

# ECMA31000: Introduction to Empirical Analysis

## Introduction; Probability; Distributions

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University of Chicago

Autumn 2021

# Logistics

**Lecture:** TuTh 2:00PM - 3:20PM in Kent Chem Lab 101.

**Discussion Section:** W 4:30PM - 5:20PM in Rosenwald Hall 015.

**Instructor:** Joseph Hardwick

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Office Hours: Via Zoom on Tuesdays 3:30PM-5:00PM and  
Wednesdays 9:30AM-11:00AM.

**Teaching Assistant:** Tanya Rajan

Email: [tanyar@uchicago.edu](mailto:tanyar@uchicago.edu)

Office Hours: Via Zoom on Mondays and Tuesdays  
10AM-11:30AM.

# Readings

My lecture slides and annotations, which will be on Canvas.

There is no required textbook, but parts of the following are useful references:

- *Probability and Statistics for Economists* (2021) and *Econometrics* (2021) by Bruce Hansen. Available for free on Bruce's website.
- *Econometric Analysis of Cross Section and Panel Data* (2010) by Jeffrey M. Wooldridge. Available online through the university library.
- *A Primer in Econometric Theory* (2016) by John Stachurski. Many solved exercises on material covered in this class. Slides/Sample Chapters/Code available online.

# Evaluation

Final grade determined by assignments, Midterm and Final. 2

Exams:

- Exam 1: October 26 in class 2PM-3:20PM; 80 mins in length.
- Exam 2: TBA, 120 mins in length. Weighting:

Problem Sets: 20% Submitted weekly on Canvas.

Lowest score dropped.

Exam 1: 30% (Covers lectures 1-8)

Exam 2: 50% (Covers lectures 1-16)

| Letter Grade  | A  | A- | B+ | B  | B- | C+ | C  | C- |
|---------------|----|----|----|----|----|----|----|----|
| Overall Score | 90 | 85 | 80 | 75 | 70 | 65 | 60 | 55 |

Any student scoring higher than the cutoff given above will earn at least that grade in the course.

# Expectations

- All lectures/sections are in person unless we must go remote.
- Please wear masks at all times. No food and drink in classrooms.
- Please ask questions in class!
- Use the discussion boards for clarifications/questions on material.
- Any questions/concerns, please reach out!

# Objective 1: Description

- Learn about some feature of the population from a finite sample.
  - Example: Rate of unemployment in the US, households sampled in Current Population Survey.
  - Example: Mean height of population of UChicago undergraduates.
- Can learn about unemployment rate/heights using sample employment/height data  $x_1, \dots, x_n$ .
  - Data are sampled from an unknown population distribution.
  - We estimate the quantity of interest e.g. Mean/Expected height,  $\mu$ , estimated by sample mean  $\frac{1}{n} \sum_{i=1}^n x_i$ .
  - Test whether mean height is 175cm. Can assess whether hypothesis is reasonable based on sample data.
- Key difference: Actual average height is constant, but sample average is random.

# Objective 1: Description

- Often interested in measuring differences in means.
- e.g. Difference in apartment prices with 3 bedrooms vs. 2 bedrooms.
- Quantity of interest:

$$E(\textit{Price}|\textit{Beds} = 3) - E(\textit{Price}|\textit{Beds} = 2).$$

- If floor area is fixed, then quantity of interest is

$$\begin{aligned} &E(\textit{Price}|\textit{Beds} = 3, \textit{Area} = 1000\textit{sqft}) \\ &- E(\textit{Price}|\textit{Beds} = 2, \textit{Area} = 1000\textit{sqft}). \end{aligned}$$

## Objective 2: Prediction

- Given a student's ACT and high school GPA, what is your best prediction of their college GPA?
- We don't think higher test scores during high school cause a high GPA, but may predict it.
- We can consider linear predictors of the form

$$\widehat{colGPA} = \beta_0 + \beta_1 ACT + \beta_2 hsGPA$$

where the  $\beta_j$  are specially chosen to give the “best” predictor of college GPA, using *ACT* and *hsGPA* as predictors.

- We use the data available to draw line/plane of best fit.



## Objective 3: Causality

- Learn the effect of changing an observable characteristic on an individual's outcome.
  - Example: The effect of an extra year of schooling on earnings
  - Example: The effect of a job training grant on firm productivity
  - Example: Effect of increasing statewide alcohol tax on wine sales.
- This is different from description:
  - Example: On average, people with 4 years of high school education earn \$X more than those with 3 years.
  - Example: On average, firms receiving job training grants have a X% higher scrap rate than those which don't.
  - Example: On average, wine sales increased by X% in year following tax increase.

# Linear Regression

- In this class we consider linear models for the purposes of description, prediction and inferring causality.
- The observed variables are the explanatory variables  $x_1, \dots, x_k$  and outcome  $y$ .
- For an individual  $i$  drawn from the population, the model/data generating process is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_k x_{ik} + u_i,$$

- “Linear” means linear in parameters  $\beta_j$ .
- e.g. if  $x_1 = r$  and  $x_2 = r^2$  for some variable  $r$ , the model  $y = \beta_0 + \beta_1 r + \beta_2 r^2 + u$  is linear in parameters even though it contains a quadratic term in  $r$ .

# Linear Regression

- Model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + u_i$$

- $u_i$  is the error term/idiosyncratic shock. In a causal model,  $u_i$  contains unobserved determinants of  $y_i$ .
- Parameters  $\beta_j$  are assumed to be unknown constants.
- Want to get a “good estimate” of the  $\beta_j$  from a sample of individuals from the population.

