Price Theory I: Problem Set 3 Question 1

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The Question

"Here we consider the opportunity costs of closing schools for a year during a pandemic. We assume that no net learning occurs during that year: students end the year with the same human capital they had at the beginning.

To systematically analyze this issue, we treat schooling as a production process with many inputs. The inputs are especially effort and attention by students, teachers, and parents, at various ages. The ultimate output is human capital that enhances earnings and household production during the students' adult lives, the present value of which is in the millions of dollars per person."

The Setup

- *Simple model built upon Ben-Porath (1967).*
- Consider the problem for a given agent (kid).
  - Not a representative agent model; simply consider the problem facing one of many agents.
- Time is continuous with $t \in [0, T^d]$.
  - Pandemic starts at $t = 0$ and ends at $t = t_0$.
  - Agent becomes adult at $t = T \in (t_0, T^d)$, and ceases to accumulate human capital.
  - Agent dies at T^d.
  - Agent discounts time at rate $r > 0$ (how impatient agent is).
    - ★ Large r means agent is impatient. Future consumption is not very good substitute for current consumption.
    - ★ $r \to \infty$: agent only values current consumption.
    - ★ $r = 0$: future consumption and current consumption are perfect substitutes.

The Setup

- **Agent:**
  - Before becoming an adult, agent accumulates human capital by going to school. During this period, agent may allocate a portion of their human capital to household production, earning a wage rate ν.
  - Agent ceases human capital accumulation (schooling) and enters workforce after reaching adulthood. Agent receives earnings from their accumulated human capital at a wage rate μ.
    - ★ These earnings could be from either working or hosuehold production.
  - Agent consumes their earnings from working and household production.
  - Agent must use human capital in order to produce more human capital (more later).
  - Agent makes human capital investment decision to maximize present value of consumptions.
    - ★ Trade-off: agent must trade off higher future consumptions against lower current consumptions (since human capital investments must be made).
    - ★ Trade-off depends on relative prices of inputs.

The Setup

- **Schooling:**
  - Schooling is a production process with joint effort by kids, teachers and parents as input.
    - Let e^{kid}, $e^{teacher}$ and e^{parent} denote effort by kids, teachers and parents respectively.
    - Joint effort by kids, teachers and parents is produced according to a composite function $e = \phi(e^{kid}, e^{teacher}, e^{parent})$.
    - Note that if e^{kid}, $e^{teacher}$ and e^{parent} are perfect substitutes, then $e = e^{kid} + e^{teacher} + e^{parent}$; if e^{kid}, $e^{teacher}$ and e^{parent} are perfect complements, then $e = \min\{e^{kid}, e^{teacher}, e^{parent}\}$.
  - Schooling D is produced according to $D = \Lambda(e)$, where $\Lambda(\cdot)$ is increasing in its argument.
  - Schooling D has a "shadow" price p, which corresponds to opportunity cost of effort.
    - E.g., during pandemic, agents stay home and there are fewer activities they could do, and thus the shadow price p is relatively low.
    - The shadow price p can be thought of as a composite price, and is a weighted average of the shadow prices facing kids, teachers and parents.

The Setup

- **Human Capital:**
  - Human capital stock is denoted by K.
  - Before reaching adulthood, the rental rate for a unit of human capital per unit of time is ν; after reaching adulthood, the rental rate is μ (human capital is elastically supplied).
  - The stock of human capital, K, deteriorates at rate $\delta = 0$ (for simplicity). That is,

    $$\dot{K}_t = Q_t$$

    where Q denotes the flow production of human capital.
  - Human capital is produced using human capital investment and schooling as input. That is,

    $$Q_t = F(\alpha, s_t K_t, D_t)$$

    where $s_t \in [0, 1]$ is the fraction of human capital stock allocated to produce human capital, and α denotes production technology. Assume $F(\alpha, 0, D_t) = 0$, and production function is homogeneous of degree less than one.

