Problem Set 4 - Question 1

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Question, I

A local government plans to levy a tax on the use of fossil fuel by its producers. Here we want to assess the effect of the tax on worldwide greenhouse gas (GHG) emissions, which are proportional to the amount of fuel consumed worldwide.
A potentially helpful tool is the Edgeworth box, with the local economy in one corner and the rest of the world (ROW) in the other. The dimensions of the box, both quantities, are fossil fuels and a composite of all other production inputs. Each economy's production exhibits constant returns in the two inputs. Until part (h), assume that both production factors are mobile between sectors.
Consumers have homothetic preferences over the local output and the output in the rest of the world. For simplicity, we represent that as a relative demand curve for the two that depends on the relative price of each area's output.

Question, II

- *a) Holding constant the dimensions of the box, what is the effect of the tax on global emissions? Don't overthink this.*
- *b) If both economies have the same production function, how is the no-tax equilibrium represented in the box? What if their production functions are different?*
- *c) How is the equilibrium with local tax represented?*

Question, III

- *d) If we were to augment the Edgeworth box with an endogenous supply of fossil fuel, and the model parameters were such that the tax reduce the ROW price of fuel relative to other production inputs, what can you say about the impact of the tax on global emissions? If more information is needed to sign the effect, explain.*
- *e) Absent the tax, how would a shift in relative demand for the local economy good affect global emissions? What model parameters determine the magnitude of the effect.*
- *f) Holding constant the relative demand curve, can you sign the effect of the local tax on global emissions? What model parameters determine its magnitude?*

Question, IV

- *g) Thus far, we have held fixed each economy's production function. If the ROW has a choice of more or less fuel-intensive production technologies, how would the local tax affect that choice?*
- *h) The Edgeworth Box approach rules out transport costs between the two areas. Discuss how your answer to (f) might be different with transport costs. Does it matter whether the transportation is fuel intensive?*

Model

- A general equilibrium model.
- Assume free trade and free mobility of production factors.
- Production:
  - Two factors.
    - Fossil fuels (g) with exogenous supply $\bar{g}$.
    - A composite input (k) with exogenous supply $\bar{k}$.
  - Competitive firms in two sectors with CRS production technology.
    - Local sector produces output x with $F(g, k)$.
    - ROW sector produces output y with $G(g, k)$.
- Consumers:
  - Homothetic preferences $u(x, y)$ over two outputs.
    - Homotheticity: $u(x, y) = f(v(x, y))$ for some strictly increasing f and v homogenous of degree one.
  - Two representative consumers in local and ROW economy.
    - Aggregation: Representative consumers exist due to homotheticity.
  - Firms are owned by consumers.
  - Endowment $g_1^e, k_1^e, g_2^e, k_2^e$ are also owned by consumers.

Assumptions

- Standard assumptions on u and F, G.
  - Convex preferences.
  - Inada conditions (to rule out corner solutions).
  - Diminishing returns.

General Equilibrium, Concept

- What is an equilibrium?
- A set of prices and allocations satisfying the following conditions:
  - Consumers maximize utility subject to budget.
  - Firms maximize profits subject to technology.
  - All markets clear.

General Equilibrium, Definition I

- An equilbrium is a set of:
  - Prices.
    - Factor wages w for fossil fuel and r for the composite input.
    - Output prices p_x for local output and p_y for ROW output.
  - Allocations.
    - Factors g_1, k_1 hired by local, and g_2, k_2 hired by ROW.
    - Outputs x_1, y_1 consumed by local, and x_2, y_2 consumed by ROW.
- Such that the following are satisfied.

General Equilibrium, Definition II

- Consumers maximize utility subject to budget.
    - Local consumers:

$$\max_{x_1, y_1} u(x_1, y_1)$$
$$\text{s.t. } p_x x_1 + p_y y_1 \leq M_1 := w g_1^e + r k_1^e + \Pi_1$$

    - ROW consumers:

$$\max_{x_2, y_2} u(x_2, y_2)$$
$$\text{s.t. } p_x x_2 + p_y y_2 \leq M_2 := w g_2^e + r k_2^e + \Pi_2$$

- FOCs pin down relative consumption by relative output prices

$$\frac{p_x}{p_y} = \frac{v_x\left(x_1/y_1, 1\right)}{v_y\left(x_1/y_1, 1\right)}; \ \frac{p_x}{p_y} = \frac{v_x\left(x_2/y_2, 1\right)}{v_y\left(x_2/y_2, 1\right)}$$

    - Recall homotheticity: $u(x, y) = f\left(v(x, y)\right)$ with f str. incr., v H.D.1.
- and ...

