Game Theory Cheat-Sheet

Basics of Game Theory

Common Proof Techniques

Want to show that $A \implies B$.
Contradiction Suppose C is a known false statement

$$\neg(A \implies B) \implies C$$

Contraposition

$$\neg A \implies \neg B$$

Exhaustion Suppose $A = \{A_1, A_2..., A_n\}$

$$A_1 \implies B$$
$$A_2 \implies B$$
$$...$$
$$A_n \implies B$$

What is a game?

Game has 3 items:

- Players
- Actions for each player
- Preferences for each player over everyone's actions

Properties of preferences

A preference relation is a method of ranking alternatives in a set X. There are 3 kinds of relations:

- Preference: $\succeq$
- Indifference: $x \sim y$ if $x \succeq y$ & $y \succeq x$
- Strict Preference $x \succ y$ if $\neg(y \succeq x)$

A relation, R, can have the following properties:
Completeness

$$\text{if } \forall x, y \in X: \text{ Either } (xRy), (yRx), \text{ or both}$$

Transitive

$$\text{if } \forall x, y, z \in X:$$
$$(xRy) \ \& \ (yRz) \implies (xRz)$$

Rational

If **Complete** & **Transitive**

Reflexive

$$xRx \ \forall \ x \in X$$

Asymmetric

$$xRy \implies \neg(yRx)$$

Anti-symmetric

$$xRy \ \& \ yRx \implies x = y$$

Negative Transitive

$$\text{if } \forall x, y, z \in X:$$
$$\neg(xRy) \ \& \ \neg(yRz) \implies \neg(xRz)$$

Total

$$xRy \text{ or } yRx \text{ whenever } x \neq y$$

Implications of Rationality

If a preference relation $\succeq$ is rational, then :

- The indifference relation $\sim$ is reflexive and symmetric
- The strict preference relation $\succ$ is asymmetric and negative transitive
- If $\succeq$ is complete and anti-symmetric then $\succ$ is total

Propositions about utility and preferences
Proposition: If $\succeq$ is represented by U then $\succeq$ is rational.
Note: The converse does not hold in general.

Rational implies U counterexample

Let $X = [0, 1]$ and we have lexicographic preferences, i.e. $(x, y) \succeq (x', y')$ if $x \geq x'$ or $x = x'$ and $y \geq y'$. By contradiction, a utility function existing would imply that the cardinality of $[0, 1]$ is weakly less than the rationals.

Useful Math Definitions

Surjective: A function $f : A \to B$ is surjective if $\forall b \in B \exists a \in A$ s.t. $f(a) = b$
Injective: A function $f : A \to B$ is injective if $\forall a \in A$, $f(a) \neq f(a')$ whenever $a \neq a'$.
Countable: A set X is countable if $\exists$ a surjective function f: $\mathbb{N} \to X$

Countability, Rationality, and Utility Functions

If a set X is countable, and $\succsim$ is a rational preference relation on X, then there exists some utility function u which represents $\succsim$.

Separability and Utility Functions

If (X, d) is a separable metric space (equipped with the metric d), $\succsim$ is rational, and for all $x \in X$ the set $\{y \in X \mid x \succ y\}$ is open, then there exists some utility function which represents $\succsim$.

Lotteries and Expected Utility

Lottery Definition

A lottery has the following elements:

- Finite Set of Outcomes Ω
- Lotteries: $x : \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} x(\omega) = 1$.
- X=set of all lotteries, or $\Delta(\Omega)$

Expected Utility Functions & Lotteries

An EUF is defined as:

$$\sum_{\omega} x(\omega) y(\omega)$$

Lottery Mixtures

Given $x, y \in X$, $\alpha \in [0, 1]$ the lottery mixture $\alpha_x + (1 - \alpha)y$ is defined by $\alpha x(\omega) + (1 - \alpha)y(\omega)$

Independence Axiom

Given $x, y, z \in X$ and $\alpha \in (0, 1]$, $x \succeq y$ iff $\alpha x + (1 - \alpha)z \succeq \alpha y + (1 - \alpha)z$.

This implies:

$$x \succ y \ \& \ \alpha \in (0, 1] \implies \tag{1}$$
$$\alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z \tag{2}$$

Continuity

$\succeq$ is **continuous** if:

$$\forall x, y, z \ s.t. \ x \succeq y \succeq z, \exists \alpha \in [0, 1] s.t. \tag{3}$$
$$\alpha x + (1 - \alpha)z \sim y \tag{4}$$

Expected Utility Theorem

$\succeq$ on X has an expected utility representation iff $\succeq$ is rational, satisfies independence, and is continuous.

Positive Affine Transformations

Given EUF's u and v, v is a positive affine transformation of u if $\exists a > 0, b$ s.t. $v(x) = a * u(x) + b$.
EUF Representation Theorem: Suppose $\succeq$ is a preference relation on x and u is an EUF that represents it. Then v is also an EUF that represents it iff v is a PAT of u.

Normal Form Games

Definitions

Normal form games take on the same definition introduced initially:

$$G = (N, A, \succeq)$$

Where players are denoted by N, actions are denoted of A, and preferences are rational on A for every player in N.

Solution Concepts and Nash Equilibrium

The best response correspondance for player i is given by:

$$B_i : A_{-i} \to 2^{A_i}$$
$$B_i : a_{-i} \to \{a_i | a_i \text{ is a BR to } a_{-i}\}$$

A Nash equilibrium of G is an $a \in A$ s.t. $a_i \in B_i(a_{-i}) \forall i$.

Mixed Strategy Games

A game with mixed strategies is a tuple

$$\left(N, \{A_i\}_{i\in N}, \{\succsim\}_{i\in N}\right),$$

where player i's actions are mixed strategies in $\Delta(A_i)$ (lotteries over A_i) and $\succsim_i$ is defined over $\Delta(A_i)$ and are expected utility preferences.
For a game $G = (N, A, \succsim)$, the game with mixed strategies

$$\tilde{G} = \left(N, \tilde{A}, \tilde{\succsim}\right)$$

is a mixed extension if $\tilde{A}_i = A_i$ and if $\succsim$ and $\tilde{\succsim}$ give the same ranking on pure actions.
Best Responses: Definition A mixed strategy profile α_i is a best response to α_{-i} if

$$u_i(\alpha_i, \alpha_{-i}) \geq u_i(\alpha_i', \alpha_{-i}), \ \forall \alpha_i'.$$

Best Responses: Proposition Fix a game with mixed strategies and a mixed strategy profile α. Then, α_i is a best response to α_{-i} if and only if a_i is a best response to α_{-i} for all a_i such that $\alpha(a_i) > 0$.

Zero Sum Games & Maxmin

Max-min Soln. Definition

Given a game, the maxmin solution for player i is the set of solutions to

$$\max_{a_i\in A_i} \min_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i}).$$

TPZSG And Maxmin

Fix a two-player zero-sum game (A_1, A_2, u_1). Suppose a maxmin solution exists for both players, with payoffs v_1^* and v_2^*. Then,

$$v_1^* \leq -v_2^*.$$

Duality Gap Definition

The **duality gap** of a two-player zero-sum game is $v_1^* + v_2^*$ (≤ 0).

Duality Gap and Nash Equilibria

The duality gap is zero if and only if a Nash equilibrium exists. If the duality gap is zero, then the set of Nash equilibria are precisely the maxmin solutions for each player.

Minimax

If G is a two-player zero-sum game with finite pure actions and mixed strategies, then the game has a value; i.e., the maxmin solution exists and the duality gap is zero.

