ECMA31000: Introduction to Empirical Analysis

Asymptotics II

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Outline

- Last week:
    - Studied 4 modes of convergence, $\xrightarrow{p}$and $\xrightarrow{a.s.}$, $\xrightarrow{d}$, $\xrightarrow{\text{r-th}}$.
- This week:
    - Continuous mapping theorem.
    - Slutsky's Theorem.
    - Law of large numbers and central limit theorem.
    - Stochastic Order notation.
    - Delta Method.

Definitions

- A sequence of $(K \times 1)$ random vectors $X_n \xrightarrow{p} X$ if $\forall \epsilon > 0$:

$$\lim_{n \to \infty} \mathrm{P}\left(\|X_n - X\| > \epsilon\right) = 0.$$

- $X_n \xrightarrow{a.s} X$ if:

$$\mathrm{P}\left(\{\omega : \|X_n(\omega) - X(\omega)\| \to 0\}\right) = 1.$$

- $X_n \xrightarrow{\text{r-th}} X$ if

$$\lim_{n \to \infty} \mathrm{E}\left(\|X_n - X\|^r\right) = 0.$$

- $X_n \xrightarrow{d} X$ if

$$F_{X_n}(x) \to F_X(x)$$

for all x such that F_X is continuous at x.

Summary of implications

Continuous Mapping Theorem

- Let $g : \mathbb{R}^k \to \mathbb{R}^n$ be a function that is continuous on a set $S \subset \mathbb{R}^k$ with $\mathrm{P}(X \in S) = 1$. Then the following hold:

$$X_n \overset{a.s.}{\to} X \implies g(X_n) \overset{a.s.}{\to} g(X);$$
$$X_n \overset{p}{\to} X \implies g(X_n) \overset{p}{\to} g(X);$$
$$X_n \overset{d}{\to} X \implies g(X_n) \overset{d}{\to} g(X).$$

- The theorem doesn't hold for $\overset{\text{r-th}}{\to}$: Take

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{n^2}; \\ 0 & \text{with probability } 1 - \frac{1}{n^2}. \end{cases}$$

Letting $g(x) = x^2$, we see $\mathrm{E}(|X_n - 0|) = \frac{1}{n} \to 0$ but, for all n:

$$\mathrm{E}(|g(X_n) - g(0)|) = \mathrm{E}\left(\left|X_n^2 - 0\right|\right) = 1.$$

Continuous Mapping theorem

- It is important that $\mathrm{P}(X \in S) = 1$. To see this, suppose
$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} X \\ c \end{pmatrix}.$$
- Consider the continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = \frac{x}{y}$. g is continuous except at points $(x, 0) \in \mathbb{R}^2$. Set
$$S = \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}.$$
If $c \neq 0$, then $\mathrm{P}((X, c) \in S) = 1$, and the CMT gives
$$g(X_n, Y_n) = \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c} = g(X, c).$$
If $c = 0$, then $\mathrm{P}((X, c) \in S) = 0$. This would lead to the nonsensical result

Example: Slutsky's Theorem

Theorem

Suppose $X_n \stackrel{d}{\to} X$ *and* $Y_n \stackrel{p}{\to} c$ *for some constant* c. *Then:*

$$X_n + Y_n \stackrel{d}{\to} X + c;$$

$$X_n Y_n \stackrel{d}{\to} Xc;$$

$$X_n / Y_n \stackrel{d}{\to} X/c \text{ provided } c \neq 0.$$

Proof.

We stated that $X_n \stackrel{d}{\to} X$ and $Y_n \stackrel{p}{\to} c$ implies

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \stackrel{d}{\to} \begin{pmatrix} X \\ c \end{pmatrix}.$$

The result follows by noting that $x + y$, xy and x/y are all continuous functions of (x, y), provided $y \neq 0$ in the last case. □

Example: Slutsky's Theorem

- It is important that Y_n converges in probability to a constant! If instead, for all n

$$Y_n = (-1)^n X_n \sim \mathcal{N}(0, 1),$$

then $X_n + Y_n = 0$ for odd n, and $X_n + Y_n \sim \mathcal{N}(0, 4)$ for even n. Thus,

$$F_{X_n + Y_n}(x)$$

does not converge for any $x \in \mathbb{R}$. This means $X_n + Y_n$ cannot converge in distribution.

Example: Sample Correlation

- Suppose $\{(X_i, Y_i)\}_{i\geq 1}$ is a sequence of (2×1) iid random vectors with $\mathrm{E}\left(X_i^2\right) < \infty$, $\mathrm{E}\left(Y_i^2\right) < \infty$.
- The sample correlation between X, Y is given by

$$\begin{aligned}\hat{\rho}_{XY} &= \frac{\sum_{i=1}^n \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sqrt{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}\sqrt{\sum_{i=1}^n \left(Y_i - \bar{Y}\right)^2}} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n X_i Y_i - \bar{X}\bar{Y}}{\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2}\sqrt{\frac{1}{n}\sum_{i=1}^n Y_i^2 - \bar{Y}^2}},\end{aligned}$$

where $\bar{X} := \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$ are the sample means.