General Equilibrium, Definition III

- Firms are maximizing profits subject to technology.
    - Local firms:

$$\max_{g_1,k_1} \Pi_1 := p_x x - w g_1 - r k_1$$
$$\text{s.t. } x = F(g_1, k_1)$$

    - ROW firms:

$$\max_{g_2,k_2} \Pi_2 := p_y y - w g_2 - r k_2$$
$$\text{s.t. } y = G(g_2, k_2)$$

- FOCs pin down factor demand by factor wages

$$p_x F_g(g_1/k_1, 1) = w;\ p_x F_k(g_1/k_1, 1) = r;$$
$$p_y G_g(g_2/k_2, 1) = w;\ p_y G_k(g_2/k_2, 1) = r.$$

    - Zero profits due to CRS (Euler's theorem).
    - F_g, F_k, G_g, G_k are H.D.0 (Euler's theorem again).
- and ...

General Equilibrium, Definition III

- All markets clear:
  - Factor markets

$$g_1 + g_2 = \bar{g}$$
$$k_1 + k_2 = \bar{k}$$

    - Will endogenize $\bar{g}$ later.
  - Output markets

$$x_1 + x_2 = F\left(g_1, k_1\right)$$
$$y_1 + y_2 = G\left(g_2, k_2\right)$$

General Equilibrium, Comments

- Convince yourself this definition makes sense:
  - The model 12 unknowns (4 prices and 8 allocations).
  - Equilibrium conditions provide 6 FOCs and 4 feasibility constraints.
- What are the two missing equations?
  - Prices are relative.
    - Normalize $p_x = 1$.
  - Homotheticity: Only the ratio x_i/y_i is determined.
    - Need one consumer budget constraint (either local or ROW).

General Equilibrium, Simplification

- Due to CRS and homotheticity, we can focus on relative quantities.
- Focus on relative prices:
  - Already did so by $p_x = 1$.
  - Go further: w/p_y and r/p_y (w/r is the ratio of the two).
- Focus on relative factor demand: g_1/k_1 and g_2/k_2
  - g_1, k_1, g_2, k_2 determined by g_1/k_1 and g_2/k_2 plus market clearing

$$g_1 + g_2 = \bar{g};\ k_1 + k_2 = \bar{k}$$

- Focus on relative consumption: x/y where $x := x_1 + x_2$; $y := y_1 + y_2$
  - The split between x_1, x_2 and y_1, y_2 determined by wealth (endowment).
  - Note that $x/y = x_1/y_1 = x_2/y_2$ due to homotheticity.

General Equilibrium, Equations

- Focus on relative quantities only:
  - Relative prices: w, r, p_y.
  - Relative quantities: $g_1/k_1, g_2/k_2, x/y$.
- Equilibrium conditions:

$$\frac{v_x(x/y,1)}{v_y(x/y,1)} = \frac{1}{p_y}; \frac{x}{y} = \frac{wg_1 + rk_1}{wg_2 + rk_2} p_y;$$
$$\frac{F_g(g_1/k_1,1)}{F_k(g_1/k_1,1)} = \frac{w}{r}; \frac{G_g(g_2/k_2,1)}{G_k(g_2/k_2,1)} = \frac{w}{r}$$
$$\frac{F_g(g_1/k_1,1)}{G_g(g_2/k_2,1)} = \frac{1}{p_y}; \frac{F_k(g_1/k_1,1)}{G_k(g_2/k_2,1)} = \frac{1}{p_y}$$

  - Only prices and relative quantities are determined.
    - Trade-offs are transparent in this form.
  - Levels determined by feasibility constraints (left out intentionally).
    - Augment the system with the left-out constraints to solve.

