Price Theory Cheat-Sheet

The Chicago Approach

Core Methodology

Price theory is **applied**: write a model, solve it, check if predictions match data. Use the **simplest model** that captures the economic force. The goal is not generality but *insight*—a two-good, two-period model that delivers a clear comparative static beats a general framework that delivers ambiguity.

Three functions of prices: (1) transmit scarcity information, (2) provide incentives to economize, (3) direct resources to highest-valued uses. When prices are prevented from adjusting, *something else* clears the market (queues, search, quality degradation).

Demand Without Rationality

Becker (1962): Random choice on a budget line $\Longrightarrow$ aggregate demand is downward-sloping, HD0, satisfies adding up and Slutsky symmetry. *Implication:* many demand properties come from the budget constraint alone. If a model only needs a demand curve, rationality is not required—budget constraints suffice.

Equilibrium Thinking Template

Never stop at the direct effect:

1. Write down direct/partial equilibrium effect
2. Identify margins of adjustment (other prices, quantities, behavior)
3. Re-solve for new equilibrium; compare with PE
4. Check: does direct effect survive? Reverse? Amplify?

Ethanol: PE: corn demand $\uparrow$. GE: feed prices $\uparrow$, benefits producers $>$ subsidy (multiplier). *Paxlovid:* PE saves lives; GE: less prevention $\Longrightarrow$ net ambiguous. *Rent control:* PE: rents $\downarrow$; GE: quality $\downarrow$, supply $\downarrow$, misallocation.

TFU Toolkit

Recurring exam patterns:

- **Stock vs. flow:** asset price $\neq$ rental price; tax equivalence holds only in SS
- **Average vs. marginal:** $MC < AC \Longrightarrow AC$ falling; monopolist targets marginal consumer
- **Pecuniary vs. real:** price changes redistribute but are *not* externalities
- **SR vs. LR:** stock fixed in SR; adjustment via entry/exit
- **Envelope theorem:** eliminates first-order behavioral adjustments

Modeling Shortcuts & Setup

Standard Simplifying Assumptions

Unless contradicted by the problem or making it trivial:

1. Assume **normal goods**
2. **Ignore income effects** (quasilinear utility)
3. Assume **continuously differentiable** utility
4. $\lim_{x\to 0} U'(x) = \infty$, $\lim_{x\to\infty} U'(x) = 0$ (interior solutions)
5. Assume **constant marginal costs**
6. **DRS** at firm level, **CRS** at industry level
7. **Representative agent** or ex-ante identical agents
8. i.i.d. shocks; ignore integer constraints
9. Use only **two goods** and/or **composite commodities**

Model Setup Checklist

1. Count equations vs. unknowns. Adding a constraint without a variable $\Longrightarrow$ over-determined. Introduce a new margin (search, rationing, wait time).

2. Choose the right numeraire. Set $p_m = 1$ or normalize a factor price. All results hold in relative prices.

3. Representative agent + market clearing. Normalize to one consumer. Aggregate = individual in equilibrium. Avoids tracking distributions when heterogeneity isn't the focus.

4. Linear production $F = AH$. Price pinned to $p = w/A$ by zero profit. Reduces to a pure consumer problem.

5. One-dimensional heterogeneity. Index types by θ_i. Optimal behavior monotone in $\theta \Longrightarrow$ cutoff strategies: $\bar\theta$ separates actions. Equilibrium reduces to finding $\bar\theta$.

Consumer Demand

Utility Maximization (Marshallian)

$$\max_x U(x) \quad \text{s.t.} \quad p\cdot x \le M$$

Lagrangian: $\mathcal{L} = U(x) + \lambda(M - p\cdot x)$. FOCs:

$$\frac{\partial U}{\partial x_i} = \lambda p_i \quad \forall i \qquad \Longrightarrow \qquad \frac{U_i'}{p_i} = \frac{U_j'}{p_j} = \lambda$$

The **equimarginal principle**: marginal utility per dollar is equalized across all goods. λ is the marginal utility of income.

Solution: **Marshallian demand** $x^M(p, M)$.

Indirect utility: $V(p, M) = U(x^M(p, M))$.

Roy's Identity: $x_i^M = -\frac{\partial V/\partial p_i}{\partial V/\partial M}$.

Expenditure Minimization (Hicksian)

$$\min_x p\cdot x \quad \text{s.t.} \quad U(x) \ge \bar u$$

Solution: **Hicksian demand** $x^H(p, \bar u)$.

Expenditure function: $e(p, \bar u) = p\cdot x^H(p, \bar u)$.

Shephard's Lemma: $x_i^H = \frac{\partial e}{\partial p_i}$.

$e(p, \bar u)$ is concave, HD1 in p, increasing in $\bar u$.

The Slutsky Equation

Connects Marshallian and Hicksian systems:

$$\frac{\partial x_i^M}{\partial p_j} = \underbrace{\frac{\partial x_i^H}{\partial p_j}}_{\text{substitution}} - \underbrace{\frac{\partial x_i^M}{\partial M} x_j^M}_{\text{income effect}}$$

In **elasticity form**: $\epsilon_{ij}^M = \epsilon_{ij}^H - s_j\eta_i$, where $s_j = p_j x_j/M$ (budget share), $\eta_i = \frac{\partial x_i}{\partial M}\frac{M}{x_i}$ (income elasticity).

Own-price: $\epsilon_{ii}^M = \epsilon_{ii}^H - s_i\eta_i$. For normal goods ($\eta_i > 0$), Marshallian demand is *more* elastic than Hicksian.

Demand Identities

These follow from the budget constraint and HD0 of Hicksian demand.

Adding Up (Engel aggregation + Cournot aggregation):

$$\sum_i s_i\eta_i = 1, \qquad \sum_i s_i\epsilon_{ij}^M + s_j = 0$$

Homogeneity of degree zero:

$$\sum_j \epsilon_{ij}^M + \eta_i = 0$$

Slutsky Symmetry: $\frac{\partial x_i^H}{\partial p_j} = \frac{\partial x_j^H}{\partial p_i}$, or equivalently:

$$s_i\epsilon_{ij}^M = s_j\epsilon_{ji}^M + s_i s_j(\eta_j - \eta_i)$$

Key: Symmetry + Homogeneity $\Longrightarrow$ Adding Up, and Symmetry + Adding Up $\Longrightarrow$ Homogeneity. The three are **not all independent**.

Additively Separable Utility

$U = \sum_i u_i(x_i)$ with $u_i'' < 0$. Key properties:

- **No inferior goods**: $\partial x_i^M/\partial M > 0$ for all i
- Demand: $x_i^M = (u_i')^{-1}(\lambda p_i)$ where λ solves $\sum_i p_i(u_i')^{-1}(\lambda p_i) = M$
- Since λ must decrease when M rises (to exhaust the budget), all goods increase $\Longrightarrow$ no good can be inferior

Application (time allocation): With $U = u_f(f) + u_\ell(\ell) + u_m(m)$ and linear production $p_f = w/A$, the FOC gives $A = u_\ell'(\ell)/u_f'(f)$—the consumption ratio is pinned by technology alone. Money supply changes M only, leaving the real allocation unchanged (money neutrality).

