Problem Set 1 - Question 1

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Question 1

Consider a market with many identical consumers, with quasiconcave utility function $u(q, x)$ and many identical producers producing at cost $c(q)$ that is increasing and weakly convex. q is the quantity per consumer and x is their consumption of other goods. Consumers pay p per each unit of good to the producers. Following Marshall, we use the $[q, p]$ plane to examine this market. We also rule out Giffen goods.

Part (a)

(a) Draw the producers' isoprofit curves in the $[q, p]$ plane. How are they related to the cost function? What can you say about their slopes and convexity/concavity?

Part (a) - Suggested Solution

- Profit function:

$$\pi(p, q) = pq - c(q).$$

- Drawing an iso-profit curve for some fixed profit $\overline{\pi}$:

$$\overline{\pi} = pq - c(q), \qquad \therefore p = \frac{\overline{\pi} + c(q)}{q}.$$

- Let's check slope:

$$\frac{dp}{dq} = \frac{qc'(q) - \overline{\pi} - c(q)}{q^2} = \frac{qc'(q) - pq}{q^2} = \frac{c'(q) - p}{q},$$

$$\therefore \frac{dp}{dq} \geq 0 \ \text{ if } \ c'(q) \geq p, \qquad \frac{dp}{dq} < 0 \ \text{ if } \ c'(q) < p.$$

Part (a) - Suggested Solution

- Checking concavity:

$$\frac{d^2p}{dq^2} = \frac{q^2(c'(q) + qc''(q) - c'(q)) - 2q(qc'(q) - \overline{\pi} - c(q))}{q^4}$$

$$= \frac{q^3c''(q) - 2q^2c'(q) + 2q(\overline{\pi} + c(q))}{q^4} = \frac{c''(q)}{q} + \frac{2}{q^2}[p - c'(q)].$$

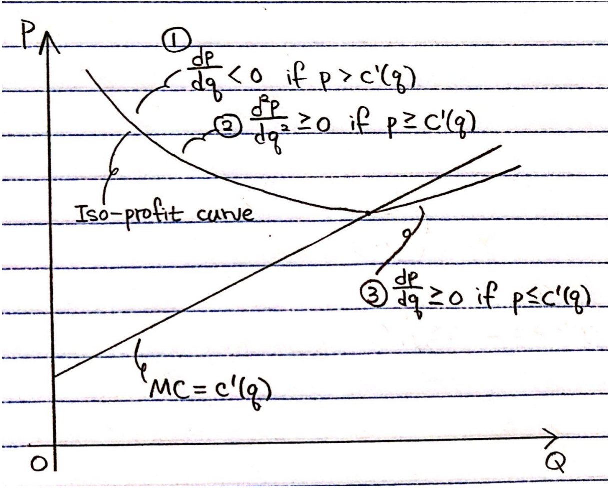
$c''(q) \geq 0$ since $c(\cdot)$ is weakly convex, so

If $p \geq c'(q)$ then $\dfrac{d^2p}{dq^2} \geq 0,$ sign undetermined otherwise.

- Both slope and concavity depend on whether price p is above or below the marginal cost $c'(q)$.

Part (a) - Suggested Solution

- Since $c(\cdot)$ is weakly convex, $c'(q)$ (i.e. MC) is weakly increasing:



Parts (b) & (c)

(b) Draw the consumers' indifference curves in the $[q, p]$ plane, assuming no income effects on the demand for q. Note that you are not drawing them in the $[q, x]$ plane. What can you say about their slopes and convexity/concavity? How do they relate to the Hicksian demand curve? The Marshallian demand curve?

(c) Repeat part (b) allowing q to be a normal good.