General Equilibrium, with Local Tax on g

- Suppose there is a tax τ levied on locally employed fossil fuel g_1.
  - Equilibrate after-tax factor wages $(1-\tau)w_1 = w_2 := w$.
  - Government revenue transferred back to local consumers.
- Equilibrium conditions:

$$\frac{v_x(x/y,1)}{v_y(x/y,1)} = \frac{1}{p_y}; \frac{x}{y} = \frac{wg_1/(1-\tau) + rk_1}{wg_2 + rk_2} p_y;$$
$$\frac{F_g(g_1/k_1,1)}{F_k(g_1/k_1,1)} = \frac{w}{(1-\tau)r}; \frac{G_g(g_2/k_2,1)}{G_k(g_2/k_2,1)} = \frac{w}{r}$$
$$\frac{F_g(g_1/k_1,1)}{G_g(g_2/k_2,1)} = \frac{1}{(1-\tau)p_y}; \frac{F_k(g_1/k_1,1)}{G_k(g_2/k_2,1)} = \frac{1}{p_y}$$

Part (a): Exogenous Total Supply

a) Holding constant the dimensions of the box, what is the effect of the tax on global emissions? Don't overthink this.

- Total emission unchanged if dimensions of the box unchanged.
  - Market clearing condition for g.
- A local carbon tax only changes the composition, not the total.

Part (b): No-Tax Equilibrium

b) If both economies have the same production function, how is the no-tax equilibrium represented in the box? What if their production functions are different?

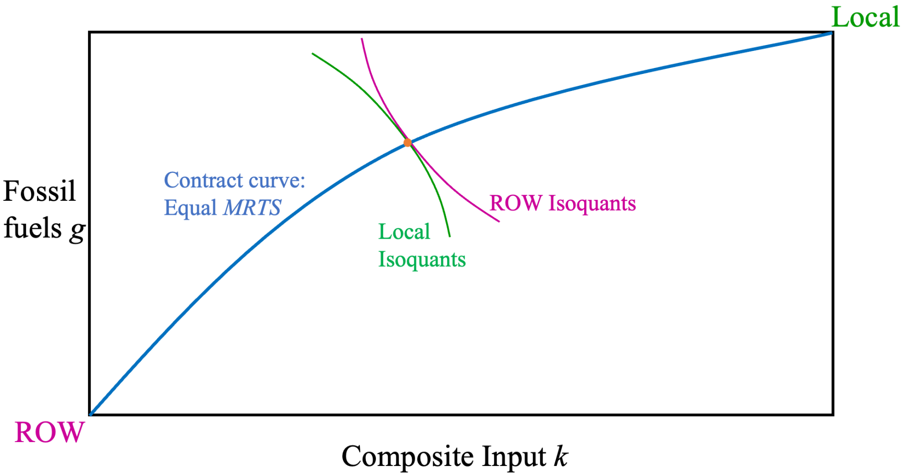
- See Edgeworth box.
  - Factor allocations g_1, k_1, g_2, k_2 presented in the box.
  - Output (and consumption) then given by production functions.
  - Prices represented by some slopes in the box. (more on this later)
- An equilibrium must be on the contract curve: Equal MRTS.

$$\frac{F_g\left(g_1/k_1, 1\right)}{F_k\left(g_1/k_1, 1\right)} = \frac{w}{r} = \frac{G_g\left(g_2/k_2, 1\right)}{G_k\left(g_2/k_2, 1\right)}$$

  - If $F = G$, then $g_1/k_1 = g_2/k_2$, diagonal contract curve.
  - If $F \neq G$, the shape of the contract curve depends on technology.
- Where on contract curve? What are the prices?
  - Depends on the endowment point.