# Why large sample theory?

- Suppose we observe a random sample of  $X_i$ , distributed as  $X_i \sim U[0, \theta]$ .
- We wish to estimate  $\theta$ . Consider the Method of Moments estimator

$$\hat{\theta} = \frac{2}{n} \sum_{i=1}^n X_i$$

What is the exact distribution of  $\hat{\theta}$ ? For  $n$  small, this is somewhat straightforward. For  $n = 100$ , it isn't! But, scaled and centered, it is approximated well by the Standard Normal Distribution, due to the central limit theorem.

- There are other ways to estimate  $\theta$ , e.g. maximum likelihood. The asymptotic distribution is non-normal in this case!

# Why large sample theory?

- In the regression example, we will generally not assume the distribution of  $u$  is known.
- Implies distribution of estimator  $\hat{\beta}$  is unknown, even if we could compute it
  - Properly scaled and centered version of  $\hat{\beta}$  still has an approximately normal distribution!
- The central limit theorem provides this result, and allows us to conduct hypothesis tests on the unknown parameter  $\beta$ .

# Overview: Part I

- Probability:
  - Sample space, Events, Probability.
  - Random variables, distributions, conditional distributions.
- Large Sample Theory
  - Modes of Convergence
  - Law of Large Numbers, CLT
  - Continuous mapping theorem, Delta Method
- Estimation
  - Definitions, finite sample and asymptotic properties
  - Maximum likelihood , (Generalized) Method of Moments

# Overview: Part II

- Hypothesis testing
- Linear Regression
  - Review of Linear Algebra, Projections
  - Interpretations and Estimation of Linear Model
  - Properties of OLS
  - Testing with Normal errors, non-normal errors
- Instrumental Variables
  - Reasons best fit line/plane doesn't accurately measure causal effect of  $x$  on  $y$  (endogeneity)
  - How IV methods can help
  - Properties of IV estimators
  - GMM
- Topic (if time permits)

# Questions?



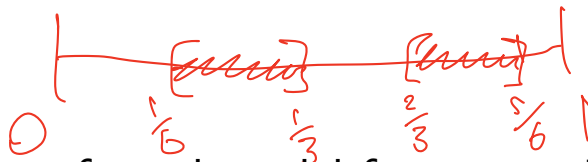
# Probability Spaces

- Probability spaces provide a formal model for uncertainty. They consist of three parts:
  - ①  $\Omega$ , the sample space, is a non-empty set containing all possible outcomes  $\omega$  of an experiment.
    - A subset of  $E \subset \Omega$  is called an event.
    - When the uncertainty is resolved, if the outcome  $\omega \in E$ , we say event  $E$  occurred.
  - ②  $\mathcal{F}$ , a sigma algebra, is a collection of events to which we can assign a probability satisfying:
    - ①  $E \in \mathcal{F} \implies E^c \in \mathcal{F}$ ;
    - ②  $E_1, E_2, \dots \in \mathcal{F} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$ ;
    - ③  $\Omega \in \mathcal{F}$ .
  - ③  $P$ , a probability measure which is a mapping  $P : \mathcal{F} \rightarrow [0, 1]$  such that
    - ①  $P(\Omega) = 1$ ;
    - ②  $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$  for disjoint events  $E_1, E_2, \dots \in \mathcal{F}$ .

# Probability Spaces

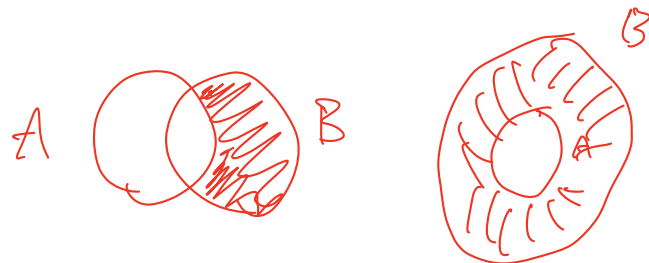
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# Properties of P



- $\mathcal{F}$  keeps track of which events we can assign a probability to.  
P tells us how likely these events are to occur.

- Let  $A, B \in \mathcal{F}$  with  $A \subset B$ . Then:

①  $P(B \setminus A) = P(B) - P(A)$ , where  $B \setminus A = B \cap A^c$  NOT

②  $P(A) \leq P(B)$

③  $P(A^c) = 1 - P(A)$

④  $P(\emptyset) = 0$

⑤  $\emptyset = \Omega^c \Rightarrow P(\Omega^c) = 1 - P(\Omega) = 0$ .  
AND

- Subadditivity: For any  $A, B \in \mathcal{F}$ ,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$  (Exercise!)

OR

③ Use ① with  $B = \Omega$ .

$P(\Omega \setminus A) = P(\Omega) - P(A) \Rightarrow P(A^c) = 1 - P(A)$ .

# Properties of $P$ : Proofs

$$P(B \setminus A) = P(B) - P(A).$$



- 1: Note that  $A$  and  $B \cap A^c$  are disjoint, and  $\underline{A \cup (B \cap A^c) = B}$ . By property 2 of probability measure,  $P(B) = P(A) + P(B \setminus A)$ .  
 $P(B) = P(A \cup [B \cap A^c])$
- 2: Since  $P$  is nonnegative,  $P(B) = P(A) + P(B \setminus A) \geq P(A)$
- 3: Follows from 1 with  $B = \Omega$ .  
 $= P(A) + P(B \cap A^c)$
- 4: Follows from 3 with  $A = \Omega$ .

# (In)dependence

with  $P(B) > 0$ .