Example: Sample Correlation

- We have already seen that $\bar{X} \xrightarrow{p} \mathrm{E}(X)$ and $\bar{Y} \xrightarrow{p} \mathrm{E}(Y)$.
- Next, note that since the vectors (X_i, Y_i) are iid, the product $X_i Y_i$ is an iid sequence of random variables.
- By the weak law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} X_i Y_i \xrightarrow{p} \mathrm{E}(XY);$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathrm{E}\left(X^2\right);$$

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^2 \xrightarrow{p} \mathrm{E}\left(Y^2\right).$$

Example: Sample Correlation

- It follows that

$$\left(\bar{X}, \bar{Y}, \frac{1}{n}\sum_{i=1}^{n} X_i Y_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2, \frac{1}{n}\sum_{i=1}^{n} Y_i^2\right)$$
$$\xrightarrow{p} \left(\mathrm{E}(X), \mathrm{E}(Y), \mathrm{E}(XY), \mathrm{E}\left(X^2\right), \mathrm{E}\left(Y^2\right)\right).$$

- Now let

$$g(x, y, s, t, w) = \frac{s - xy}{\sqrt{t - x^2}\sqrt{w - y^2}}.$$

- g is continuous at all points except where $t = x^2$ and $w = y^2$. Provided neither X nor Y are constant random variables (if they were the sample variances would also be 0!) we get

$$\mathrm{E}\left(X^2\right) > \mathrm{E}(X)^2; \qquad \mathrm{E}\left(Y^2\right) > \mathrm{E}(Y)^2,$$

so g is continuous at
$\left(\mathrm{E}(X), \mathrm{E}(Y), \mathrm{E}(XY), \mathrm{E}\left(X^2\right), \mathrm{E}\left(Y^2\right)\right)$.

Example: Sample Correlation

- It follows by the continuous mapping theorem that:

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \bar{X}\bar{Y}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2}} \xrightarrow{p} \frac{\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)}{\sqrt{\mathrm{E}(X^2) - \mathrm{E}(X)^2}\sqrt{\mathrm{E}(Y^2) - \mathrm{E}(Y)^2}} = \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}},$$

which is the population correlation coefficient.

Example

- A form of the following result will appear several times when we analyse OLS/IV estimators.
- Let $A_n \in \mathbb{R}^{P \times K}$ be a sequence of matrices converging in probability to a constant matrix A.
    - This is just the same as vector convergence: Stack the columns on top of each other!
- Let B_n be a sequence of $(K \times 1)$ random vectors such that

$$B_n \xrightarrow{d} \mathcal{N}(\mu, \Sigma).$$

Then:

$$A_n B_n \xrightarrow{d} A\mathcal{N}(\mu, \Sigma) \sim \mathcal{N}\left(A\mu, A\Sigma A'\right).$$

Example

Proof.

Since the vector consisting of the columns of A_n, denoted $vec\left(A_n\right)$ converges in probability to $vec\left(A\right)$, a constant vector, we obtain

$$\begin{pmatrix} B_n \\ vec\left(A_n\right) \end{pmatrix} \stackrel{d}{\to} \begin{pmatrix} \mathcal{N}\left(\mu, \Sigma\right) \\ vec\left(A\right) \end{pmatrix}.$$

By the continuous mapping theorem,

$$A_n B_n \stackrel{d}{\to} A\mathcal{N}\left(\mu, \Sigma\right).$$

Since $\mathrm{E}\left(AX\right) = A\mathrm{E}\left(X\right)$, and we showed in problem set 1 that $Var\left(AX\right) = AVar\left(X\right)A'$, we conclude that

$$A\mathcal{N}\left(\mu, \Sigma\right) \sim \mathcal{N}\left(A\mu, A\Sigma A'\right)$$

because linear transformations of multivariate normals are also (multivariate) normal. □

Questions?

Strong Law of Large Numbers (SLLN)

- In Lecture 3, we proved that if $\{X_i\}_{i\geq 1}$ is an iid sequence with $\mathrm{E}\left(X_i^2\right) < \infty$, then

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mathrm{E}\left(X_1\right).$$

- Called this a "weak" law of large numbers because it establishes $\xrightarrow{p}$, not $\xrightarrow{a.s.}$.
- The SLLN delivers a stronger result under a weaker condition:

Theorem

If $\{X_i\}_{i\geq 1}$ is an iid sequence with $\mathrm{E}\left(X_i\right) = \mu$, then

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mathrm{E}\left(X_1\right).$$

Strong Law of Large Numbers (SLLN)

- Result also holds for a sequence of $(K \times 1)$ iid random vectors $\{X_i\}_{i \geq 1}$ such that $\mathrm{E}(X_i) = \mu$, since

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} X_{i1} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} X_{iK} \end{pmatrix},$$

and $\{X_{ij}\}_{i \geq 1}$ are iid sequences of random variabes with $\mathrm{E}(X_{ij}) = \mu_j$, for each $j = 1, \ldots, K$.