Parts (b) & (c) - Suggested Solution

- Denote income as m and normalize the price of good x as 1. By choosing to purchase q, the consumer purchases $x = m - pq$:

$$u(q, m - pq) = \overline{U}, \qquad F(p, q) = u(q, m - pq) - \overline{U} = 0.$$

- By Implicit Function Theorem,

$$\frac{dp}{dq} = -\frac{\partial F/\partial q}{\partial F/\partial p} = -\frac{\frac{\partial u}{\partial q} + \frac{\partial u}{\partial x} \cdot (-p)}{\frac{\partial u}{\partial x} \cdot (-q)} = \frac{\frac{\partial u}{\partial q} - p \cdot \frac{\partial u}{\partial x}}{q \cdot \frac{\partial u}{\partial x}}.$$

$$\therefore q\frac{dp}{dq} = \frac{\partial u/\partial q}{\partial u/\partial x} - p = MRS - p.$$

- Therefore, slopes become (regardless of income effects)

$$\frac{dp}{dq} \geq 0 \text{ if } p \leq \frac{\partial u/\partial q}{\partial u/\partial x}, \qquad \frac{dp}{dq} < 0 \text{ if } p > \frac{\partial u/\partial q}{\partial u/\partial x}.$$

Parts (b) & (c) - Suggested Solution

- The indifference curve peaks on the (Marshallian) demand curve (identical to Hicksian demand curve assuming no income effect).
- Assuming continuously differentiable indifference curves, our results for slopes already imply that the indifference curves are locally concave at the demand curve.
- Let q be the quantity on the demand curve. Then, $\frac{dp}{dq} > 0$ for $q - \epsilon$ and $\frac{dp}{dq} < 0$ for $q + \epsilon$, so $\frac{d^2p}{dq^2} < 0$ at q.
- It turns out that the indifference curve is strictly concave not just on the demand curve, but also everywhere below the demand curve (but not necessarily above the demand curve).
- Income effect does not affect the characterization of the indifference curve (both slopes and shapes). Nevertheless, assuming no income effect (e.g. quasi-linear utility functions) simplifies the calculations.

Parts (b) & (c) - Suggested Solution

- Example with no income effect:

$$u(q,x) = f(q) + x, \qquad \therefore \frac{\partial u}{\partial x} = 1.$$

- Slope becomes

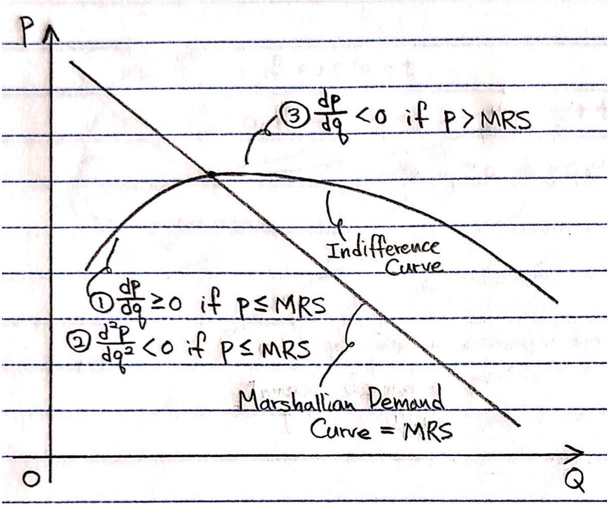
$$\frac{dp}{dq} = \frac{\frac{\partial u}{\partial q} - p}{q},$$

$$\therefore \frac{dp}{dq} \geq 0 \quad \text{if} \quad p \leq \frac{\partial u}{\partial q}, \qquad \frac{dp}{dq} < 0 \quad \text{if} \quad p > \frac{\partial u}{\partial q}.$$

- Concavity becomes

$$\frac{d^2 p}{dq^2} = \frac{q \frac{\partial^2 u}{\partial q^2} - 2\left[\frac{\partial u}{\partial q} - p\right]}{q^2} < 0 \ \text{ if } \ p \leq \frac{\partial u}{\partial q}.$$

Parts (b) & (c) - Suggested Solution



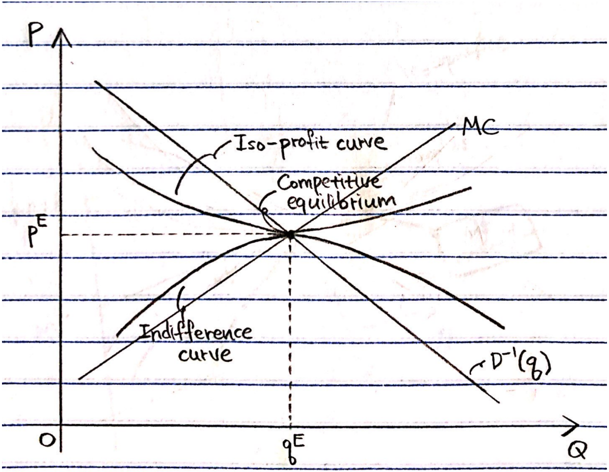
Part (d)

(d) How is the competitive equilibrium related to the indifference curves of consumers and producers? Are there other $\{q, p\}$ combinations that result in the same joint consumer-producer surplus?