Quasilinear Utility

$U = v(x) + m$ where m is the numeraire. Then $v'(x) = p$ determines demand independently of income. No income effects; $CS = \int_0^{x^*} v'(t)\,dt - px^*$ is an exact welfare measure.

Application: Any partial equilibrium welfare analysis (DWL of a tax, monopoly surplus) is exact under quasilinearity. Use this as default assumption unless the question specifically involves income effects.

Cobb-Douglas Demand

$U = x_1^\alpha x_2^{1-\alpha}$. **Fixed budget shares:** $p_1x_1 = \alpha M$, $p_2x_2 = (1-\alpha)M$.

Demand: $x_i = \frac{\alpha_i M}{p_i}$. Budget shares are invariant to prices and income. $\epsilon_{ii}^M = -1$, $\eta_i = 1$.

CES Demand

$U = \left(\sum_i \alpha_i x_i^\rho\right)^{1/\rho}$, $\rho \le 1$. Elasticity of substitution $\sigma = \frac{1}{1-\rho}$.

$$x_i = \frac{\alpha_i^\sigma p_i^{-\sigma}}{\sum_j \alpha_j^\sigma p_j^{1-\sigma}} M$$

As $\sigma \to 1$: Cobb-Douglas. As $\sigma \to 0$: Leontief. As $\sigma \to \infty$: perfect substitutes.

Price Indices

Decompose income change $M^1/M^0 = (1+\Pi)(1+\Xi)$ into price and quantity:

Laspeyres (base-period bundle at new prices): $\Pi_L = \frac{\sum X_i^0 P_i^1}{\sum X_i^0 P_i^0} - 1$.
Overstates cost increase (ignores substitution toward cheaper goods).

Paasche (new bundle at base prices): $\Pi_P = \frac{\sum X_i^1 P_i^1}{\sum X_i^1 P_i^0} - 1$.
Understates cost increase.

True cost of living lies between Paasche and Laspeyres: $\Pi_P \le \Pi_{\text{true}} \le \Pi_L$.
The bias arises because Laspeyres uses the old bundle (no substitution) while Paasche uses the new (full substitution).
Chained indices reduce bias by updating the base period frequently.

Supply & Production

Cost Minimization

$$\min_x w \cdot x \quad \text{s.t.} \quad f(x) \ge q$$

FOCs: $w_i = \mu f_i(x) \implies \text{MRTS}_{ij} = f_i/f_j = w_i/w_j$.
Cost function: $C(w,q) = w \cdot x^*(w,q)$. Concave and HD1 in w.
Shephard's Lemma: $x_i^*(w,q) = \partial C/\partial w_i$.
Application: To find how a wage increase affects input mix, differentiate C rather than re-solving the optimization. Shephard's Lemma gives conditional factor demand directly.

Returns to Scale

$f(\lambda x) = \lambda^k f(x)$: $k>1$ IRS, $k=1$ CRS, $k<1$ DRS.
CRS: $C(w,q) = qc(w)$, $MC = AC = c(w)$, zero profit. Euler: $f = \sum x_i f_i$ (output exhausted by marginal products).
DRS at firm, CRS at industry: Each firm has upward-sloping MC (supply), but free entry pins $P = \min AC$ in the long run. Industry supply is horizontal at P^{LR}.
Application: Positive demand shock $\implies$ SR: P rises, $\pi > 0$. LR: entry drives P back to $\min AC$, quantity adjusts through number of firms N.

Profit Maximization

$\max_q \pi = pq - C(q)$. FOC: $p = MC(q)$. SOC: $MC'(q) > 0$.
Hotelling's Lemma: $q^S(p) = \partial \pi(p)/\partial p$ where $\pi(p) = pq^S(p) - C(q^S(p))$.

Short-Run vs. Long-Run

Some inputs fixed in SR ($\bar{K}$): $C^{SR}(q;\bar{K}) \ge C^{LR}(q)$, equality at q^* where $\bar{K}$ is optimal. $\implies$ LR supply more elastic (Le Chatelier).
Application: A tax on capital has a larger long-run effect on output than short-run, because firms can adjust capital stock in the long run.

Cobb-Douglas Production

$f(K,L) = AK^\alpha L^\beta$ with $\alpha + \beta = 1$ (CRS).
Factor demands: $wL/rK = \beta/\alpha$ (fixed factor cost shares).
$C = \kappa w^\beta r^\alpha q$.
Application (Harberger): With CD technology, factor shares $s_K = \alpha$, $s_L = \beta$ are constant regardless of factor prices. This pins down GE tax incidence (see Harberger model).

Industry vs. Firm Substitution

Even with **Leontief firms** ($\sigma_{\text{firm}} = 0$), the *industry* substitutes: factor price changes cause high-cost firms to exit while low-cost firms expand $\implies$ industry $\sigma > 0$ always. This is why industry-level regressions consistently find substitution even in industries where individual firms use fixed proportions.

Competitive Equilibrium & Welfare

Market Equilibrium

$Q^D(p) = Q^S(p)$. With inverse demand $P^D(Q)$ and inverse supply $P^S(Q)$:

$$P^D(Q^*) = P^S(Q^*)$$

Welfare

$CS = \int_0^{Q^*} P^D(Q)\,dQ - P^*Q^*$. $\quad PS = P^*Q^* - \int_0^{Q^*} P^S(Q)\,dQ$.
$W = CS + PS$. Competitive equilibrium maximizes W (First Welfare Theorem).

Tax Incidence (Partial Equilibrium)

Per-unit tax t. Let $\epsilon_D < 0$, $\epsilon_S > 0$:

$$\frac{dp^D}{dt} = \frac{\epsilon_S}{\epsilon_S - \epsilon_D}, \qquad \frac{dp^S}{dt} = \frac{\epsilon_D}{\epsilon_D - \epsilon_S}$$

The more **inelastic** side bears more. Incidence is independent of statutory assignment.
Application: Payroll tax "paid by employers" vs. "paid by workers" has the same incidence: what matters is ϵ_S vs. ϵ_D, not statutory labels.
Deadweight loss (Harberger triangle):

$$DWL \approx \frac{1}{2}\frac{\epsilon_S|\epsilon_D|}{\epsilon_S - \epsilon_D}\frac{t^2}{P}Q$$

DWL grows with $t^2 \implies$ broad low taxes dominate narrow high taxes.
Key implications:

1. Doubling tax rate *quadruples* DWL
2. Two 5% taxes create $\approx \frac{1}{2}$ the DWL of one 10% tax
3. Tax most heavily goods with inelastic demand/supply
4. DWL exists even with inelastic labor supply if non-cash compensation (fringe benefits) is untaxed—the tax distorts the *form* of compensation

Pigouvian correction: Tax $t = MC_{\text{external}}$ restores efficiency. Unlike ordinary taxes, this *reduces* DWL. Target the externality directly—tax emissions, not electricity (see §Externalities).

