Problem 2.2

Olivier Kooi

University of Chicago

October 2021

Grading

- Point distribution
  1. 1 point for questions a,b,c
  2. 2 points for questions d,e,f
  3. 1 last point for clarity/quality above expectation
- Common point deductions/grading decisions
  1. Confuse ad-valorem and excise taxes (allocated by 0.5 point reduced in q1)
  2. No mention concavity cost function
  3. No derivation for any property of DWL/missing relevant properties
  4. No information on $\frac{dp_f}{dt_x}$ other than a sign (you get points for some insight into magnitudes)
  5. Missing part of the DWL square/ interpret square in demand for f as tax revenue
  6. Vague reference convexity DWL function to argue taxes should differ (given 0.5 points)
  7. People often asserted rather than showed certain conditions in f
  8. Very few people provided clear derivation in f. Usually no points given for unclear derivation

Next Slide

Overview

- Here we consider a three-good model, with quantities consumed c, x, and y. Preferences are described by the utility function $u(c, F(x, y))$, where F is a homogenous function. An excise tax is levied on good x at rate t.
- Let us start with some notation:
  1. Prices of c, x, y are fixed and denoted p_c, p_x, p_y
  2. Demand function are denoted X_i
  3. Expenditure function: $e(p_c + t, p_x, p_y, u)$

Next Slide

Question a

Question: How does the expenditure function vary with t?

- Write $q_c = p_c + t$ and note $\frac{\partial e}{\partial q_c} = \frac{\partial e}{\partial t}$
- We can get the properties of the expenditure function from the textbook:
    1. Homogenous of degree 1 in prices
    2. Increasing in t
    3. $\frac{\partial e}{\partial t} = X_c^h$
    4. Concave in t

Next Slide

Question b

Question: We define deadweight loss as the difference between (i) the effect of t on cost and (ii) the tax revenue obtained by taxing good x. Describe some properties of the deadweight loss function.

- Defition DWL:

$$DWL(t) \equiv e(p_c + t, p_x, p_y, u) - e(p_c, p_x, p_y, u) - tX_c^h$$

- Take a second order taylor expansion

$$MDWL(t) = \frac{dDWL}{dt}\Delta t + \frac{1}{2}\frac{d^2DWL}{dt}(\Delta t)^2$$

Next Slide

Question b

- First order term (applying envelope results from book):

$$\frac{dDWL}{dt} = X_c^h + t\frac{dX_c^h}{dt} - X_c^h = -t\frac{dX_c^h}{dt}$$

- Second order term:

$$\frac{d^2DWL}{dt^2} = -\frac{dX_c^h}{dt} - t\frac{d^2X_c^h}{dt^2}$$

- Putting things together: generalized Harberger triangle

$$MDWL(t) = -t\frac{dX_c^h}{dt}(\Delta t) - \frac{1}{2}(\frac{dX_c^h}{dt} + t\frac{d^2X_c^h}{dt^2})(\Delta t)^2$$

Next Slide

Question b

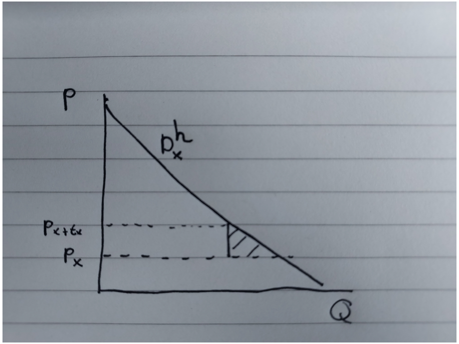
- The two equations
  1. $DWL(t) \equiv e(p_c + t, p_x, p_y, u) - e(p_c, p_x, p_y, u) - tX_c$
  2. $MDWL(t) = -t\frac{dX_c^h}{dt}(\Delta t) - \frac{1}{2}(\frac{dX_c^h}{dt} + t\frac{d^2X_c^h}{dt^2})(\Delta t)^2$
- Note the following
  1. DWL(0) $= 0$
  2. $MDWL(0) = -\frac{1}{2}\frac{dX_c^h}{dt}(\Delta t)^2$
  3. MDWL typically increasing in t

Next Slide

Question c

Question: c) Use the demand-for-x curve to illustrate the deadweight loss. Are you using Hicksian or Marshallian demand?