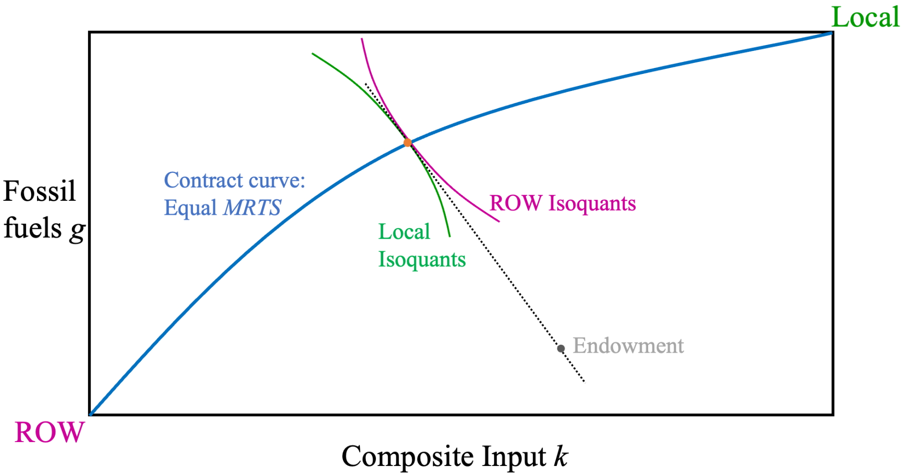
Part (b): Illustration, I

- Contract curve: Equal MRTS, firm optimality.
  - Firm optimality is necessary for an equilibrium.
  - Output x and y given by production function.



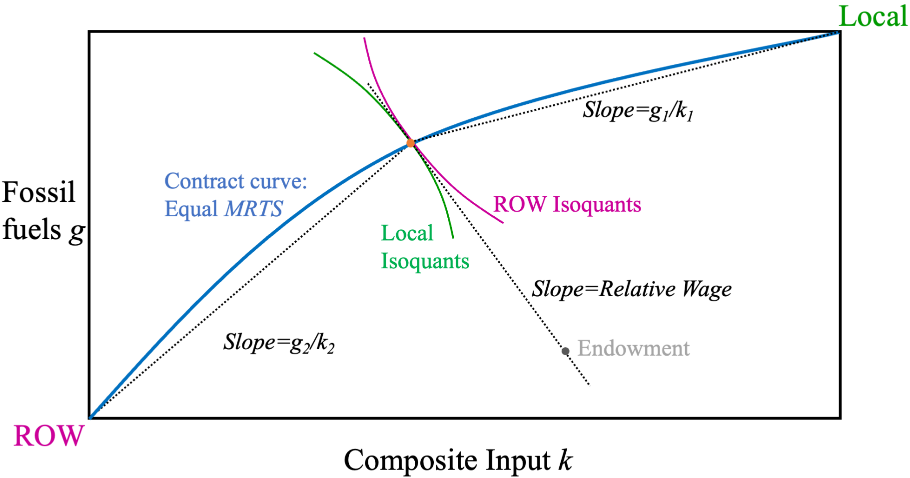
Part (b): Illustration, II

- Equilibrium point: Determined by the endowment point.
  - The two economies start with endowment, and trade factors to reach an equilibrium.



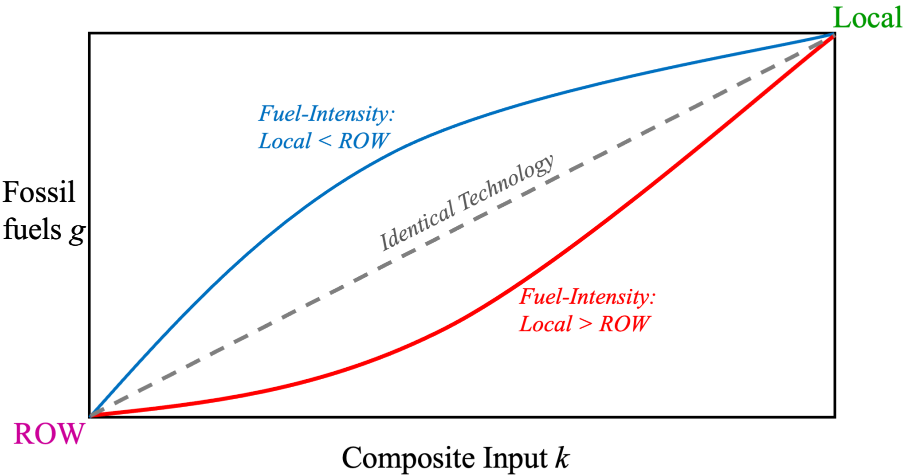
Part (b): Illustration, III

- Slopes: Relative prices and relative factor demand.
  - From endowment to equilibrium: Slope is relative wage.
  - From Origin to equilibrium: Slope is relative factor demand.



