Exam 1 Practice Questions

1 Questions

Question 1: Find a sequence of random variables X_n and a random variable X such that $\mathrm{E}\left(X_n - X\right) \to 0$ but $\mathrm{E}\left(|X_n - X|\right) \not\to 0$ and $X_n \overset{p}{\not\to} X$. Prove it.

Question 2: a) Suppose $Z_n \xrightarrow{p} Z$ and $Z_n \xrightarrow{p} W$. Show that $\mathrm{P}\left(Z = W\right) = 1$.
b) Suppose $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, where all random variables $\left(\{X_n, Y_n\}_{n\geq 1}, X, Y\right)$ exist on the same probability space. Show that $X_n - Y_n \xrightarrow{p} 0$ iff $X = Y$ with probability 1.

Question 3: Prove that $O_p\left(1\right) \times O_p\left(1\right) = O_p\left(1\right)$. Is it true that $\frac{o_p(1)}{O_p(1)} = O_p\left(1\right)$?

Question 4: Suppose that $Z_n \sim t_n$. Then the distribution of Z_n can be represented by

$$Z_n \sim \frac{\mathcal{N}\left(0,1\right)}{\sqrt{\frac{\chi_n^2}{n}}},$$

where the $\mathcal{N}\left(0,1\right)$ and χ_n^2 are independent.
a) Prove that $Z_n \xrightarrow{d} \mathcal{N}\left(0,1\right)$.
b) It can be shown that the $1-\alpha$ quantile of t_n, denoted $t_{n,1-\alpha}$, converges to $z_{1-\alpha}$, the $1-\alpha$ quantile of the standard normal distribution. Prove that if $T_n \xrightarrow{d} \mathcal{N}\left(0,1\right)$, then

$$\mathrm{P}\left(T_n \geq t_{n,1-\alpha}\right) \to \alpha.$$

Question 5: Given a sample space $\Omega := \{1,2,3\}$, let $A := \{1\}$, $B := \{2\}$, and $C := \{3\}$. Let $\mathrm{P}(A) = P(B) = \frac{1}{3}$. Compute $P(C)$, $P(A \cup B)$, $P(A \cap B)$, $P(A^c)$, $P(A^c \cup B^c)$. Are A and C independent?

Question 6: Let $\{X_i\}_{i\geq 1}$ be an independent sequence of random variables where $X_i \sim \mathcal{N}\left(0, 2^{-i}\right)$. Define:

$$\Gamma_n \coloneqq \max_{n \leq i \leq 2n} X_i.$$

Prove that, as $n \to \infty$, we have $\Gamma_n \xrightarrow{\text{p}} 0$.

Question 7: Let $X_1, \ldots, X_n$ be an iid sequence of random variables with CDF F.

a) Show that $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i \le x\}}$ converges in probability to $F(x)$.

b) Show for any fixed value of x that:

$$\sqrt{n}\left(\hat{F}_n(x) - F(x)\right) \xrightarrow{\text{d}} N\left(0, \sigma^2(x)\right).$$

Provide an expression for $\sigma^2(x)$. Use the sample analogue principle to construct an estimator $\hat{\sigma}_n^2(x)$ of $\sigma^2(x)$. Does $\hat{\sigma}_n^2(x) \xrightarrow{p} \sigma^2(x)$? Justify your answer.

Question 8: Let $X_1, X_2, \ldots, X_n$ be an iid sequence of random variables such that $\mathbb{E}[X_i] = 0$ and $\operatorname{Var}[X_i] < \infty$. Let

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n-1}(X_i + X_{i+1}).$$

a) What is the limit in probability of $\hat{\mu}_n$? Provide formal justification.

b) What is the limiting distribution of $\sqrt{n}\hat{\mu}_n$? Again, provide formal justification.

2 Solutions

Q1: Suppose that for all n:

$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}.$$

Clearly $\mathrm{E}\left(X_n - 0\right) = 0$ for all n but $\mathrm{E}\left(|X_n - 0|\right) = 1$ for all n, and $\mathrm{P}\left(|X_n - 0| > \epsilon\right) = 1$ for all n whenever $\epsilon < 1$. Could also use $X_n = -X$ for all n where $X \sim \mathcal{N}(0,1)$ or X has the same distribution as X_n.