TPZS, values, and Nash equilibria

Every symmetric, two-player zero-sum game with mixed strategies has a value; i.e. it has a Nash equilibrium and $v_1^* = -v_2^*$.

Farkas' Lemma

Given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, exactly one of the following holds:

(i) $\exists x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

(ii) $\exists y \in \mathbb{R}^m$ such that $yA \geq 0$ and $yb < 0$

Farkas' Lemma: Variants

One variant is as follows: given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, exactly one of the following holds:

(i) $\exists x \in \mathbb{R}^n$ such that $Ax \leq b$

(ii) $\exists y \in \mathbb{R}^m$ such that $yA = 0$, $y \geq 0$, and $yb < 0$

Finally, an equivalent statement known as Gordan's Lemma gives the following two mutually exclusive possibilities for A and b:

(i) $\exists x \in \mathbb{R}^n$ such that $Ax < 0$

(ii) $\exists y \in \mathbb{R}^m$ such that $yA = 0$ with $y \geq 0$ and $y_i > 0$ for some i.

Rationalizability

Strict Dominance: Definition

Strictly Dominated Given a game w/ mixed strategies, $\alpha_i \in \Delta(A_i)$ is strictly dominated if $\exists \alpha_i' \in \Delta(A_i)$ s.t $\forall a_{-i}$, $U_i(\alpha_i', a_{-i}) > U_i(\alpha_i, a_{-i})$. Note: This is the same as if we said $\forall \alpha_{-i}$. **Strictly Dominant** An action α_i is strictly dominant if $\forall \alpha_i' \neq \alpha_i$, α_i' is strictly dominated by α_i

Never-Best Responses: Definition

A never best response is an action $\alpha_i \in \Delta(A_i)$ such that there does not exist a mixed strategy $\alpha_j \in \Delta(A_i)$ such that $a_i \in B_i(\alpha)$.
A set-based definition is as follows:
Given a product set of actions $\tilde{A} = \Pi_{i=1}^N \tilde{A}_i$, $\tilde{A}_i \subseteq A_i$,
$N_i(\tilde{A}) = \{a_i \in A_i | \neg\exists\mu \in \Delta(\tilde{A}_{-i}) \text{ s.t. } a_i \in B_i(\mu)\}$

NBR and SD Correspondence Proposition

The key proposition for Strict Dominance and NBR is their direct correspondence.
Proposition:
α_i is strictly dominated IFF α_i is not a best reply for any $\mu \in \Delta(A_{-i})$ or $\alpha_i \notin B_i(\mu)\forall\mu$.
NOTE: This definition includes correlated equilibrium. a_{-i} can be correlated but a_i is independent of a_{-i}.

Deletion Sequence Definition

A deletion sequence is a sequence of product sets of action profiles $\{A^k\}_{k=0}^K$, with K either finite or infinite, such that

1. $A_i^0 = A_i$ (the set of all actions),
2. $A^{k+1} \subsetneq A^k$ for all $k < K$
3. $A_i^k - A_i^{k+1} \subseteq N_i(A^k)$ for all i and all $k < K$
4. $N_i(A^\infty) \cap A_i = \emptyset$, where $A_i^\infty = \bigcap_{k\geq 0} A_i^k$

Greedy Deletion Sequence

The greedy deletion sequence is $A_i^0 = A_i$, and
$A_i^k = A_i^{k-1} - N_i(A_i^{k-1})$.

Rationalizability Theorem

Suppose $G = (N, A, u)$ is a finite game with mixed strategies, then $\exists$ a prod set $A^* \neq \emptyset$ s.t. for every deletion sequence $\{A^k\}$, $A^* = \bigcap_{k\geq 0} A^k$

Hammer Lemma

For any finite game with mixed strategy, player i's best response correspondence is nonempty for any strategy profile $\mu \in \Delta(A_{-i})$; that is, $B_i(\mu) \neq \emptyset$.

Smiley Lemma

$N_i(\tilde{A})$ is decreasing in $\tilde{A}$, ie, if $\tilde{A} \subseteq \tilde{A}'$ then $N_i(\tilde{A}') \subseteq N_i(\tilde{A})$

Rainbow Lemma

Every seq $\{A^k\}$ satisfying 1-3 definitions of a deletion sequence has finite length.

Star Lemma

The limit of any deletion sequence is non-empty.

Best Reply Property

A product set has the best reply property if $\forall i$ and $a_i \in \tilde{A}_i$, $\exists\mu \in \Delta(\tilde{A}_{-i})$ s.t. $a_i \in B_i(\mu)$.
An equivalent definition is as follows:
$N_i(\tilde{A}) \cap \tilde{A}_i = \emptyset \forall i$.
Corollary:
If $\tilde{A}$ then that set has the best reply property.

Swirl Lemma

If a product set $\tilde{A}$ has the best response property, then $\tilde{A} \subseteq \tilde{A}'$ for any $\tilde{A}'$ that survives iterated deletion.

Correlated Equilibrium (CE)

Correlation Device Definition

A correlation device is a collection $D = (\{\Omega_i\}_{i\in N}, \pi)$ where $\Omega = \prod_{i\in N} \Omega_i$ and $\pi \in \Delta(\Omega)$. IE: Ω_i is a finite set of signaled actions that player i observes and π is the joint distribution of signals.

Augmented Game Definition

Given a game $G = (N, A, u)$ w/ exp utility prefs and a correlation device D, the augmented game (G, D) is a game in which player i's actions are behavioral strategies

$$\sigma_i : \Omega_i \to \Delta(A_i)$$

We write $\sigma_i(a_1|w_i)$ for prob of a_i given ω_i, and $\sigma(a \mid \omega) = \prod_{i\in N} \sigma_i(a_i \mid \omega_i)$ for $a \in A_1$, $\omega \in \Omega$.
Player i's payoff from a profile σ is

$$U_i(\sigma) = \sum_{\omega\in\Omega}\sum_{a\in A} u_i(a)\sigma(a \mid \omega)\pi(\omega)$$

Correlation Device Induction

Fix a game $\sigma = (N, A, u)$ and $D = (\Omega, \pi)$ and strategies σ. Then (D, σ) induce a distribution $\mu \in \Delta(A)$ a according to

$$\mu(a) = \sum_{\omega\in\Omega} \pi(\omega)\sigma(a/\omega)$$

CE Definition

A correlated equilibrium is a μ such that $\forall i, a_i, a_i'$

$$\sum_{a_{-i}} \mu(a_i, a_{-i})(u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})) \geq 0$$

IE: The set of signals is such that a player will never benefit from deviating from "obedient" play to the signal.

CE Induction Theorem

A distribution $\mu \in \Delta(A)$ is induced by some (D, σ) s.t. σ is a Nash Equilibrium of (G, D) IFF μ is a correlated equilibrium.

CE Existence Theorem

Fix a finite game w/ exp utility preferences $G(N, A, u)$ then $\exists \mu \in \Delta(A)$ such that μ is a CE.

CE Loss Lemma

For any Fixed σ $\exists \mu \in \Delta(\{1, \ldots, m\})$ such that $\forall u_i : \{1, \ldots, m\} \to \mathbb{R}$,

$$L(\mu, u) = \sum_{j,k} \mu(j)\sigma(k \mid j)(u(j) - u(k)) \geq 0$$

Nash's Theorem and Lemke-Howson Algorithm

Nash's Theorem

Every finite game with mixed strategies has a Nash Equilibrium

Non-Degenerate Symmetric 2 Player Game Definition

Define $G = (\{1, 2\}, A, u)$ is **non-degenerate** if $\forall i$ and mixed strategy α_i the number of **pure** best responses to α_i for player j is at most $|\{a_i \mid \alpha_i(a_i) > 0\}|$. Typically m denotes this quantity of pure best responses, and $A_i = \{1, ..., m\}$ denotes each best response as a numbering.