The Setup - Comments

- Key assumptions:
  - Agent's utility does not depend on time as an input. In each period, a fixed amount of time is devoted to activities that produce earnings and human capital accumulation.
    - We abstract agent's work-leisure decision away from the model. Instead, the tradeoff is captured by the shadow price of effort.
  - The stock of human capital does not enter into agent's utility function.
  - Human capital is elastically supplied in the rental market. Agents are price takers.
  - T^d is assumed with certainty to be the end of life. Assume pandemic does not affect life expectancy.
    - Plausible in COVID-19 pandemic, given that children are low-risk.
    - T^d could also be thought of as expected end of life.
  - Kids can only control their human capital investment and their own effort. They take teachers and parents' effort as given.

Agent's Problem

- Suppose at T (when agent becomes an adult), their human capital stock is K_T, then the discounted value of future earnings at time T is:

$$E_T = \int_T^{T^d} e^{-r(\tau - T)} \mu K_T d\tau = \frac{\mu}{r}\left[1 - e^{-r(T^d - T)}\right] K_T$$

  - E_T is linearly increasing in K_T.
  - Thus, we can focus on K_T and solve agent's problem before time T (when they become adult).

Agent's Problem

- Cost of investing in human capital is:

$$I_t = \nu s_t K_t + pD_t \tag{1}$$

  - First term denotes foregone earnings and second term denotes shadow cost of schooling.
  - Agent minimizes cost of human capital investment subject to human capital production constraint:

$$\min_{s_t, e^{kid}} \quad I_t = \nu s_t K_t + pD_t \quad \text{s.t.} \quad Q_t = F(\alpha, s_t K_t, D_t)$$

  FOCs (where λ is Lagrangian multiplier): where $\Lambda'(e) = \partial\Lambda(e)/\partial e^{kid}$ (return to effort)

$$\left.\begin{array}{l} \nu = \lambda F_K(\alpha, s_t K_t, D_t) \\ p = \lambda F_D(\alpha, s_t K_t, D_t)\Lambda'(e_t) \end{array}\right\} \implies \frac{p}{\nu} = \frac{F_D(\alpha, s_t K_t, D_t)\Lambda'(e_t)}{F_K(\alpha, s_t K_t, D_t)} \tag{2}$$

Agent's Problem

- Using (1) and (2) and the constraint $Q_t = F(\alpha, s_t K_t, D_t)$, express I_t in terms of prices p and ν, technology α, flow production of human capital Q_t, and return to effort $\Lambda'(e)$:

$$I_t = \Psi(\alpha, p, \nu, Q_t, \Lambda'(e_t)) \tag{3}$$

- At time t, agent maximizes present value of consumptions:

$$W_t = \int_t^T e^{-r(\tau - t)} \left[\nu K_\tau - I_\tau \right] d\tau + e^{-r(T-t)} E_T \tag{4}$$

- Three cases:
  - $s \in (0, 1)$ the existing human capital is large enough so that the agent allocates a fraction but not all human capital stock to human capital production.
  - $s = 0$ the existing human capital is so large that it is optimal to divest human capital.
  - $s = 1$ the existing human capital is not enough for optimal human capital production even when fully allocated for that purpose.

Agent's Problem

- Consider case $s \in (0,1)$.
  - Marginal cost of human capital production:

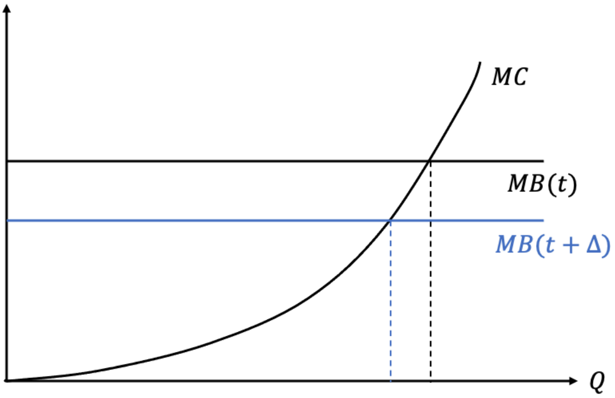
$$MC_t = \frac{\partial I_t}{\partial Q_t} = \frac{\Psi(\alpha, p, \nu, Q_t, \Lambda'(e_t))}{\partial Q_t}$$

  - Marginal benefit of human capital is the present value of additional earnings the agent can get by acquiring an additional unit of human capital (net of deterioration):

$$MB_t = \frac{\nu}{r}\left[1 - e^{-r(T-t)}\right] + e^{-r(T-t)}\frac{\mu}{r}\left[1 - e^{-r(T^d-T)}\right]$$

  - Optimal policy is $MC_t = MB_t$, from which we can solve for optimal human capital production Q_t for any given time t.