Non-Price Market Clearing

When prices cannot adjust:

- **Wait time:** Kidney market — excess demand cleared by queue. Wait time τ acts as a "shadow price": $P_{eff} = P_{ceiling} + v\tau$
- **Search costs:** Min wage — effective leisure price $p_\ell = w - p_s$. Ratio $p_\ell/p_f < A \implies$ less food, more leisure, plus DWL from search
- **Quality adjustment:** Rent control — landlords reduce maintenance until effective price matches equilibrium

Modeling: Replace constrained price with shadow price $P_{eff} = P_{\text{cap}} + \text{non-price cost}$, then re-solve.

Monopoly & Market Power

Static Monopoly

$\max_q \pi = P(q)q - C(q)$. FOC:

$$MR = P + P'q = MC \qquad \iff \qquad P\left(1 + \frac{1}{\epsilon_D}\right) = MC$$

Lerner index: $\frac{P - MC}{P} = -\frac{1}{\epsilon_D}$. More elastic demand $\implies$ smaller markup.

Linear demand $P = \alpha - \beta q$: $q^* = \frac{\alpha - MC}{2\beta}$, $P^* = \frac{\alpha + MC}{2}$.

$DWL = \frac{(\alpha - MC)^2}{8\beta}$ (one quarter of competitive surplus is lost).

Two-Period Monopoly with Copying

Inventor sells in period 1; each unit spawns n copies in period 2. Inverse demand $v(c)$, cost w/A.

Interior: When n is small, the two-period FOC collapses to the static monopoly result—the inventor fully captures period-2 surplus through period-1 pricing.

Corner ($c_{i2} = 0$, linear demand):

$$c_{i1}^* = \frac{(n+1)(\alpha - w/A)}{(n(n+1)+2)\beta}, \qquad c_2^* = nc_{i1}^*$$

As $n \to \infty$: $c_{i1}^* \to 0$, $c_2^* \to 2c^*_{\text{comp}}$. Perfect copying destroys creation incentives but yields competitive allocation.

Application (IP policy): n parameterizes enforcement. Weak IP (high n) $\implies$ more consumption but less creation. The optimal n trades off static efficiency against dynamic incentives.

Price Discrimination

1st degree: Charge WTP. Efficient but extracts all CS.

2nd degree: Menu of bundles; distort low-type quantity to prevent high-type mimicking. Self-selection via IC constraints.

3rd degree: Segment markets. FOC per market: $P_i(1 + 1/\epsilon_i) = MC$. Higher price where demand is less elastic.

Application: Airline pricing charges business travelers (inelastic) more and leisure travelers (elastic) less. Formally: $P_{bus}/P_{lei} = (1 + 1/\epsilon_{lei})/(1 + 1/\epsilon_{bus}) > 1$ when $|\epsilon_{bus}| < |\epsilon_{lei}|$.

Price Controls on Monopoly

A **price ceiling below monopoly price** can *increase* output: the monopolist faces a kinked demand curve (horizontal at the ceiling, then original demand). If ceiling is between MC and P^M: output rises toward competitive level.

Quality deterioration: Under price controls, firms reduce quality until effective price matches unconstrained equilibrium.

Industrial Organization

Dominant Firm (Price Leadership)

One large firm (share S), competitive fringe. Dominant firm faces **residual demand**: $D_R = D - S_f$.

Residual demand elasticity:

$$\epsilon^* = \frac{\epsilon_D}{S} - \frac{(1-S)}{S}\epsilon_S^{\text{fringe}}$$

Markup: $\frac{P-MC}{P} = -\frac{1}{\epsilon^*}$. Market power is **highly nonlinear** in share:

Share S	10%	30%	50%	100%
Markup	1.8%	5.3%	14.3%	100%

(With $\epsilon_D = -1$, $\epsilon_S^f = 1$.) A firm with 10% share has negligible pricing power.

Application (AT&T/Time Warner): Vertical merger gives control of content. Market power depends on whether rivals can access substitutes—fringe supply elasticity matters more than market share.

Worked example: Market demand $P = 100 - Q$, fringe supply $Q_f = P$, dominant firm $MC = 20$; share $S \approx 50\%$. Residual demand: $Q_R = (100 - P) - P = 100 - 2P$, so $P = 50 - Q_R/2$. $MR_R = 50 - Q_R$. Set $MR_R = 20 \implies Q_R = 30$, $P = 35$, fringe supplies 35, total $Q = 65$. Markup: $(35 - 20)/35 \approx 43\%$.

Collusion

Stigler's conditions (collusion harder when):

- Many sellers (harder to monitor)
- Heterogeneous products or costs
- Large, infrequent orders (incentive to cheat)
- Low barriers to entry (profits attract entrants)

Cheating incentive: Deviating firms earn more per unit (free-ride on collusive price without restricting output).

Long-run: Entry reduces shares $\implies$ ϵ^* rises $\implies$ markup falls $\implies$ collusion collapses.

Vertical Integration

Double marginalization: Upstream monopolist charges $P_U > MC_U$; downstream monopolist adds its own markup. Final price has *two* markups $\implies$ higher than integrated monopoly price.

Vertical integration eliminates one markup: integrated firm sets $MR = MC_U + MC_D$. Output rises, price falls, **both firms and consumers gain**. This is why vertical mergers are often pro-competitive.

Cournot effect for complements: n independently priced complements: each monopolist ignores effect of own price on others' demand. Integration internalizes complementarity $\implies$ lower bundle price.

Application: Vertical merger eliminates double marginalization on content licensing.

Network Effects & Social Multiplier

Demand depends on others' consumption: $x_i = f(p, \bar{x})$ where $\bar{x}$ is aggregate.

Social multiplier: If $\partial x_i/\partial \bar{x} = \phi$, the equilibrium response to a price change is:

$$\left.\frac{dx}{dp}\right|_{\text{eq}} = \frac{\partial x/\partial p}{1-\phi}$$

With $\phi > 0$ (conformity), the equilibrium response exceeds the individual response by factor $1/(1-\phi)$.

Multiple equilibria possible with S-shaped adoption curves; small interventions can tip between them.

Advertising (Dorfman-Steiner)

$$\frac{A}{PQ} = \frac{\epsilon_A}{|\epsilon_D|}$$

Advertise more when advertising elasticity is high relative to demand elasticity.

Price Discrimination: Welfare

3rd degree: Welfare effect is ambiguous. Deadweight loss falls in elastic-demand market (output expands) but rises in inelastic market (output falls). **Total output must increase** for PD to improve welfare.

Spence quality distortion: A monopolist designs quality for the *marginal* consumer, not the average. If marginal consumer values quality less than average, quality is underprovided. If more (e.g., attracting new customers), quality is overprovided.

Factor Demand & Marshall's Laws

Factor demand ties consumer theory to production: the same substitution-vs-scale decomposition that drives Slutsky drives Marshall's Laws. Every consumer result has a producer dual (see Duality Table in §Math Toolkit).

Derived Demand

Demand for an input is *derived* from demand for the output it produces. A wage increase $w\uparrow$ affects L^* through two channels:

Substitution effect (hold output fixed, reoptimize inputs): $\partial L^c/\partial w < 0$.

Scale effect (higher cost $\implies$ higher price $\implies$ lower output): reinforces the substitution effect for normal inputs.

Two-Input Model

Firm produces $q = f(K, L)$, sells at price p. Cost minimization: $\frac{f_L}{f_K} = \frac{w}{r}$.

