(Non-Examinable) Addendum to Lecture 6

Recall that a square $n \times n$ matrix A is invertible iff $\det(A) \neq 0$. Since the determinant is a continuous function of the elements of A, if A is invertible then a small perturbation of the elements of A will not break invertibility because the determinant will remain non-zero. In other words, there is an $\epsilon > 0$ such that $\|B - A\| \leq \epsilon$ implies B is invertible, where $\|\cdot\|$ denotes the frobenius norm (same as the euclidean norm after stacking the columns in a single vector). Define

$$B_\epsilon(A) := \left\{D \in \mathbb{R}^{n \times n} \,|\, \|D - A\| \leq \epsilon\right\}$$

and

$$F_A := \max_{D \in B_\epsilon(A)} \left\|D^{-1}\right\|.$$

The maximum is attained because the norm composed with the inverse is a continuous function on the compact set $B_\epsilon(A)$.

Now consider $[A + C_n]^{-1}$, where A is invertible and $C_n = o_p(1)$, meaning $\|C_n\| \xrightarrow{p} 0$. The idea is to trap all but finitely many terms of the sequence $A + C_n$ in $B_\epsilon(A)$ with high probability, since we know on this set the matrix norm is bounded above. We have for any $\delta > 0$ that $\exists N_\delta$ such that $\forall n \geq N_\delta$:

$$\mathrm{P}(A + C_n \in B_\epsilon(A)) = \mathrm{P}(\|C_n\| \leq \epsilon) \geq 1 - \delta.$$

Moreover, $A + C_n \in B_\epsilon(A)$ implies

$$\left\|[A + C_n]^{-1}\right\| \leq F_A.$$

Therefore, if $n \geq N_\delta$,

$$\mathrm{P}\left(\left\|[A + C_n]^{-1}\right\| \leq F_A\right) \geq 1 - \delta.$$

Finally, for $k = 1, \ldots, N_\delta$, choose F_A^k such that

$$\mathrm{P}\left(\left\|[A + C_k]^{-1}\right\| \leq F_A^k\right) \geq 1 - \delta,$$

and set $G = \max\left\{F_A, F_A^1, \ldots, F_A^k\right\}$. Provided each $A + C_k$ is invertible a.s. we have for all n that $\mathrm{P}\left(\left\|[A + C_n]^{-1}\right\| \leq G\right) \geq 1 - \delta$, so $[A + o_p(1)]^{-1} = O_p(1)$.