Agent's Problem



Example - Cobb-Douglas Production Function

- Suppose human capital production function is of Cobb-Douglas form ($\beta_1 + \beta 2 < 1$):

$$Q_t = \alpha(s_t K_t)^{\beta_1} D_t^{\beta_2} \tag{5}$$

- (2) becomes:

$$\frac{p}{\nu} = \frac{\beta_2}{\beta_1} \frac{s_t K_t}{D_t} \Lambda'(e_t) \tag{6}$$

- Substitute (5) and (6) into (1):

$$I_t = \left(\frac{\nu}{\beta_1}\right) \left(\frac{Q_t}{\alpha}\right)^{\frac{1}{\beta_1+\beta_2}} \left(\frac{\beta_1 p}{\beta_2 \nu \Lambda'(e_t)}\right)^{\frac{\beta_2}{\beta_1+\beta_2}} \left(\beta_1 + \beta_2 \Lambda'(e_t)\right)$$

  - $MC_t = MB_t$ occurs at $Q_t^* > 0$. (True for all production functions that are homogeneous of degree less than one.)

Return to Schooling

- Mincer (1958) return to schooling:
  - Increase in earnings associated with an extra year of schooling.
  - In this model, if schooling ends (agent enters adulthood) at $T+\Delta$ instead of T, incremental human capital stock is:

$$\Delta K_T = K_{T+\Delta} - K_T = \int_T^{T+\Delta} Q_\tau d\tau$$

  - Incremental earnings:

$$\Delta E_T = \frac{\mu}{r} K_T \left[e^{-r(T^d - T)} - e^{-r(T^d - T - \Delta)} \right] + \frac{\mu}{r} \left[1 - e^{-r(T^d - T - \Delta)} \right] \int_T^{T+\Delta} Q_\tau d\tau$$

    - ★ Since Q_t is decreasing with time t, when T is large enough, $\Delta E_T < 0$.

Part (a)

"Do you expect children to "make up for" the lost schooling after the pandemic ends? That is, would the pandemic reduce long-run human capital, or just perturb its time path during and near the pandemic? What factors of tastes and technology determine the answer?"

Part (a)

- Consider the agent's decision at t_0 (end of pandemic). Analysis for $t > t_0$ is analogous.
  - If school did not close (benchmark): agent's human capital stock at $t = t_0$ is K_{t_0}, where K follows dynamic $\dot{K}_t = Q_t$.
    - ★ Let $Q^*_{t_0}$ be such that $MC_{t_0} = MB_{t_0}$.

$$Q_{t_0} = \begin{cases} Q^*_{t_0} & \text{if} \quad Q^*_{t_0} \in \Big(F(\alpha, 0, \Lambda(e)), F(\alpha, K_{t_0}, \Lambda(e))\Big) \quad s \in (0,1) \text{ case} \\ F(\alpha, K_{t_0}, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \geq F(\alpha, K_{t_0}, \Lambda(e)) \quad s = 1 \text{ case} \\ F(\alpha, 0, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \leq F(\alpha, 0, \Lambda(e)) \quad s = 0 \text{ case} \end{cases}$$

  - If school closed: agent's human capital stock at $t = t_0$ is K_0 instead.

$$\tilde{Q}_{t_0} = \begin{cases} Q^*_{t_0} & \text{if} \quad Q^*_{t_0} \in \Big(F(\alpha, 0, \Lambda(e)), F(\alpha, K_0, \Lambda(e))\Big) \\ F(\alpha, K_0, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \geq F(\alpha, K_0, \Lambda(e)) \\ F(\alpha, 0, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \leq F(\alpha, 0, \Lambda(e)) \end{cases}$$

Part (a)