Q2: a) If $Z_n \xrightarrow{p} Z$ and $Z_n \xrightarrow{p} W$ but $\mathrm{P}\left(Z = W\right) \neq 1$, there exist constants $\epsilon, \delta > 0$ such that

$$\begin{aligned} \delta < \mathrm{P}\left(|Z - W| > \epsilon\right) &\leq \mathrm{P}\left(|Z - Z_n| + |Z_n - W| > \epsilon\right) \\ &\leq \mathrm{P}\left(|Z_n - Z| > \epsilon/2\right) + \mathrm{P}\left(|Z_n - W| > \epsilon/2\right). \end{aligned}$$

But both terms on the RHS converge to 0 since $Z_n \xrightarrow{p} Z$ and $Z_n \xrightarrow{p} W$, a contradiction.

b) Suppose $X = Y$ with probability 1. Then

$$\begin{aligned} \mathrm{P}\left(|X_n - Y_n| \geq \epsilon\right) &\leq \mathrm{P}\left(|X_n - X| + |X - Y| + |Y - Y_n| > \epsilon\right) \\ &\leq \mathrm{P}\left(|X_n - X| > \epsilon/3\right) + \mathrm{P}\left(|Y_n - Y| > \epsilon/3\right) \\ &+ \mathrm{P}\left(|X - Y| > \epsilon/3\right) \end{aligned}$$

The first and second term on the RHS converge to 0 because $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. Finally, $\mathrm{P}\left(|X - Y| > \epsilon/3\right) = 0$ because $X = Y$ with probability 1.

Now suppose $X_n - Y_n \xrightarrow{p} 0$. By Slutsky's Theorem, we also know $X_n - Y_n \xrightarrow{p} X - Y$. By part a), this means $\mathrm{P}\left(X - Y = 0\right) = \mathrm{P}\left(X = Y\right) = 1$.

Q3: If $X_n = O_p(1)$ and $Y_n = O_p(1)$, then for all $\delta_X, \delta_Y > 0$, $\exists B_X, B_Y$ such that for all n:

$$\begin{aligned} \mathrm{P}\left(|X_n| \geq B_X\right) &\leq \delta_X, \\ \mathrm{P}\left(|Y_n| \geq B_Y\right) &\leq \delta_Y. \end{aligned}$$

Let $\delta > 0$ be given. Choose $\delta_X + \delta_Y < \delta$. It follows that

$$\begin{aligned} \mathrm{P}\left(|X_n Y_n| \geq B_X B_Y\right) &\leq \mathrm{P}\left(\{|X_n| \geq B_X\} \cup \{|Y_n| \geq B_Y\}\right) \\ &\leq \mathrm{P}\left(|X_n| \geq B_X\right) + \mathrm{P}\left(|Y_n| \geq B_Y\right) \\ &\leq \delta_X + \delta_Y < \delta. \end{aligned}$$

The statement $\frac{o_p(1)}{O_p(1)} = O_p(1)$ is false: Take $X_n = \frac{1}{n}$, $Y_n = \frac{1}{n^2}$. Then $X_n = o_p(1)$, $Y_n = O_p(1)$, but

$$\frac{X_n}{Y_n} = n \to +\infty,$$

so for every $M > 0$,

$$\mathrm{P}(|X_n| \geq M) = 1$$

for $n \geq M$. Therefore, $X_n/Y_n \neq O_p(1)$.

Q4: Let $Z_n \sim t_n$. Then the distribution of Z_n can be represented by

$$Z_n \sim \frac{\mathcal{N}(0,1)}{\sqrt{\frac{\chi_n^2}{n}}},$$

where the $\mathcal{N}(0,1)$ and χ_n^2 are independent. Therefore t_n has the same distribution as

$$\frac{X}{\sqrt{\frac{1}{n}\sum_{i=1}^n Y_i^2}}$$

where $X \sim \mathcal{N}(0,1)$ is independent of $\frac{1}{n}\sum_{i=1}^n Y_i^2$, and each of these random variables is defned on the same probability space. Take, for example,

$$\begin{pmatrix} X \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} \sim \mathcal{N}(\mathbf{0}_{n+1}, I_{n+1})$$