Elasticity of substitution: $\sigma = \frac{d\ln(K/L)}{d\ln(w/r)}\Big|_{q\text{ fixed}}$

$\sigma = 0$: Leontief (no substitution). $\sigma = 1$: Cobb-Douglas. $\sigma = \infty$: linear technology.

Marshall's Laws of Derived Demand

Demand for a factor is **more elastic** when:

1. **Elasticity of substitution** σ is high
2. **Demand for output** $|\epsilon_D|$ is high
3. Factor's **cost share** s is large (caveat: when $\sigma < |\epsilon_D|$, larger share makes demand *less* elastic)
4. **Supply of other factors** ϵ_S^{other} is high

Two-factor formula:

$$\eta_L = \frac{(1-s_L)\sigma\epsilon_S^K + s_L|\epsilon_D|\sigma + (1-s_L)|\epsilon_D|\epsilon_S^K}{(1-s_L)\epsilon_S^K + s_L\sigma + (1-s_L)|\epsilon_D|}$$

With $\epsilon_S^K = \infty$ (perfectly elastic capital supply):

$$\eta_L = (1-s_L)\sigma + s_L|\epsilon_D|$$

A weighted average: substitution effect (weight $1-s_L$) and scale effect (weight s_L).

Cross-price elasticity: $\beta_{ij} = s_j\epsilon_D + s_j\sigma_{ij}$

Application (education production): Inputs: student time (sK), purchased materials (D). Remote learning lowers productivity of sK $\implies$ MC rises $\implies$ scale effect reduces all inputs. Substitution: shift toward purchased inputs (online tools). Decompose via Marshall's Laws: scale dominates when σ is small, substitution dominates when σ is large.

Application (min wage): w forced above w^*: substitution replaces workers with capital; scale reduces output. Total effect depends on σ and $|\epsilon_D|$ via Marshall's formula.

Isoprofit Curves

For a firm choosing (w, L) along an isoprofit: $\pi = pf(K, L) - wL - rK$. The isoprofit curve in (L, w) space is:

$$w = \frac{pf(K,L) - rK - \bar{\pi}}{L}$$

Higher isoprofits are closer to the origin. The firm's labor demand curve traces out the tangencies of isoprofit curves with the wage line.

General Equilibrium & Tax Incidence

Edgeworth Box

Two consumers (A, B), two goods, fixed endowments ω.

- **Feasible:** $x^A + x^B = \omega^A + \omega^B$
- **Pareto optimal:** $\text{MRS}^A = \text{MRS}^B$ (contract curve)
- **Competitive eq.:** Utility maximization on budget sets at prices p, markets clear

1st Welfare Thm: competitive eq. is Pareto optimal. 2nd Welfare Thm: any Pareto optimum is a competitive eq. with lump-sum transfers.

Application: To check whether trade improves on autarky, verify $\text{MRS}^A \neq \text{MRS}^B$ at the endowment point. If they differ, gains from trade exist.

2×2 GE Model (Jones, 1965)

Two goods (X, Y), two factors (K, L), CRS, competitive markets. Hat notation: $\hat{z} = dz/z$.

Zero-profit: $a_{LX}w + a_{KX}r = p_X, \quad a_{LY}w + a_{KY}r = p_Y$.

Full-employment: $a_{LX}X + a_{LY}Y = L, \quad a_{KX}X + a_{KY}Y = K$.

Stolper-Samuelson (price → factor returns): $\hat{w} > \hat{p}_X > \hat{p}_Y > \hat{r}$ when X is L-intensive.

Factor returns are *magnified*: the factor used intensively gains more than the price increase; the other factor loses.

Rybczynski (endowment → output): $\hat{X} > \hat{L} > 0 > \hat{Y}$ when X is L-intensive.

Application (trade): A country abundant in L exports X (the L-intensive good). Opening trade raises $p_X \implies$ raises w and lowers r (Stolper-Samuelson). Predicts that trade liberalization hurts the scarce factor.

Application (immigration): An increase in L expands the L-intensive sector and contracts the K-intensive sector (Rybczynski). Output adjusts, but factor prices remain unchanged if both goods are produced (factor price insensitivity).

Harberger Tax Incidence

Tax on capital in sector X at rate t. Capital earns $r_X = r + t$ in X, $r_Y = r$ in Y (mobile capital, net return r).

Symmetric Cobb-Douglas case: If both sectors have identical factor shares and $\sigma = 1$: capital bears 100% of the tax: $dr/dt = -1$.

Intuition: Tax drives K from X to Y, lowering r everywhere. With symmetric technologies, GE adjustment exactly offsets PE incidence.

General case: Who bears the tax depends on: (1) factor intensities, (2) σ in each sector, (3) product demand elasticities. PE incidence can be misleading—a tax appearing to fall on consumers may fall entirely on a factor in GE.

Modeling tip: In Harberger models, there are 4 unknowns ($\hat{w}$, $\hat{r}$, $\hat{X}$, $\hat{Y}$) and 4 equations (two zero-profit, two full-employment). Log-linearize and solve the 4×4 system.

CRS Industry & Factor Prices

Four-Equation CRS Model

A competitive industry with CRS firms, output Q, inputs (L, K):

(1) Zero profit:	$p = c(w, r)$
(2) Labor demand:	$L = c_w(w, r) \cdot Q$
(3) Capital demand:	$K = c_r(w, r) \cdot Q$
(4) Output demand:	$Q = D(p)$

With factor supplies $L^S(w)$, $K^S(r)$: six equations, six unknowns (p, w, r, Q, L, K).

Solution order: (1) pins p given factor prices → (4) gives Q → (2),(3) give L, K.

Hat Calculus

Log-differentiate the four equations ($\hat{z} = dz/z$):

$$\hat{p} = s_L\hat{w} + s_K\hat{r}$$
$$\hat{L} = -s_K\sigma\hat{w} + s_L\sigma\hat{r} + \hat{Q}$$
$$\hat{K} = s_L\sigma\hat{w} - s_K\sigma\hat{r} + \hat{Q}$$
$$\hat{Q} = \epsilon_D\hat{p}$$

where s_L, s_K are factor cost shares and σ is the elasticity of substitution.

Lump-Sum vs. Per-Unit Tax

Per-unit tax (t per unit): shifts MC up by t. Effects:

- SR (identical firms): $P\uparrow$, $q\downarrow$, $\pi < 0$
- LR: exit $\implies P = \min AC + t$, firms remaining produce at same q, fewer firms
- Identical firms LR: consumers bear 100% (supply horizontal at $\min AC + t$)
- Non-identical: shared between consumers and infra-marginal firms

Lump-sum tax (F per firm): raises AC but **not** MC. Effects:

- SR: no change in output or price (MC unchanged)
- LR: exit until $P = \min AC'$ (new, higher min AC). Each surviving firm is *larger* (min AC at higher q). Output per firm $\uparrow$, number of firms $\downarrow$

Changes in Factor Prices

Surprising result: A wage increase can *raise* profits of a competitive industry if demand is inelastic and there are fixed factors (revenue increase from $P\uparrow$ exceeds cost increase).

Monopolist: A factor price increase *never* raises monopolist profits (already optimizing).

