INTRODUCTION TO CATEGORY THEORY

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Abstract. Module Theory.

0. MATH 697 Homework Zero

Exercise 0.1. Prove Theorem 4 (Isomorphism Theorems):

(1) (*The First Isomorphism Theorem for Modules*) Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then $\ker\varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proof. Let M, N be R-modules and let $\varphi : M \to N$ be an R-modules homomorphism. Then by definition $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$ for all $x, y \in M$, $r \in R$. We want to show that $\ker\varphi = \{m \in M : \varphi(m) = 0\}$ is a submodule. Observe that since M is a module then M is an abelian group by definition so there exists $0 \in M$ such that $m+0 = m$ for all $m \in M$. In particular $\varphi(0) = \varphi(0+0) = \varphi(0)+\varphi(0)$ implying $\varphi(0) = 0$. Conclude that $0 \in \ker\varphi \neq \emptyset$. Now let $r \in R$, $x, y \in \ker\varphi$. Observe that $\varphi(x + ry) = \varphi(x) + \varphi(ry) = \varphi(x) + r\varphi(y) = 0 + r \cdot 0 = 0 + 0 = 0$. Hence $x + ry \in \ker\varphi$. Conclude by the submodule criterion that $\ker\varphi$ is in fact a submodule.

Now define $\Phi : M/\ker\varphi \to \varphi(M)$ by $\Phi(m + \ker\varphi) = \varphi(m)$. We want to show that this mapping is a well-defined bijective homomorphism. We first show well-definedness. Suppose $m + \ker\varphi = m' + \ker\varphi$ it follows by property of cosets that $m - m' \in \ker\varphi$, in particular $\varphi(m - m') = \varphi(m) - \varphi(m') = 0$ and hence $\varphi(m) = \varphi(m')$. But since $\varphi(m) = \Phi(m + \ker\varphi)$ and $\varphi(m') = \Phi(m' + \ker\varphi)$ we have $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Conclude that Φ is in fact well-defined.

Suppose that $\Phi(m + \ker\varphi) = \Phi(m' + \ker\varphi)$. Then it follows that $\varphi(m) = \varphi(m')$ and so $\varphi(m - m') = 0$ and so $m - m' \in \ker\varphi$. By property of cosets it follows that $m + \ker\varphi = m' + \ker\varphi$ and hence Φ is injective.

Let $n \in \varphi(M)$. Then by definition of image of φ there exists $m \in M$ such that $n = \varphi(m)$. It is immediate that $m + \ker\varphi \in M/\ker\varphi$ and we can conclude that Φ is surjective.

Now we must show that Φ is an R-module homomorphism. Let $x, y \in M/\ker\varphi$ where $x = m + \ker\varphi$ and $y = m' + \ker\varphi$ for some $m, m' \in M$ and let $r \in R$. Observe that

$$\begin{aligned}
\Phi(x+y) &= \Phi(m + m' + \ker\varphi) \\
&= \varphi(m + m') \\
&= \varphi(m) + \varphi(m') \\
&= \Phi(m + \ker\varphi) + \Phi(m' + \ker\varphi) \\
&= \Phi(x) + \Phi(y)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(rx) &= \Phi(r(m + \ker\varphi)) \\
&= \Phi(rm + \ker\varphi) \\
&= \varphi(rm) \\
&= r\varphi(m) \\
&= r\Phi(m + \ker\varphi) \\
&= r\Phi(x)
\end{aligned}$$

Hence we have shown that Φ is a well-defined bijective homomorphism and thus we can conclude by definition of R-module isomorphism that $M/\ker\varphi \cong \varphi(M)$. ☐

(2) (*The Second Isomorphism Theorem*) Let A, B be submodules of the R-module M. Then $(A+B)/B \cong A/(A \cap B)$.

Proof. Define $\varphi : A \to (A+B)/B$ by $\varphi(a) = a + B$. This mapping is clearly well-defined. We want to show that φ is a homomorphism. Let $r \in R$, $a, a' \in A$ and observe that

$$\varphi(a + a') = a + a' + B$$

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$$= a + B + a' + B$$
$$= \varphi(a) + \varphi(a')$$

and

$$\begin{aligned}\varphi(ra) = ra + B \\ &= r(a+B) \\ &= r\varphi(a)\end{aligned}$$

and so φ is an R-module homomorphism by definition. Observe that $\ker\varphi = \{a \in A : \varphi(a) = 0\} = \{a \in A : a + B = 0\} = \{a \in A : a \in B\} = A \cap B$. Now let $x \in (A+B)/B$ then $x = a + b + B$ for some $a \in A$, $b \in B$. But observe that $a + b + B = a + B$ by absorbption. So φ is immediately surjective. In particular we have $\varphi(A) = (A+B)/B$. Conclude by the First Isomorphism Theorem for Modules that $A/\ker\varphi = A/(A \cap B) \cong (A+B)/B = \varphi(A)$. □

(3) (*The Third Isomorphism Theorem*) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then $(M/A)/(B/A) \cong M/B$.