Slack Variables Definitions and Notation

Let $U \in \mathbb{R}^{mxm}$ where $U_{ij} = u_1(i, j)$. IE U_{ij} is the payoff matrix for player 1's ith action as a response to player 2 playing j.
Thus we can represent a mixed strategy as $x \in \mathbb{R}^m$ with traditional probability properties.
Let $U_x = (u(a_1, x), ..., u(a_j, x))$ and $v = max_j(U_x)_j$ be the best response payoff to j. Then x is a symmetric NE if $x_i > 0 \implies U_{x_i} = v$. Thus we can define

$$y = x/v \qquad \text{strategy that divides x by BR payout}$$
$$z = \vec{1} - U_y \qquad \text{ratio of payoff gap between x and BR}$$

Now let M be $[UI]$ thus $[UI](y, z) = \vec{1}$. Using this we can define:

$$F = \{(y, z) \mid y \geq 0, z \geq 0, Mf = 1\}$$

and $f = (y, z) \in F$.

Slack Variables Mixed Strategy Proposition

For any $f = (y, z) \in F$, $y_i > 0$ iff i is played w/ positive probability in the corresponding mixed strategy: $x = \frac{y}{\sum y_i}$ where $z_i = 1 - (U_y)_i = 0$ iff i is a best response.

Slack Variable Definition of Nash Equilibrium

$(y, z) \in F$ is an NE if $y_i > 0 \implies z_i = 0$ and $y_i \neq 0$.

Best Response Support Definition

$f = (y, z)$ is supported on best responses if $\forall i$ $y_i z_i = 0$. IE: Either i is not played or its a BR.

Best Response Support NE Lemma

$f = (y, z) \in F$ is an NE iff $y \neq 0$ and (y, z) is supported on BR.

I-Almost Supported Definition and Notation

f is i-almost supported on best responses if $y_j z_j = 0$ $\forall j \neq i$. Note:

$$F^* = \{f \in F \textbf{ supp on BR}\}$$
$$F_i = \{f \in F \textbf{ i-almost supp on BR}\}$$
$$F^* \subseteq F_i \forall i$$

Basis Definition and notation

Given $B \subseteq \{1, ..., 2m\}$ then
$M_B =$ submatrix of M w/ columns in B.
Given $f \in F$ $f_B =$ sub vector of f w/ indices in B.
We call B a **Basis** if $|B| = m$

Basis Hammer Lemma

If G is non-degenerate, let B be a basis and $f \in F$ with $f_i = 0$ $\forall i \notin B$ Then:

- Number of actions played w/ positive probability under f is equal to the number of BR.
  $f = (y, z) |\{i|y_i > 0\}| = |\{i|z_i = 0\}|$
- $f_i > 0 \forall i \in B$
- M_B is invertible and $f_B = (M_B)^{-1} * \vec{1}$

Basic

We say $f \in F$ is **Basic** if $|B(f)| = m$

Basis Ankh Lemma

Suppose $f = (y, z) \in F_i$ is basic and not supported on BR, then $y_i > 0, z_i > 0$ and $\forall j \neq i$ $y_j z_j = 0$ Moreover, $\exists$ exactly one $j \neq i$ with $y_j = z_j = 0$

Pivot Operation Definition

Step 1: Start w/ basic point f. Pick $i \notin B(f) = B$ and make f_i positive. Define a new solution $f(t)$ where $f_i(g) = t$. Now let t^* be the smallest t such that:

$$f_j(t) = 0 = (M_B^{-1}(1 - M_i t))$$
$$t^* = (M_B^{-1} * 1)_j - (M_B^{-1} M_i)_j t$$
$$t_j = \frac{(M_B^{-1} * 1)_j}{M_B^{-1} M_i}$$

New point $f' = f(t^*)$ where this operation is called pivoting from f on i which leads to f'.

Reversible Pivot Lemma

Suppose $f = (y, z) \in F$ is basic then pivoting from f on $i \notin B(f)$ leads to a basic point f' s.t. $|B(f) \cap B(f')| = m - 1$. Moreover, pivoting from f' on $j \in B(f)$ $B(f')$ leads to f.

I-Adjacency Definition

Two Basic points, f, f' are $i - adjacent$ if $\exists$ a pivot from f that leads to f' and vice versa such that the path from $f(t)$ from f to f' is in F_i

I-Adjacency Lemma

Suppose $f \in F_i$ is basic. If f is supported on best responses, $\exists$ exactly one $f' \in F_i$, $f' \neq f$ s.t. f' is i-adjacent to f.
If f is not supported on BR's then there are exactly two f', f'' $f' \neq f'' \neq f$ that are i-adjacent to f.

I-Path Definition Lemma

Let $f \in F_i$ be basic and supported on best responses. $\exists!$ sequence of basic points (called an i-path) $\{f^\ell\}_{\ell=0}^L$ with $L > 0$ such that

- $f^0 = f$
- $f^\ell \neq f^{\ell'} \forall \ell \neq \ell'$
- f^ℓ is i-adj to $f^{\ell+1} \forall \ell = 0, ..., L - 1$
- f^L is also supported on best responses

Parity of Nash equilibria

Any non-degenerate, two-player symmetric game has an odd number of (symmetric) Nash equilibria. (This follows from the pairwise matching of best response supported basic points we established in the construction of Lemke-Howson, together with the fact that the artificial equilibrium is a basic point)

Perturbations to Degenerate Games

Let $G = (\{1, 2\}, A = \{1, 2, \ldots, m\}, u)$ be any symmetric, two-player game. For any $\epsilon \geq 0$, define

$$G^\epsilon = (\{1, 2\}, A = \{1, 2, \ldots, m\}, u^\epsilon),$$

where $u_i^\epsilon(a) = \frac{1}{1+\epsilon^{a_i}} u_i(a)$. Then, there exists some $\hat{\epsilon} > 0$ such that $\epsilon \in (0, \hat{\epsilon})$ implies G^ϵ is non-degenerate.

Coincidence of Nash Equilibria Between Degenerate and Perturbed Game

For any symmetric, two-player game G, there exists some $\bar{\epsilon} > 0$ such that $\epsilon \in (0, \bar{\epsilon})$ implies G^ϵ is nondegenerate, and if f is a (non-artificial) basic solution for G^ϵ with basis B, then there exists some feasible and non-artificial f' which is a solution for G with $B(f') \subseteq B$.

Dropping Non-degeneracy

For all two-player finite games with mixed strategies, there exists a Nash equilibrium.

Bolzano-Weierstrass

Recall that, for any $N \in \mathbb{N}$ and any bounded sequence $\{x^k\}_{k=0}^\infty$ in $\mathbb{R}^N$, we have that some subsequence $\{x^{k_l}\}_{l=0}^\infty$ which converges.