for each n. We know by the SLLN that $\frac{1}{n}\sum_{i=1}^n Y_i^2 \xrightarrow{a.s.} \mathrm{E}(Y_i^2) = 1$. Since $X_n = X$ trivially converges in distribution to $\mathcal{N}(0,1)$, it follows by Slutsky's theorem that

$$\frac{X}{\sqrt{\frac{1}{n}\sum_{i=1}^n Y_i^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

b) Since $T_n \xrightarrow{d} \mathcal{N}(0,1)$ and $t_{n,1-\alpha} \to z_{1-\alpha}$, by Slutsky's Theorem, $T_n - t_{n,1-\alpha} \xrightarrow{d} \mathcal{N}(0,1) - z_{1-\alpha}$, so since 0 is a continuity point of the distribution of $\mathcal{N}(0,1)$,

$$\begin{aligned} \mathrm{P}(T_n - t_{n,1-\alpha} \geq 0) &\to \mathrm{P}(\mathcal{N}(0,1) - z_{1-\alpha} \geq 0) \\ &= 1 - \Phi(z_{1-\alpha}) = \alpha, \end{aligned}$$

as desired.

Q5: First, $P(C) = \frac{1}{3}$, since $1 = P(\Omega) = P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{3} + P(C) \implies$

$P(C) = \frac{1}{3}$. Moreover, $P(A \cup B) = \frac{2}{3}$, $P(A \cap B) = 0$, $P(A^c) = \frac{2}{3}$, $P(A^c \cup B^c) = P((A \cap B)^c) = P(\Omega) = 1$. Finally, because we found that $P(A \cap C) = 0 \neq \frac{1}{9} = P(A)P(C)$, it follows that A and C are not independent.

Q6: We wish to show that:

$$P\left(|\Gamma_n| > \varepsilon\right) = P\left(\left|\max_{n \le i \le 2n} X_i\right| > \varepsilon\right) \to 0.$$

Note that $|\max_{n \le i \le 2n} X_i| \le \max_{n \le i \le 2n} |X_i|$, which means that:

$$\begin{aligned}
P\left(\left|\max_{n \le i \le 2n} X_i\right| > \varepsilon\right) &\le P\left(\max_{n \le i \le 2n} |X_i| > \varepsilon\right) \\
(1) &= P\left(\bigcup_{i=n}^{2n} |X_i| > \varepsilon\right) \\
(2) &\le \sum_{i=n}^{2n} P\left(|X_i| > \varepsilon\right) \\
(3) &\le \sum_{i=n}^{2n} \frac{\mathrm{E}\left(X_i^2\right)}{\epsilon^2} \\
&= \frac{1}{\epsilon^2} \sum_{i=n}^{2n} 2^{-i} \\
&\le \frac{1}{\epsilon^2} \sum_{i=n}^{\infty} 2^{-i} = \frac{1}{2^{n-1}} \to 0
\end{aligned}$$

as $n \to \infty$, where (1) follows since $\{\max_{n \le i \le 2n} |X_i| > \varepsilon\}$ is equivalent to the the event that at least one of the $|X_i|$ is greater than ε; (2) follows by Problem Set 2 Q1b (also known as Boole's Inequality); and (3) follows from Chebyshev's inequality. Therefore, $P\left(|\max_{n \le i \le 2n} X_i| > \varepsilon\right) \to 0$ as $n \to \infty$, as required.

Q7: a) Since:

$$\mathbb{E}\left[\mathbf{1}_{\{X_i \le x\}}\right] = P\left(X_i \le x\right) < \infty,$$

and X_i's are iid, which, in turn, implies that $\mathbf{1}_{\{X_i \le x\}}$'s are iid, by WLLN, $\hat{F}_n(x) \xrightarrow{\mathrm{p}} P\left(X_i \le x\right) = F(x)$.