Inferior Factors

A factor is **inferior** if increased output reduces demand for it. With two factors:

$$\hat{L} = (\sigma s_K + \epsilon_D s_L)(\hat{w}/s_L)$$

Factor L is inferior iff $\sigma s_K < -\epsilon_D s_L$ (strong scale effect dominates substitution).

Application: Unskilled labor in a sector with elastic demand and low substitutability with capital. Output expansion from a demand shift may reduce unskilled employment if the scale effect induced by capital deepening is large enough.

Total Factor Productivity

Primal approach: $\hat{Q} = s_L\hat{L} + s_K\hat{K} + \widehat{TFP}$.

Dual approach: $\hat{p} = s_L\hat{w} + s_K\hat{r} - \widehat{TFP}$.

Dual says: productivity growth *either* lowers output prices (holding factor prices fixed) *or* raises factor prices (holding output price fixed).

Bias: If TFP growth is labor-augmenting, at constant factor prices: K/L falls, $w\uparrow$, r unchanged. Capital-augmenting: opposite.

Application: Computing $\widehat{TFP} = \hat{Q} - s_L\hat{L} - s_K\hat{K}$ as a residual attributes all non-input-growth to productivity. Mismeasured inputs (quality changes) bias TFP estimates.

Worked example (hat calculus): Industry with $s_L = 0.6$, $s_K = 0.4$, $\sigma = 1$ (Cobb-Douglas), $\epsilon_D = -2$. Wage rises 10% ($\hat{w} = 0.1$), capital price fixed ($\hat{r} = 0$). $\hat{p} = 0.6(0.1) = 0.06$ (price up 6%).
$\hat{Q} = -2(0.06) = -0.12$ (output down 12%).
$\hat{L} = -0.4(1)(0.1) + (-0.12) = -0.16$ (labor down 16%).
$\hat{K} = 0.6(1)(0.1) + (-0.12) = -0.06$ (capital down 6%). Substitution away from labor (4pp) plus scale contraction (12pp).

Durable Goods & Stock-Flow

The Four Equations

A durable good (housing, cars, capital) has a **stock** S and a **flow** of new production I. Rental price R and asset price P are linked:

(1) Demand:	$S = D(R)$
(2) Asset pricing:	$R = (r + \delta)P - \dot{P}$
(3) Supply of new:	$I = I(P)$
(4) Stock evolution:	$\dot{S} = I - \delta S$

Eq. (2) is the **user cost of capital**: rental = interest + depreciation − capital gains.

Equivalently: $P = \sum_{k=0}^{\infty} R_{t+k}\frac{(1-\delta)^k}{(1+r)^k}$ (present value of future rents).

Steady State

Set $\dot{P} = 0$, $\dot{S} = 0$:

$$S^* = D(R^*), \qquad P^* = \frac{R^*}{r + \delta},$$
$$I^* = I(P^*), \qquad I^* = \delta S^*$$

Solution order: Work backward from rental market. Given $S^* \to R^*$ from (1), $R^* \to P^*$ from (2), $P^* \to I^*$ from (3), verify $I^* = \delta S^*$ from (4).

Comparative Statics: Construction Cost Increase

SR (stock fixed): S unchanged $\implies R$ unchanged $\implies P$ unchanged. But higher construction costs reduce I at given P.

Transition: $I < \delta S \implies S$ falls $\implies R$ rises $\implies P$ rises. P must *overshoot* new SS to sustain $I = \delta S^*_{new}$ with higher costs.

New SS: $S^*\downarrow$, $R^*\uparrow$, $P^*\uparrow$, $I^*\downarrow$.

Policy Applications

Rent control (R capped below R^*): SR rental $\downarrow$, $P\downarrow$ immediately. Lower $P \implies I\downarrow \implies S$ falls over time $\implies$ effective rent rises via queuing/quality decline.

Property tax (tax τ on value P): SR rental unchanged, P drops (capitalizes tax). User cost: $R = (r + \delta + \tau)P$. LR: lower $P \implies I\downarrow \implies S\downarrow \implies R\uparrow$, P partially rebounds.

Construction subsidy: SR: I expands, no immediate R change. LR: $S\uparrow$, $R\downarrow$, P may fall (new SS has lower rental).

Rental vs. investment tax: In SS, a tax on rental income R and a tax on the return to investment $(r + \delta)P$ are *equivalent* since $R = (r + \delta)P$.

Convergence Speed

Adjustment to new SS is faster when:

- Demand for rental services is **inelastic** (small S changes $\implies$ large R changes)
- Supply of new construction is **elastic** (strong I response to P)
- Depreciation rate δ is **high** (stock turns over faster)

Application: Housing (δ low, supply inelastic in dense cities) adjusts slowly; cars (δ moderate, elastic supply) adjust faster. Rent control distortions persist for decades in housing.

Worked example (property tax): Housing demand $R = 100 - S$, construction $I = 2P$, $r = 0.05$, $\delta = 0.02$. Initial SS: $R^* = (r+\delta)P^*$, $I^* = \delta S^*$. With $\tau = 0.01$ property tax: user cost becomes $R = (0.05 + 0.02 + 0.01)P = 0.08P$. New SS: $P^*_{\text{new}} = R^*/0.08$ vs. old $P^* = R^*/0.07$. Immediate effect: P drops (capitalizes tax); then $I \downarrow \implies S \downarrow \implies R \uparrow$ until new SS.

Durable Goods Monopoly (Coase Conjecture)

Monopolist selling durable competes with own future sales. With patient consumers and no commitment: $P \to MC$. Solutions: leasing, planned obsolescence, capacity constraints.

Intertemporal & Exhaustible Resources

Savings & Borrowing

Two-period model: income (Y_1, Y_2), consumption (C_1, C_2), interest rate r.

$$\max U(C_1, C_2) \quad \text{s.t.} \quad C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

FOC: $\frac{U_1'(C_1)}{U_2'(C_2)} = 1 + r$ (MRS = price ratio).

Interest rate increases:

- **Saver** ($S_1 > 0$): substitution $\implies C_1 \downarrow$, income $\implies$ can afford more of both. C_2 rises unambiguously; C_1 ambiguous (but saving increases)
- **Borrower** ($S_1 < 0$): both effects reduce C_1, borrowing falls

Life-Cycle / Euler Equation

With time preference ρ: $U'(C_t) = \frac{1+r}{1+\rho} U'(C_{t+1})$.

- $r > \rho$: consumption rises over time (patient agent)
- $r < \rho$: consumption falls (impatient agent)
- $r = \rho$: flat consumption path

Application (retirement): Consumption drops at retirement because the **price of leisure falls** ($w \to 0$), causing substitution toward time-intensive goods and reducing market expenditure.

Tree-Cutting / Wine-Selling

Tree value $W(t)$ grows over time. Optimal harvest: $W'(t^*)/W(t^*) = r$.

Rule: Cut when the **growth rate of value equals the interest rate**. Before t^*: tree grows faster than r (keep). After t^*: capital earns more invested elsewhere.

Equivalently: $P'(t)/P(t) = r$ for wine. Hold asset until capital gains rate = opportunity cost.