Imitation Games

Fix $G = (N, Z, u)$. Define a sequence of *imitation games* given by $\{G^m\}_{m=0}^\infty$ by

$$G^m = (\{1, 2\}, X^m \times X^m, \tilde{u}_i),$$

where $X^m = \prod_{i \in N} x_i^m$, $x_i^m \overset{\text{fin.}}{\subseteq} \Delta(A_i)$. That is, the set of actions for each player in the imitation game is the product set of some finite collections of mixed strategies for each player in the original game. We construct the utilities as

$$\tilde{u}_1(\mu_1, \mu_2) = \sum \mu^1(\alpha_1)\mu^2(\alpha^2) \sum_{i \in N} u_i(\alpha_i^1, \alpha_{-i}^2)$$

Extensive Form Games

Components of an Extensive Form Game

An extensive form game is comprised of

1. Players: $N = \{1, \ldots, n\}$
2. Actions: A_i for each $i \in N$; for all $\tilde{N} \subseteq N$, define $A_{\tilde{N}} = \prod_{i\in\tilde{N}} A_i$, and we must have $A = \bigcup_{\tilde{N}\subseteq N} A_{\tilde{N}}$.
3. Terminal histories: H, where an element $h \in H$ is a sequence $\{a^t\}_{t=0}^T$, for T finite or infinite.
4. Player correspondence: $P : S^*(H) \to 2^N$
5. Feasibility correspondence: $F : H_i \to 2^{A_i}$
6. Preferences over H, $\succsim_i$,

where (H, P, F) are consistent, meaning $F(h) = \{a \,|\, (h, a) \in S(H)\}$, with $S(H)$ the set of subhistories of histories in H, $S^*(H)$ the set of proper subhistories, and H_i the set of histories at which player i can act (that is, $H_i = \{h \in S^*(H) \,|\, i \in P(h)\}$).
We say that an extensive form game is finite if N, A, and H are all finite.

Strategies, Nash Equilibria, and Sequential Rationality

For an extensive form game of complete information (EFGCI) G, a strategy for player i is a mapping $\sigma_i : H_i \to A_i$ such that $\sigma_i(h) \in F_i(h)$. Given a strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ and a non-terminal history h, we define

$$\omega(\sigma, h) = (h, h'),$$

where $h' = \{\sigma_i(h^k)\}_{i\in P(h)}$. That is, each subsequent entry in the history is determined by the strategy σ's choice. We define $\omega(\sigma, \emptyset)$ to be the **path of play**.

Nash Equilibrium

A strategy profile σ is a Nash Equilibrium of the game if

$$\omega(\sigma_i, \sigma_{-i}, \emptyset) \succsim_i \omega(\sigma_i', \sigma_{-i}, \emptyset)$$

for all $i \in N$ and σ_i'.

Normal Form

Given an extensive form game G, the normal form is defined as

$$(N, \{\tilde{A}_i\}, \{\tilde{\succsim}_i\}),$$

with $\tilde{A}_i$ the set of all of player i's strategies σ_i and $\sigma \tilde{\succsim}_i \sigma'$ if and only if $\omega(\sigma, \emptyset) \succsim_i \omega(\sigma', \emptyset)$.

Sequential Rationality

A strategy σ_i is **sequentially rational** against σ_{-i} if, for all $h \in S(H)$ and σ_i', we have

$$\omega(\sigma_i, \sigma_{-i}, h) \succsim_i \omega(\sigma_i', \sigma_{-i}, h).$$

Notice that this must hold not just at the initial node ($\emptyset$), but at all subhistories. If every σ_i in $\sigma = (\sigma_1, \ldots, \sigma_n)$ is sequentially rational against σ_{-i}, then σ is a **sequential equilibrium**.

One-Shot Deviation Principle

Given σ_i, a **one-shot deviation** is a σ_i' such that there is a unique h with $\sigma_i(h) \neq \sigma_i'(h)$.

Now, let G be a finite EFGCI. Then, σ is a sequential equilibrium if and only if

$$\omega(\sigma, h) \succsim_i \omega(\sigma_i', \sigma_{-i}, h)$$

for all i, all $h \in S^*(H)$, and one-shot deviation σ_i to σ_i'.

Existence of Sequential Equilibria

An EFGCI has **purely sequential moves** if $|P(h)| = 1$ for all $h \in S^*(H)$.

Every finite EFGCI with purely sequential moves has a sequential equilibrium.

Subjective Utility

Basic Definitions

A state of the world is $s \in S$ where S is finite.
An act is a function, $f : s \to \mathbb{R}_+$ the set of acts is denoted as F
Note that acts have the following properties:

$$\forall f, g \in F : h = f + g$$
$$h(s) = f(s) + g(s) \forall s$$
$$\forall f \in F, \alpha \in \mathbb{R}_+,\ h = \alpha f$$
$$h(s) = \alpha f(s)\ \forall s$$

Subjective Utility Representation (SEU Reps) of Preferences

$\succeq$ has a subjective utility representation $p \in \Delta(S)$ if

$$f \succeq g \iff \sum_s p(s)f(s) \geq \sum_s p(s)g(s)$$

Where each sum is denoted as a subjective expected utility function.

Note the following proposition: If p and q are SEU reps for $\succeq$ then p=q.

SEU Axioms

SEU Existence Theorem $\succeq$ on F has a SEU rep iff it is rational, monotonic, additive, non-trivial, and continuous. Note that $\succeq$ must be rational.
1. Monotonicity
$f(s) \geq g(s) \forall s \implies f \succeq g$
2. Additivity
$f \succeq g \iff f + h \succeq g + h \forall h$
3. Non-triviality
$\exists f, g,\ s.t.\ f \succ g$
4. Continuity
If $f \succeq g \succeq h$, then $\exists\ \alpha \in [0, 1]$ s.t. $\alpha f + (1-\alpha)h \sim g$

The Basics of Knowledge

Aumann Knowledge Model Definitions

Let S be a set of finite states, and $E \subseteq S$ is an event.
Let a knowledge partition K_i be defined as a set of states that an agent in the model cannot distinguish between being the true state of the world or not: IE a partition of S.
Let a knowledge function $k_i(s) = E \in K_i$ that contains s.
Note the following property of the knowledge function:

$$s \in k_i(s)\ \&\ s' \in k_i(s) \implies k_i(s) = k_i(s')$$

i knows E at s if $k_i(s) \subseteq E$

Mutual Knowledge

Definition
$E \subseteq S$ is mutual knowledge at $s \in S$ if $k_i(s) \subseteq E \forall i$
Mutual Knowledge Function Definition: Given E, $m(E) = \{s | \text{E is mutual knowledge at s}\}$
Mutual Knowledge Induction Proposition:
If $E \subseteq E'$ then $\forall\ \ell \geq 1$, $m^\ell(E) \subseteq m^\ell(E')$

Common Knowledge and Self-Evidence

Common Knowledge Definition:
E is common knowledge at s if $s \in m^\ell(E) \forall \ell \geq 1$
Self-Evident Knowledge Definitions:
E is self evident if $\forall s \in E$, $k_i(s) \subseteq E$. Equivalently:
E is mutual knowledge at every $s \in E$
$E = m(E)$
Common Knowledge and Self-Evidence Correspondence Theorem:
E is common knowledge at s if $\exists$ a self evident event E' s.t. $s \in E' \subseteq E$
Self-Evident Union Proposition: E is self evident iff $\forall i$, $\exists \tilde{K_i} \subseteq K_i$ s.t. $E = \cup_{E' \in \tilde{K_i}} E'$

Aumann's Agreement Theorem

Suppose $|N| = 2$ and there is a common prior. Also, suppose that, for some E, η_1 and η_2, the event

$$\tilde{E} = \{s' \,|\, p_1(E|k_1(s')) = \eta_1 \text{ and } p_2(E|k_2(s')) = \eta_2\}$$

is common knowledge at some s. Then, $\eta_1 = \eta_2$.