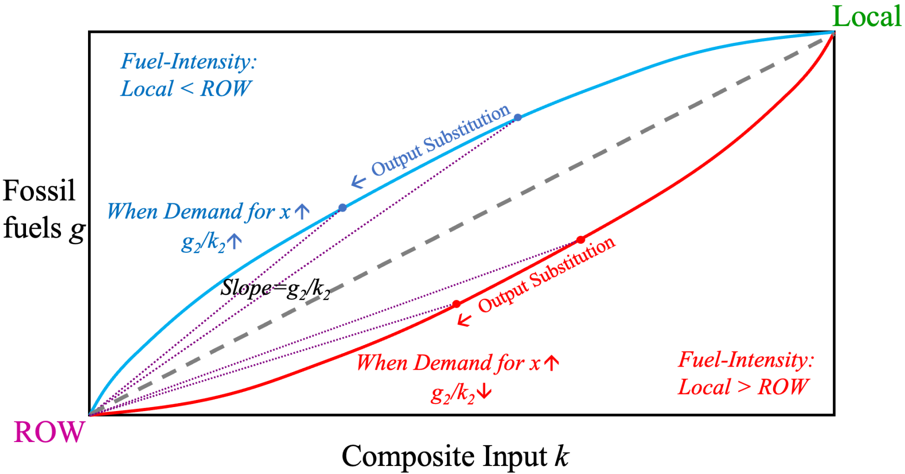
Part (b): Illustration, IV

- If $F \neq G$, the shape of the contract curve depends on technology.
  - Contract curve above diagonal if ROW is more fuel intensive than local.
  - Contract curve below diagonal if ROW is less fuel intensive than local.



Part (b): Illustration, V

- Movement on the contract curve: If demand for $x \uparrow$.
  - More x to be produced. How does w/r change?
  - Depends on relative intensity: g_2/k_2, and hence w/r, can go either way.



Part (c): Equilibrium with Tax

c) How is the equilibrium with local tax represented?

- See Edgeworth box.
- Without tax, we had

$$\frac{F_g(g_1/k_1, 1)}{F_k(g_1/k_1, 1)} = \frac{w}{r} = \frac{G_g(g_2/k_2, 1)}{G_k(g_2/k_2, 1)}$$

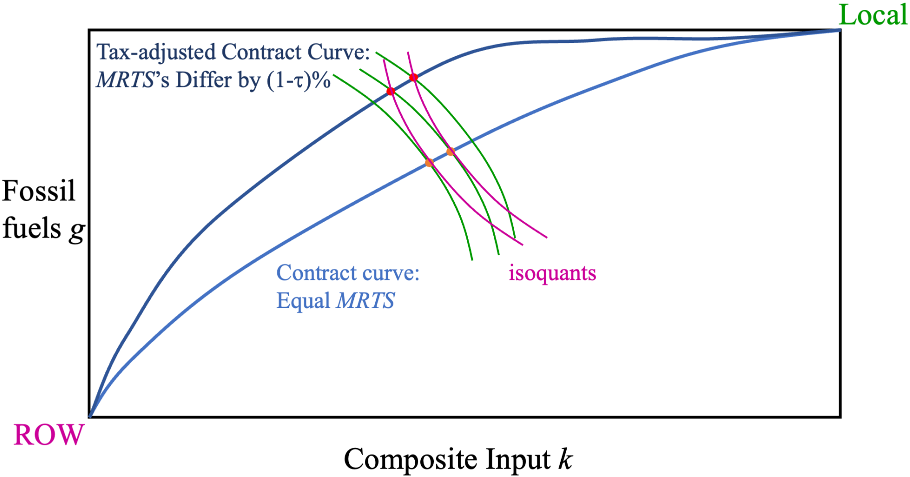
- Equilibrium on tax-adjusted contract curve: MRTS's differ by $(1-\tau)$.

$$(1-\tau)\frac{F_g(g_1/k_1, 1)}{F_k(g_1/k_1, 1)} = \frac{w}{r} = \frac{G_g(g_2/k_2, 1)}{G_k(g_2/k_2, 1)}$$

- How does the tax on g_1 affect w/r? (This is useful later.)
  - Local must choose a lower relative factor demand, $g_1/k_1 \downarrow$.
  - Local must release both factors to ROW due to lower output, g_1/k_1?.
  - Trick: With tax on g_1, difficult to sign $\Delta w/r$ with g_1/k_1.
    - Instead, use the movement of g_2/k_2 to sign $\Delta w/r$.

