetric.
- Let  $L, M, N \in X$ . Suppose  $L$  is isomorphic to  $M$ , then by definition there exists  $\varphi : L \rightarrow M$  a bijective  $R$ -module homomorphism. Suppose  $M$  is isomorphic to  $N$ , then there exists  $\Phi : M \rightarrow N$  a bijective  $R$ -module homomorphism. Observe that  $\varphi \circ \Phi : L \rightarrow N$  is again a bijective  $R$ -module homomorphism by property of composition of mappings. Hence by definition  $L$  is isomorphic to  $N$ .

Conclude that the relation “is  $R$ -module isomorphic to” is an equivalence relation on any set of  $R$ -modules.  $\square$

**Exercise 0.4.** (DF §10.2 Exercise 3): Give an explicit example of a map from one  $R$ -module to another which is a group homomorphism but not an  $R$ -module homomorphism.

*Solution.* Consider the Quaternions  $\mathbb{H} = R$ ; they form a commutative group under addition and a noncommutative group under multiplication. Hence  $\mathbb{H}$  is a noncommutative ring with unity. In particular  $\mathbb{H}$  is an  $R$ -module over itself. Define  $\varphi : \mathbb{H} \rightarrow \mathbb{H}$  by  $\varphi(h) = ih$ . This is a group homomorphism since  $\varphi(h + h') = i(h + h') = ih + ih' = \varphi(h) + \varphi(h')$ . But note that  $\varphi(j \cdot 1) = \varphi(j) = ij = k \neq -k = ji = j(i \cdot 1) = j\varphi(1)$ . Conclude that  $\varphi$  is not an  $R$ -module homomorphism since the definition is not satisfied.  $\blacktriangleleft$

**Exercise 0.5.** (DF §10.2 Exercise 4): Let  $A$  be a  $\mathbb{Z}$ -module, let  $a$  be any element of  $A$  and let  $n$  be a positive integer. Prove that the map  $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  given by  $\varphi(\bar{k}) = ka$  is a well-defined  $\mathbb{Z}$ -module homomorphism if and only if  $na = 0$ . Prove that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$ , where  $A_n = \{a \in A : na = 0\}$  (So  $A_n$  is the annihilator in  $A$  of the ideal  $(n)$  of  $\mathbb{Z}$ ).

*Proof.* Suppose that the map  $\varphi_a : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  given by  $\varphi(\bar{k}) = ka$  is a well-defined  $\mathbb{Z}$ -module homomorphism. Then by definition if  $\bar{m} = \bar{k}$  then  $\varphi(\bar{m}) = \varphi(\bar{k})$  or equivalently  $ma = ka$ . Moreover  $\varphi(a + b) = \varphi(a) + \varphi(b)$  and  $\varphi(ra) = r\varphi(a)$  for all  $a, b \in A$  and  $r \in \mathbb{Z}$ . Observe that  $\bar{0} = \bar{k}$  so by hypothesis  $\varphi(\bar{0}) = \varphi(\bar{k})$  but observe that  $\varphi(\bar{0}) = 0 \cdot a = 0$  and  $\varphi(\bar{k}) = ka$ . Hence by equality  $ka = 0$ . Conversely suppose that  $na = 0$ . We want to show that  $\varphi : \mathbb{Z}/n\mathbb{Z} \rightarrow A$  defined by  $\varphi(\bar{k}) = ka$  is a well-defined  $R$ -module homomorphism. Say  $\bar{k} = \bar{m}$  then by property of cosets  $k - m \in n\mathbb{Z}$  and so  $n \mid k - m$ . By definition  $n \mid k - m$  and hence there exists  $t \in \mathbb{Z}$  such that  $k - m = nt$ . Observe that

$$\begin{aligned} k - m &= nt \\ (k - m)a &= nta \\ ka - ma &= (na)t \\ ka - ma &= 0 \\ ka &= ma \end{aligned}$$

Thus we have  $\varphi(\bar{k}) = \varphi(\bar{m})$  and we can conclude that  $\varphi$  is in fact a well-defined  $R$ -module homomorphism.