Part (d) - Suggested Solution

- With consumers and producers in the economy, the competitive equilibrium allocation is at the intersection of MC and (inverse) demand curves: $c'(q^E) = MRS(q^E) = D^{-1}(q^E) = p^E$.
- In the parts (a) and (b), we found that consumer indifference curve peaks on demand and producer iso-profit curve bottoms on MC.
- Since $\{p^E, q^E\}$ is simultaneously on both the demand curve and the MC curve, consumer indifference curve peaks and the producer iso-profit curve bottoms at the competitive equilibrium.
- In fact, if both consumer and producer indifference curves were continuously differentiable, they would be tangent at $\{p^E, q^E\}$.

Part (d) - Suggested Solution



Part (d) - Suggested Solution

- Joint consumer-producer surplus:

$$S = \Big[u(q) - pq\Big] + \Big[pq - c(q)\Big] = u(q) - c(q),$$

$$\therefore \frac{\partial S}{\partial p} = 0.$$

  Such a result makes sense since pq is simply a transfer of surplus from consumers to producers, without increasing a total surplus.

- Therefore, other $\{q, p\}$ combinations that result in the same joint surplus as $\{q^E, p^E\}$ make up a vertical line on the $[q, p]$ plane:

$$\Big\{\ (q, p)\ \mid\ q = q^*,\ p \in \mathbb{R}^+\ \Big\}.$$

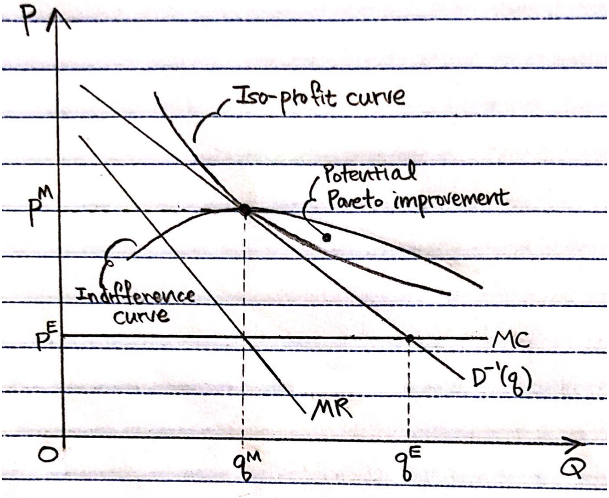
Part (e)

(e) Suppose that (i) producers could jointly determine their quantities and prices and (ii) consumers are free to purchase as much or little as they want at the price selected by the producers. How is this equilibrium related to the indifference curves of consumers and producers?

Part (e) - Suggested Solution

- Since producers can now jointly determine their quantities and prices, they act as a monopoly (and share profits among themselves).
- Monopolist finds the intersection that makes $MR = MC$, and then sets the price p^M so that $D(p^M)$ is the quantity at which $MR = MC$. Given the price p^M, consumers demand $q^M = D(p^M)$. The corresponding quantity (q^M) is lower than the quantity at the competitive equilibrium (q^E).
- At the new monopoly allocation $\{q^M, p^M\}$, the indifference curves of consumers and producers are no longer tangent.
- Consequently, there exists an allocation at which both consumers and producers are better off (i.e. the new allocation under monopolist is not Pareto optimal). However, there needs to be some negotiation/commitment for consumers and producers to achieve such Pareto improvement.