Strong Law of Large Numbers (SLLN)

- By the SLLN for random variables,

$$\frac{1}{n}\sum_{i=1}^{n} X_{ij} \stackrel{a.s.}{\to} \mathrm{E}\left(X_{1j}\right)$$

for each $j = 1, \ldots, K$. It follows that

$$\frac{1}{n}\sum_{i=1}^{n} X_{i} \stackrel{a.s.}{\to} \mathrm{E}\left(X_{1}\right).$$

Central Limit Theorem

- Let $\{X_i\}_{i\geq 1}$ be an iid sequence with $\mathrm{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2$ and consider the distribution of the sample mean $\bar{X}_n$.
- We know that

$$\mathrm{E}\left(\bar{X}_n\right) = \mu; \quad Var\left(\bar{X}_n\right) = \frac{\sigma^2}{n}.$$

Central Limit Theorem

- It follows that

$$\mathrm{E}\left(\sqrt{n}\left(\bar{X}_n - \mu\right)\right) = 0; \quad Var\left(\sqrt{n}\left(\bar{X}_n - \mu\right)\right) = \sigma^2.$$

- Note that if $X_i \sim \mathcal{N}\left(0, \sigma^2\right)$, then for all n:

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \sim \mathcal{N}\left(0, \sigma^2\right).$$

- The Central Limit Theorem provides an approximation to the distribution of $\sqrt{n}\left(\bar{X}_n - \mu\right)$ when X_i are non-normal.
- In "large" samples, $\sqrt{n}\left(\bar{X}_n - \mu\right)$ is approximately normally distributed, sometimes written as

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \overset{a}{\sim} \mathcal{N}\left(0, \sigma^2\right).$$

Central Limit Theorem

- Notice that if instead of $\sqrt{n}$, we choose n^r, for some $r > 0$:

$$Var\left(n^r\left(\bar{X}_n - \mu\right)\right) = \frac{\sigma^2 n^{2r}}{n} \to \begin{cases} 0 & \text{if } r < \frac{1}{2} \\ \sigma^2 & \text{if } r = \frac{1}{2} \\ +\infty & \text{if } r > \frac{1}{2} \end{cases}.$$

- By Problem Set 2 Q3, for $r < \frac{1}{2}$, $\mathrm{E}\left(n^r\left(\bar{X}_n - \mu\right)\right) = 0$ and $Var\left(n^r\left(\bar{X}_n - \mu\right)\right) \to 0$ implies $n^r\left(\bar{X}_n - \mu\right) \xrightarrow{p} 0$.
- We say the limiting distribution of $n^r\left(\bar{X}_n - \mu\right)$ is degenerate, because it has the same distribution as a constant random variable (in this case, 0).
- We now consider the cases $r = \frac{1}{2}, r > \frac{1}{2}$.

Central Limit Theorem

- We say a sequence of random variables $\{Y_n\}_{\geq 1}$ converges in distribution to a non-degenerate limit if:
  - Y_n converges in distribution to some random variable Y,
  - Y is not (almost surely) constant.
- If $r = \frac{1}{2}$, The distribution of $\sqrt{n}\left(\bar{X}_n - \mu\right)$ converges to the distribution of $\mathcal{N}\left(0, \sigma^2\right)$, no matter what the initial distribution of the X_i! This is the classical Central Limit Theorem (CLT):

Theorem

(Lindeberg-Lévy CLT) Let $\{X_i\}_{i\geq 1}$ *be an iid sequence of random variables with* $\mathrm{E}\left(X_i\right) = \mu$ *and* $Var\left(X_i\right) = \sigma^2$ *. Then*

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right).$$

Central Limit Theorem

- Next, denote $Y_n^r := n^r \left(\bar{X}_n - \mu\right)$ and let $r > \frac{1}{2}$.
- For any $M > 0$, and $n \geq M^{2/(r-1/2)}$:

$$
\begin{aligned}
\mathrm{P}\left(Y_n^r \leq M\right) &= \mathrm{P}\left(\sqrt{n}\left(\bar{X}_n - \mu\right) \leq \frac{M}{n^{r-1/2}}\right) \\
\left(\text{since } n^{r-1/2} \geq M^2\right) &\leq \mathrm{P}\left(\sqrt{n}\left(\bar{X}_n - \mu\right) \leq \frac{M}{M^2}\right) \\
&\to \Phi\left(\frac{1}{M}\right)
\end{aligned}
$$

as $n \to \infty$.

Central Limit Theorem

- If it held that $Y_n^r \xrightarrow{d} Y$ for some Y, the definition of convergence in distribution and the above would imply:

$$F_Y(M) = \lim_{n\to\infty} F_{Y_n}(M) \leq \Phi\left(\frac{1}{M}\right),$$

for all continuity points M of F_Y.

- Since F_Y is a distribution function, we obtain

$$1 = \lim_{M\to\infty} F_Y(M) \leq \lim_{M\to\infty} \Phi\left(\frac{1}{M}\right) = \frac{1}{2},$$

a contradiction. Therefore, Y_n^r cannot converge in distribution for $r > \frac{1}{2}$.

Questions?