Now we want to show that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n = \{a \in A : na = 0\}$ . *Note:* We are making the assumption that we want to show this in an isomorphism of  $R$ -modules as exercise does not specify group, ring or module isomorphism. Define  $\Phi : A_n \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$  by  $\Phi(a) = \varphi_a$ . We will show that this is in fact an  $R$ -module homomorphism, then that is a bijection.

Let  $a, a' \in A_n, r \in \mathbb{Z}$ . Observe that

$$\begin{aligned} \Phi(a + a')(\bar{k}) &= \varphi_{a+a'}(\bar{k}) \\ &= (a + a')k \\ &= ak + a'k \\ &= \varphi_a(\bar{k}) + \varphi_{a'}(\bar{k}) \end{aligned}$$

$$= \Phi(a)(\bar{k}) + \Phi(a')(\bar{k})$$

So  $\Phi(a + a') = \Phi(a) + \Phi(a')$  by definition. Moreover

$$\begin{aligned} \Phi(ra)(\bar{k}) &= \varphi_{ra}(\bar{k}) \\ &= rak \\ &= r\varphi_a(\bar{k}) \\ &= r\Phi(a)(\bar{k}) \end{aligned}$$

Hence  $\Phi(ra) = r\Phi(a)$  by definition. Conclude by definition that  $\Phi$  is in fact an  $R$ -module homomorphism.

Recall that  $\ker \Phi = \{a \in A_n : \Phi(a) = 0\}$  and observe that

$$\begin{aligned} \ker \varphi &= \{a \in A_n : \Phi(a) = 0\} \\ &= \{a \in A_n : \varphi_a(\bar{k}) = 0 \text{ for all } k \in \mathbb{Z}/n\mathbb{Z}\} \\ &= \{0\} \end{aligned}$$

So we conclude that  $\ker \Phi = \{0\}$  and hence  $\Phi$  is injective.

Let  $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A_n)$ . Define  $a = \varphi(\bar{1})$  and hence  $na = n\varphi(\bar{1}) = \varphi(n\bar{1}) = \varphi(\bar{0}) = 0$  and hence  $a \in A_n$ . Observe that for all  $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$  it follows that  $\varphi(\bar{k}) = \varphi(\bar{k} \cdot \bar{1}) = \varphi(k\bar{1}) = k\varphi(\bar{1}) = ka = \varphi_a(\bar{k})$ . Thus by definition  $\varphi = \varphi_a$ . So for any  $\varphi \in \text{Hom}(\mathbb{Z}/n\mathbb{Z}, A_n)$  we can find  $a \in A_n$  such that  $\Phi(a) = \varphi_a = \varphi$ . Conclude by definition that  $\Phi$  is surjective.

Now applying the First Isomorphism Theorem for  $R$ -modules we can conclude that  $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, A_n) \cong A_n$ .  $\square$

**Exercise 0.6.** (DF §10.2 Exercise 5): Exhibit all  $\mathbb{Z}$ -module homomorphisms from  $\mathbb{Z}/30\mathbb{Z}$  to  $\mathbb{Z}/21\mathbb{Z}$ . By previous exercise we know  $\text{Hom}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/21\mathbb{Z}) \cong (\mathbb{Z}/21\mathbb{Z})_{30}$  where  $(\mathbb{Z}/21\mathbb{Z})_{30} = \{a \in \mathbb{Z}/21\mathbb{Z} : 30a = 0\} = A_{30}$ . Observe that  $(\mathbb{Z}/21\mathbb{Z})_{30} = 7\mathbb{Z}/30\mathbb{Z} = \{0, 7, 14\}$  and it has three elements. Hence we know that there are three homomorphisms from  $\mathbb{Z}/30\mathbb{Z}$  to  $\mathbb{Z}/21\mathbb{Z}$ . The three homomorphisms are the ones defined by the trivial homomorphism,  $\varphi_7(\bar{x}) = 7x$ ,  $\varphi_{14}(\bar{x}) = 14x$ .

**Exercise 0.7.** (DF §10.2 Exercise 6): Prove that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$ .

**Exercise 0.8.** (DF §10.2 Exercise 7): Let  $z$  be a fixed element of the center of  $R$ . Prove that the map  $m \rightarrow zm$  is an  $R$ -module homomorphism from  $M$  to itself. Show that for a commutative ring  $R$  the map from  $R$  to  $\text{End}_R M$  given by  $r \rightarrow rI$  is a ring homomorphism (where  $I$  is the identity endomorphism.)