- Since $F(\alpha, K, D)$ is homogeneous of degree less than one, and $F(\alpha, 0, \Lambda(e)) = 0$, then

$$Q_{t_0} = \begin{cases} Q^*_{t_0} & \text{if} \quad Q^*_{t_0} \in \left[0, F(\alpha, K_{t_0}, \Lambda(e))\right) \\ F(\alpha, K_{t_0}, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \geq F(\alpha, K_{t_0}, \Lambda(e)) \end{cases}$$

$$\tilde{Q}_{t_0} = \begin{cases} Q^*_{t_0} & \text{if} \quad Q^*_{t_0} \in \left[0, F(\alpha, K_0, \Lambda(e))\right) \\ F(\alpha, K_0, \Lambda(e)) & \text{if} \quad Q^*_{t_0} \geq F(\alpha, K_0, \Lambda(e)) \end{cases}$$

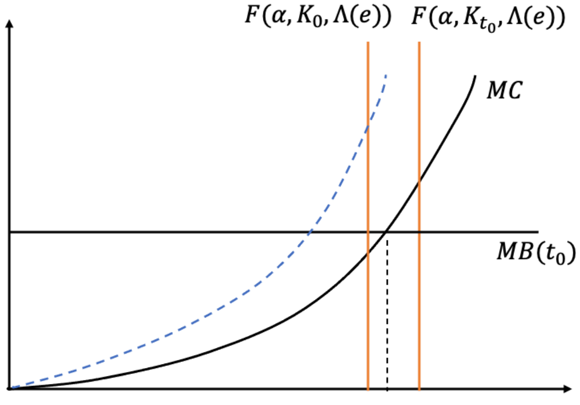
Part (a)

- Since production function homogeneous of degree less than one implies that $Q_t \geq 0$, $\forall t$, $F(\alpha, K_{t_0}, \Lambda(e)) > F(\alpha, K_0, \Lambda(e))$.
  - If $Q^*_{t_0} \geq F(\alpha, K_{t_0}, \Lambda(e))$, then $Q^*_{t_0} > F(\alpha, K_0, \Lambda(e))$. Thus, $\tilde{Q}_{t_0} < Q_{t_0}$.
    - ★ Agent does not make up for lost schooling, due to constraint of lower human capital stock.
  - If $Q^*_{t_0} \in \Big(F(\alpha, K_0, \Lambda(e)), F(\alpha, K_{t_0}, \Lambda(e))\Big)$, then $\tilde{Q}_{t_0} = F(\alpha, K_0, \Lambda(e)) < Q^*_{t_0} = Q_{t_0}$.
    - ★ Again, agent does not make up for lost schooling.
  - If $Q^*_{t_0} \geq F(\alpha, K_0, \Lambda(e))$, then $\tilde{Q}_{t_0} = Q_{t_0} = Q^*_{t_0}$.
    - ★ Agent does not make up for lost schooling and produces the same amount of human capital.

Part (a)

- Preferences r shifts MB curve.
  - If preferences are such that the agent is impatient (future consumption not very good substitute for current consumption), marginal benefit of investing in human capital decreases, and thus $Q^*_{t_0}$ decreases.
    - Less likely to hit corner. Agent tends to produce the same amount of human capital instead of making up for lost schooling.
  - If preferences are such that the agent is very patient (future consumption very good substitute for current consumption), marginal benefit of investing in human capital increases, and thus $Q^*_{t_0}$ increases.
    - More likely to hit corner. Agent produces less human capital (compared to benchmark case without school closing).
- Technology α in production of human capital shifts $F(\alpha, K, \Lambda(e))$.
  - For example, when technology is such that agent produces human capital less efficiently, it is less likely to hit corner. Agent thus tends to produce the same amount of human capital instead of making up for lost schooling.

Part (a)



Part (b)

"To the extent that persons of schooling age during the pandemic later enter adulthood with x less human capital (present value of future earnings) than they would have had, how would you value that increment to human capital?"

Part (b)

- From Part (a), at time T, human capital differential (difference in present value of future earnings) is:

$$x = E_T - \tilde{E}_T = \frac{\mu}{r}\left[1 - e^{-r(T^d - T)}\right](K_T - \tilde{K}_T)$$

  - Since $\dot{K}_t = Q_t$, we have

$$K_T = \int_0^T Q_\tau d\tau + K_0; \quad \tilde{K}_T = \int_{t_0}^T \tilde{Q}_\tau d\tau + K_0$$

  - Q_τ and $\tilde{Q}_\tau$ are determined by $MC_\tau = MB_\tau$.

