INTRODUCTION TO HOMOLOGICAL ALGEBRA

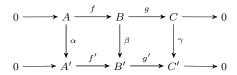
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1. MATH 697 Notes

R Proposition 2.18:

- (1) A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective.
- (2) A sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective.
- (3) A sequence $0 \to A \xrightarrow{h} B \to 0$ is exact if and only if h is an isomorphism.

DF §10.5 Proposition 24: (The Short Five Lemma) Let α, β, γ be homomorphisms of short exact sequences:



(1) If α and γ are injective then so is β

Proof. Let $b \in B$ such that $\beta(b) = 0$. We want to show that b = 0. Observe that $g'(\beta(b)) = g'(0) = 0$. By commutativity we have $\gamma(g(b)) = g'(\beta(b)) = g'(0) = 0$. Since γ is injective we know g(b) = 0 so $b \in \ker g$ but since we are in an exact sequence we have im $f = \ker g$ and hence $b \in \operatorname{im} f$. By definition there exists $a \in A$ with f(a) = b. Now $f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$. Since f' is injective, it follows that $\alpha(a) = (f')^{-1}(0) = 0$. Now we have $a = \alpha^{-1}(0)$ and so a = 0. So 0 = f(a) = b.

(2) If α and γ are surjective then so is β .

Proof. Let $b' \in B'$ then $g'(b') \in C'$. Since γ is surjective there excists $c \in C$ such that $\gamma(c) = g'(b')$. Since this is an exact sequence, g is surjective so there exists $b \in B$ such that g(b) = c. By equality we have $\gamma(c) = \gamma(g(b)) = g'(b')$. Now observe that

$$g'(b' - \beta(b)) = g'(b') - g'(\beta(b)) = g'(b') - \gamma(g(b)) = g'(b') - g'(b') = 0$$

So in particular $b' - \beta(b) \in \ker g'$ but by exactness im $f' = \ker g'$ so there exists $a' \in A'$ such that $f'(a') = b' - \beta(b)$. But since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Now $f'(a') = f'(\alpha(a)) = b' - \beta(b)$. By commutativity $f'(\alpha(a)) = \beta(f(a)) = b' - \beta(b)$ so $\beta(f(a)) + \beta(b) = b'$ and we have $\beta(f(a) + b) = b'$.

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphism).

Proof. Follows from (1) and (2).

R Proposition 2.72: (Five Lemma) Consider the commutative diagram with exact rows.

$$A_{1} \xrightarrow{J} A_{2} \xrightarrow{g} A_{3} \xrightarrow{h} A_{4} \xrightarrow{k} A_{5}$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma} \qquad \downarrow^{\delta} \qquad \downarrow^{\epsilon}$$

$$B_{1} \xrightarrow{f'} B_{2} \xrightarrow{g'} B_{3} \xrightarrow{h'} B_{4} \xrightarrow{k'} B_{5}$$

- (1) If h_2 and h_4 are surjective and h_5 is injective, then h_3 is surjective.
- (2) If h_2 and h_4 are injective and h_1 is surjective, then h_3 is injective.
- (3) If h_1, h_2, h_4 and h_5 are isomorphisms, then h_3 is an isomorphism.

AM Proposition 2.10: (Snake Lemma) Let

$$0 \longrightarrow M' \xrightarrow{g} M \xrightarrow{h} M'' \longrightarrow 0$$

$$\downarrow f' \qquad \qquad \downarrow f \qquad \qquad \downarrow f''$$

$$0 \longrightarrow N' \xrightarrow{g'} N \xrightarrow{h'} N'' \longrightarrow 0$$

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be a commutative diagram of R-modules and homomorphisms, with the rows exact. Then there exists a sequence

$$0 \to \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d} \operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \to 0$$

in which $\overline{u}, \overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

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