Bayesian Games

Type Space Definition

A type space consists of the following:

- Payoff relevant state of the world $\theta \in \Theta$
- Set of players N
- Player i's information is described by a **type** $t_i \in T_i$
- $T = \prod_i T_i$ is the set of type profiles $t = (t_1, ..., t_n)$
- Beliefs denoted: $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ notated formally by $\pi_i(\theta, t_{-i}|t_i)$

Relationship to Aumann

There is a direct correspondence between type spaces and the Aumann Knowledge model, namely:

$$S = \Theta \times T$$
$$k_i(s) = \{(\theta, t')|t'_i = t_i\}$$
$$p_i(s'|s) \iff \pi(\theta, t_{-i}|t_i)$$

Common Prior Definition

A belief $\pi \in \Delta(\theta \times T)$ is a common prior for the type space $(\Theta, N, T, \{\pi_i\})$ if $\forall i, t_i,\ \pi(\theta, t) = \pi_i(\theta, t_{-i}|t_i) * (\sum_{\theta', t'_{-i}} \pi(\theta', t_i, t'_{-i}))$

Bayesian Game Definition

A bayesian game consists of the regular parts of a game:

- Players, N
- Actions A_i for player i, $A = \pi_i A_i$
- Utility functions $u_i :\ A \times \Theta \to \mathbb{R}$
- types, T

Where the addition of types and the subtle manipulations of the old game definitions and the addition of a type space delineate it from normal mixed-strategy games.

Behavioral Strategies

A behavioral Strategy for player i is a mapping: $\sigma_i :\ T_i \to \Delta(A_i)$.
Σ_i is a set of strategies for player i, whereas:
$\Sigma = \pi_i \Sigma_i$ strategy profiles of $\sigma = (\sigma_1, ..., \sigma_n)$.

Type Preferences

Given (T, G) player i's type t_i preferences over Σ are represented by:

$$U_i(t_i, \sigma) = \sum_{\theta} \sum_{t_{-i}} \pi_i(\theta, t_{-i}|t_i)\sigma(a|t_i)u_i(a, \theta)$$

Bayes Nash Equilibrium

σ is a bayes nash equilibrium if:

$$u_i(t_i, \sigma) \geq u_i(t_i, \sigma'_i, \sigma_{-i}) \forall i, t_i, \sigma'_i$$

Finite Bayesian Game NE Existence Theorem

Suppose (T, G) is a finite bayesian game then a nash equilibrium exists.

Value of Knowledge

Experiment Game Setup

- Let (T, π), $\pi \in \Delta(\Theta \times T)$
- $\pi(\theta, t) = \mu(\theta) * \rho(t|\theta)$
- E=(T, ρ) is an **experiment**

Note: the prior is given by μ and ρ is the cond'l dist of t given θ.

Informative Definition

Given $E = (T, \rho)$, and $E' = (T', \rho')$, we say E is **more informative** than E' if $\exists$ a function
$\lambda : T \to \Delta(T')\ s.t.\ \forall \theta, t',\ \rho'(t'|\theta) = \sum_t \lambda(t'|t)\rho(t|\theta)$.
Where λ is a garbling from $E \to E'$

Informativeness Theorem

E is more informative than E' iff $\forall$ game forms $G = (A, u)$ and prior μ

$$\max_{\sigma : T \to \Delta(A)} \sum_{\theta, t, a} u(a, \theta)\sigma(a|t)\rho(t|\theta)\mu(\theta) \geq$$
$$\max_{\sigma' : T' \to \Delta(A)} \sum_{\theta, t', a} u(a, \theta)\sigma'(a|t')\rho'(t'|\theta)\mu(\theta)$$

Jury Games

Jury Game Set-up

- $\Theta = \{Innocent, Guilty\}$
- $A_i = \{Convict, Acquit\}$

$$u_i(a, \theta) = \begin{cases} -(1-y) & \text{if } \theta = G \text{ and } |\{i|a_i = c\}| < k \\ -y & \text{if } \theta = I \text{ and } |\{i|a_i = c\}| \geq k \\ 0 & \text{o.w} \end{cases}$$

for some $k \in N = \{1, ..., n\}$ and $y \in (0, 1)$.
Where y is the threshold probability of guilt s.t. conviction is preferred.

Jury Game: Odd Majority Rule

If every juror is truthful, then I only consider my decision when it is pivotal. IE: $\{\#j \neq i|a_j = c\} = \frac{n-1}{2}$. Thus the posterior of guilt is:

$$\pi(\theta = G|t_i = G, \text{pivotal}) =$$
$$\frac{\frac{1}{2}x * x^{\frac{n-1}{2}}(1-x)^{\frac{n-1}{2}}}{\frac{1}{2}x * x^{\frac{n-1}{2}}(1-x)^{\frac{n-1}{2}} + \frac{1}{2}(1-x) * (1-x)^{\frac{n-1}{2}}(x)^{\frac{n-1}{2}}}$$
$$= x$$
$$\pi(\theta = I|t_i = I, \text{pivotal})$$

Jury Game: Unanimity Equilibrium

We are looking for a setup where:

$$\sigma_i(c|G) = 1$$
$$\sigma_i(c|I) = \gamma\ \in (0, 1)$$

This implies:

$$y = \pi(G|t_i = I,\ \text{pivotal})$$
$$= \frac{\frac{1}{2}(1-x)(x + (1-x)\gamma)^{n-1}}{\frac{1}{2}(1-x)(x + (1-x)\gamma)^{n-1} + \frac{1}{2}x(1 - x + x\gamma)^{n-1}}$$

This comes out to:

$$y = \frac{xz^{\frac{1}{n-1}} - (1-x)}{x - (1-x)z^{\frac{1}{n-1}}}$$

Where $z = (\frac{1-x}{x}\frac{1-y}{y})^{\frac{1}{n-1}}$.
Notice that, under unanimity, truthful voting is **not** an equilibrium. In fact, the equilibrium under unanimity requires mixing when a juror receives an innocent signal.

Jury Game: Even Majority Rule

In this case, an agent is pivotal only when $k - 1 = N/2$ of the other agents vote to convict and the remaining $N/2$ - 1 other agents vote to acquit. In this event, using Bayes' rule, the posterior probability that the defendant is guilty conditional on a signal of $t_i = G$ is

$$\pi(G \mid G) = \frac{(1/2)x^{(N/2+1)}(1-x)^{(N/2-1)}}{(1/2)x^{(N/2+1)}(1-x)^{(N/2-1)} + (1/2)x^{(N/2-1)}(1-x)^{(N/2+1)}}$$
$$= \frac{x^2}{x^2 + (1-x)^2}$$

Note that

$$x < \tfrac{x^2}{x^2 + (1-x)^2}$$
$$\iff x > x^2 + (1-x)^2$$
$$\iff \frac{1}{2} < x < 1$$

As $x \in (1/2, 1)$ and $x \geqslant y \geqslant 1 - x$ the agent will convict when they observe $t_i = G$. When they observe $t_i = I$ their posterior is

$$\pi(I \mid I) = \frac{(1/2)x^{(N/2)}(1-x)^{(N/2)}}{(1/2)x^{(N/2)}(1-x)^{(N/2)} + (1/2)x^{(N/2)}(1-x)^{(N/2)}} = \frac{1}{2}$$

Thus $\pi(G \mid I) = 1/2$ and the agent will only acquit if $y < \frac{1}{2}$.

E-Mail Games

Setup

Two Players:

1. $N = \{1, 2\}$
2. $\Theta = \{G, B\}$
3. Prior $p < 1/2$ that $\theta = G$
4. $A_i = \{F, R\}$

The utility matrices are given by:

State $\theta = B$

	R	F
R	$(2, 2)$	$(1, -3)$
F	$(-3, 1)$	$(0, 0)$

State $\theta = G$

	R	F
R	$(0, 0)$	$(1, -3)$
F	$(-3, 1)$	$(2, 2)$

Full Information

Now suppose that both generals observe the state with probability 1. Then:

$$\pi(\theta, t) = \begin{cases} p & \text{if } \theta = t_1 = t_2 = G \\ 1-p & \text{if } \theta = t_1 = t_2 = B \\ 0 & \text{o.w} \end{cases}$$

Thus when the state is bad, retreat is dominant and when the state is good there are two pure equilibria but fighting when the state is good is a BNE with the highest payoff.