- For any events  $A, B \in \mathcal{F}$ , the conditional probability of  $A$  given  $B$  is

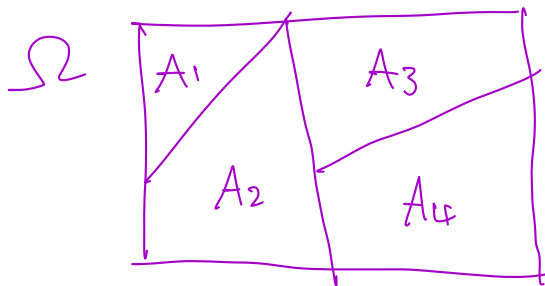
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- Events  $A_1, \dots, A_N \in \mathcal{F}$  are independent if

$$P\left(\bigcap_{n=1}^N A_n\right) = \prod_{n=1}^N P(A_n).$$

- If  $A$  and  $B$  are independent events,  $P(A|B) = P(A)$ .

# Law of total probability



- A partition of  $\Omega$  is a countable collection of events  $\{B_n\}_{n=1}^{\infty}$  such that
  - ①  $B_n \cap B_m = \emptyset$  when  $n \neq m$  (events are disjoint)
  - ②  $\bigcup_{n=1}^{\infty} B_n = \Omega$  (events cover the sample space)
- (LOTP) Let  $\{B_n\}_{n=1}^{\infty}$  be a partition of  $\Omega$  and  $A$  be any event. If  $P(B_m) > 0$  for all  $m$ , then

$$P(A) = \sum_{m \geq 1} \underbrace{P(A|B_m) \cdot P(B_m)}_{"}$$

Property 2 of prob meas:

$$P(A) = P\left(\bigcup_{m=1}^{\infty} (A \cap B_m)\right)$$

because  $\{A \cap B_m\}_{m \geq 1}$  forms a partition of  $A$ .

$$\frac{P(A \cap B_m)}{P(B_m)} \cdot P(B_m) = P(A \cap B_m).$$

Bayes' Law

$$= \sum_{m \geq 1} P(A \cap B_m).$$

- (Bayes' Law) By definition of the conditional probabilities  $P(A|B), P(B|A)$ :

$$P(B|A) = P(A \cap B) / P(A).$$

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$

$$(LOTP) = \frac{P(B|A) P(A)}{P(B|A) P(A) + P(B|A^c) P(A^c)}.$$



## Example: Screening for health conditions

- A test for a certain condition indicates either +, a signal the patient has a condition (C), or −, a signal they don't have the condition (NC).
- Base rate of the condition in the population is

$$P(C) = 0.001.$$

- Test has a false positive rate of 0.01. There are no false negatives:

$$P(+|NC) = 0.01,$$

$$P(-|C) = 0.$$

## Example: Screening for health conditions

- The test does not often produce false positives or false negatives. How likely is it that a patient has the condition given a positive result?

$$\begin{aligned} P(C|+) &= \frac{P(+|C)P(C)}{P(+|C)P(C) + P(+|NC)P(NC)} \\ &= \frac{1 \times 0.001}{1 \times 0.001 + 0.01 \times 0.999} \approx \frac{1}{10}. \end{aligned}$$

- Although false positives are rare, the base rate of the condition is so low that even if a positive result is found it is still unlikely the patient has it.

# Random Variables

- A random variable  $X$  maps outcomes from a sample space  $\Omega$  to real numbers:

$$X : \Omega \rightarrow \mathbb{R}$$

e.g. We observe the outcome of 2 coin flips, want to know how many heads observed:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$P(\{\omega\}) = 1/4 \text{ for all } \omega \in \Omega.$$

$\mathcal{P}$  is "power set" of  $\Omega$ , i.e. set of all subsets of  $\Omega$ .

For  $\omega \in \Omega$ , define

$$X_1(\omega) = \begin{cases} 0 & \omega \in \{(T, T)\} \\ 1 & \omega \in \{(H, T), (T, H)\} \\ 2 & \omega \in \{(H, H)\} \end{cases}$$

# Indicator Functions

$$\mathbf{1}_A: \Omega \rightarrow \mathbb{R}$$
$$\Omega \rightarrow \{0, 1\}.$$

- A binary random variable which indicates that some event  $A$  has occurred is called an indicator function:

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

- A look ahead: Hypothesis tests are indicator functions! We reject the null hypothesis if a statistic  $T$  is larger than some critical value  $c$ .
  - The event  $A = \{\omega : T(\omega) > c\}$  is the set of sample outcomes that produce a test stat larger than  $c$ .
  - Write  $\phi(\omega) = \mathbf{1}_A(\omega)$ . We reject if and only if we observe  $\phi = 1$ .

↑  
"Test Function".

# Random Vectors

- If  $X$  maps  $\Omega$  onto  $\mathbb{R}^N$  for  $N > 1$  we call  $X$  a random vector.
- E.g. If I collect a random sample of heights, the collection of observations is a random vector.
- E.g. (Coin flips) I care about number of heads  $X_1$  AND binary rv indicating whether the second flip is a tail ( $X_2$ ):

$$X(\omega) = \begin{pmatrix} X_1(\omega) \\ X_2(\omega) \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \end{pmatrix}' & \omega = (T, T) \\ \begin{pmatrix} 1 & 1 \end{pmatrix}' & \omega = (H, T) \\ \begin{pmatrix} 1 & 0 \end{pmatrix}' & \omega = (T, H) \\ \begin{pmatrix} 2 & 0 \end{pmatrix}' & \omega = (H, H) \end{cases}$$

- In this case: knowledge of both  $X_1$  and  $X_2$  implies knowledge of the actual outcome.