Example: Gamma Distribution

- We now demonstrate that there is no good rule of thumb for how large n should be to invoke the CLT.
- Let X_i be iid with $\mathrm{E}(X_i) = \mu$, $Var(X_i) = \sigma^2$ and skewness $\kappa = \mathrm{E}\left(\left(\frac{X_i - \mu}{\sigma}\right)^3\right)$.
- For symmetric distributions (e.g. normal), $\kappa = 0$.
- Can show that the skewness of the sample mean is

$$\mathrm{E}\left(\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^3\right) = \frac{\kappa}{\sqrt{n}},$$

which decreases with n, as we may expect.

Example: Gamma Distribution

- Suppose $X_i \sim Gamma\left(k, \theta\right)$, where k, θ are shape and scale parameters respectively.
- Conveniently $\sum_{i=1}^{n} X_i \sim Gamma\left(kn, \theta\right)$ and $\bar{X}_n \sim Gamma\left(kn, \frac{\theta}{n}\right)$.
- We have $\mathrm{E}\left(X_i\right) = k\theta$ and $Var\left(X_i\right) = k\theta^2$.
- The skewness of the gamma distribution depends only on the shape parameter:

$$\kappa = \mathrm{E}\left(\left(\frac{X_i - k\theta}{\sqrt{k\theta^2}}\right)^3\right) = \frac{2}{\sqrt{k}}.$$

Example: Gamma Distribution

- For the sample mean:

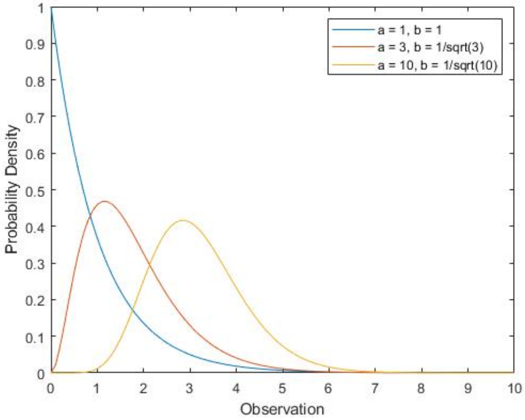
$$\mathrm{E}\left(\left(\frac{\bar{X}_n - k\theta}{\sqrt{\left(k\theta^2\right)/n}}\right)^3\right) = \frac{\kappa}{\sqrt{n}} = \frac{2}{\sqrt{nk}}.$$

  Therefore, the skewness is only close to 0 if $\sqrt{nk}$ is large.

- So if we have $n = 1000$, but X_i happens to be drawn from $\mathit{Gamma}\,(0.001, \theta)$, distribution of $\bar{X}_n$ still positively skewed:

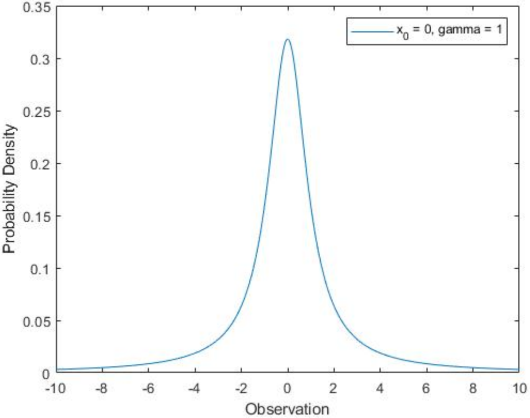
Example: Gamma Distribution

- Can show $\frac{\bar{X}_n}{\sqrt{(k\theta^2)/n}} \sim Gamma\left(nk, \frac{1}{\sqrt{nk}}\right)$:



Example: Cauchy Distribution

- We consider X_i iid with $f_X(t) = \frac{1}{\pi t^2 + 1}$.
- Can show that $\bar{X}_n$ has pdf f_X for all n! This occurs because the Cauchy distribution does not have mean or variance.



Multivariate CLT

Theorem

Let $\{X_i\}_{i\geq 1}$ be an iid sequence of $(K \times 1)$ random vectors with mean μ and finite variance matrix Σ. Then

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \Sigma\right).$$

Multivariate CLT

Proof.

By the Cramér-Wold Theorem, it suffices to show that for all $t \in \mathbb{R}^K$,

$$t' \left(\sqrt{n} \left(\bar{X}_n - \mu \right) \right) \xrightarrow{d} t' \mathcal{N} \left(0, \Sigma \right).$$

Note that

$$t' \left(\sqrt{n} \left(\bar{X}_n - \mu \right) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} t' \left(X_i - \mu \right),$$

where $\{ t' \left(X_i - \mu \right) \}_{i \geq 1}$ is an iid sequence with mean 0 and variance $t' \Sigma t$, so by the univariate CLT:

$$t' \left(\sqrt{n} \left(\bar{X}_n - \mu \right) \right) \xrightarrow{d} \mathcal{N} \left(0, t' \Sigma t \right),$$

which has the same distribution as $t' \mathcal{N} \left(0, \Sigma \right)$. $\square$

Example

- Let $\{X_i\}_{i\geq 1}$ be an iid sequence of random variables with $\mathrm{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 > 0$.
- By the CLT
$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right).$$
- Suppose we wish to test $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$ at significance level α.
    - Since σ^2 is typically unknown, how do we pick a test stat.?
    - Since the distribution of $\sqrt{n}\left(\bar{X}_n - \mu\right)$ is unknown, how do we pick a critical value?