Hotelling Rule (Exhaustible Resources)

Resource stock S, extraction q_t, price P_t, zero extraction cost.

$$P_t = P_0(1+r)^t$$

Price rises at the rate of interest. If it rose faster, owners delay extraction (excess supply today); if slower, extract now (no future supply).

With extraction cost c: $(P_t - c)$ rises at rate r. The *rent* (price minus cost) grows at r, not the price.

Monopolist vs. competitive: With *constant elasticity* demand, the monopolist's extraction path is *identical* to competitive (markups cancel). With linear demand, monopolist conserves more (slower extraction).

Application: OPEC behavior. If oil prices are expected to rise faster than r, owners restrict supply. Changes in r (e.g., from monetary policy) affect extraction incentives.

Investment & Present Value

Project with cash flows $\{C_t\}$: $NPV = \sum_t C_t/(1+r)^t$.

Investment rule: Accept iff $NPV > 0$, equivalently iff **internal rate of return** $> r$.

Application (PS5 2019 — light bulb durability): Competitive firms choose bulb lifespan T to maximize NPV of bulb sales. Optimal T: marginal cost of durability $= PV$ of one period's rental value. A monopolist chooses *shorter* lifespan to increase replacement demand (if commitment is possible; cf. Coase conjecture otherwise).

Capital Tax: Short Run vs. Long Run

SR: capital supply inelastic $\implies$ tax falls on capital (after-tax return $\downarrow$).

LR: capital supply perfectly elastic at R^* (empirically $\approx$ 6–7%). Tax is fully passed to labor via higher pre-tax return $\implies$ higher output prices $\implies$ lower real wages.

Optimal LR capital tax is zero (with CRS): eliminating the capital tax raises labor income by *more* than the lost revenue (the DWL triangle is recaptured). Fund the lost revenue with a labor tax instead.

Uncertainty, Insurance & Crime

Expected Utility

Agent faces states $s \in \{1, \dots, S\}$ with probabilities π_s. Chooses to maximize:

$$\mathbb{E}[U] = \sum_s \pi_s U(C_s)$$

Risk aversion: $U'' < 0 \implies$ agent prefers $\mathbb{E}[C]$ with certainty to the gamble. **Risk premium** ρ: $U(\mathbb{E}[C] - \rho) = \mathbb{E}[U(C)]$.

Arrow-Pratt: $r_A = -U''/U'$ (absolute), $r_R = -U''C/U'$ (relative).

Insurance

Income Y, loss L with probability p. Insurance: pay premium γ for coverage q.

Fair insurance: $\gamma = p$ per unit. At fair price, full insurance is optimal: $C_{\text{loss}} = C_{\text{no loss}}$ (smooth consumption across states).

Unfair insurance: $\gamma > p$. Partial coverage optimal. FOC:

$$\frac{U'(C_{\text{no loss}})}{U'(C_{\text{loss}})} = \frac{p(1-\gamma)}{(1-p)\gamma}$$

More risk-averse agents buy more insurance despite unfair pricing.

State-Dependent Preferences

If utility depends on state (e.g., health): $U_s(C_s) \neq U_{s'}(C_{s'})$ even at same C.

Application: If marginal utility of consumption is **lower** when sick ($U'_{\text{sick}} < U'_{\text{healthy}}$ at same C), optimal insurance provides *less* than full coverage—transfer income to healthy state where it's more valued.

Application (disability): Optimal disability insurance may not fully replace income if disability reduces ability to enjoy consumption.

Moral Hazard

Insurance reduces the cost of risky behavior $\implies$ behavioral response.

Model: Probability of loss $p(e)$ depends on effort e, unobserved by insurer. Full insurance $\implies e = 0 \implies p$ maximized. Optimal contract trades off risk-sharing against incentives.

Application (health): Generous health insurance increases medical utilization (RAND experiment: \$0 copay $\implies$ 30% more visits). Deductibles and copays restore incentives at the cost of risk exposure.

Crime & Punishment (Becker)

Individual commits crime iff:

$$B > p \cdot f + (1-p) \cdot 0 = pf$$

where B = benefit, p = probability of punishment, f = fine.

Optimal deterrence: For a given expected penalty pf:

- **Risk-averse criminals**: higher f with lower p deters the same expected cost more cheaply (exploit concavity: a large fine hurts more than its expected value)
- **Risk-loving criminals**: higher p is more effective (they discount large but unlikely fines)

Enforcement is costly: raising p requires police, courts. Raising f is nearly free $\implies$ **Becker's result:** optimal policy sets f maximal and p minimal (for risk-averse offenders).

Application (traffic): Speed cameras (high p, low f) vs. license suspension (low p, high f). The optimal mix depends on offenders' risk attitudes and enforcement costs.

Peltzman Effect

Safety regulation reduces the cost of accidents $\implies$ agents take more risk. Net effect on safety is ambiguous:

$$\text{Total accidents} = \underbrace{\text{accidents per risk unit}}_{\downarrow(\text{regulation})} \times \underbrace{\text{risk-taking}}_{\uparrow(\text{behavioral})}$$

Application: Seatbelt laws reduce driver deaths but increase pedestrian deaths (drivers drive faster). Airbags, ABS: offsetting behavior partially erodes engineering gains.

Health Policy & Liability

Reassigning liability shifts behavioral responses:

PS4 2018: Restaurant liability for food poisoning $\implies$ restaurants invest more in safety, consumers eat more risky food (moral hazard). Net effect depends on relative elasticities.

PS5 2018: Cigarette externalities: Pigouvian tax = marginal external cost is first-best.

Human Capital & Dynamic Investment

Human capital investment is an application of durable goods theory: the *stock* of skills depreciates, the *flow* of investment has diminishing returns, and the optimal path is front-loaded because the payoff horizon shrinks with age.

Ben-Porath Model

Agent lives $[0, T_d]$, works from T onward. Human capital: $\dot{K}_t = Q_t - \delta K_t$.
Production: $Q_t = F(\alpha, s_t K_t, D_t)$ (ability α, time fraction s_t, purchased inputs D_t).
Returns: νK_t during school, μK_t during work ($\mu > \nu$).
Marginal benefit at time t:

$$MB_t = \frac{\nu}{r}[1 - e^{-r(T-t)}] + e^{-r(T-t)}\frac{\mu}{r}[1 - e^{-r(T_d - T)}]$$

Optimal rule: $MC_t = MB_t$. Since MB_t decreases in t: investment is **front-loaded**.
Comparative statics:

- $\alpha \uparrow$ (higher ability): MC shifts down $\implies$ more investment at every age
- $T_d \uparrow$ (longer life): MB shifts up $\implies$ more investment (explains education $\uparrow$ with life expectancy)
- $r \uparrow$: MB falls (future returns discounted more) $\implies$ less investment

Cobb-Douglas Case

With $Q_t = \alpha(s_t K_t)^{\beta_1} D_t^{\beta_2}$ and $\beta_1 + \beta_2 < 1$ (DRS):

$$MC_t \propto Q_t^{\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}$$

MC is increasing in $Q \implies$ uniquely defined Q_t^* from $MC_t = MB_t$. With $\beta_1 + \beta_2 = 1$: constant MC, and the agent invests maximally or not at all (bang-bang solution).