*Proof.* Recall that the center of  $R$  is  $\{z \in R : zr = rz \text{ for all } r \in R\}$ . Let  $z$  be in the center of  $R$  then by definition  $zr = rz$  for all  $r \in R$ . Define  $\varphi : M \rightarrow M$  by  $\varphi(m) = zm$ . We claim that this is an  $R$ -module homomorphism. Let  $m, m' \in M$ ,  $r \in R$ . Observe that  $\varphi(m + m') = z(m + m') = zm + zm' = \varphi(m) + \varphi(m')$  and  $\varphi(rm) = zrm = rz m = r\varphi(m)$ . Conclude by definition that  $\varphi$  is in fact an  $R$ -module homomorphism.

*Note:* We are making the assumption that “ $I$  being the identity endomorphism” means *multiplicative identity*. Now let  $R$  be a commutative ring and define  $\Phi : R \rightarrow \text{End}_R(M)$  by  $\Phi(r) = rI$  where  $I : M \rightarrow M$ , defined by  $I(m) = m$ , is the identity endomorphism. We want to show that  $\Phi$  is a ring homomorphism. Let  $r, s \in R$  and observe that for all  $m \in M$

$$\begin{aligned} \Phi(r + s)(m) &= (r + s)I(m) \\ &= (r + s)m \\ &= rm + sm \\ &= rI(m) + sI(m) \\ &= \Phi(r)(m) + \Phi(s)(m) \end{aligned}$$

So by definition  $\Phi(r + s) = \Phi(r) + \Phi(s)$ . Moreover,

$$\begin{aligned} \Phi(rs) &= rsI(m) \\ &= rsI(m)I(m) \\ &= rI(m) \cdot sI(m) \\ &= \Phi(r)(m)\Phi(s)(m) \end{aligned}$$

And hence  $\Phi(rs) = \Phi(r)\Phi(s)$  and we can conclude by definition that  $\Phi$  is a ring homomorphism.  $\square$

**Exercise 0.9.** (DF §10.2 Exercise 8): Let  $\varphi : M \rightarrow N$  be an  $R$ -module homomorphism. Prove that  $\varphi(\text{Tor}(M)) \subseteq \text{Tor}(N)$ .

*Proof.* Recall that  $\text{Tor}(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$ . Now it follows that  $\varphi(\text{Tor}(M)) = \{n \in N : n = \varphi(m) \text{ for some } m \in \text{Tor}(M)\}$ . Let  $n \in \varphi(\text{Tor}(M))$  then  $n = \varphi(m)$  for some  $m \in \text{Tor}(M)$  by definition. Since  $m \in \text{Tor}(M)$  there exists  $0 \neq r \in R$  such that  $rm = 0$ . Hence  $rn = r\varphi(m) = \varphi(rm) = \varphi(0) = 0$ . Conclude that  $n \in \text{Tor}(N)$  and hence  $\varphi(\text{Tor}(M)) \subseteq \text{Tor}(N)$ .  $\square$

**Exercise 0.10.** (DF §10.2 Exercise 9): Let  $R$  be a commutative ring. Prove that  $\text{Hom}_R(R, M)$  and  $M$  are isomorphic as left  $R$ -modules.

*Proof.* Define  $\Phi : \text{Hom}_R(R, M) \rightarrow M$  by  $\Phi(\varphi) = \varphi(1)$ . We must first show that this is in fact an  $R$ -module homomorphism. Observe that for all  $\varphi, \xi \in \text{Hom}_R(R, M)$  and all  $r \in R$  it follows that

$$\begin{aligned}\Phi(\varphi + \xi) &= (\varphi + \xi)(1) \\ &= \varphi(1) + \xi(1) \\ &= \Phi(\varphi) + \Phi(\xi)\end{aligned}$$

and also by Proposition 2 we have

$$\begin{aligned}\Phi(r\varphi) &= (r\varphi)(1) \\ &= r\varphi(1) \\ &= r\Phi(\varphi)\end{aligned}$$

Hence we can now conclude that  $\Phi$  is an  $R$ -module homomorphism.