# Transformations of Random Variables

- If  $X : \Omega \rightarrow \mathbb{R}$  is a random variable, what about  $g(X)$  for some function  $g$ ? For our purposes, yes:

$$g(X(\omega)) : \Omega \rightarrow \mathbb{R}$$

is still a map from the sample space to  $\mathbb{R}$ .

- Requires:  $X$  is a random variable,  $g$  is “measurable”.
- Note: All continuous functions are “measurable”, as are indicator functions of most sets of interest.
- Hypothesis tests are usually transformations of random variables/vectors:
  - We observe  $X$ , and  $T(\omega)$  is in fact  $T(X(\omega))$ . Then  $\phi(\omega)$  is in fact  $\phi(T(X(\omega)))$ .

# Questions?

# Distributions

- With coin flips, we observe the sample outcome  $\omega$  and compute  $X(\omega)$  .. but:
- In many cases we only ever observe  $X$ , so we never fully specify experiment  $(\Omega, \mathcal{F}, P)$ .
  - Not a problem if  $X$  is object of interest.
  - Goal is to learn about features of the distribution  $P_X$  of  $X$ .
- For  $B \subset \mathbb{R}$ , we define

$$P_X(X \in B) := P(\omega : X(\omega) \in B).$$

The probability measure  $P_X$  is the distribution of  $X$ .



# Distribution Function

- The distribution function of  $X$ ,  $F_X$ , is found by evaluating  $P_X(B)$  on all sets  $B$  of the form

$$B = (-\infty, c]; \quad c \in \mathbb{R},$$

so that

$$F_X(c) = P_X((-\infty, c]).$$

Notice that the event  $\{X \in (-\infty, c]\}$  is just  $\{X \leq c\}$ .

- Writing down  $F_X$  much easier than specifying  $P_X$ , since far fewer sets  $B$  to consider.
- Good news:  $P_X$  is completely determined by  $F_X$ !

# Properties of distribution functions

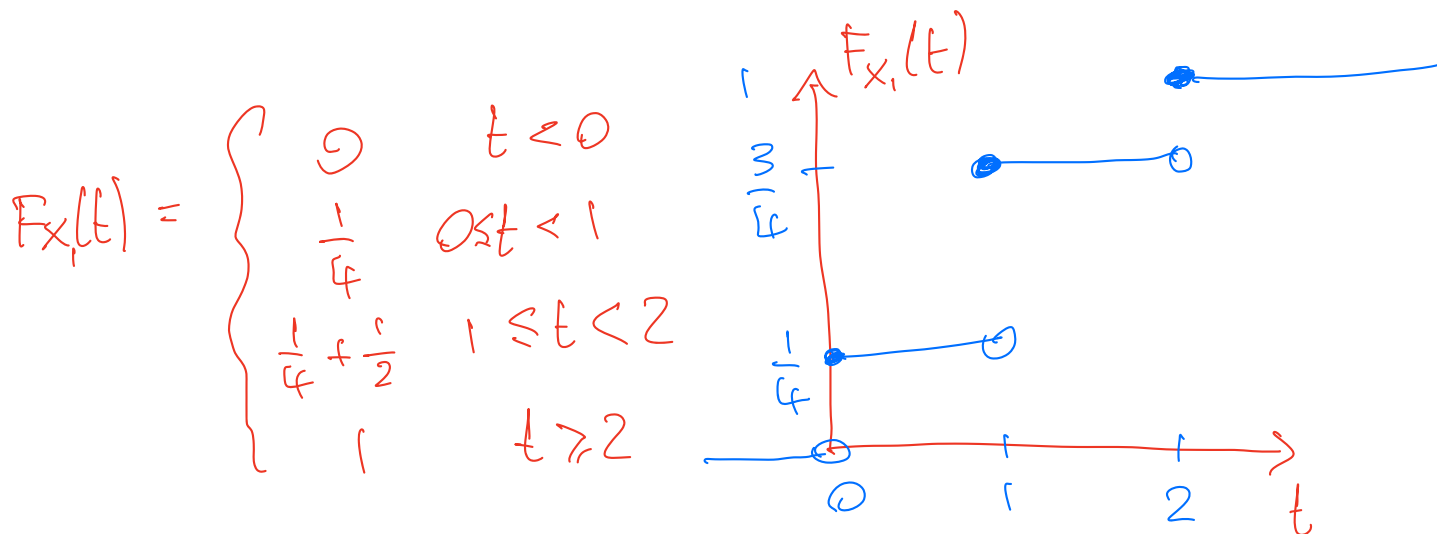
- Properties of  $F_X$ :
  - Weakly increasing:  $F_X(x) \leq F_X(y)$  if  $x \leq y$ .
  - Right continuous:  $F_X(x_n) \rightarrow F_X(x)$  if  $x_n \geq x$  for all  $n$  and  $x_n \rightarrow x$ .
  - $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F_X(x) = 1$ .

## Example (Coin Flips)

- Recall 'number of heads' in 2 consecutive coin flips described by

$$X_1(\omega) = \begin{cases} 0 & \omega \in \{(T, T)\}, \\ 1 & \omega \in \{(H, T), (T, H)\}, \\ 2 & \omega \in \{(H, H)\}. \end{cases}$$

- Suppose the coin is fair. Draw  $F_{X_1}$ :



# Expectation

- Recall  $X : \Omega \rightarrow \mathbb{R}$ . We say  $X$  induces a distribution on  $\mathbb{R}$ , because  $P_X$ , the distribution of  $X$ , is determined by:
  - $P$ , the prob. measure on the underlying space  $(\Omega, \mathcal{F}, P)$ ,
  - The random variable  $X$ :

$$P_X(B) = P(\omega : X(\omega) \in B).$$

- Expectation of  $X$  can be computed directly from  $P_X$ . e.g. If  $X$  is continuous:

$$\begin{aligned} E(X) &= \int x dP_X \\ &= \int x f_X(x) dx. \end{aligned}$$

- This means  $E(X)$  is a feature of  $P_X$ : Can “learn”  $E(X)$  by observing draws from  $P_X$ .