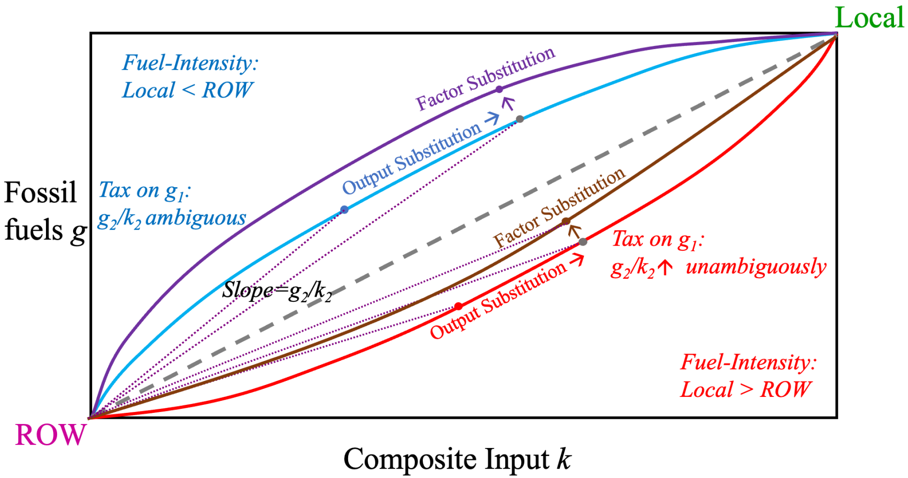
Part (c): Illustration, I

- Tax-adjusted contract curve:
  - MRTS's differ by $(1 - \tau)$.
  - Contract curve moves northwest. (Depends on how you set up the box)



Part (c): Illustration, II

- Tax on g_1: $\Delta w/r$ ambiguous if ROW more fuel-intensive.
  - Factor substitution: $g_1/k_1 \downarrow \Rightarrow g_2/k_2 \uparrow$.
  - Output substitution (scale effect): tax$\Rightarrow p_x \uparrow \Rightarrow$output $x \downarrow \Rightarrow g_2/k_2$?



Part (d): Endogenous Fuel Extraction

d) If we were to augment the Edgeworth box with an endogenous supply of fossil fuel, and the model parameters were such that the tax reduce the ROW price of fuel relative to other production inputs, what can you say about the impact of the tax on global emissions? If more information is needed to sign the effect, explain.

- According to Part (a), we need to endogenize $\bar{g}$ in order to discuss the effects on total emissions.

Endogenous Fuel Extraction: Additional Setup

- Suppose consumers are only endowed with $\bar{k}$ units of composite input.
- Fossil fuel has to be extracted using the composite input $g = H(k)$.
  - "Real" costs to extract fuel: Some resources have to be spent.
  - Technology H satisfies $H(0) = 0, H'(0) = +\infty, H'(k) > 0, H''(k) < 0$.
  - Non-linear technology, otherwise we force a constant relative wage.
- An additional sector for fossil fuel extraction.
  - k_g units of composite input employed for fossil fuel extraction.
  - These firms also owned by consumers.
    - Need to specify how profits are split between local and ROW.
    - Assume it is proportional to endowments k_1^e, k_2^e.
    - Such split is not the focus of this question.
  - Firms' problem in the fuel extraction sector

$$\max_{k_g} w\bar{g} - rk_g$$
$$\text{s.t. } \bar{g} = H(k_g)$$

Endogenous Fuel Extraction: General Equilibrium

- Equilibrium quantities:
  - Relative prices: w, r, p_y.
  - Relative quantities: $g_1/k_1, g_2/k_2, x/y$.
  - Composite input employed for fuel extraction: k_g.
- Equilibrium conditions:
  - Existing conditions