We must now show that  $\Phi$  is injective. Suppose that  $\Phi(\varphi) = \Phi(\xi)$  then by definition  $\varphi(1) = \xi(1)$  or equivalently  $\varphi(1) - \xi(1) = 0$  and hence  $(\varphi - \xi)(1) = 0$ . But since  $\varphi - \xi \in \text{Hom}_R(R, M)$  it is an  $R$ -module homomorphism so  $(\varphi - \xi)(x) = (\varphi - \xi)(x \cdot 1) = x(\varphi - \xi)(1) = x \cdot 0 = 0$  for all  $x \in R$ . Conclude that  $\varphi(x) = \xi(x)$  for all  $x \in R$  and hence by definition  $\varphi = \xi$ . Conclude that  $\Phi$  is injective.

We must now show that  $\Phi$  is surjective. Let  $m \in M$  be arbitrary. We want to show that there exists  $\varphi \in \text{Hom}_R(R, M)$  such that  $\Phi(\varphi) = m$ . Let us define  $\varphi : R \rightarrow M$  by  $\varphi(x) = xm$ . We need to show that  $\varphi \in \text{Hom}_R(R, M)$ . Observe that  $\varphi(x + y) = (x + y)m = xm + ym = \varphi(x) + \varphi(y)$  for all  $x, y \in R$  and  $\varphi(rx) = rxm = r\varphi(x)$  for all  $x \in R, r \in R$ . Hence we have shown that  $\varphi$  is in fact an  $R$ -module homomorphism. Now observe that  $\Phi(\varphi) = \varphi(1) = 1 \cdot m = m$ . Conclude by definition that  $\Phi$  is surjective.

We have shown that  $\Phi$  is a bijective  $R$ -module homomorphism. Conclude that  $\text{Hom}_R(R, M) \cong M$ .  $\square$

**Exercise 0.11.** (DF §10.2 Exercise 10): Let  $R$  be a commutative ring. Prove that  $\text{Hom}_R(R, R)$  and  $R$  are isomorphic as rings.

*Proof.* Define  $\Phi : \text{Hom}_R(R, R) \rightarrow R$  by  $\Phi(\varphi) = \varphi(1)$ . We will first show that  $\Phi$  is a ring homomorphism. Observe that for all  $\varphi, \xi \in \text{Hom}_R(R, R)$  and all  $r \in R$  by Proposition 2 we have

$$\begin{aligned}\Phi(\varphi + \xi) &= (\varphi + \xi)(1) \\ &= \varphi(1) + \xi(1) \\ &= \Phi(\varphi) + \Phi(\xi)\end{aligned}$$

and also by property of commutativity

$$\begin{aligned}\Phi(\varphi \circ \xi) &= (\varphi \circ \xi)(1) \\ &= \varphi(\xi(1)) \\ &= \varphi(\xi(1) \cdot 1) \\ &= \xi(1)\varphi(1) \\ &= \varphi(1)\xi(1) \\ &= \Phi(\varphi)\Phi(\xi)\end{aligned}$$

Hence we can conclude that  $\Phi$  is in fact a ring homomorphism.

We must now show that  $\Phi$  is injective. Suppose that  $\Phi(\varphi) = \Phi(\xi)$  then by definition  $\varphi(1) = \xi(1)$  or equivalently  $\varphi(1) - \xi(1) = 0$  and hence  $(\varphi - \xi)(1) = 0$ . But since  $\varphi - \xi \in \text{Hom}_R(R, R)$  it is an  $R$ -module homomorphism so  $(\varphi - \xi)(x) = (\varphi - \xi)(x \cdot 1) = x(\varphi - \xi)(1) = x \cdot 0 = 0$  for all  $x \in R$ . Conclude that  $\varphi(x) = \xi(x)$  for all  $x \in R$  and hence by definition  $\varphi = \xi$ . Conclude that  $\Phi$  is injective.

