Problem Set 3 - Question 1

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Question 1

Consider the production function $F(L, K_1, K_2, ..., K_N)$ which gives output as a function of labor and N different capital inputs, $K_1, K_2, ..., K_N$. Typically we desire to aggregate these capital inputs into a single capital stock measure, K.

Part (a)

(a) If the production function can be written as $F(L, G(K_1, ..., K_N))$, where G is homothetic, define a capital aggregate, K, and an associated capital input rental rate, R, that will allow us to deal with a single capital good.

Part (a) - Suggested Solution

- Since G is homothetic, it can be written as a monotonic transformation of a function that is homogeneous of degree 1:

$$G(K_1, ..., K_N) = g\Big(H(K_1, ..., K_N)\Big).$$

- Let $K = H(K_1, ..., K_N)$. Then, the production function becomes

$$F(L, G(K_1, ..., K_N)) = F(L, g(K)).$$

- The economic interpretation is that given N different capital inputs $K_1, ..., K_N$, there exists a production technology represented by $H(\cdot)$ that uses those N inputs to create K "capital aggregates".

- Then, the final output only depends only on K regardless of how many of individual inputs $K_1, ..., K_N$ were used to produce such K.

Part (a) - Suggested Solution

- Recall that $H(\cdot)$ is homogeneous of degree 1:

$$K = H(K_1, ..., K_N), \quad \therefore 1 = H(\frac{K_1}{K}, ..., \frac{K_N}{K}).$$

- Therefore, denoting rental rates for N different capital inputs as $R_1, ..., R_N$, the total rent needed to produce a single unit of the capital aggregate (which equals the associated rental rate) becomes

$$R = \frac{K_1}{K} \cdot R_1 + ... + \frac{K_N}{K} \cdot R_N.$$

- To summarize, the capital aggregate K and the associated capital input rental rate R can be defined as

$$K = H(K_1, ..., K_N), \quad R = \underbrace{\frac{K_1}{K}}_{w_1} R_1 + ... + \underbrace{\frac{K_N}{K}}_{w_N} R_N,$$

where $H(\cdot)$ is some function that is homogeneous of degree 1.

(b) Suppose further that each capital rental rate reflects a constant rate tax on the input amount. Each input has its own tax rate. What do your findings from (a) suggest about defining an aggregate tax rate on capital inputs?

Part (b) - Suggested Solution

- Denote tax rates on N individual inputs as $\tau_1, ..., \tau_N$. Then, their after-tax rental rates become $R_1(1 + \tau_1), ..., R_N(1 + \tau_N)$.

- Recall from the Part (a) that

$$K = H(K_1, ..., K_N), \quad \therefore 1 = H(\frac{K_1}{K}, ..., \frac{K_N}{K}).$$

- Therefore, an aggregate tax rate τ satisfies

$$R(1 + \tau) = \frac{K_1}{K} \cdot R_1(1 + \tau_1) + ... + \frac{K_N}{K} \cdot R_N(1 + \tau_N),$$

$$\therefore \tau = \underbrace{\frac{K_1 R_1}{KR}}_{w_1} \tau_1 + ... + \underbrace{\frac{K_N R_N}{KR}}_{w_N} \tau_N \quad (\text{using } \sum_{i=1}^{N} K_i R_i = KR),$$

where K and R are as defined in the Part (a).

Part (c)

(c) If the relative rental rates of the capital goods are fixed and the production function has the general form $F(L, K_1, K_2, ..., K_N)$, define a capital aggregate and a capital rental rate measure that will allow us to deal with a single capital good.

Part (c) - Suggested Solution

- The relative rental rates of the capital goods are fixed:

$$R_1 = \theta_1 R, \quad R_2 = \theta_2 R, \quad ..., \quad R_N = \theta_N R$$

where $\theta_1, ..., \theta_N$ are constants.

- Then, we can write the firm's profit maximization problem as

$$\max_{L, K_1, ..., K_N} PF(L, K_1, ..., K_N) - WL - R_1 K_1 - ... - R_N K_N$$

or equivalently,

$$\max_{L, K_1, ..., K_N} PF(L, K_1, ..., K_N) - WL - R(\theta_1 K_1 + ... + \theta_N K_N).$$

Part (c) - Suggested Solution

- Introduce a new production function $G(L, K)$ as the following:

$$G(L, K) = \max_{K_1, \ldots, K_N} F(L, K_1, \ldots, K_N)$$

such that

$$\theta_1 K_1 + \ldots + \theta_N K_N = K.$$

- Then, the firm's (nested) profit maximization problem becomes

$$\max_{L, K} PG(L, K) - WL - RK.$$

- The capital rental rate R satisfies

$$R = \frac{R_1 K_1}{\theta_1 K_1} = \ldots = \frac{R_N K_N}{\theta_N K_N} = \underbrace{\frac{K_1}{K}}_{w_1} R_1 + \ldots + \underbrace{\frac{K_N}{K}}_{w_N} R_N.$$

Part (d)

(d) Under what conditions, if any, would the aggregate capital rental rate measure put negative weight on the rental rate of one of the capital inputs?