Two-Period Investment

Invest I today, return $R(I)$ tomorrow with $R' > 0$, $R'' < 0$.
FOC: $R'(I^*) = 1 + r$. With borrowing constraint $I \le \bar{I}$: if $R'(\bar{I}) > 1 + r$, the agent underinvests.
Application: Credit constraints $\implies$ high-ability agents invest too little. This justifies student loans/subsidies: the distortion is not from a tax but from a missing market (credit for human capital).

Household Production

Household production extends consumer theory by recognizing that utility comes not from market goods directly but from *commodities* produced by combining goods and time. This framework explains why wages affect non-market behavior: they change the shadow price of time-intensive activities.

The Model

Agents produce commodities Z from market goods x and time t_h:

$$Z = f(x, t_h), \quad \text{e.g., } Z = x^\alpha t_h^{1-\alpha} \text{ (Cobb-Douglas)}$$

Budget: $px = wH + A$. Time: $T = H + t_h + \ell$. Merging:

$$px + wt_h = wT + A - w\ell \equiv \text{full income} - w\ell$$

The **shadow price** of time in home production is w (opportunity cost).

Two-Stage Optimization

Stage 1: Minimize cost of producing Z:

$$\min_{x, t_h} px + wt_h \quad \text{s.t.} \quad f(x, t_h) = Z$$

FOC: $\frac{f_{t_h}}{f_x} = \frac{w}{p}$. Cost function: $C(p, w, Z)$.
With CRS Cobb-Douglas: $C = \kappa p^\alpha w^{1-\alpha} Z$ where $\kappa = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$. **Shadow price of Z:** $\pi_Z = \kappa p^\alpha w^{1-\alpha}$.
Stage 2: Choose Z and ℓ to maximize $U(Z, \ell)$ subject to $\pi_Z Z + w\ell = wT + A$.

Comparative Statics

Wage increase ($w \uparrow$): Raises π_Z (if time-intensive) and raises full income. Net effect on Z: substitution away from time-intensive commodities, income toward all normal goods.
Technology shock in home production ($A_h \uparrow$): Shadow price π_Z falls $\implies$ more Z produced, possibly less market work.
Application (temperature & insulation): PS2 2018. Home comfort $Z = f(\text{heating}, \text{insulation})$. Colder location $\implies$ marginal product of heating rises $\implies$ substitution toward heating, π_Z rises $\implies$ less comfort consumed (scale effect).

Comparative Advantage in Home Production

With two household members, efficient allocation assigns tasks based on comparative advantage:

$$\frac{w_1}{MP_{h,1}} > \frac{w_2}{MP_{h,2}} \implies \text{Person 1 works in market, Person 2 at home}$$

The person with lower opportunity cost of home time specializes in household production.
Application (PS3 2018): Two-earner household chooses home vs. market production of meals. If w_H/w_L rises, comparative advantage sharpens: high-wage spouse works more in market, low-wage spouse does more cooking. Corner solution: one spouse fully specializes.

Rationing and Household Production

When a market good is rationed ($x \le \bar{x}$), the shadow price of the ration exceeds the market price. Key result: **rationing one good lowers the own-price elasticity of all other goods**, because the consumer loses a substitution margin.

Allocation of Time (Becker)

Full income: $\text{FI} = wT + A$. Time spent on activity j has shadow price w (opportunity cost). Optimal allocation: $\frac{MP_j}{p_j} = \frac{MP_\ell}{w}$ across all activities.
Wage increase: Substitution toward market-intensive goods (eat out more, clean less). Income effect raises demand for all normal goods. Net: time-intensive leisure falls, market work rises (empirically dominant for primary earners).
Application: Rising female wages $\implies$ substitution from home to market goods (less cooking, more childcare markets, higher female LFP).

Search & Information

Sequential Search

Agent draws offers from $F(w)$ at cost c per search. Accept offer iff $w \ge \bar{w}$ (reservation value). Optimal $\bar{w}$ solves:

$$c = \int_{\bar{w}}^{\infty} (w - \bar{w})\, dF(w)$$

LHS is constant; RHS is decreasing in $\bar{w} \implies$ unique solution.
Comparative statics: $c \downarrow \implies \bar{w} \uparrow$ (cheaper search $\implies$ more selective). Mean of F shifts up $\implies \bar{w} \uparrow$. Spread of F increases $\implies \bar{w} \uparrow$ (more upside from waiting).
Application (labor): Unemployment benefits lower the effective search cost $\implies \bar{w}$ rises $\implies$ longer unemployment spells but better matches.

Search in Asset Markets

Investors hold 0 or 1 unit. Preference shocks at rate γ; dealer meetings at rate λ. Nash bargaining (dealer power θ).
Value of holding: $\Delta V(\delta) = V_1(\delta) - V_0(\delta)$ solves:

$$(r + \gamma + \lambda(1-\theta))\Delta V(\delta) = \delta + \gamma \int \Delta V(\delta') dF(\delta') + \lambda(1-\theta)P$$

Bid/ask: $B(\delta) = \theta \Delta V(\delta) + (1-\theta)P, \quad A(\delta) = (1-\theta)\Delta V(\delta) + \theta P.$
Bid-ask spread: $\frac{\theta(\epsilon + \epsilon')}{r + \gamma + \lambda(1-\theta)} > 0$ — positive from **search frictions**, not market power.
Frictionless limit: $\lambda(1-\theta) \to \infty \implies P \to \delta^*/r$, spread $\to 0$.
Application: The model explains why identical assets trade at different prices in OTC vs. exchange markets, and why spreads widen in crises (when λ drops).

Externalities & Equilibrium Effects

When Private $\neq$ Social Cost

Agent i chooses x_i to maximize private benefit, ignoring effect on others.
Pigouvian tax: $t = MC_{external}$ at the optimum $\implies P + t = MC_{social}$.
Coase: With well-defined property rights and zero transaction costs, bargaining achieves efficiency regardless of initial allocation. Pigouvian taxes become unnecessary.
Application: Pollution tax = marginal external damage. But if transaction costs are low (a factory and one neighbor), Coasian bargaining may suffice.
Pecuniary vs. real externalities: Price changes that transfer surplus (e.g., a new Walmart lowering local prices) are *pecuniary*—they redistribute but do not create inefficiency. Only **real** externalities (unpriced physical effects) justify intervention. *The price mechanism is not an externality.*
Application (Coase): Efficient allocation is independent of initial property rights assignment (with zero transaction costs). Distribution differs: free workers capture surplus, slaves do not.

Externalities as Unpriced Inputs

Key insight: An externality is an input with zero price. Overconsumption is the natural result.
Application (health): Hospitalization capacity is not priced. Individuals don't internalize their impact on aggregate hospital severity s_h:

$$\text{Treatment threshold:} \quad \bar{\theta} = \frac{p_t}{s_h c_d}$$

When p_t falls (cheap treatment), fewer people prevent, raising s_h. But the social cost of s_h is unpriced $\implies$ overconsumption of risky behavior.
Price elasticity of treatment demand: $\frac{dD}{dp_t}\frac{p_t}{D} = -\frac{g(\bar{\theta})}{1-G(\bar{\theta})} \cdot \frac{p_t}{c_d \eta}$ (inverse Mills ratio of the type distribution).