We must now show that  $\Phi$  is surjective. Let  $r \in R$  be arbitrary. We want to show that there exists  $\varphi \in \text{Hom}_R(R, R)$  such that  $\Phi(\varphi) = r$ . Let us define  $\varphi : R \rightarrow R$  by  $\varphi(x) = xr$ . We need to show that  $\varphi \in \text{Hom}_R(R, R)$ . Observe that  $\varphi(x + y) = (x + y)r = xr + yr = \varphi(x) + \varphi(y)$  for all  $x, y \in R$  and  $\varphi(sx) = sxr = s\varphi(x)$  for all  $s \in R$  and all  $x \in R$ . Hence we have shown that  $\varphi$  is in fact an  $R$ -module homomorphism. Now observe that  $\Phi(\varphi) = \varphi(1) = 1 \cdot r = r$ . Conclude by definition that  $\Phi$  is surjective.

We have shown that  $\Phi$  is a bijective ring homomorphism. Conclude that  $\text{Hom}_R(R, R) \cong R$ .  $\square$

**Exercise 0.12.** (DF §10.2 Exercise 11): Let  $A_1, A_2, \dots, A_n$  be  $R$ -modules and let  $B_i$  be submodules of  $A_i$  for each  $i = 1, 2, \dots, n$ . Prove that

$$(A_1 \times A_2 \times \cdots \times A_n) / (B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

*Proof.* Define  $\varphi : A_1 \times A_2 \times \cdots \times A_n \rightarrow (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$  by  $\varphi(a_1, a_2, \dots, a_n) = (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n)$ . We want to show that this is an  $R$ -module homomorphism and that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$  and that the mapping is surjective. Then the first isomorphism theorem for modules yields the result.

Let  $x, y \in A_1 \times A_2 \times \cdots \times A_n$  where  $x = (a_1, a_2, \dots, a_n)$  and  $y = (a'_1, a'_2, \dots, a'_n)$ . Observe that

$$\begin{aligned} \varphi(x+y) &= \varphi((a_1, a_2, \dots, a_n) + (a'_1, a'_2, \dots, a'_n)) \\ &= \varphi(a_1 + a'_1, a_2 + a'_2, \dots, a_n + a'_n) \\ &= (a_1 + a'_1 + B_1, a_2 + a'_2 + B_2, \dots, a_n + a'_n + B_n) \\ &= (a_1 + B_1 + a'_1 + B_1, a_2 + B_2 + a'_2 + B_2, \dots, a_n + B_n + a'_n + B_n) \\ &= (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) + (a'_1 + B_1, a'_2 + B_2, \dots, a'_n + B_n) \\ &= \varphi(a_1, a_2, \dots, a_n) + \varphi(a'_1, a'_2, \dots, a'_n) \\ &= \varphi(x) + \varphi(y) \end{aligned}$$

and for  $r \in R$

$$\begin{aligned} \varphi(rx) &= \varphi(r(a_1, a_2, \dots, a_n)) \\ &= \varphi(ra_1, ra_2, \dots, ra_n) \\ &= (ra_1 + B_1, ra_2 + B_2, \dots, ra_n + B_n) \\ &= (r(a_1 + B_1), r(a_2 + B_2), \dots, r(a_n + B_n)) \\ &= r(a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) \\ &= r\varphi(a_1, a_2, \dots, a_n) \\ &= r\varphi(x) \end{aligned}$$

Thus by definition we can conclude that  $\varphi$  is an  $R$ -module homomorphism.

Now we want to show that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$ . Observe that

$$\begin{aligned} \ker \varphi &= \{x \in A_1 \times A_2 \times \cdots \times A_n : \varphi(x) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : \varphi(a_1, a_2, \dots, a_n) = 0\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n) = (0, 0, \dots, 0)\} \\ &= \{(a_1, a_2, \dots, a_n) \in A_1 \times A_2 \times \cdots \times A_n : a_1 \in B_1, a_2 \in B_2, \dots, a_n \in B_n\} \end{aligned}$$

Hence  $\ker \varphi \subseteq B_1 \times B_2 \times \cdots \times B_n$  and trivially  $B_1 \times B_2 \times \cdots \times B_n \subseteq \ker \varphi$  by construction of  $\varphi$ . Conclude that  $\ker \varphi = B_1 \times B_2 \times \cdots \times B_n$ .