# Discrete Distributions

- A random variable  $X$  is discrete if there is a countable set of values  $\{x_j\}_{j \geq 1}$  such that  $P_X(X \in \{x_j\}_{j \geq 1}) = 1$ .
- For each  $j$ , denote  $P_X(x_j) = p_j$ . The probability mass function is

$$p_X(x) = \begin{cases} p_j & x = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

- The CDF  $F_X$  is determined by summing probability masses:

$$F_X(x) = \sum_{j \geq 1} p_j \mathbf{1}(x_j \leq x).$$

and for a transformation  $h(X)$ :

$$E(X) = \sum_{j \geq 1} p_j h(x_j).$$

# Continuous distributions

- A density function is a nonnegative function on  $\mathbb{R}$  which integrates to 1.
- $X$  is (absolutely) continuous if there exists a density function  $f_X$  such that

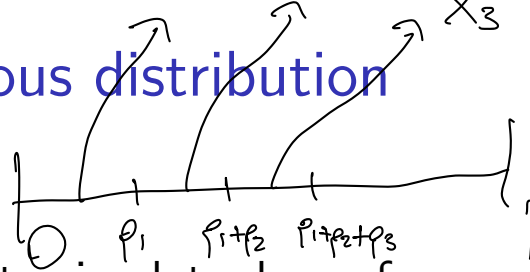
$$F_X(x) = \int_{-\infty}^x f_X(t) dt.$$

for all  $x \in \mathbb{R}$ .

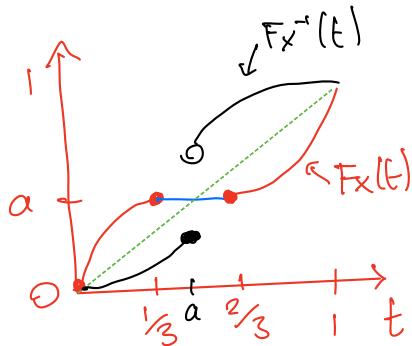
- In contrast to discrete distributions, absolutely continuous distributions assign probability 0 to any countable set.
- For any transformation  $h(X)$ :

$$E(h(X)) = \int_{\mathbb{R}} h(x) f_X(x) dx.$$

# Simulating a continuous distribution



- Suppose we want to simulate draws from continuous  $F_X$ , but our software only produces independent draws  $Y_i \sim U[0, 1]$ .
- Can turn these into draws from  $F_X$  simply by applying  $F_X^{-1}$  to the outcome:



Continuous distributions have continuous CDFs because any discontinuity would come from a discrete component. The CDF may have a flat region ( $\frac{1}{3} \leq t \leq \frac{2}{3}$ ), so  $F_X^{-1}(a)$  could be defined as any  $t \in [\frac{1}{3}, \frac{2}{3}]$ . We make this choice explicit by setting

$$F_X^{-1}(s) := \inf \{ b \in \mathbb{R} \mid F_X(b) \geq s \}.$$

This gives a strictly monotonic inverse function satisfying  $F_X F_X^{-1}(b) = b$  for all  $b \in (0, 1)$ .

Note:  $F_X^{-1}(Y) \leq t \Rightarrow F_X F_X^{-1}(Y) \leq F_X(t)$   
 $\Leftrightarrow Y \leq F_X(t).$

$Y \leq F_X(t) \Rightarrow F_X^{-1}(Y) \leq F_X^{-1} F_X(t)$   
 $\leq t.$

So:  $F_X^{-1}(Y) \leq t \Leftrightarrow Y \leq F_X(t).$

Hence:  $P(F_X^{-1}(Y) \leq t) = P(Y \leq F_X(t)) = F_X(t) = P(X \leq t).$  So  $X$  and  $F_X^{-1}(Y)$  have the same distrib.

# Distribution of a Random Vector

- Distribution of  $X = (X_1, \dots, X_n)$  characterized by the joint distribution function  $F_X$ , defined by:

$$F_X(x_1, \dots, x_n) := P_X(X_1 \leq x_1, \dots, X_n \leq x_n).$$

- As in scalar case,  $X$  is continuous if there exists a density function  $f_X : \mathbb{R}^N \rightarrow \mathbb{R}_+$  such that

$$F_X(x_1, \dots, x_n) = \int_{s_1=-\infty}^{x_1} \cdots \int_{s_n=-\infty}^{x_n} f_X(s_1, \dots, s_n) ds_1 \dots ds_n.$$

- $X$  is discrete if there is a countable set of  $(n \times 1)$  vectors  $\{x_j\}_{j \geq 1}$  such that  $P_X(X \in \{x_j\}_{j \geq 1}) = 1$ .
- If  $P_{X_1}, \dots, P_{X_n}$  are the same univariate distribution, we say  $X_1, \dots, X_n$  are identically distributed.



# Questions?

# Properties of Expectation

- Expectation is Linear: If  $E(X)$  and  $E(Y)$  exist then for any constants  $a, b$ ,

$$E(aX + bY) = aE(X) + bE(Y).$$

- If  $P(\omega : X(\omega) \leq Y(\omega)) = 1$ , then  $E(X) \leq E(Y)$ .
- (Cauchy-Schwarz Inequality) If  $E(X^2)$  and  $E(Y^2)$  exist:

$$|E(XY)|^2 \leq E(X^2) E(Y^2),$$

with equality iff  $X = aY$  for some constant  $a$ .