Part (e) - Suggested Solution

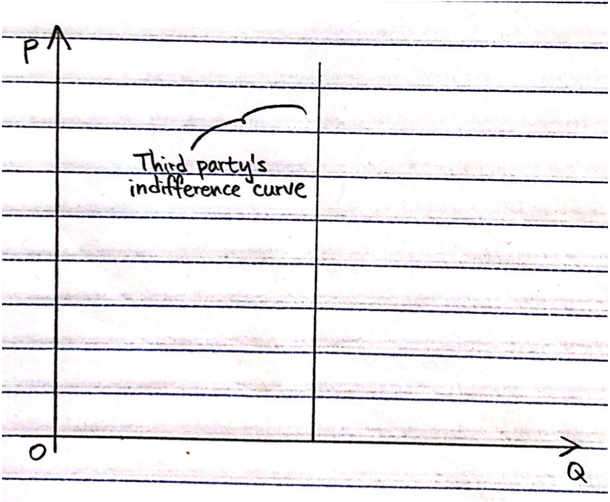


Part (f)

(f) Now suppose that a third party - neither a producer nor a consumer - is harmed by the activity in this industry. Their harm, measured in units of revenue, is tq, where $t > 0$ is a constant. Draw their indifference curves in the $[q, p]$ plane.

Part (f) - Suggested Solution

- The third party's harm tq only depends on the quantity q:



Part (g)

(g) Draw the iso-surplus curves for the combined surplus of the producers and the third parties. What can you say about their slopes and convexity/concavity?

Part (g) - Suggested Solution

- Combined surplus of the producers and the third parties:

$$S = \Big[pq - c(q)\Big] + \Big[-tq\Big].$$

- Drawing an iso-surplus curve for some fixed surplus $\overline{S}$:

$$\overline{S} = pq - c(q) - tq, \quad \therefore p = \frac{\overline{S} + c(q) + tq}{q}.$$

- Let's check slope:

$$\frac{dp}{dq} = \frac{q[c'(q) + t] - [\overline{S} + c(q) + tq]}{q^2} = \frac{c'(q) + t - p}{q},$$

$$\therefore \frac{dp}{dq} \geq 0 \text{ if } p \leq c'(q) + t, \qquad \frac{dp}{dq} < 0 \text{ if } p > c'(q) + t.$$

Part (g) - Suggested Solution

- Checking concavity:

$$\frac{d^2p}{dq^2} = \frac{q^2[c'(q) + t + qc''(q) - c'(q) - t] - 2q[qc'(q) - \overline{S} - c(q)]}{q^4}$$

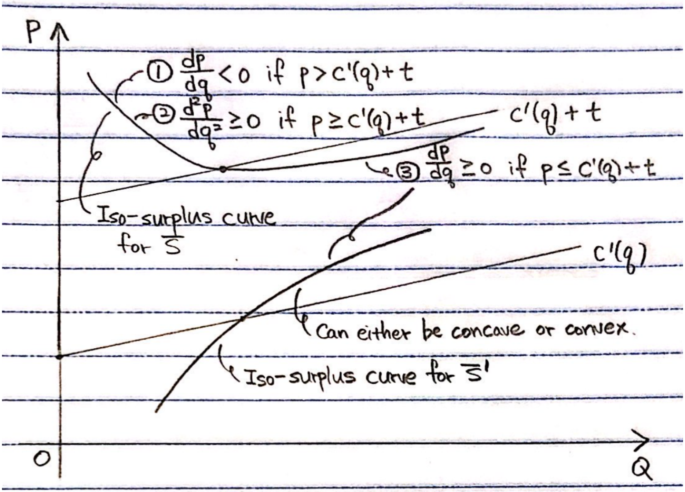
$$= \frac{q^3c''(q) - 2q^2c'(q) + 2q[\overline{S} + c(q)]}{q^4} = \frac{c''(q)}{q} + \frac{2}{q^2}[p - c'(q) - t].$$

$c''(q) \geq 0$ since $c(\cdot)$ is weakly convex, so

If $p \geq c'(q) + t$ then $\dfrac{d^2p}{dq^2} \geq 0$, sign undetermined otherwise.

- Both slope and concavity depend on whether price p is above or below the marginal cost plus constant per-unit negative externality, $c'(q) + t$. Note that the results are almost identical to the part (a), except that we now compare p with $c'(q) + t$ instead of just $c'(q)$.