The mapping is trivially surjective. Applying first Isomorphism theorem yields the result.  $\square$

**Exercise 0.13.** (DF §10.3 Exercise 3): Show that the  $F[x]$ -modules in Exercises 18 and 19 of Section 1 are both cyclic.

*Proof.* For Problem 18:  $F = \mathbb{R}$ ,  $V = \mathbb{R}^2$ ,  $T : V \rightarrow V$  is defined by  $T(x, y) = (y, -x)$ . Let  $(a, b) \in \mathbb{R}^2$  be arbitrary. Observe that  $(ax + b)(0, 1) = aT(0, 1) + b(0, 1) = a(1, 0) + b(0, 1) = (a, b)$ . Hence it follows by definition that  $V = \mathbb{R}(0, 1)$ . Moreover it can also be written as  $V = \mathbb{R}(1, 0)$  with  $p(x) = a - bx$ , so the representation is not unique.

For Problem 19:  $F = \mathbb{R}$ ,  $V = \mathbb{R}^2$ ,  $T : V \rightarrow V$  is defined by  $T(x, y) = (0, y)$ . Let  $(a, b) \in \mathbb{R}^2$  be arbitrary. Observe that  $(a + (b - a)x)(1, 1) = (a, a) + (b - a)T(1, 1) = (a, a) + (0, b - a) = (a, b)$ . Hence it follows by definition that  $V = \mathbb{R}(1, 1)$ .  $\square$

**Exercise 0.14.** (DF §10.3 Exercise 4): An  $R$ -module  $M$  is called a *torsion* module if for each  $m \in M$  there is a nonzero element  $r \in R$  such that  $rm = 0$ , where  $r$  may depend on  $m$  (i.e.,  $M = \text{Tor}(M)$  in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion  $\mathbb{Z}$ -module. Give an example of an infinite abelian group that is a torsion  $\mathbb{Z}$ -module.

*Proof.* Let  $M$  be a finite abelian group. Let  $m \in M$ . We want to show that there exists  $0 \neq r \in R = \mathbb{Z}$  such that  $rm = 0$ . Consider  $1m, 2m, 3m, 4m, \dots$ . These are not all distinct, because if they were we would have infinitely many, a contradiction. So we are assured  $km = lm$  for some  $k, l \in \mathbb{Z}$  nonzero with  $k \neq l$ . It follows that  $km - lm = 0$  and hence by property of modules,  $(k - l)m = 0$ . Finally observe that  $k - l \neq 0$  so  $m \in \text{Tor}(M)$ . Conclude that  $M = \text{Tor}(M)$ .

As for the example. Let  $n \in \mathbb{Z}$  be greater than 1. Consider  $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \cdots$  and observe that this is an infinite abelian group. This can be seen as a  $\mathbb{Z}$ -module. Let  $(a_1, a_2, \dots) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \cdots$  be arbitrary. Observe that  $n(a_1, a_2, \dots) = (na_1, na_2, \dots) = (0, 0, \dots) = \mathbf{0}$ .  $\square$

**Exercise 0.15.** (DF §10.3 Exercise 5): Let  $R$  be an integral domain. Prove that every finitely generated torsion  $R$ -module has a nonzero annihilator i.e., there is a nonzero element  $r \in R$  such that  $rm = 0$  for all  $m \in M$  – here  $r$  does not depend on  $m$  (the annihilator of a module was defined in Exercise 9 of Section 1). Give an example of a torsion  $R$ -module whose annihilator is the zero ideal.

*Proof.* Let  $M$  be a finitely generated torsion  $R$ -module where  $R$  is an integral domain. By definition  $M = Rm_1 + \cdots + Rm_n$  for some  $m_1, m_2, \dots, m_n \in M$ . Let  $m \in M$ . Since  $M$  is finitely generated there exist  $r_1, r_2, \dots, r_n \in R$  such that  $m = r_1m_1 + \cdots + r_nm_n$ . Since  $M$  is torsion, there exists  $0 \neq \bar{r}_i \in R$  such that  $\bar{r}_im_i = 0$  for  $i = 1, 2, \dots, n$ . Define  $r = \bar{r}_1\bar{r}_2 \cdots \bar{r}_n$ . Now observe that