# Jensen's Inequality

- (Jensen's inequality) If  $E(X)$  and  $E(g(X))$  exist and  $g$  is convex:

$$E(g(X)) \geq g(E(X)).$$

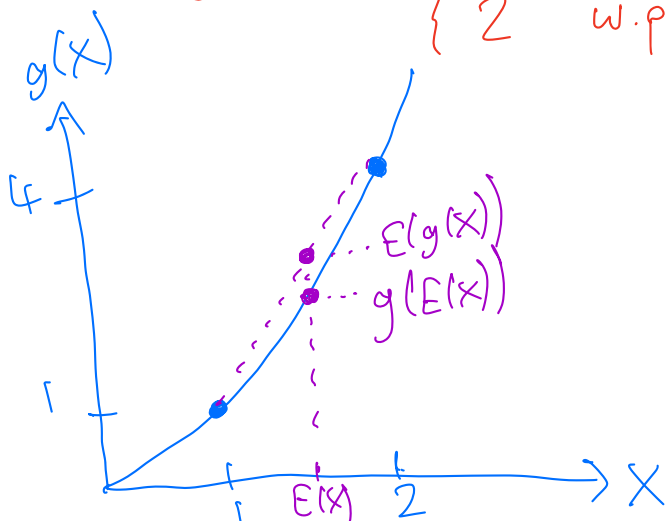
- The inequality is strict if  $X$  is not almost surely constant and  $g$  is strictly convex.

eg.  $X = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 2 & \text{w.p. } \frac{1}{2} \end{cases}$

$$g(x) = x^2,$$

$$E(g(X)) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 4 = \frac{5}{2}$$

$$g(E(X)) = g\left(\frac{3}{2}\right) = \frac{9}{4}.$$



# Moments

- If  $E(X^k)$  exists, then:
  - $E(X^k)$  is the  $k$ -th moment of  $X$ .
  - $E[(X - E(X))^k]$  is the  $k$ -th central moment of  $X$ .
  - $k = 2$  gives the Variance of  $X$ .
- The covariance of  $X$  and  $Y$  is given by

$$\text{Cov}(X, Y) := E[(X - E(X))(Y - E(Y))].$$

- The correlation between  $X$  and  $Y$  is given by

$$\text{Corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

# Correlation

- $\text{Corr}(X, Y)$  measures the strength of a linear relationship.  $X$  and  $Y$  are uncorrelated if  $\text{Corr}(X, Y) = 0$ .
- In general:

$$-1 \leq \text{Corr}(X, Y) \leq 1.$$

The correlation is 1 or  $-1$  iff  $X = a + bY$  for some constants  $a, b$ .

## Proof.

By the Cauchy Schwarz inequality:

$$|\mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]|^2 \leq \mathbb{E}[(X - \mathbb{E}(X))^2] \mathbb{E}[(Y - \mathbb{E}(Y))^2].$$

or

$$|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X) \text{Var}(Y)}.$$

with equality iff  $X - \mathbb{E}(X) = b(Y - \mathbb{E}(Y))$  for some constant  $b$ , which holds iff  $X = a + bY$  for some constants  $a, b$ . □

# Chebyshev's Inequality

- Bounds the probability that  $X$  is large using expectations.
- Suppose  $X^r$  is a non-negative integrable random variable for some  $r > 0$ . Then for any  $\delta > 0$ :

$$\mathbb{P}(X \geq \delta) \leq \frac{\mathbb{E}(X^r)}{\delta^r}.$$

## Proof.

Write  $X^r \geq X^r \mathbf{1}_{X < \delta} + \delta^r \mathbf{1}_{X \geq \delta} \geq \delta^r \mathbf{1}_{X \geq \delta}$ . Taking expectations on both sides of the inequality yields

$$\mathbb{E}(X^r) \geq \delta^r \mathbb{P}(X \geq \delta).$$

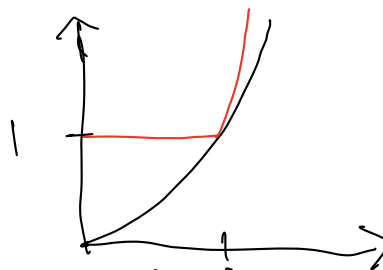


# Existence of Moments

- Suppose  $E(|X|^k) < \infty$  for some  $k > 0$ . Then for  $0 < r < k$ ,  $E(|X|^r) < \infty$ .

Proof.

First note



$$|X|^r \leq 1 \cdot \mathbf{1}_{|X| < 1} + |X|^k \mathbf{1}_{|X| \geq 1}.$$

Recall that if  $X \leq Y$ ,  $E(X) \leq E(Y)$ . So, if RHS of the inequality has finite expectation,  $E(|X|^r) < \infty$  also. We have:

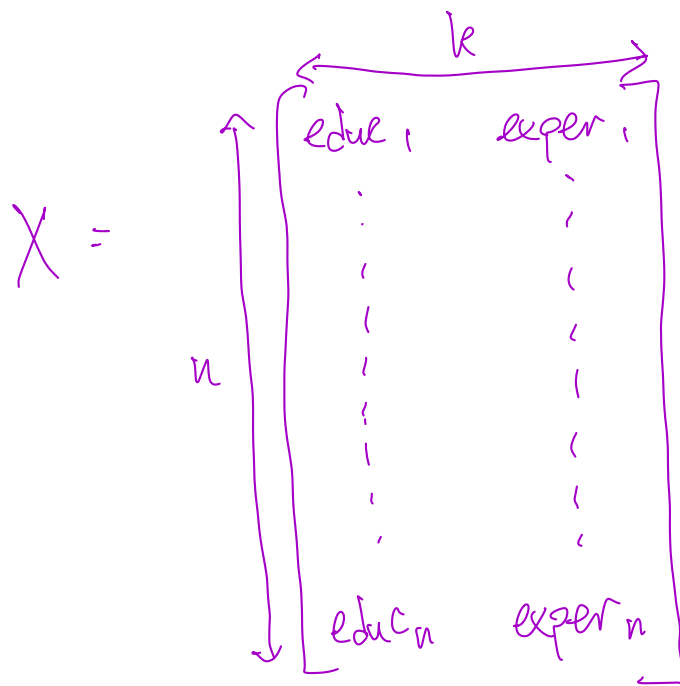
$$\begin{aligned} E\left(\underbrace{1 \cdot \mathbf{1}_{|X| < 1}}_{E(\mathbf{1}_{|X| < 1})} + |X|^k \mathbf{1}_{|X| \geq 1}\right) &= P(|X| < 1) + E(|X|^k \mathbf{1}_{|X| \geq 1}) \\ &\leq P(|X| < 1) + E(|X|^k) \\ &< \infty. \end{aligned}$$

$$\begin{aligned} &= 1 \cdot P(|X| < 1) + 0 \cdot P(|X| \geq 1) \\ &= P(|X| < 1). \end{aligned}$$

□

# Matrices

- We will collect information about individuals together in a single matrix.
- e.g. Information about education levels and years of experience can be collected in a matrix  $X$ :



- $X$  has dimensions  $n \times k$ .
- $n$  is the sample size.
  - $k$  is the number of regressors

Each row contains covariate information for a single observation.  
Each column contains all observations of a single covariate.



# Expectation of Random Matrices

- If  $X \in \mathbb{R}^{N \times M}$  is a random matrix, its expectation is defined as

$$E(X) := \begin{pmatrix} E(X_{11}) & E(X_{12}) & \cdots & E(X_{1M}) \\ E(X_{21}) & E(X_{22}) & \cdots & E(X_{2M}) \\ \vdots & \vdots & \ddots & \vdots \\ E(X_{N1}) & E(X_{N2}) & \cdots & E(X_{NM}) \end{pmatrix}.$$

- $M = 1$  corresponds to a random vector.
- The variance of an  $(N \times 1)$  random vector is the matrix

$(N \times N)$   $Var(X) = E \left[ (X - E(X))(X - E(X))' \right],$

with  $(i, j)$  element equal to  $Cov(X_i, X_j)$ . Note  
 $Cov(X_i, X_i) = Var(X_i)$ .

# (Co)variance of Random Vectors

- Let  $X$  be a random vector such that  $\text{Var}(X)$  exists. Let  $A$  be a matrix of constants and  $b$  a vector of constants. Then

$$\text{Var}(AX + b) = A\text{Var}(X)A'$$

- If  $X \in \mathbb{R}^N$  and  $Y \in \mathbb{R}^M$ , the covariance of  $X$  and  $Y$  is the  $N \times M$  matrix

$$\text{Cov}(X, Y) := \mathbb{E} \left[ (X - \mathbb{E}(X))(Y - \mathbb{E}(Y))' \right].$$

# Independence

- The elements of  $X$  are independent if

$$F_X(x_1, \dots, x_n) = \prod_{j=1}^n F_{X_j}(x_j)$$

for all  $(x_1, \dots, x_n)$ , where  $F_{X_j}$  is the distribution of  $X_j$ .

- A countable collection of random variables  $\{X_i\}_{i \geq 1}$  is independently and identically distributed (iid) if:
  - $F_{X_j} = F_{X_1}$  for all  $j \geq 1$ . (Identical distribution)
  - Any finite combination of the  $X_i$  forms a vector of independent random variables.
- e.g. A stationary time series features identically distributed but not necessarily independent observations.

# Marginal Distributions

- If  $X = (X_1, \dots, X_n)$  has joint distribution  $P_X$ , define marginal distribution of  $X_1$  as:

$$P_{X_1}(X_1 \in B) = P_X(X_1 \in B, X_2 \in \mathbb{R}, \dots, X_n \in \mathbb{R}).$$

- It also holds that

$$F_{X_1}(x_1) = \lim_{x_2, \dots, x_n \rightarrow +\infty} F_X(x_1, \dots, x_n).$$

# Marginal Distributions

- If  $(X, Y)$  is continuously distributed, then for all  $x$ ,

$$\begin{aligned} F_X(x) &= \lim_{y \rightarrow +\infty} F_{XY}(x, y) \\ &= \lim_{y \rightarrow +\infty} \int_{s=-\infty}^x \int_{t=-\infty}^y f_{XY}(s, t) \, ds \, dt \\ &= \int_{s=-\infty}^x \int_{t=-\infty}^{\infty} f_{XY}(s, t) \, dt \, ds \end{aligned}$$

This shows that

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{XY}(s, t) \, dt$$

since densities are unique (up to a set of lebesgue measure 0).

# Questions?

# Conditional Distributions

- Let  $(X, Y)$  have joint density  $f_{XY}$ . The conditional density of  $Y$  given  $X$  is

$$f_{Y|X}(y|x) := \frac{f_{XY}(x, y)}{f_X(x)}$$

and it follows that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx$$

- View this as continuous version of LOTP.
- Discrete case follows as in definition of conditional probability to give conditional pmf  $p_{Y|X}$ .