Tractable Case - Cobb-Douglas Production Function

- Cobb-Douglas production:

$$Q_t = \alpha(s_t K_t)^{\beta_1} D_t^{\beta_2}$$

- Further suppose $\Lambda'(e)$ is constant.
- MC_t and MB_t are respectively:

$$MC_t = \frac{\nu}{\alpha\beta_1} \frac{\beta_1 + \beta_2 \Lambda'(e)}{\beta_1 + \beta_2} \left(\frac{\beta_1 p}{\beta_2 \nu \Lambda'(e)}\right)^{\frac{\beta_2}{\beta_1+\beta_2}} \left(\frac{Q_t}{\alpha}\right)^{\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}$$
$$MB_t = \frac{\nu}{r}\left[1 - e^{-r(T-t)}\right] + e^{-r(T-t)} \frac{\mu}{r}\left[1 - e^{-r(T^d - T)}\right]$$

Part (b)

- Can solve Q_t in closed form by $MC_t = MB_t$:

$$
\begin{aligned}
Q_t =& \alpha\left[\frac{\alpha\beta_1}{\nu}\frac{\beta_1+\beta_2}{\beta_1+\beta_2\Lambda'(e)}\left(\frac{\beta_2\nu\Lambda'(e)}{\beta_1 p}\right)^{\frac{\beta_2}{\beta_1+\beta_2}}\right]^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}} \\
& \times\left\{\frac{\nu}{r}\left[1-e^{-r(T-t)}\right]+e^{-r(T-t)}\frac{\mu}{r}\left[1-e^{-r(T^d-T)}\right]\right\}^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}} \\
=& H\left\{\frac{\nu}{r}+\frac{\mu-\nu}{r}e^{-r(T-r)}-\frac{\mu}{r}e^{-r(T^d-t)}\right\}^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}}
\end{aligned}
$$

- Take into account corner solutions:

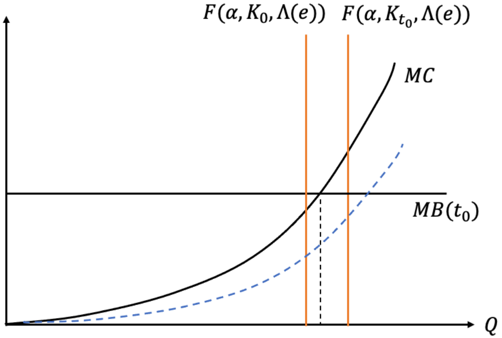
$$
Q_t = \min\left\{H\left\{\frac{\nu}{r}+\frac{\mu-\nu}{r}e^{-r(T-r)}-\frac{\mu}{r}e^{-r(T^d-t)}\right\}^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}}, \alpha K_t^{\beta_1}D_t^{\beta_2}\right\}
$$

Part (c)

"How does the value of the x increment to human capital relate to the opportunity cost of closing schools during a nonpandemic year? How does it relate to the opportunity cost of closing schools during the pandemic?"

Part (c)

- Pandemic reduces shadow cost of schooling p.
  - During pandemic, agents have fewer activities they could do, so the shadow cost p is relatively low, and thus the opportunity cost of closing schools is relatively low.
    - Since MC is increasing in p, smaller p shifts MC curve down, more likely to hit corner. $x \uparrow$.

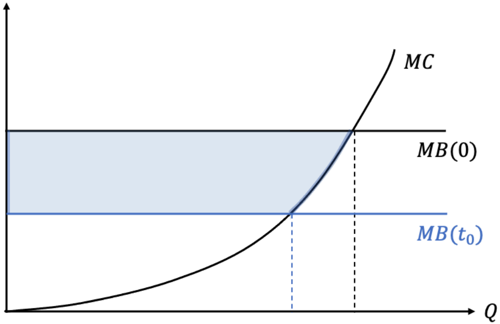


Part (d)

"How could you estimate the value of the x increment using data on the amount of market inputs (such as teacher salaries) and student time that normally goes into schooling?"