Example

- To solve first issue: estimate σ^2 by

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_i^2 - \bar{X}_n^2\right).$$

- The denominator $1/(n-1)$ is used because

$$\mathrm{E}\left(S_n^2\right) = \sigma^2.$$

- From the SLLN:

$$\frac{1}{n}\sum_{i=1}^{n}X_i^2 \overset{a.s.}{\to} \mathrm{E}\left(X_i^2\right); \qquad \bar{X}_n \overset{a.s.}{\to} \mathrm{E}\left(X_i\right)$$

and by ordinary analysis, $\frac{n}{n-1} \to 1$. It follows that:

$$\left(\frac{n}{n-1}, \frac{1}{n}\sum_{i=1}^{n}X_i^2, \bar{X}_n\right) \overset{a.s.}{\to} \left(1, \mathrm{E}\left(X_i^2\right), \mathrm{E}\left(X_i\right)\right).$$

Example

- By the continuous mapping theorem, with

$$g\left(x, y, z\right) = \frac{1}{\sqrt{x\left(y - z^2\right)}}$$

we obtain (since $\sigma^2 > 0$):

$$\frac{1}{\sqrt{S_n^2}} \stackrel{a.s.}{\to} \frac{1}{\sqrt{\sigma^2}} = \frac{1}{\sigma}.$$

Example

- By Slutsky's theorem, since $\frac{1}{S_n} \xrightarrow{p} \frac{1}{\sigma}$, and $\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$, it follows that

$$\frac{1}{S_n} \times \sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \frac{1}{\sigma}\mathcal{N}\left(0, \sigma^2\right),$$

  which has the standard normal distribution.

- Therefore, under H_0:

$$\frac{\sqrt{n}\left(\bar{X}_n - \mu_0\right)}{S_n} \xrightarrow{d} \mathcal{N}(0, 1).$$

Example

- By definition of $\xrightarrow{d}$:

$$\mathrm{P}\left(\frac{\sqrt{n}\left(\bar{X}_n - \mu_0\right)}{S_n} > z_{1-\alpha}\right) \to \alpha,$$

  where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution.

- Therefore, comparing our test statistic with the $1-\alpha$ quantile of $\mathcal{N}\left(0,1\right)$ produces <u>asymptotically</u> correct null rejection probability, although the finite sample significance level of this test will usually <u>not</u> be α.

Questions?

Berry-Esseen Theorem

- We know that no fixed n is large enough to guarantee a good normal approximation. But.. the example used really highly skewed distributions. Berry-Esseen gives a finite sample bound on how far away from normality the distribution actually is:

Theorem

(Berry-Esseen) Let $\{X_i\}_{i\geq 1}$ be an iid sequence of random variables with $\mathrm{E}(X_i) = \mu$, $0 < Var(X_i) < \infty$ and $\mathrm{E}\left(|X_i - \mu|^3\right) = \rho < \infty$. Then

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}\left(\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \leq x\right) - \Phi(x) \right| \leq \frac{C\rho}{\sigma^3\sqrt{n}},$$

where Φ is the standard normal CDF and C is a constant that does not depend on n or the distribution of the X_i.

Example: Bernoulli Distribution

- If $X_i \sim Bernoulli(p)$, then $\mathrm{E}(X_i) = p$, $Var(X_i) = p(1-p)$, and

$$\frac{\rho}{\sigma^3} = \mathrm{E}\left(\left|\frac{X_i - \mathrm{E}(X_i)}{\sqrt{Var(X_i)}}\right|^3\right) = \frac{p^2 + (1-p)^2}{\sqrt{p(1-p)}}.$$

- Berry-Esseen theorem tells us that

$$\sup_{x \in \mathbb{R}} \left| \mathrm{P}\left(\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \leq x\right) - \Phi(x) \right| \leq C \frac{p^2 + (1-p)^2}{\sqrt{np(1-p)}}.$$

Tightness

- There are cases in which a sequence of random vectors does not have a limiting distribution, but it does remain "bounded" in a probabilistic sense, so that none of the probability mass escapes to $+\infty$ or $-\infty$.
- For example, $X_n = (-1)^n$ is a sequence of constant random variables that does not converge in distribution, but for all n:

$$\mathrm{P}\left(|X_n| \leq 1\right) = 1.$$

Tightness

- We call a sequence of random vectors $\{X_n\}_{n \geq 1}$ tight if for all $\epsilon > 0$ there exists $B_\epsilon > 0$ such that for all n,

$$\mathrm{P}\left(\|X_n\| \leq B_\epsilon\right) \geq 1 - \epsilon.$$

- Tightness requires that for any ϵ, the entire sequence can be contained in a ball of radius B_ϵ with probability at least $1 - \epsilon$, so a tight sequence is sometimes called "bounded in probability".