b) Since:

$$\begin{aligned}
\mathrm{Var}\left[\mathbf{1}_{\{X_i \le x\}}\right] &= \mathbb{E}\left[\left(\mathbf{1}_{\{X_i \le x\}}\right)^2\right] - \mathbb{E}\left[\mathbf{1}_{\{X_i \le x\}}\right]^2 \\
&= \mathbb{E}\left[\left(\mathbf{1}_{\{X_i \le x\}}\right)\right]\left(1 - \mathbb{E}\left[\mathbf{1}_{\{X_i \le x\}}\right]\right) \\
&= F(x)\left(1 - F(x)\right) < \infty,
\end{aligned}$$

and X_i's are iid, by CLT, $\sqrt{n}\left(\hat{F}_n(x) - F(x)\right) \xrightarrow{\text{d}} N(0, F(x)(1 - F(x)))$. Next, using the sample analogue principle, let:

$$\hat{\sigma}_n^2(x) = \hat{F}_n(x)\left(1 - \hat{F}_n(x)\right).$$

Since $\hat{F}_n(x) \xrightarrow{\text{p}} F(x)$, by Continuous Mapping Theorem, letting $g(a) = a(1-a)$, we find that $\hat{\sigma}_n^2(x) = g\left(\hat{F}_n(x)\right) \xrightarrow{\text{p}} g(F(x)) = \sigma^2(x)$.

Q8: a) Since $(X_i + X_{i+1})$'s are not iid, we cannot simply apply the WLLN. But, we can express $\hat{\mu}_n$ as:

$$\begin{aligned}
\hat{\mu}_n &= \frac{1}{n}\sum_{i=1}^{n-1} X_i + \frac{1}{n}\sum_{i=1}^{n-1} X_{i+1} \\
&= \left(\frac{1}{n}\left(-X_n + \sum_{i=1}^{n} X_i\right)\right) + \frac{1}{n}\left(-X_1 + \sum_{i=1}^{n} X_i\right) \\
&= 2\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \frac{1}{n}(X_n + X_1).
\end{aligned}$$

Since $\text{Var}[X_i]$ exists (which implies that $\mathbb{E}[|X_i|]$ exists), by WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{p}} \mathbb{E}[X_i]$. By Chebyshev's Inequality, we know that, for any $\varepsilon > 0$,

$$\begin{aligned}
P\left(\left|\frac{1}{n}(X_n + X_1)\right| > \varepsilon\right) &= P(|X_n + X_1| > n\varepsilon) \\
&\leq \frac{\mathbb{E}\left[|X_n + X_1|^2\right]}{n^2\varepsilon^2} \\
&= \frac{2\text{Var}[X_i]}{n^2\varepsilon^2},
\end{aligned}$$

where the last equality follows since $X_n \perp X_1$ and $\mathbb{E}[X_i] = 0$. Moreover, $\text{Var}[X_i] < \infty$ means that the right-hand side tends to zero as $n \to \infty$, so that $\frac{1}{n}(X_n + X_1) \xrightarrow{\text{p}} 0$. Since marginal convergence in probabilities imply joint convergence in probabilities, and by CMT, $\hat{\mu}_n \xrightarrow{\text{p}} 2\mathbb{E}[X_i] - 0 = 0$.

b) From the previous part, we know that:

$$\begin{aligned}
\hat{\mu}_n &= 2\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \frac{1}{n}(X_n + X_1) \\
\Rightarrow \sqrt{n}\hat{\mu}_n &= 2\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \frac{1}{\sqrt{n}}(X_n + X_1).
\end{aligned}$$

Since X_i's are iid and $\text{Var}[X_i] < \infty$, by CLT,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X_i]\right) \xrightarrow{\text{d}} N(0, \text{Var}[X_i]).$$

By Chebyshev's Inequality, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\left(X_n + X_1\right)\right| > \varepsilon\right) &= \mathbb{P}\left(\left|X_n + X_1\right| > \sqrt{n}\varepsilon\right) \\
&\leq \frac{\mathbb{E}\left[\left|X_n + X_1\right|^2\right]}{n\varepsilon^2} \\
&= \frac{2\mathrm{Var}\left[X_i\right]}{n\varepsilon^2} \to 0.
\end{aligned}$$

Hence, by Slutsky's Theorem,

$$\sqrt{n}\hat{\mu}_n \xrightarrow{\mathrm{d}} 2N\left(0, \mathrm{Var}\left[X_i\right]\right) + 0 \overset{\mathrm{d}}{=} N\left(0, 4\mathrm{Var}\left[X_i\right]\right).$$