# Part (d) - Suggested Solution

- Possible when we have an inferior input. Definition from Chapter 10: K_1 is an *inferior input* if and only if $\frac{\partial^2 C}{\partial R_1 \partial Y} < 0$.
- In other words, marginal cost of production (which is equal to the rental rate for aggregate capital at optimum) is decreasing in R_1. Therefore, R would put negative weight on R_1.
- Slutsky equation for the firm (can be derived using cost function):

$$\frac{\partial K_i}{\partial R_j} = \underbrace{\frac{\partial^2 C}{\partial R_i \partial R_j}}_{\text{substitution effect}} - \underbrace{\left(\frac{\partial^2 C}{\partial R_i \partial Y} \frac{\partial^2 C}{\partial Y \partial R_j}\right) \Big/ \frac{\partial^2 C}{\partial Y^2}}_{\text{scale effect}}.$$

- Example from the book: "Maybe a firm is using shovels to do small digging projects. But then shovels get more expensive, so the firm switches to digging with an excavator machine. As long as the firm has the machine, the firm digs more".

Part (d) - Suggested Solution

- Cannot have inferior inputs with homothetic production function:
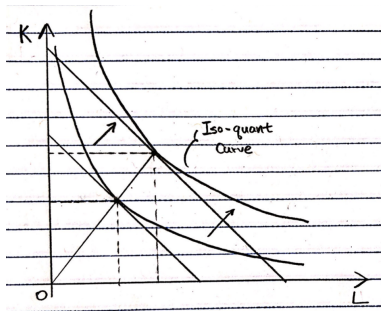
- Takeaway: Marginal cost of production may change due to behavioral shifts, such as changing the major production input to the one better suited for large-scale production. But with homothetic production function, need to increase all factor inputs by the same proportions to increase the production scales (holding factor prices constant).

(e) Under what conditions, if any, would the owner of the firm, who rents the various inputs and sells the output, be better off when one of the capital inputs became more expensive?

Part (e) - Suggested Solution

- Let's continue to consider a standard firm maximization problem:

$$\max_{K_1,...,K_N} PF(K_1,...,K_N) - R_1K_1 - R_2K_2 - ... - R_NK_N.$$

- Owner is "better off" when the (maximized) profit becomes higher.
- At optimum:

$$\pi^* = PF(K_1^*,...,K_N^*) - R_1K_1^* - ... - R_NK_N^*.$$

By Envelope Theorem:

$$\frac{\partial \pi^*}{\partial R_i} = -K_i^* < 0.$$

- Therefore, in general, the owner of the firm cannot be better off when one of the capital inputs become more expensive. But may be possible if price is a function of rental rates (e.g. oil companies).

(f) Can you find a single empirical strategy that would work if you knew that either the conditions in part (a) or part (c) were true but did not know which was correct (i.e. is there a general theory of constructing a capital stock measure)?

Part (f) - Suggested Solution

- Given some base time $t = 0$, we can decompose expenditure growth as

$$\frac{E_t}{E_0} = \frac{\sum_{i=1}^{N} R_i^t K_i^t}{\sum_{i=1}^{N} R_i^0 K_i^0} = \underbrace{\frac{\sum_{i=1}^{N} R_i^t K_i^0}{\sum_{i=1}^{N} R_i^0 K_i^0}}_{\text{price index}} \times \underbrace{\frac{\sum_{i=1}^{N} R_i^t K_i^t}{\sum_{i=1}^{N} R_i^t K_i^0}}_{\text{quantity index}}.$$

- A price index is a first order approximation to the cost function and a quantity index is a first order approximation to the production function (or in this case aggregation function $H(K_1, ..., K_N)$, which is what we want to approximate) - similar intuition as Chapter 4.

- Therefore, construct the quantity index as above (or construct the price index and back out the corresponding quantity index using the relationship above), and then use such first order approximations at various data points to see how N different capital inputs are aggregated into K.