Tightness

- Suppose $X_n \xrightarrow{d} X$, with distribution function F_X. We know that

$$\lim_{t \to +\infty} F_X(t) = 1,$$
$$\lim_{t \to -\infty} F_X(t) = 0.$$

It follows that if $X_n = X$ for all n, then trivially $X_n \xrightarrow{d} X$, but also $\exists B_\epsilon$ such that for any $\epsilon > 0$:

$$\mathrm{P}(|X_n| \leq B_\epsilon) = \mathrm{P}(|X| \leq B_\epsilon) \geq 1 - \epsilon,$$

since B_ϵ can be chosen large enough that

$$F_X(B_\epsilon) - F_X(-B_\epsilon) \geq 1 - \epsilon.$$

- In fact, convergence in distribution always implies tightness.

Tightness

Theorem

If $\{X_n\}_{n\geq 1}$ is a sequence of random vectors such that $X_n \xrightarrow{d} X$, then $\{X_n\}_{n\geq 1}$ is tight.

Proof.

(For random variables). Suppose $X_n \xrightarrow{d} X$, with distribution function F_X. We know that $\exists B_\epsilon > 0$ such that

$$\mathrm{P}\left(|X| > B_\epsilon\right) \leq \epsilon/3,$$

where $B_\epsilon, -B_\epsilon$ are continuity points of F_X. We have $F_{X_n}\left(B_\epsilon\right) \to F_X\left(B_\epsilon\right)$, so choose n large enough that

$$\left|F_{X_n}\left(B_\epsilon\right) - F_X\left(B_\epsilon\right)\right| < \epsilon/3,$$
$$\left|F_{X_n}\left(-B_\epsilon\right) - F_X\left(-B_\epsilon\right)\right| < \epsilon/3.$$

Tightness

Proof.

Now note that

$$
\begin{aligned}
\mathrm{P}\left(|X_n| > B_\epsilon\right) &= \mathrm{P}\left(X_n > B_\epsilon\right) + \mathrm{P}\left(X_n < -B_\epsilon\right) \\
&= \mathrm{P}\left(X_n > B_\epsilon\right) - \mathrm{P}\left(X > B_\epsilon\right) \\
&+ \mathrm{P}\left(X > B_\epsilon\right) + \mathrm{P}\left(X < -B_\epsilon\right) \\
&+ \mathrm{P}\left(X_n < -B_\epsilon\right) - \mathrm{P}\left(X < -B_\epsilon\right) \\
&\leq \left|1 - F_{X_n}\left(B_\epsilon\right) - \left[1 - F_X\left(B_\epsilon\right)\right]\right| \\
&+ \mathrm{P}\left(|X| > B_\epsilon\right) \\
&+ \left|F_{X_n}\left(-B_\epsilon\right) - F_X\left(-B_\epsilon\right)\right| \\
&< \epsilon.
\end{aligned}
$$

$\square$

Stochastic order notation

- Let $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ be sequences of real numbers.
- Write $x_n = o(y_n)$ if

$$\left|\frac{x_n}{y_n}\right| \to 0 \text{ as } n \to \infty.$$

- Write $x_n = O(y_n)$ if $\exists B < \infty$ such that for all n:

$$\left|\frac{x_n}{y_n}\right| \leq B.$$

- Note that $x_n = o(y_n) \implies x_n = O(y_n)$.

Stochastic order notation

- Let $\{X_n\}_{n\geq 1}, \{Y_n\}_{n\geq 1}$ be sequences of random variables.
- Write $X_n = o_p(Y_n)$ if

$$\left|\frac{X_n}{Y_n}\right| \xrightarrow{p} 0.$$

- Write $X_n = O_p(Y_n)$ if for all $\epsilon > 0$, $\exists B_\epsilon > 0$ such that $\forall n$:

$$\mathrm{P}\left(\left|\frac{X_n}{Y_n}\right| \leq B_\epsilon\right) \geq 1 - \epsilon.$$

- This last condition just says that $|X_n/Y_n|$ is tight.
- Note that $X_n = o_p(Y_n) \implies X_n = O_p(Y_n)$.

Example

- $\{X_n\}_{n\geq 1}$ is tight is equivalently written as $X_n = O_p(1)$.
- $X_n \xrightarrow{p} 0$ is equivalently written as $X_n = o_p(1)$.
- Some properties:
    - $o_p(1) + o_p(1) = o_p(1)$
    - $o_p(1) + O_p(1) = O_p(1)$
    - $o_p(1)\, O_p(1) = o_p(1)$
    - $(A + o_p(1))^{-1} = O_p(1)$ for an invertible matrix A.
- By Slutsky's Theorem, if $X_n \xrightarrow{d} X$ and $Y_n = o_p(1)$, then $X_n + Y_n \xrightarrow{d} X$.