$$\frac{v_x\left(x/y,1\right)}{v_y\left(x/y,1\right)} = \frac{1}{p_y}; \frac{x}{y} = \frac{wg_1/(1-\tau)+rk_1}{wg_2+rk_2}p_y;$$

$$\frac{F_g\left(g_1/k_1,1\right)}{F_k\left(g_1/k_1,1\right)} = \frac{w}{(1-\tau)r}; \frac{G_g\left(g_2/k_2,1\right)}{G_k\left(g_2/k_2,1\right)} = \frac{w}{r}$$

$$\frac{F_g\left(g_1/k_1,1\right)}{G_g\left(g_2/k_2,1\right)} = \frac{1}{(1-\tau)p_y}; \frac{F_k\left(g_1/k_1,1\right)}{G_k\left(g_2/k_2,1\right)} = \frac{1}{p_y}$$

  - Additional constraints from endogenous fuel extraction

$$\frac{w}{r} = \frac{1}{H'\left(k_g\right)};$$

$$\bar{g} = H\left(k_g\right); \bar{k} = k_1 + k_2 + k_g$$

Part (d): Endogenous Fuel Extraction, Solution

- Fuel extraction sector's optimality condition

$$\frac{w}{r} = \frac{1}{H'(k_g)}$$

- In this model, effects on global emissions boil down to relative wage w/r.
- When $w/r \downarrow \Rightarrow k_g \downarrow$, global emissions $\downarrow$.

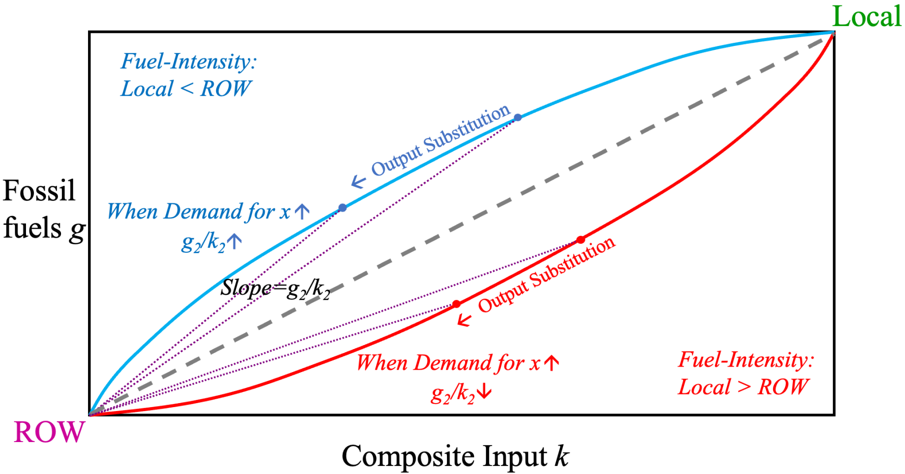
Part (e): Demand Shift on Emissions

e) Absent the tax, how would a shift in relative demand for the local economy good affect global emissions?

- Suppose demand for x goes up ($v_x \uparrow$). (Symmetric for $v_x \downarrow$.)
- Holding fixed k_g for now, sign $\Delta w/r$.
- Given the sign of $\Delta w/r$, fuel extraction firms adjust k_g accordingly.
  - If $w/r \uparrow$, global emissions $\uparrow$.
  - If $w/r \downarrow$, global emissions $\downarrow$.
- How to sign $\Delta w/r$?
  - Discussed in Part (b), this is a movement on the contract curve.

Part (e): Demand Shift on Emissions, Illustration

- Movement on the contract curve: If demand for $x \uparrow$.
  - More x to be produced. How does w/r change?
  - Depends on relative intensity: g_2/k_2, and hence w/r, can go either way.