# Conditional Expectation

- Compute  $E(Y|X)$  as

$$E(Y|X = x) = \begin{cases} \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy & (X, Y) \text{ continuous,} \\ \sum_y y p_{Y|X}(y|x) & (X, Y) \text{ discrete.} \end{cases}$$

- $E(Y|X)$  is itself a function of  $X$ , so it is a random variable.
- Properties:
  - (Linearity)  $E(aY + bZ|X) = aE(Y|X) + bE(Z|X)$ ;
  - (Law of Iterated Expectation)  $E(Y) = E(E(Y|X))$ ;
  - (Taking out what is known)  
 $E(f(X) + g(X)Y|X) = f(X) + g(X)E(Y|X)$ .



# Conditional Variance

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$
$$\text{Var}(Y|X) = E(Y^2|X) - [E(Y|X)]^2.$$

- Define

$$\begin{aligned}\text{Var}(Y|X) &:= E([Y - E(Y|X)]^2 | X) \\ &= E(Y^2|X) - E(Y|X)^2.\end{aligned}$$

- Same as variance, but with all expectations now conditioned on  $X$ .
- Can show:

$$\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)).$$

$$E(Y) = E(E(Y|X))$$

# Conditional Mean Independence

- $Y$  is mean independent of  $X$  if  $E(Y|X) = E(Y)$ .
- The following implications hold:

$$\begin{aligned} X \text{ independent of } Y &\implies Y \text{ mean independent of } X \\ &\implies \text{Cov}(X, Y) = 0 \end{aligned}$$

- Note that independence implies

$$P_{Y|X}(Y \in B | X \in A) = P_Y(Y \in B)$$

i.e. the conditional distribution of  $Y$  does not depend on  $X$ , so any feature of it (mean, variance) doesn't either.

# Conditional Mean Independence

$$E(Y|X) = E(Y) \Rightarrow \text{Cov}(X, Y) = 0.$$

- Second implication follows from LIE:

$$\begin{aligned}\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= E(XE(Y|X)) - E(X)E(Y) \\ &= E(XE(Y)) - E(X)E(Y) \\ &= E(X)E(Y) - E(X)E(Y) \\ &= 0.\end{aligned}$$

$$\begin{aligned}E(XY) \\ &= E(E(XY|X)) \\ &= E(XE(Y|X))\end{aligned}$$

- Reverse implications don't generally hold, except when  $(X, Y)$  are jointly normal.

# Linear Regression

- Suppose the vector  $(Y, X_1, \dots, X_k)$  has distribution  $P_{YX}$ , where  $X = (X_1, \dots, X_k)$ .
- Suppose  $E(Y^2) < \infty$  and  $E(X_j^2) < \infty$  for each  $j = 1, \dots, n$ .
- A feature of this joint distribution is the function  $g^*$  which minimizes

$$E(Y - g(X))^2.$$

That is

$$g^* \in \arg \min_{g \in L^2(X)} E(Y - g(X))^2,$$

where  $L^2(X) = \{f : E[f(X)^2] < \infty\}$ .

- Why learn this feature?  $g^*$  is the best predictor of  $Y$  under square loss.

# Linear Regression

- Now show that  $g^*(X)$  is given by  $E(Y|X)$ : Write

$$\begin{aligned} E(Y - g(X))^2 &= E(Y - E(Y|X) + E(Y|X) - g(X))^2 \\ &= E(Y - E(Y|X))^2 + E(E(Y|X) - g(X))^2 \\ &\quad + 2E[(Y - E(Y|X))(E(Y|X) - g(X))]. \end{aligned}$$

Use the LIE to show last term is 0! What remains is

$$\begin{aligned} E(Y - g(X))^2 &= E(Y - E(Y|X))^2 + E(E(Y|X) - g(X))^2 \\ &\geq E(Y - E(Y|X))^2. \end{aligned}$$

so  $g(X) = E(Y|X)$  is the best predictor of  $Y$  under square loss.

# Linear Regression

- We can restrict  $L^2(X)$  to a smaller subset

$$H(X) = \{f : f(X) = X'a \text{ for some } a \in \mathbb{R}^N\}$$

- Then, the best linear predictor of  $Y$  given  $X$  is found by solving

$$\min_{b \in \mathbb{R}^N} E(Y - X'b)^2$$

- Expanding the square gives

$$\min_{b \in \mathbb{R}^N} E(Y^2) - 2b'E(XY) + b'E(XX')b.$$

# Linear Regression

$$E(Y^2) - 2b' E(XY) + b' E(XX') b.$$

- Differentiate wrt.  $b$  to yield FOC:

$$2E(XX') b^* - 2E(XY) = 0.$$

$$E(XX') b^* = E(XY)$$
$$b^* = E(XX')^{-1} E(XY)$$

- Provided  $X_j$  are not linearly dependent random variables,  $E(XX')$  is full rank, so

$$b^* = E(XX')^{-1} E(XY).$$

- Define the prediction error  $U = Y - X'b^*$ . Then, by construction,

$$Y = X'b^* + U; \quad E(XU) = 0,$$

since

$$E(XU) = E(XY) - E(XX') b^* = 0$$

is the first order condition.

## Questions?

⊗ By the law of iterated expectation:

$$\begin{aligned} & E[(Y - E(Y|X))(E(Y|X) - g(X))] \\ \stackrel{\text{LIE}}{=} & E\left[ E\left[ (Y - E(Y|X))(E(Y|X) - g(X)) \mid X \right] \right] \\ & \quad \text{depends only on } X. \text{ Take out what is known:} \\ & = E\left[ (E(Y|X) - g(X)) E[Y - E(Y|X) \mid X] \right] \\ & \quad = E[E(Y|X)] - E(Y|X) \\ & = 0. \end{aligned}$$

$$= 0.$$