Player 1 Informed/ Player 2 Uninformed

If player 1 is informed and player 2 is uninformed then the priors given are the same as in the full information model, but player 2's signal is uninformative. Now if the state is bad, then retreating is strictly dominant for player 1. Thus the payoff for player 2 of fighting and retreating is given by:

$$U_2(F) \leq (1-p)(-3) + 2p$$
$$U_2(R) \geq 2(1-p)$$

However, since we assumed $p < 1/2$ so retreating is the unique best response, and so both generals always retreat.

Approximate Knowledge

Now suppose that the first general observes the state, and both generals get a number of confirmations that are sent back and forth. The new set of priors is that:

$$\pi(t,\theta) = \begin{cases} 1-p \text{ if } t=(B,0) \text{ and } \theta = B \\ p(1-\varepsilon)^{2k}\varepsilon \text{ if } t=(k,k) \text{ and } \theta = G \\ p(1-\varepsilon)^{2k+1}\varepsilon \text{ if } t=(k,k+1) \text{ and } \theta = G \\ 0 \text{ o.w} \end{cases}$$

From this set of priors, we can note that for an epsilon sufficiently high there is no number of messages that can be passed that causes both generals to want to attack.

Extensive Form Games with Incomplete Information

Set-up and Notation

- Players: $i = 1, ..., N$
- Nature $i = 0$
- Actions: $A: \ i = 0, ..., N$ and $a = \prod_{i=0}^{N} A_i$
- Histories: $H \subseteq \bar{H} := \bigcup_{t=0}^{\infty} A^t$
- Given h, h', h follows h' if $h = (h', h'')$ for some $h'' \in \bar{H}$ this is denoted $h \geq h'$ and $h > h'$ if $h \geq h'$ and $h \neq h'$.
- $Z \subseteq H$ is the set of terminal histories. $h \in Z \implies \neg\exists\ h' \in H$ s.t. $h' > h$.
- Players have EU Prefs over Z represented by $u_i : Z \to \mathbb{R}$
- $\forall h \in H \setminus Z$, $A_i^h \subset A_i$ is player i's actions at h and $A^h = \prod_i A_i^h$ is the set of action profiles at h.

The Structural Assumption on H is given as:

$$\forall h \in H \setminus Z,$$
$$\{(h,a) \in H \mid a \in A\} = \{(h,a) \mid a \in A^n\}$$

Information in Sequential Games

$\forall i$ P_i is a partition of H_i.
Note the following definitions:

1. Let $p_i(h)$ be the cell of P_i containing $h \in H \setminus Z$.
2. Given $h = (a^0, a^1, ...a^T)$, let $t_1 < ... < t_K$ be the times at which $(a^0, a^1, ...a^t) \in H^i$ for $0 \leq t \leq T$
3. Player $i's$ experience at h is given by: $x_i(h) = (p_i(a^0, ..., a^{t_k}, a_i^{t_k+1})_{k=1}^{K}$

Furthermore,
These knowledge partitions have the following two properties:

1. $h, h' \in p_i \in P_i \implies A_i^h = A_i^{h'}$ (people know their actions)
2. $\forall h, h' \in H \setminus Z$, $X_i(h) \neq x_i(h') \implies p_i(h) \neq p_i(h')$ (Perfect Recall)

Perfect Recall Lemma

$\forall h \in Z, p_i \in P_i$ there is at most one $h' \in H$ s.t. $h > h'$ and $h' \in P_i$

Perfect Recall Equivalency Proposition

Under the maintained assumption that the game has perfect recall. Then: $\forall \sigma_i \in \Sigma_i$, $\exists \alpha \in \Delta\left(\Sigma_i^p\right)$ s.t. α and σ_i are equivalent.

Kuhn's Theorem

Suppose the game has perfect recall, then for any $\alpha_i \in \Delta(\Sigma_i^P)$, $\exists \sigma_i \in \Sigma_i$ s.t. α_i and σ_i are equivalent. This goes in the opposite direction of the previous proposition.

Total Probability and Some Definitions

- $A_i^{P_i} = A_i^h$ for $h \in p_i \in P_i$
- A behavioral strategy is $\sigma_i : P_i \to \Delta(A_i)$ s.t. $\sigma(a_i|p_i) \geq 0 \implies a_i \in A_i^{P_i}$
- Σ_i = set of $i's$ behav. strat
- $\Sigma = \Pi_{i \geq 1} \Sigma_i$
- $\sigma \in \Sigma$ define $\forall h \in H \setminus Z$, $a \in A$,

$$\tilde{\sigma}(a|h) = \sigma_0(a_0)|h \prod_{\{i|h \in H^i\}} \sigma_i(a_i|p_i(h))$$

Define $\mu_i(h'|h.\sigma_i)$ as the probability i's behaviors are consistent with reaching h' from h. Which is equivalent to the following if we let $(h, a^1, ...a^T) = \phi$,.

$$\mu_i\left(h' \mid h, \sigma_i\right) = \begin{cases} 1 & \text{if } h \geq h' \\ \Pi_{\{0 \leq t < T|(\phi)} \sigma_i(a^{t+1}|p_i(\phi)) \in H^i\} & \text{if } h' = (\phi) \\ 0 & \text{o.w.} \end{cases}$$

Expected Utility

Player i's expected utility from σ starting at h

$$U_i(\sigma, h) = \sum_{h' \in Z} u_i\left(h'\right) \mu\left(h' \mid h, \sigma\right)$$

Nash Equilibrium

σ is a Nash equilibrium if $\forall$ i, $\sigma_i' \in \Sigma_i$

$$u_i(\sigma, \emptyset) \geq u_i\left(\sigma_i', \sigma_{-i}, \emptyset\right).$$

Interim Beliefs and Assessments

An interim belief for player i is a mapping

$$\beta_i : P_i \to \Delta(H) \quad \text{s.t.} \quad \forall p_i \in P_i, \quad \sum_{h \in p_i} \beta_i\left(h \mid p_i\right) = 1$$

(Player i believes what they know)
An assessment is a pair of strategies and interim beliefs, (σ, β)

Consistency of assessments

An assessment (σ, β) is consistent if $\exists$ a set $(\sigma^k)_{k=0}^{\infty}$ of totally mixed strategies such that $\sigma^k \to \sigma$. This is equivalent to:
$\forall i, \ p_i, \ a_i \in A_i, \ \sigma_i^k(a_i|p_i) \to \sigma_i(a_i|p_i)$.
And

$$\beta_i(h|p_i) = \lim_{k \to \infty} \frac{\mu(h|\sigma^k, \emptyset)}{\sum_{h' \in P_i} \mu(h'|\sigma^k, \emptyset)}$$

Sequential Rationality

An assessment is sequentially rational if $\forall i, \ p_i \in P_i, \ \sigma_i' \in \Sigma_i$:

$$\sum_{h \in P_i} u_i(\sigma, h)\beta_i(h|p_i) \geq \sum_{h \in p_i} u_i(\sigma_i', \sigma_{-i}, h)\beta_i(h|p_i)$$

Sequential Equilibrium

A sequential equilibrium is any assessment that is both consistent and sequentially rational. By the sequential equilibrium existence theorem, this exists.