Proof: $o_p(1)\, O_p(1) = o_p(1)$

- Let $X_n = o_p(1)$ and $Y_n = O_p(1)$. Need to show $\mathrm{P}(|X_n Y_n| > \epsilon) \to 0$ for any $\epsilon > 0$.
- Choose $\gamma > 0$ such that $\mathrm{P}(|Y_n| \geq B_\gamma) \leq \gamma$.
- Write

$$
\begin{aligned}
|X_n Y_n| &= |X_n Y_n|\, \mathbf{1}_{|Y_n| < B_\gamma} + |X_n Y_n|\, \mathbf{1}_{|Y_n| \geq B_\gamma} \\
&\leq B_\gamma |X_n| + |X_n Y_n|\, \mathbf{1}_{|Y_n| \geq B_\gamma}.
\end{aligned}
$$

- It follows that

$$
\begin{aligned}
\mathrm{P}(|X_n Y_n| > \epsilon) &\leq \mathrm{P}\left(B_\gamma |X_n| + \left|X_n Y_n \mathbf{1}_{|Y_n| \geq B_\gamma}\right| > \epsilon\right) \\
&\leq \mathrm{P}(B_\gamma |X_n| > \epsilon/2) + \mathrm{P}\left(\left|X_n Y_n \mathbf{1}_{|Y_n| \geq B_\gamma}\right| > \epsilon/2\right) \\
&\leq o(1) + \gamma.
\end{aligned}
$$

Since $\gamma > 0$ was chosen arbitrarily, $\mathrm{P}(|X_n Y_n| > \epsilon) \to 0$.

Questions?

Delta Method

- So far we have developed the CLT, SLLN and CMT, now add Delta Method.
- CLT gives:

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right).$$

- CMT gives:

$$g\left(\sqrt{n}\left(\bar{X}_n - \mu\right)\right) \xrightarrow{d} g\left(\mathcal{N}\left(0, \sigma^2\right)\right)$$

- What can we say about

$$\sqrt{n}\left(g\left(X_n\right) - g\left(\mu\right)\right)?$$

Delta Method

Theorem

Let $\{X_n\}_{n\geq 1}$ be a sequence of $(K \times 1)$ random vectors and suppose that

$$n^r (X_n - c) \xrightarrow{d} X$$

for some $r > 0$ and constant vector c. Let $g : \mathbb{R}^K \to \mathbb{R}^d$ be differentiable at the point c. Let $Dg(c)$ be the $d \times k$ matrix of partial derivatives evaluated at c. Then

$$n^r (g(X_n) - g(c)) \xrightarrow{d} Dg(c) X.$$

In particular, if $X \sim \mathcal{N}(0, \Sigma)$, then

$$n^r (g(X_n) - g(c)) \xrightarrow{d} \mathcal{N}\left(0, Dg(c) \Sigma Dg(c)'\right).$$

Delta Method

- $Dg\left(c\right)$ is the following matrix of partial derivatives:

$$Dg\left(c\right)=\begin{bmatrix}\frac{\partial g_1}{\partial x_1}\left(c\right) & \frac{\partial g_1}{\partial x_2}\left(c\right) & \cdots & \frac{\partial g_1}{\partial x_k}\left(c\right)\\ \frac{\partial g_2}{\partial x_1}\left(c\right) & \frac{\partial g_2}{\partial x_2}\left(c\right) & \cdots & \frac{\partial g_2}{\partial x_k}\left(c\right)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial g_d}{\partial x_1}\left(c\right) & \frac{\partial g_d}{\partial x_2}\left(c\right) & \cdots & \frac{\partial g_d}{\partial x_k}\left(c\right)\end{bmatrix}.$$

Delta Method

Proof.

By Taylor's theorem:

$$g\left(x\right)=g\left(c\right)+Dg\left(c\right)\left(x-c\right)+h_1\left(x\right)\left(x-c\right),$$

for some function $h_1\left(x\right)$ with $\lim_{x\to c}h_1\left(x\right)=h_1\left(c\right)=0$. It follows that

$$n^r\left(g\left(X_n\right)-g\left(c\right)\right)=Dg\left(c\right)n^r\left(X_n-c\right)+h_1\left(X_n\right)n^r\left(X_n-c\right).$$

Since $n^r\left(X_n-c\right)\xrightarrow{d}X$, by Slutsky's Theorem:

$$Dg\left(c\right)n^r\left(X_n-c\right)\xrightarrow{d}Dg\left(c\right)X.$$

Now show that $h_1\left(X_n\right)n^r\left(X_n-c\right)=o_p\left(1\right)$:

Delta Method

Proof.

Since $n^r(X_n - c) \xrightarrow{d} X$, we have $n^r(X_n - c) = O_p(1)$ and $X_n \xrightarrow{p} c$ (See Problem Set 3). Since h_1 is continuous at c by construction,

$$h_1(X_n) \xrightarrow{p} h_1(c) = 0$$

by the CMT. Therefore,

$$h_1(X_n)\, n^r(X_n - c) = o_p(1) \cdot O_p(1) = o_p(1),$$

so

$$\begin{aligned} n^r(g(X_n) - g(c)) &= Dg(c)\, n^r(X_n - c) + o_p(1) \\ &\xrightarrow{d} Dg(c)\, X. \end{aligned}$$

□

Delta Method

- Note that if $Dg\left(c\right)=0$,

$$n^r\left(g\left(X_n\right)-g\left(c\right)\right)\xrightarrow{d}0,$$

  which is a degenerate limiting distribution, (so $\xrightarrow{p} 0$ also).
- If g has higher order derivatives, we can derive an alternate form of the Delta Method when $Dg\left(c\right)=0$.
- Suppose $\left\{X_n\right\}_{n\geq 1}$ is a sequence of random variables and $g:\mathbb{R}\rightarrow\mathbb{R}$ has 2 derivatives.
- Taylor's theorem implies

$$g\left(x\right)=g\left(c\right)+g'\left(c\right)\left(x-c\right)+\frac{g''\left(c\right)}{2}\left(x-c\right)^2+h_2\left(x\right)\left(x-c\right)^2,$$

  where $\lim_{x\rightarrow c}h_2\left(x\right)=h_2\left(c\right)=0$.