- One should be using the Hicksian
  1. Why not include income effects in behavior response (why not Marshallian)?
  2. Not a distortion, not a sign of failure to conduct mutually beneficial transaction



Next Slide

Question d

How does t affect the price of F? Illustrate the deadweight loss using the demand-for-F curve.

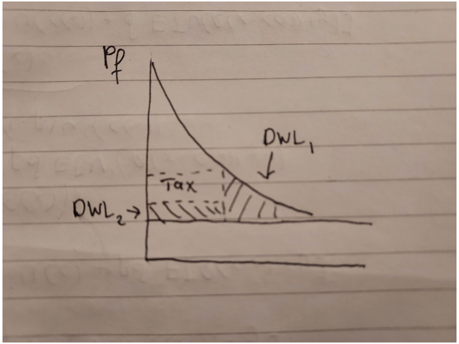
- Think of a new commodity: d produced through $F(x, y)$
- Consider minimization problem: min $p_x X_x + p_y X_y$ s.t $F(x, y) = \bar{d}$
- Resulting cost function: $C(p_x + t, p_y, \bar{d})$
- By homotheticity: $G(d)C(p_x + t, p_y, 1)$
- Change in marg cost: $\frac{dMC}{dt} = G'(d)\frac{\partial C}{\partial t} = G'(d)X_x(p_x + t, p_y, 1)$
- Writing as an elasticity $\epsilon_{p_c}^{MC} = \frac{p_c X_c}{C(p_x+t, p_y, 1)} = s_x$
- Notice that in "industry equilibrium" $p_d = MC$

Next Slide

Question d

The price for f is pushed up, reflecting both tax revenue and distortions between x,y

- Two types of DWL (clarification from question in class)
  1. One between cf, this is reflected by the triangle DWL_1
  2. One between x,y reflected by the rectangle DWL_2



Next Slide

Question e - Intuitive answer

- Efficient division of consumption between x, y determined by
$$\frac{F_x(\frac{x}{y},1)}{F_u(\frac{x}{y},1)} = \frac{p_x}{p_y}$$
- Homogeneity implies absense of Corrlet-Hague reasons to distorts bundle (no heterogeneity in cross-elasticities)
- So optimal consumption taxes on the same ray
$$\frac{F_x(\frac{x}{y},1)}{F_u(\frac{x}{y},1)} = \frac{p_x}{p_y} = \frac{p_x + t_x}{p_y + t_y}$$
- Little bit of algebra yields this expression
$$\frac{t_x}{p_x + t_x} = \frac{t_y}{p_y + t_y}$$
- Tax both but unless $p_x = p_y$ not at same rate

Next Slide

Question e - mathy answer I

item e) Use your results for (c) and (d) to show how the same tax revenue could be raised more efficiently using also levying an excise tax on good y. Should x and y be taxed at the same rate?

- Consumer strictly better off with lower p_d. So lets minimize p_d subject to enough revenue

$$L = C(p_c + t_c, p_d + t_d, 1) - \lambda \sum_{c,d} X_i^h t_i$$

- The first order condition yields

$$C_i = \lambda(X_i^h + \sum_{c,d} X_i^h \frac{\partial X_j^h}{\partial t_i}) = X_i^h$$

Next Slide

Question e - mathy answer II

- Note that $\frac{\partial X_i^h}{\partial t_j} = S_{ij}$, where S denotes Hicksian price derivates
- Denoting compensated elasticties by $\epsilon^{ij} = \frac{p_j + t_j}{X_i} S_{ij}$
- Expanding the equation yields $\frac{1-\lambda}{\lambda} X_i = t_i S_{ii} + t_j S_{ij}$
- Combining these we obtain

$$\frac{t_x}{p_x + t_x} = \frac{t_y}{p_y + t_y} \frac{\epsilon_{xy} - \epsilon_{yy}}{\epsilon_{yx} - \epsilon_{xx}}$$

- Sum of cross price elasticities: $\sum_{j=c,x,y} (p_i + t_i) S_{ij} = 0$
- Implies $\epsilon_{cx} + \epsilon_{cc} + \epsilon_{cd} = 0$
- Also: $\epsilon_{yc} + \epsilon_{yx} + \epsilon_{yy} = 0$

Next Slide

Question e - mathy answer III

- Combining all of the above to obtain

$$\frac{t_x}{p_x + t_x} = \frac{t_y}{p_y + t_y} \frac{-(\epsilon_{xx} + \epsilon_{yy}) - \epsilon_{cx}}{-(\epsilon_{xx} + \epsilon_{yy}) - \epsilon_{cy}}$$