Part (d)

- Idea comes from Mincer equation.
  - Panel data of earnings, student time , teacher salaries etc. and run reduced form regressions.
  - Get coefficients and plug in the fact that no schooling occurs during pandemic to estimate lost earnings.
- Can also do structural estimation.



Part (d)

- Production function:
  - Suppose Cobb-Douglas production function in logs

$$q_{it} = \beta_k k_{it} + \beta_d d_{it} + \omega_{it} + \epsilon_{it}$$

  where ω_{it} is technology shock and evolves exogenously following first-order Markov process $p(\omega_{it+1}|\mathcal{F}_{it}) = p(\omega_{it+1}|\omega_{it})$.

- Assume effort (d) is non-dynamic input, while human capital (k) is dynamic input subject to an investment process:

$$k_{it+1} = k_{it} + q_{it}$$

- Agent's optimal human capital investment is strictly increasing in current productivity ω_{it}:

$$q_{it} = f_t(\omega_{it}, k_{it}) \Longrightarrow \omega_{it} = f^{-1}(q_{it}, k_{it})$$

Part (d)

- Substitute into production function:

$$q_{it} = \beta_k k_{it} + \beta_d p_{it} + f_t^{-1}(q_{it}, k_{it}) + \epsilon_{it}$$

$$\implies \quad k_{it+1} = (1+\beta_k)k_{it} + \beta_d d_{it} + f_t^{-1}(q_{it}, k_{it}) + \epsilon_{it} = \beta_d d_{it} + \Phi_t(q_{it}, k_{it}) + \epsilon_{it}$$

- First stage:
  - Use student time to proxy human capital stock and flow investment; estimate effort using teacher salary etc.
  - Treat f^{-1} non-parametrically, estimate coefficient on effort β_d and composite term $\Phi_t(q_{it}, k_{it}) = (1+\beta_k)k_{it} + f_t^{-1}(q_{it}, k_{it})$. Denote estimates by $\hat{\beta}_d$ and $\hat{\Phi}_{it}$ respectively.

Part (d)

- Second stage: Given $p(\omega_{it+1}|\mathcal{F}_{it}) = p(\omega_{it+1}|\omega_{it})$, and estimates from first stage, write

$$\omega_{it} = \mathbb{E}[\omega_{it}|\mathcal{F}_{it-1}] + \xi_{it} = \mathbb{E}[\omega_{it}|\omega_{it-1}] + \xi_{it}$$

  where ξ_{it} is innovation component of ω_{it}.
- By properties of conditional expectation, since $k_{it} \in I_{it-1}$,

$$\mathbb{E}[\xi_{it}|\mathcal{F}_{it-1}] = 0 \Longrightarrow \mathbb{E}[\xi_{it}|k_{it}] = 0 \quad \text{or} \quad \mathbb{E}[\xi_{it}k_{it}] = 0$$

- Then use GMM.

Part (e)

“Suppose that remote learning (the mode used while schools are closed) reduces the productivity of student and teacher time and effort in terms of producing learning. Use Marshall’s laws of derived demand to discuss the effect of closed schools on student and teacher time and effort. Is this assumption about remote learning consistent with our earlier assumption that no net learning occurred during the year?”

Part (e)

- Now suppose there is remote learning.
  - Remote learning reduces the productivity of student and teacher time and effort in terms of producing learning, thus, $\Lambda'(e)$ decreases.
    - ★ This has the effect of shifting MC curve up (marginal cost increases).
  - During pandemic, shadow cost of effort decreases due to fewer outside activities.
    - ★ This has the effect of shifting MC curve down (marginal cost decreases).
  - Q_t may decrease or increase, depending on which effect dominates.
  - Similar to the "light bulb" example in class.

References

Ben-Porath, Yoram, "The Production of Human Capital and the Life Cycle of Earnings," *Journal of Political Economy*, 1967, *75* (4), 352–365.

Mincer, Jacob, "Investment in Human Capital and Personal Income Distribution," *Journal of Political Economy*, 1958, *66* (4), 281–302.