ECMA31000: Introduction to Empirical Analysis
Exam 1

Tuesday, October 26, 2021

INSTRUCTIONS: This is an 80 minute exam. There are 4 questions and a total of 80 points. There is no choice, so answer all questions. You may use any results stated/proven in class without re-proving them, but you must justify all your answers. Write your answers to Questions 1+2 in one blue book and to Questions 3+4 in the other. Good Luck!

Question 1 (16 points)

a) (5 points) Find a sequence of random variables $\{X_n\}_{n\geq 1}$ and a random variable X such that $\mathrm{E}\left([X_n - X]^2\right) \to 0$ as $n \to \infty$ but $\mathrm{E}\left([X_n - X]^4\right) \not\to 0$ as $n \to \infty$.

b) (6 points) Suppose $X_n \xrightarrow{p} X$ and $Y_n - X_n \xrightarrow{d} 0$. Prove that $Y_n \xrightarrow{p} X$.

c) (5 points) Suppose $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{d} Y$. Prove that $\mathrm{P}(X = Y) = 1$ or find a counterexample.

Question 2 (24 points) Our friend from PSET 1 is throwing darts at a dartboard again. The area of the dartboard is 1, and the location of each of their throws is <u>uniformly distributed</u> across the board. <u>Each throw is independent of all others</u>. For each n, let A_n be a subset of the board and define

$$X_n = \begin{cases} 1 & \text{if dart } n \text{ lands in } A_n, \\ 0 & \text{if dart } n \text{ lands outside } A_n. \end{cases}$$

a) (6 points) Explain in non-technical terms what it means for $X_n \xrightarrow{p} 0$ and what it means for $X_n \xrightarrow{a.s.} 0$ in the context of this game. It may help you to write the mathematical descriptions first.

In parts **b)** and **c)**, assume A_n has area $\frac{1}{n}$ for each n.

b) (4 points) Prove that $X_n \xrightarrow{p} 0$.

c) (6 points) Prove that

$$\mathrm{P}\left(\cap_{k=n}^{\infty} \{X_k = 0\}\right) = 0$$

for all n. Use your answer to part a) to conclude that $X_n \overset{a.s.}{\not\to} 0$.

Note: You may use without proof the fact that if $B_n, B_{n+1}, B_{n+2}, \ldots$ are independent events, then

$$\mathrm{P}\left(\cap_{k=n}^{\infty} B_k\right) = \prod_{k=n}^{\infty} \mathrm{P}\left(B_k\right).$$

d) (8 points) Now suppose only that A_n has been chosen to satisfy $X_n \xrightarrow{p} 0$ (but not necessarily $X_n \xrightarrow{a.s.} 0$). Define

$$Y_n = \begin{cases} 1 & \text{if } X_k = 1 \text{ for all } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $Y_n \xrightarrow{a.s.} 0$. Hint: Argue that $\{Y_n = 1\} = \cup_{k=n}^{\infty} \{Y_k = 1\}$.

Question 3 (15 points)

a) (5 points) Suppose $X_n = o_p(n)$ and $Y_n = O_p(n^{-1})$. Prove that $X_n Y_n = o_p(1)$ is true or provide a counterexample.

b) (10 points) Suppose that $X_n = O_p(a_n)$ and $Y_n = O_p(b_n)$ for some sequences of strictly positive numbers $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$. Show that $X_n + Y_n = O_p(\max\{a_n, b_n\})$.

Question 4 (25 points) Suppose we have an iid sample $\{X_i\}_{i=1}^n$ drawn from a distribution represented by the pdf

$$f_\lambda(x) = \begin{cases} (\lambda + 1)\, x^\lambda & 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\lambda > -1$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

a) (5 points) Find $\mathrm{E}(X)$ and show that

$$\hat{\lambda}_n = \frac{1 - 2\bar{X}_n}{\bar{X}_n - 1}$$

is a method of moments estimator of λ.

b) (5 points) Show that $\hat{\lambda}_n$ is a strongly consistent estimator of λ.

c) (10 points) Find constants $r \geq 0$ and c such that $n^r\left(\hat{\lambda}_n - c\right) \xrightarrow{d} Y$ for some non-degenerate random variable Y.

Note: If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - h'(x)g(x)}{h(x)^2}$.

d) (5 points) Use your answer to part c) to find a function f such that

$$\frac{n^r\left(\hat{\lambda}_n - c\right)}{f\left(\hat{\lambda}_n\right)} \xrightarrow{d} \mathcal{N}(0, 1),$$

and prove this convergence.