- Homogeneity of aggregator F implies $\epsilon_{cx} = \epsilon_{cy}$

$$\frac{t_x}{p_x + t_x} = \frac{t_y}{p_y + t_y}$$

- Hence it is efficient to tax both x, y but not at the same excise tax rate

Next Slide

Question f - setup

- Note that the problem has this recursive structure
- Think of it as two related two by two demand systems
- Lets denote again $d = F(x, y)$ and think of demand systems over c, d and x, y
- Note that homogeneity implies that $\frac{x^m(p,M)}{y^m(p,M)} = c$ for any M
- Rewriting this in terms of elasticities implies $\eta_x = \eta_y$

Next Slide

Question f - demand system for x,y

- Denote budget shares for subproblem by s_i^d
- Let us assume we know s_x^d and ϵ_{xx}
- Recover s_y from adding up: $s_y^d = 1 - s_x^d$
- Recover η_i^d from Engel aggregation and $\eta_x^d = \eta_y^d$
- From here on you can follow page 43 from book
  - By homogeneity: $\epsilon_{xy}^M = -\epsilon_{xx}^M - \eta_x$
  - By symmetry: $\epsilon_{yx}^M = \frac{s_x^d \epsilon_{xy}^M}{s_y}$
  - By homogeneity: $\epsilon_{yy}^M = -\epsilon_{yx}^M - \eta_y$

Next Slide

Question f - demand system for c,d

- Denote budget shares for subproblem by s_i
- Let us assume we know s_c, ϵ_{cc} and η_{cc}
- Recover s_d from adding up: $s_d = 1 - s_c$
- By Engel aggregation: $\eta_2 = \frac{1 - s_c \eta_c}{s_d}$
- From here on you can follow page 43 from book
  - By homogeneity: $\epsilon^M_{cd} = -\epsilon^M_{cc} - \eta_c$
  - By symmetry: $\epsilon^M_{dc} = \frac{s^d_c \epsilon^M_{cd}}{s_d}$
  - By homogeneity: $\epsilon^M_{dd} = -\epsilon^M_{dc} - \eta_d$

Next Slide

Question f - demand system for c,x,y

- Can we recover the original demand system from parameters we have so far?
- Use $s_c + s_d = s_c + s_d(s_x^d + s_y^d) = s_c + s_x + s_y$
- By engel Aggregation: $s_c\eta_c + \eta_x(s_x + s_y) = 1$ to recover $\eta_x = \eta_y$
- By Engel aggregation: $\eta_d = \frac{1-s_c\eta_c}{s_d}$
- Think about the restriction on cross elasticities:
  - $\frac{dc}{dp_x} = \frac{\partial c}{\partial p_d}\frac{\partial p_d}{\partial p_c}$
  - Multiply by $\frac{p_x}{c}$ and $\frac{p_d}{p_d}$
  - Obtain $\epsilon_{cx} = \epsilon_{cd}\epsilon_{p_x}^{p_d} = \epsilon_{cd}s_x^d$
  - Same thing for ϵ_{cy}

Next Slide

Question f - demand system for c,x,y

- We recovered $s_c, s_x, s_y,\ \eta_c, \eta_x, \eta_y,\ \epsilon_{cc}, \epsilon_{cx}, \epsilon_{cy}$
- Use symmetry $s_i \epsilon_{ij}^M = s_j \epsilon_{ji}^M + s_i s_j (\eta_j - \eta_i)$
- Recover ϵ_{xc} and ϵ_{yc}
- Combine adding up, homogeneity and symmetry to get the last four elasticities
  - Obtain ϵ_{xx} from $\epsilon_{xc} + \epsilon_{xx} + \epsilon_{xy} = 1$ and $s_x \epsilon_{xy} + s_y \epsilon_{yy} + s_c \epsilon_{cy} = -s_y$
  - Obtain ϵ_{xy} from homogeneity using ϵ_{xx}
  - Obtain ϵ_{yx} from symmetry using ϵ_{xy}
  - Obtain ϵ_{yy} from homogeneity using ϵ_{yx}
- Note that we started with 5 parameters. We reduced the degrees of freedom by 2 from the usual 7.
- Some people brought in additional information on prices and the ratio $\frac{x}{y}$. You obtain 3 and this is correct also

Next Slide