Proof. Define $\varphi M/A \to M/B$ by $\varphi(m + A) = m + B$. We need to show φ is well-defined. Suppose $m + A = m' + A$ then $m - m' \in A \subseteq B$ by property of cosets. It also follows that $m + B = m' + B$. Hence $\varphi(m + A) = m + B = m' + B = \varphi(m' + A)$ and hence φ is well-defined.

Now we must show φ is an R-module homomorphism. Let $m, m' \in M$ and $r \in R$. Observe that

$$\begin{aligned}\varphi((m + A) + (m' + a)) &= \varphi(m + m' + A) \\ &= m + m' + B \\ &= (m + B) + (m' + B) \\ &= \varphi(m + A) + \varphi(m' + A)\end{aligned}$$

and

$$\begin{aligned}\varphi(r(m + A)) &= \varphi(rm + A) \\ &= rm + B \\ &= r(m + A) \\ &= r\varphi(m + A)\end{aligned}$$

and hence we can conclude by definition that φ is an R-module homomorphism.

Observe that $\ker\varphi = \{x \in M/A : \varphi(x) = 0\} = \{m + A"\varphi(m + A) = m + B = 0\} = \{m + A : m \in B\} = B/A$. Let $m + B \in M/B$. Clearly $\varphi(m + A) = m + B$ and hence φ is surjective. Now by the First Isomorphism Theorem for Modules we have $(M/A)/\ker\varphi = (M/A)/(B/A) \cong M/B = \varphi(M/A)$. □

(4) (*The Fourth or Lattice Isomorphism Theorem*) Let N be a submodule of the R-module M. There is a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow A/N$, for all $A \supseteq N$. The correspondence cummutes with the processes of taking sums and intersections (i.e., is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N).

Proof. Let N be a submodule of M. Define $S = \{K : K \text{ is a submodule of } M, N \subseteq K\}$, $T = \{L : L \text{ is a submodule of } M/N\}$. Define $\varphi : S \to T$ by $\varphi(K) = K/N$. We want to show that this mapping is bijective.

Let $K_1, K_2 \in S$ and suppose that $\varphi(K_1) = \varphi(K_2)$. Then $K_1/N = K_2/N$. We want to show that $K_1 = K_2$. Let $x \in K_1$, then $x + N \in K_1/N = K_2/N$, in particular there exists $y \in K_2$ such that $x + N = y + N$. By property of cosets it follows that $x - y \in N$. But since $N \subseteq K_2$ by construction $x - y \in K_2$. Since K_2 is a submodule of M, it is closed under addition and so $(x - y) + y = x \in K_2$. Conclude that $K_1 \subseteq K_2$. By symmetric argument $K_2 \subseteq K_1$ and hence $K_1 = K_2$. Thus by definition φ is injective.

Let L be a submodule of M/N. Consider the natural projection map $\pi : M \to M/N$ defined by $\pi(m) = m + N$. We want to show that there exists $K \in S$ such that $\varphi(K) = L$. To do this we will show that $\pi^{-1}(L)$ is a submodule of M and that $N \subseteq \pi^{-1}(L)$. Recall that $\pi^{-1}(L) = \{m \in M : \pi(m) \in L\}$. Observe that $0 \in \pi^{-1}(L)$ since $\pi(0) = 0$ and hence $\pi^{-1}(L) \neq \emptyset$. Let $x, y \in \pi^{-1}(L)$ and $r \in R$. Observe that $\pi(x + ry) = \pi(x) + r\pi(y)$. Since $\pi(y) \in L$ by definition and L is a submodule of M/N, it follows that since scalar multiplication is closed $r\pi(y) \in L$. Thus it follows that $\pi(x) + r\pi(y) \in L$ and hence $x + ry \in \pi^{-1}(L)$. Thus by the submodule criterion we can conclude that $\pi^{-1}(L)$ is in fact a submodule. Now let $n \in N$ and observe that $\pi(n) = n + N = 0 + N \in L$ so by definition it follows that $n \in \pi^{-1}(L)$. Conclude that $N \subseteq \pi^{-1}(L)$ and hence φ is surjective.

Conclude φ is bijective and result follows. □

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