Part (g) - Suggested Solution



Part (h)

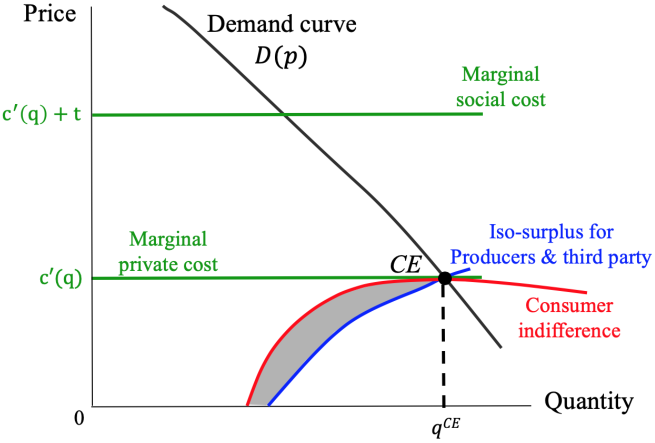
(h) Taking into account the third parties and assuming they can cooperate with the producers, what $\{q, p\}$ allocations are Pareto improvements on the competitive equilibrium? How does your finding compare to the recommendation that negative externalities should be taxed?

Part (h) - Suggested Solution

- In the competitive equilibrium, consumers and producers end up at the intersection of the marginal "private" cost, $c'(q)$, and the demand curve (point CE in the figure). That's because they don't account for the negative externality imposed to the third parties.
- At such CE, consumer indifference curve peaks and producer iso-profit curve bottoms. However, as shown in the Part (g), iso-surplus curve for combined surplus of the producers and the third parties continues to increase below and above the marginal private cost as long as the price is less than the marginal social cost $c'(q) + t$.
- All else equal, consumers are better off when the prices are lower. Therefore, lower consumer indifference curve corresponds to higher consumer utility. On the other hand, producers and the third parties together are better off when the prices are higher. Therefore, higher iso-surplus curve corresponds to higher joint surplus.

Part (h) - Suggested Solution

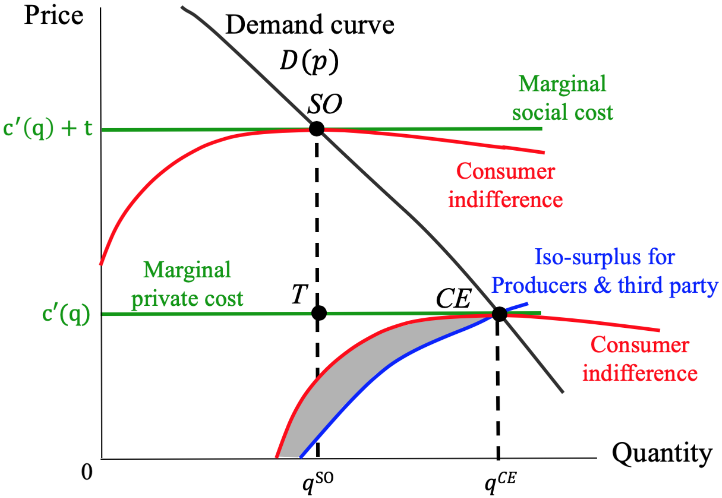
- Therefore, the shaded region corresponds to the allocations that are Pareto improvements to the competitive equilibrium:



Part (h) - Suggested Solution

- How do such Pareto improvement allocations compare to taxation?
- Pigouvian approach is to tax producers by t per sale so that their marginal cost becomes $c'(q) + t$ (i.e. internalize the externality).
- The new competitive equilibrium with taxation is now at the intersection of the demand curve and the marginal "social" cost (point SO in the figure). The new equilibrium quantity q^{SO} is socially optimum and less than q^{CE} (typical in negative externality).
- Nevertheless, consumers are much worse off at SO than at CE. Therefore, a typical way to compensate is by rebating the excise tax revenue to consumers in a way that is independent of their consumption choices.
- The new allocation after taxation and rebate is the allocation T in the figure. Note that it still is not Pareto optimal relative to CE since consumers are worse off. That's caused by excess burden/deadweight loss of the tax.

Part (h) - Suggested Solution



References

1. Jaffe, Sonia, Robert Minton, Casey B. Mulligan, and Kevin M. Murphy. Chicago price theory (Mostly Chapter 13). Princeton University Press, 2019.
2. Mulligan, Casey B. “Beyond Pigou: externalities and civil society in the supply–demand framework.” Public Choice (2023): 1-18.

Both of them are on the reading list!