$$\begin{aligned} rm &= r(r_1m_1 + \cdots + r_nm_n) \\ &= (r_1\bar{r}_2 \cdots \bar{r}_n)\bar{r}_1m_1 + \cdots + (\bar{r}_1 \cdots \bar{r}_{n-1}r_n)\bar{r}_nm_n \\ &= (r_1\bar{r}_2 \cdots \bar{r}_n) \cdot 0 + \cdots + (\bar{r}_1 \cdots \bar{r}_{n-1}r_n) \cdot 0 \\ &= 0 \end{aligned}$$

hence it follows that  $0 \neq r \in \text{Ann}(m)$  and thus  $\text{Ann}(m) \neq 0$ .

As for the example: recall that  $\mathbb{Q}$  is not finitely generated, and hence  $\mathbb{Q}/\mathbb{Z} = m$  is also not finitely generated. observe that  $m$  is torsion since for any rational, non-integer number  $x$ , multiplication by its denominator, which is an integer, yields an integer, which in this case would be 0 in  $M$ . Suppose by way of contradiction that there is a nonzero annihilator, say  $0 \neq a \in R = \mathbb{Z}$ . Choose  $b \in \mathbb{Z}$  such that  $b \nmid a$ . Now  $a \cdot 1/b = 0$  by property of being annihilator so  $a/b = k$  is an integer. But then  $a = bk$  and hence  $b \mid a$ , a contradiction. So there are no nonzero annihilators.  $\square$

**Exercise 0.16.** (DF §10.3 Exercise 9): An  $R$ -module  $M$  is called *irreducible* if  $M \neq 0$  and if 0 and  $M$  are the only submodules of  $M$ . Show that  $M$  is irreducible if and only if  $M \neq 0$  and  $M$  is a cyclic module with any nonzero element as its generator. Determine all the irreducible  $\mathbb{Z}$ -modules.

**Exercise 0.17.** (DF §10.3 Exercise 10): Assume  $R$  is commutative. Show that an  $R$ -module  $M$  is irreducible if and only if  $M$  is isomorphic (as an  $R$ -module) to  $R/I$  where  $I$  is a maximal ideal of  $R$ . [By the previous exercise, if  $M$  is irreducible there is a natural map  $R \rightarrow M$  defined by  $r \mapsto rm$ , where  $m$  is any fixed nonzero element of  $M$ .]

**Exercise 0.18.** (DF §10.3 Exercise 15): An element  $e \in R$  is called a *central idempotent* if  $e^2 = e$  and  $er = re$  for all  $r \in R$ . If  $e$  is a central idempotent in  $R$ , prove that  $M = eM \oplus (1 - e)M$ . [Recall Exercise 14 in Section 1.]

**Exercise 0.19.** (DF §10.3 Exercise 16): For any ideal  $I$  of  $R$  let  $IM$  be the submodule defined in Exercise 5 of Section 1. Let  $A_1, \dots, A_k$  be any ideals in the ring  $R$ . Prove that the map

$$M \rightarrow M/A_1M \times \cdots \times M/A_kM \text{ defined by } m \mapsto (m + A_1M, \dots, m + A_kM)$$

is an  $R$ -module homomorphism with kernel  $A_1M \cap A_2M \cap \cdots \cap A_kM$ .

**Exercise 0.20.** (DF §10.3 Exercise 22): Let  $R$  be a Principal Ideal Domain, let  $M$  be a torsion  $R$ -module (cf. Exercise 4) and let  $p$  be a prime in  $R$  (do not assume  $M$  is finitely generated, hence it need not have a nonzero annihilator – cf. Exercise 5). The  *$p$ -primary component* of  $M$  is the set of all elements of  $M$  that are annihilated by some positive power of  $p$ .

- (1) Prove that the  $p$ -primary component is a submodule. [See Exercise 13 in Section 1.]
- (2) Prove that this definition of  $p$ -primary component agrees with the one given in Exercise 18 when  $M$  has a nonzero annihilator.
- (3) Prove that  $M$  is the (possibly infinite) direct sum of its  $p$ -primary components, as  $p$  runs over all primes of  $R$ .