Trembling Hand Equilibrium

A trembling hand, or perfect equilibrium if there exists a totally mixed strategy profile $\bar{\sigma}$ and a sequence $(\sigma^k)_{k=1}^{K}$ of strategies such that:

1. σ is the limit of σ^k
2. σ^k is a nash equilibrium of the game with payoffs $U_i((1-1/k)\sigma' + (1/k)\bar{\sigma})$

Basics of Auction Theory

General Auctions

Single item up for auction.

- Players (bidders): indexed by $i = 1, \ldots, N$
- Bidder i has associated value v_i for the item.
  - If bidder i observes their and only their value, we call these **private values**.
  - Usually, we consider these values $v_1, \ldots, v_N$ as being random variables. When they are statistically independent from each other and private, we call them **independent private values** (IPV).

- Actions: bids for each bidder i, $b_i \in \mathbb{R}_+$.
- Allocation function $q_i(b)$, which gives the probability of bidder i winning the item when bidding profile b is played.
- Transfer function $t_i(b)$, which gives the (expected) net payment from bidder i to the seller.
- Preferences given by utility function $u_i(v_i, b) = v_i q_i(b) - t_i(b)$, which we modify to account for uncertainty in b_{-i} by defining

$$u_i(v_i, b_i) = \mathbb{E}_{b_{-i}}\left[u_i(v_i, b_i, b_{-i})\right]$$

Basic Examples

In the following examples, we will assume that v_i are independently and identically distributed according to $U[0, 1]$, the standard uniform distribution.

Second Price Auctions: Definitions

The setup for a SPA is that a price, beginning at zero, rises. Each bidder chooses a cutoff, their bid, which sets the price at which they drop out. The price will continue rising until $N-1$ bidders have dropped out. The final price is, then, $b^{(2)}$, the second-highest bid. We define the winning correspondence as

$$W(b) = \{i \mid b_i = b^{(1)}\},$$

so that we can define the allocation by

$$q_i^{\text{SPA}}(b) = \begin{cases} \frac{1}{|W(b)|} & i \in W(b) \\ 0 & \text{o.w.} \end{cases}$$

and the transfer by $t_i^{\text{SPA}}(b) = b^{(2)} q_i^{\text{SPA}}(b)$.

Second Price Auctions: Equilibrium Bid

An equilibrium (not unique!) is given by each bidder bidding their value; that is, $b_i = v_i$.

Second Price Auctions: Equilibrium Revenue

The equilibrium revenue is

$$R^{\text{SPA}} = \mathbb{E}_v[b^{(2)}] = \mathbb{E}_v[v^{(2)}] = \frac{N-1}{N+1}$$

First Price Auction: Definitions

In a FPA, we can understand the bidding process as being a sealed-bid auction in which the highest bidder wins and pays their bid. Hence, the final price is $b^{(1)}$. Winning and allocation are the same as in SPA (that is, $q_i^{\text{SPA}} = q_i^{\text{FPA}}$). Transfers are now given by $t_i^{\text{FPA}}(b) = q_i^{\text{FPA}} b_i$.

First Price Auctions: Equilibrium Bid

An equilibrium is given by each bidder **bid shading** (that is, intentionally bidding less than their private value) by the appropriate amount; in particular, $b_i = \frac{N-1}{N} v_i$.

First Price Auctions: Equilibrium Revenue

In the end, the revenue is the same as in the SPA; we have $R^{\text{FPA}} = \frac{N-1}{N+1} = R^{\text{SPA}}$.

Durable Goods Monopoly

Set-up

In the durable goods monopoly,

- **Actions:** at each period t, the seller makes a take-it-or-leave-it offer at price p_t and the consumer chooses to buy or not buy
- **Preferences:** Sellers get 0 if the consumer doesn't buy, and p_t if they buy at period t. Consumers have a value distributed according to $F(v) = v^\alpha$, and receive $p_t - v$ when they purchase at period t.

Commitment Solution

Let δ be the discount factor from one period to the next; then, the profit associated with selling at t is $\delta^t p_t$, and the payoff for purchasing at t is $\delta^t(v - p_t)$. If the seller could credibly commit to a price, they could simply set $p_t = p_0$ for all t, so that no transactions would occur after the initial period.

Coase Conjecture

The Coase Conjecture states that, if waiting is nearly costless, then the outcome is close to the competitive outcome. (*The intuition here is that the monopoly is competing with its future self*). Turns out it's true! Taking the limit as $\delta \to 1$:

Non-Commitment Solution

We search for a solution of the form $p_t = (\gamma\beta)^t \beta$ for some β and γ, where β is the per-period reduction in cost and δ determines the consumer's willingness to purchase.

From optimality conditions, we get that

$$\delta = \frac{1}{1 - \delta(a - \beta)}, \quad \textit{(buyer optimality)}$$

$$\frac{\alpha}{1+\alpha} = \frac{1 - (\gamma\beta)^\alpha}{1 - \delta(\gamma\beta)^{1+\alpha}}, \quad \textit{(seller optimality).}$$

The latter condition is derived from using the envelope theorem on the revenue,

$$\frac{d}{dv} R(v) = \left[\frac{\partial}{\partial v} R(v, p) + \frac{\partial}{\partial p} R(v, p)\right]_{p=\beta v} = \left.\frac{\partial R(v, p)}{\partial p}\right|_{p=\beta v}.$$

We also have, from $R(v) = \beta v^{\alpha+1}(1 - (\gamma\beta)^\alpha) \sum_{t=0}^{\infty} (\delta(\gamma\beta)^{\alpha+1})^t$, we have

$$\beta = \frac{1-\delta}{\frac{1}{x} - \delta},$$

where $x = \gamma\beta$. Combining these, we find that, in the case where $\alpha = 1$ (uniform distribution),

$$\beta = \frac{\sqrt{1-\delta}}{1+\sqrt{1-\delta}} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\delta}}.$$

Reputation

Chain Store Game Set-up

In the Chain Store game, there are K (potential) entrants who decide in series, with one incumbent. The entire history is observable to all players. At each step, the incumbent prefers no entry to acquiescing, and acquiescing to fighting. The entrant prefers entering to not entering, if and only if the incumbent acquiesces.
By backwards induction, the unmodified game has an equilibrium of (In, A) at all steps. Clearly this model is missing something...

Incumbent Types

We imagine that there are two types for the incumbent: the normal type, which behaves as we're used to, and the aggressive type, which has preference for fighting above all else. Suppose that the incumbent is aggressive with probability $\epsilon > 0$. **Now,** there is no sequential equilibrium where entrant always enters and I acquiesces. This is because, for any $\epsilon \leq \frac{1}{2}$, the first entrant will play in. But in the case where the incumbent is aggressive, they will fight, informing the second entrant that they're aggressive, and the second entrant will play out.

Incumbent Minimum Payoff Theorem

In any sequential equilibrium of the Chain Store game with K entrants, the incumbent's total payoff is at least $3\left(K + \frac{\log(2\epsilon)}{\log(2)}\right)$. This is because, starting from any normal-type strategy σ_{I}, the incumbent could always deviate to $\sigma_{\text{I}}'(F|h) = 1$, mimicking the aggressive type. This forces the entrants to update their beliefs about the incumbent's type; if one of them plays In starting from history h, the posterior probability that the incumbent is aggressive after being fought is

$$\frac{\epsilon(h)}{\epsilon(h) + (1 - \epsilon(h))\sigma_{\text{I}}(F|h, \text{In})},$$

where $\epsilon(h)$ is the posterior probability of the incumbent being aggressive at history h. From this, we see that an entrant will only continue to play In so long as $\epsilon(h, \text{In}, \text{F}) \geq 2\epsilon(h)$.
After k iterations of (In, F), this becomes the condition $\epsilon(h) \geq 2^k \epsilon$. Implied by this condition is that $k \geq -\frac{\log(2\epsilon)}{\log(2)}$.