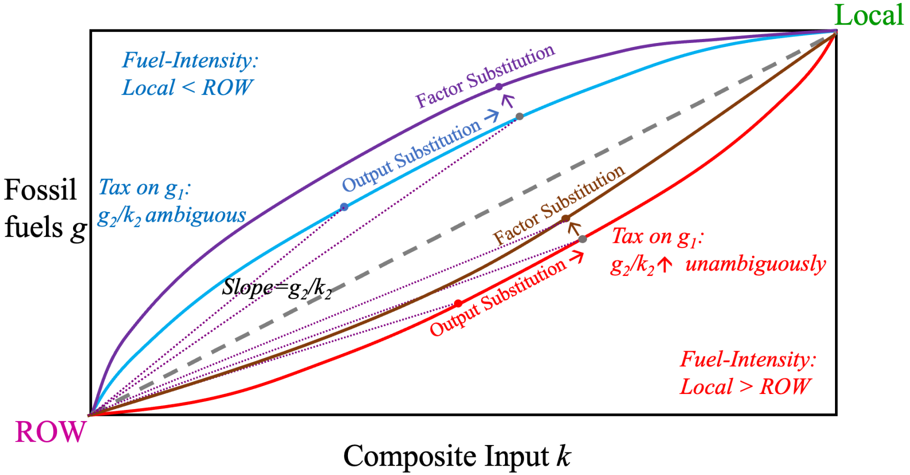
Part (f): Local Tax on Emissions

f) Holding constant the relative demand curve, can you sign the effect of the local tax on global emissions? What model parameters determine its magnitude?

- Holding fixed k_g for now, sign $\Delta w/r$.
- Given the sign of $\Delta w/r$, fuel extraction firms adjust k_g accordingly.
  - If $w/r \uparrow$, global emissions $\uparrow$.
  - If $w/r \downarrow$, global emissions $\downarrow$.
- How to sign $\Delta w/r$?
  - Discussed in Part (b), output substitution and factor substitution.
  - Magnitude determined by:
    - Shape of utility
    - Relative intensity
- You can also work out numerical examples. E.g.,
  - Local has a Leontief technology, and ROW has Cobb-Douglas.
    - Leontief shuts down factor substitution. Scale effect dominates.
  - Two Cobb-Douglas should also work.
    - More computation, though.

Part (f): Local Tax on Emissions, Illustration

- Tax on g_1: $\Delta w/r$ ambiguous if ROW more fuel-intensive.
  - Output substitution: Price of $x \uparrow$ due to tax $\Rightarrow$ output $x \downarrow \Rightarrow g_2/k_2$?
  - Factor substitution: $g_1/k_1 \downarrow \Rightarrow g_2/k_2 \uparrow$.



Part (g): Endogenous Production Technology

g) If the ROW has a choice of more or less fuel-intensive production technologies, how would the local tax affect that choice?

- Suppose tax on $g_1 \Rightarrow w/r \downarrow$. (Similar for tax$\Rightarrow w/r \uparrow$.)
- Suppose ROW firms were using less fuel-intensive G and are considering a switch to fuel-intensive G', where $G'_g/G'_k > G_g/G_k$.
    - Switching to G' achieves the same MRTS with a higher g_2/k_2.
    - The switch is cost-saving for any output level when $w/r \downarrow$.
    - A profit maximizing firm is necessarily cost minimizing.
- ROW more likely to choose fuel-intensive technology.

Part (h): Transport Costs, I

h) The Edgeworth Box approach rules out transport costs between the two areas. Discuss how your answer to (f) might be different with transport costs. Does it matter whether the transportation is fuel intensive?

- Assume it is costly to transport fuel, but k is still mobile.
  - Transporting g should cost some k (and potentially g).
  - To be precise, add a transport sector with production function $T(g, k)$.
- Key difference:
  - MRTS not equilibrated.
  - MRTS can differ by at most the transport costs.
- No need to completely solve the model. (bonus point if you did!)

Part (h): Transport Costs, I

- To fix ideas, assume the transport cost is infinity.
  - That is, an infinite amount of k or g is needed to move g.
  - Fuel can only be produced where it is employed.
  - It matters now where the fuel reserves are
    - If fuel can only be extracted locally, taxing g_1 clearly reduces k_g.
    - If fuel can only be extracted from ROW, taxing g_1 only works through the output substitution channel.
- Suppose tax on $g_1 \Rightarrow w_1/r \downarrow$.
  - Local: Less fuel extracted. Less k_1 employed due to reduced output.
  - ROW: More k_2 employed. Potentially more fuel extracted.
  - MRTS not equilibrated. Overall effect ambiguous.