Delta Method

- Repeating the argument in the proof of the original Delta Method:

$$n^{2r}\left(g\left(X_n\right)-g\left(c\right)\right)=g'\left(c\right)n^{2r}\left(X_n-c\right)+\frac{g''\left(c\right)}{2}n^{2r}\left(X_n-c\right)^2 \\ +h_2\left(X_n\right)n^{2r}\left(X_n-c\right)^2.$$

- We have

$$\begin{aligned} g'\left(c\right)n^{2r}\left(X_n-c\right)&=0\\ \frac{g''\left(c\right)}{2}n^{2r}\left(X_n-c\right)^2&\xrightarrow{d}\frac{g''\left(c\right)}{2}X^2\\ h_2\left(X_n\right)n^{2r}\left(X_n-c\right)^2&=o_p\left(1\right)O_p\left(1\right)=o_p\left(1\right). \end{aligned}$$

- In summary, if $g'\left(c\right)=0$,

$$n^{2r}\left(g\left(X_n\right)-g\left(c\right)\right)\xrightarrow{d}\frac{g''\left(c\right)}{2}X^2.$$

Example: Sample variance

- We will tie together the concepts we have learned so far to find the asymptotic distribution of

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2.$$

- Let X_i be iid random variables with $\mathrm{E}\left(X_i\right) = \mu$, $Var\left(X_i\right) = \sigma^2$ and $\mathrm{E}\left(X_i - \mu\right)^4 = \kappa$.
- We are looking for an $r > 0$, constant c and random variable X such that

$$n^r\left(S_n^2 - c\right) \xrightarrow{d} X,$$

for some non-degenerate X.

Example: Sample variance

- We have already shown that $S_n^2 \xrightarrow{p} \sigma^2$, so we must take $c = \sigma^2$.
- Unfortunately, S_n^2 is not in a form where we can apply the CLT directly:

$$n^{1/2}\left(S_n^2 - \sigma^2\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(X_i - \bar{X}_n\right)^2 - \sigma^2\right].$$

- If we could replace $\bar{X}_n$ with μ, would consider

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[(X_i - \mu)^2 - \sigma^2\right].$$

Example: Sample variance

- Note that $\left\{(X_i-\mu)^2\right\}_{i\geq 1}$ is an iid sequence with $\mathrm{E}\,(X_i-\mu)^2=\sigma^2$ and

$$Var\left((X_i-\mu)^2\right)=\mathrm{E}\left[(X_i-\mu)^4\right]-\left[\mathrm{E}\,(X_i-\mu)^2\right]^2$$
$$=\kappa-\sigma^4.$$

- Therefore, by the CLT:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(X_i-\mu)^2-\sigma^2\right]\xrightarrow{d}\mathcal{N}\left(0,\kappa-\sigma^4\right).$$

- It remains to show that

$$n^{1/2}\left(S_n^2-\sigma^2\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(X_i-\mu)^2-\sigma^2\right]+o_p\left(1\right).$$

Example: Sample variance

- Note that

$$
\begin{aligned}
\sqrt{n}\left(S_n^2-\sigma^2\right) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(X_i-\bar{X}_n\right)^2-\sigma^2\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(X_i-\mu-\left(\bar{X}_n-\mu\right)\right)^2-\sigma^2\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(X_i-\mu\right)^2-\sigma^2\right]+\sqrt{n}\left(\bar{X}_n-\mu\right)^2 \\
&- 2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(X_i-\mu\right)\left(\bar{X}_n-\mu\right)\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(X_i-\mu\right)^2-\sigma^2\right]-\sqrt{n}\left(\bar{X}_n-\mu\right)^2 .
\end{aligned}
$$

Example: Sample variance

- Finally, since

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right),$$
$$\left(\bar{X}_n - \mu\right) = o_p\left(1\right),$$

we get

$$\sqrt{n}\left(\bar{X}_n - \mu\right)^2 = o_p\left(1\right) O_p\left(1\right) = o_p\left(1\right).$$

- In summary:

$$n^{1/2}\left(S_n^2 - \sigma^2\right) \xrightarrow{d} \mathcal{N}\left(0, \kappa - \sigma^4\right).$$

Example: Sample variance

- We are not done yet: The limiting distribution is non-degenerate iff
$$\kappa - \sigma^4 > 0.$$
- Jensen's inequality gives
$$\kappa = \mathrm{E}\left[(X_i - \mu)^4\right] \geq \left[\mathrm{E}\,(X_i - \mu)^2\right]^2 = \sigma^4,$$
with equality if and only if the random variable $(X_i - \mu)^2$ is constant almost surely.
- In this case, this does NOT imply X_i is constant, since
$$(X_i - \mu)^2 = \sigma^2$$
will have two solutions: $X_i = \mu \pm \sigma$ when $\sigma > 0$.