ECMA31000: Introduction to Empirical Analysis
Exam 1 2020

Question 1 (10 points) Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $A, B \in \mathcal{F}$. Prove that if $\mathrm{P}(B) > 0$ and $\mathrm{P}(A|B) = \mathrm{P}(A|B^c)$, then A and B are independent events.

Question 2 Let $\{X_n\}_{n\geq 1}$ and X be random variables defined on the same probability space.

a) (5 points) Suppose $X_n \overset{a.s.}{\to} X$. Is it true that $\mathrm{E}\left(|X_n - X|^2\right) \to 0$?

b) (10 points) Suppose $\mathrm{E}\left(|X_n - X|^2\right) = \frac{1}{n^2}$. Is it true that $X_n \overset{a.s.}{\to} X$?

c) (10 points) Let $\{X_i\}_{i\geq 1}$ be an iid sequence with $\mathrm{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Does $\mathrm{E}\left(\left|\bar{X}_n - \mu\right|\right) \to 0$ as $n \to \infty$?

Question 3 (15 points) Let $\{X_i\}_{i\geq 1}$ be an iid sequence with $\mathrm{E}(X_i) = \mu$ and $Var(X_i) = 1$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Find constants $r \geq 0$ and c such that $n^r\left(\bar{X}_n^2 - c\right) \overset{d}{\to} Y$ for some non-degenerate random variable Y.

Question 4 a) (5 points) Is it true that $X_n = O_p(1)$ implies $X_n = o_p(1)$?

b) (10 points) Consider a sequence of random variables $\{X_n\}_{n\geq 1}$ such that $X_n > 0$ for all n. Prove that $X_n = O_p\left(\mathrm{E}\left(|X_n|^r\right)^{1/r}\right)$ for any $r > 0$ such that $\mathrm{E}\left(|X_n|^r\right)$ exists.

c) (15 points) Show that if $\frac{X_n}{Y_n} \overset{p}{\to} 0$ and $Y_n = O_p(1)$, then $X_n \overset{p}{\to} 0$.

Hint: First show that $|X_n| \leq B_\delta \left|\frac{X_n}{Y_n}\right| \mathbf{1}\left(|Y_n| < B_\delta\right) + |X_n|\,\mathbf{1}\left(|Y_n| \geq B_\delta\right)$.

Question 5 Consider the following example from Lecture 7. We wish to learn the population mean wage (or earnings potential), $\mathrm{E}(W)$. For each individual i in our sample, we observe whether or not they are employed. That is, we observe

$$D_i = \begin{cases} 1 & \text{if individual } i \text{ is employed,} \\ 0 & \text{if individual } i \text{ is unemployed.} \end{cases}$$

Assume that $0 < \mathrm{P}(D_i = 1) < 1$. We observe the wage of individual i, W_i, if and only if they are employed. The wage of unemployed individuals is not observed. That is, we observe

$$Z_i = \begin{cases} W_i & \text{if individual } i \text{ is employed,} \\ 0 & \text{if individual } i \text{ is unemployed.} \end{cases}$$

We assume $\mathrm{E}(W)$ exists, and we observe an iid sample $\{Z_i, D_i\}_{i=1}^n$.

a) (10 points) Is $\mathrm{E}(W)$ point identified? Explain using the definition of point identification.

b) (10 points) We try to estimate $\mu_W = \mathrm{E}(W)$ using the following estimator:

$$\hat{\mu}_W = \frac{\sum_{i=1}^n Z_i}{\sum_{i=1}^n D_i}.$$

Find the almost sure limit of $\hat{\mu}_W$ and justify your answer. Write your answer as a feature of the joint distribution of (W, D). Derive a condition under which this almost sure limit equals $\mathrm{E}(W)$.