Mechanism Design

We now expand our analysis of the durable goods monopoly, by extending the range of monopoly strategies to include any combination (q, p) of allocation rules q and prices p. Formally, we define a **mechanism** $\mathcal{M}$ as

1. A set of messages M that the consumer can send to the monopolist
2. An allocation rule $q(M)$
3. A price $p(M)$.

In this framework, the strategies for a consumer are $\sigma_i(\underline{v}, \overline{v}) \to \Delta(M)$ (that is, lotteries over the set of possible messages). Preferences are given by

$$U(\sigma, \mathcal{M}) = \int_v \int_m (vq(m) - p(m))\sigma(dm|v)F(dv),$$

and revenue

$$R(\sigma, \mathcal{M}) = \int_v \int_m p(m)\sigma(dm|v)F(dv).$$

Revelation Principle

It is without loss of generality to examine mechanisms for which

$$M = [\underline{v}, \overline{v}], \qquad \sigma(\{v\}|v) = 1,$$

wherein the consumer just honestly reports their value. This is called a *truthful direct mechanism*.

Incentive Compatibility and Individual Rationality

For truth-telling to be optimal, the p and q from the mechanism must satisfy

$$vq(v) - p(v) \geq vq(v') - p(v'), \quad \forall v, v'. \qquad \text{IC}$$

This condition is equivalent to requiring that q must be non-decreasing, provided that $p(v)$ takes the form

$$p(v) = vq(v) - \int_{\underline{v}}^{v} q(x)\,dx - U(\underline{v}).$$

To regularize the monopolist's behavior, we further impose the constraint

$$U(v) \geq 0, \quad \forall\, v, \qquad \text{IR}$$

so that the consumer is willing to participate at any value. (In particular, with IC, this implies that $U(\underline{v}) = 0$.

Posted Price

A posted price mechanism takes the form

$$q^r(v) = \begin{cases} 1 & v \geq r, \\ 0 & \text{o.w.} \end{cases}$$

and

$$p^r(v) = \begin{cases} 0 & x \leq r \\ r & x > r, \end{cases}$$

where $r \in [0,1]$ is the posted price.

Multiple Buyers

We now assume that there are a collection of N consumers, all with values independently distributed according to $v_i \sim F_i(v)$ with support a subset of $[\underline{v}, \overline{v}]$.
The mechanism we consider is characterized by $q_i : M \to [0,1]$ and $p_i : M \to \mathbb{R}$, where $\sum_i q_i(m) \leq 1$. Preferences are given by the general definition for mechanism utility. Revenue is

$$R(v) = \int_v \int_m \sum_i p_i(m)\sigma(dm|v)f(v)\,dv$$

Revelation Principle

For any mechanism $\mathcal{M}$ and Bayes Nash Equilibrium σ, there is a direct mechanism satisfying IC and IR with equivalent revenue.
This can be intuitively understood as, for whichever mechanism you pick, instead just reporting your type to the mechanism and having it "play for you".
Under such a mechanism, where $Q_i(v_i') = q_i(v_i', v_{-i})$ and $P_i(v_i') = p_i(v_i', v_{-i})$, revenue is

$$R = \sum_i \int_{v_i} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right] q_i(v)f(v)\,dv - \sum_i U_i(\underline{v}).$$

We call $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ the *virtual value* for consumer i.

Revenue Equivalence

For any mechanism $\mathcal{M}$ and Bayes Nash Equilibrium σ, expected revenue is equal to the expected virtual value of the bidder allocated the good, minus the utilities of the lowest types.
As a corollary, if $\mathcal{M}$ and $\mathcal{M}'$ implement the same allocation in some BNE for each, and give the same lowest utility to the lowest types, they are revenue equivalent. Examples include

- First- and Second-Price auctions with symmetric distributions $F_i = F_j$
- First- and Second-Price auctions with a reserve price.

Optimal Auctions

For the design of an auction to be revenue-optimal, we must have that $U(\underline{v}) = 0$. We say that the distributions F_i are regular if φ_i is non-decreasing. Then, under symmetry and regularity, expected revenue is maximized by SPA/FPA with reserve price $r^* = \min\{x|\varphi(x) \geq 0\}$.

Tips and Tricks

Properties of payoff functions

Let α_1, α_2 represent two mixed strategies, then note the following properties of utility functions:

$$\begin{aligned} u(\alpha_1, \alpha_2) &= \sum_{a_2 \in A_2} \alpha_2(a_2)u(\alpha_1, a_2) \\ &= \sum_{a_1 \in A_1} \alpha_1(a_1)u(a_1, \alpha_2) \\ &= \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} \alpha_1(a_1)\alpha_2(a_2)u(a_1, a_2) \end{aligned}$$

Correlated Equilibrium Tricks

A useful case of correlated strategies for proving correlated equilibrium is the following deviation: Fix any $i \in N, \hat{a}_i, \hat{a}_i' \in A_i$. Define σ_i' as:

$$\sigma_i'(a_i|\omega_i) = \begin{cases} 0 \text{ if } a_i = \hat{a}_i \\ \sigma_i(\hat{a}_i|\omega_i) + \sigma(\hat{a}_i'|\omega_i) \text{ if } a_i = \hat{a}_i' \\ \sigma_i(\hat{a}_i|\omega_i) \text{ otherwise} \end{cases}$$

Where σ is a Nash equilibrium of the augmented game.
Note also that given a behavioral strategy σ, we can derive the payoff difference between σ and a deviation σ' (from the definition of σ_i' above) as:

$$\begin{aligned} U_i(\sigma) - U_i(\sigma_i', \sigma_{-i}) &= \sum_{\omega \in \Omega} \sum_{a \in A} (\sigma(a|\omega) - (\sigma_i'\sigma_{-i})(a|\omega))\pi(\omega)u_i(a) \\ &= \sum_{a_{-i}} \mu(a_i, a_{-i})(u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})) \end{aligned}$$

A lot of other tricks can be noted just by the definitions of all the relevant variables: e.g.

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}) &= \sum_{\omega \in \Omega} \sum_{a \in A} \pi(\omega)\sigma_i(a_i|\omega_i)\sigma_{-i}(a_{-i}|\omega)u_i(a) \\ &= \sum_{a \in A} \mu(a)u_i(a), \end{aligned}$$

where μ is the correlated equilibrium induced by σ (and the correlation device). Furthermore, for any particular $\hat{a}_i \in A_i$, we have

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}) &= \sum_{a \in \hat{A}} \mu(a)u_i(a) + \sum_{a \in \hat{A}^c} \mu(a)u_i(a) \\ &= \sum_{a_{-i}} \mu(\hat{a}_i, a_{-i})u_i(\hat{a}_i, a_{-i}) + \sum_{a \in \hat{A}^c} \mu(a)u_i(a), \end{aligned}$$

where $\hat{A} := \{a : a_i = \hat{a}_i\}$.

Tricks with TPZSG's

Recall that, if u_1 represents the utility of the maximizer, we can always write $u_2(a_1, a_2) = -u_1(a_1, a_2)$.

Sequential Equilibria

- If an assessment (β, σ) is consistent, it is sequentially rational if and only if it has no one-shot deviations. Here, we understand an OSD to be a strategy σ_i' which differs from σ_i at exactly one information set.