Fixed-Point Equilibria

Actions depend on aggregate s; s depends on actions. **To show existence:**

1. Define $s = \Phi(s)$ where Φ maps aggregate s through individual best responses
2. At max s: best responses imply lower s. At min s: higher s
3. By IVT (or Brouwer), fixed point exists

Application: In the prevention model, s_h is a fixed point. If s_h is high, everyone prevents $\implies$ s_h should be low. If s_h is low, nobody prevents $\implies$ s_h should be high. Existence guaranteed; may have multiple equilibria.

Adding a Market

Introducing voluntary exchange when a market is missing (weakly) improves welfare: participants gain from trade, non-participants unaffected $\implies$ Pareto improvement.
Application (kidneys): Cash market increases supply, reduces wait time. Welfare rises if new supply $\gg$ crowded-out altruistic donors.

Compensating Differentials & Location

When markets are competitive and agents mobile, price differences across goods, jobs, or locations must reflect real differences in attributes—otherwise arbitrage would eliminate them.

Hedonic Pricing

Goods differ along attributes $z = (z_1, \ldots, z_n)$. Price function $P(z)$ in equilibrium:

$$\frac{\partial P}{\partial z_k} = \text{MRS}_{z_k,m} = MC_{z_k}$$

Hedonic gradient identifies the *marginal* buyer's WTP (not average). Full demand identification requires instruments (Rosen's second step).

Compensating Wage Differentials

Workers sort across jobs with different amenities. In equilibrium:

$$w(a) = w^* - v(a)$$

where $v(a)$ is the monetary value of amenity a to the marginal worker.
Application (VSL): Jobs with higher fatality risk Δp pay compensating differentials Δw. The value of a statistical life: $VSL = \Delta w/\Delta p$. If a job with 1/10,000 extra death risk pays \$800 more: $VSL = \$8M$.

Rent Gradient

Workers commute to CBD at cost $t \cdot d$. Spatial equilibrium equalizes utility:

$$R(d) = R(0) - t \cdot d$$

Land at city edge: $R(\bar{d}) = R_a$ (agricultural rent) $\implies$ $R(0) = R_a + t\bar{d}$.
$\bar{d}$ increases with population and productivity; decreases with t.
With heterogeneous wages: High-wage workers value time more $\implies$ $R'(d) = -w$ (savings from living closer = wage × commute time saved). Higher $w \implies$ steeper gradient $\implies$ **convex** rent gradient across worker types. High earners live near CBD.
Application: A productivity shock raises CBD wages. Workers bid up nearby rents until $R(d)$ adjusts to equalize utility. Landowners, not workers, capture location-specific surplus.
Application (Uber): Lower commuting cost t flattens the rent gradient: city-center rents fall, suburban rents rise, city expands ($\bar{d}\uparrow$).

Spatial Equilibrium

$V(w_j, r_j, a_j) = \bar{V}$ across all locations j. Higher wages offset by higher rents or worse amenities. **Cannot infer welfare from wages alone.**

Matching, Sorting & Education

Matching theory asks: when individuals differ and interact, who pairs with whom? The answer depends on whether types are complements (PAM) or substitutes (NAM) in production.

Assortative Matching

Two sides of a market (e.g., workers/firms, spouses). Match output $f(x, y)$ for types x and y.
Positive assortative matching (PAM): High types match with high types iff f is **supermodular**:

$$f_{xy} > 0 \iff \text{PAM is efficient}$$

Supermodularity means the marginal product of one type increases with the other's type $\implies$ complementarity in production.
Negative assortative matching (NAM): $f_{xy} < 0 \implies$ high matches with low (substitutability).

Marriage Market

With transferable utility: $f(x, y)$ is total surplus. Equilibrium payoffs $u(x)$, $v(y)$ satisfy:

$$\begin{aligned} u(x) + v(y) &= f(x, y) \quad \text{for matched pairs} \\ u(x) + v(y) &\geq f(x, y) \quad \text{for unmatched pairs (stability)} \end{aligned}$$

PAM when $f_{xy} > 0$: Matching the top with the top maximizes total surplus. Rearranging assignments to create mismatches reduces total output.
Application: Investment in own quality has **spillover benefits**—improving x raises the matched partner's equilibrium payoff $v(y)$, attracting a better match.

Acquired Comparative Advantage

Even identical individuals may optimally specialize.
Learning-by-doing creates comparative advantage: small initial differences → specialization → large skill gaps. Returns to education may be below r if schooling yields direct utility u_E: invest until $\text{MB} = r + \delta - u_E$.

Education / Human Capital

Higher-ability workers have lower cost of skill acquisition $\implies$ sort into skill-intensive jobs.

- **General HC:** raises productivity everywhere; worker pays (employer can't capture returns)
- **Specific HC:** raises productivity only at current firm; firm and worker share costs

Addiction (Becker-Murphy)

Adjacent complementarity: past consumption C_t raises current marginal utility via habit stock S_t:

$$S_{t+1} = (1-\delta)S_t + C_t, \qquad \frac{\partial^2 U}{\partial C_t \partial S_t} > 0$$

Myopic vs. rational addict: A *myopic* addict treats S as exogenous—only current MU matters. A *rational* addict internalizes that C_t raises future S, increasing future consumption desire. The Euler equation for the rational addict links C_{t-1}, C_t, and C_{t+1}:

$$C_t = \theta C_{t-1} + \beta\theta C_{t+1} + \theta_1 p_t + \theta_2 e_t$$

where θ captures the degree of adjacent complementarity and β is the discount factor. *Key test:* $\beta\theta > 0$ means future prices affect current consumption—myopic addicts would show $\beta\theta = 0$.
Steady-state analysis: In SS, $C^* = C_{t-1} = C_t = C_{t+1}$ and $S^* = C^*/\delta$. The SS consumption level satisfies:

$$U_C(C^*, C^*/\delta) = p + \frac{\delta}{r+\delta}\left[-U_S(C^*, C^*/\delta)\right]$$

The second term is the *future cost of addiction*: each unit of current consumption adds $1/\delta$ units to the steady-state stock, and U_S captures the harmful effect of the stock (health damage, withdrawal). Rational agents *partially* internalize this cost; myopic agents ignore it entirely.
Key results:

- Rational addicts respond to *future* price changes (a pre-announced tax reduces *current* consumption)
- Demand is *more elastic* in the long run than the short run: SR, S is fixed; LR, S adjusts downward with C, amplifying the initial response
- A monopolist may **price below MC** initially to build S, then extract rents once hooked
- Cold-turkey quitting is optimal when adjustment costs are convex and S is high—gradual reduction would prolong withdrawal
- **Tax comparative static:** A permanent tax τ reduces SS consumption by more than the PE effect suggests, because lower $C \implies$ lower $S \implies$ lower desire $\implies$ further C reduction (multiplier via the habit loop)

Application (PS5 2018): Prohibition. Banning an addictive good raises its price $\implies$ heavy users (high S) substitute to black market or worse substitutes. Light users quit. Net welfare depends on distribution of S in population. The rational model predicts that even *announcing* a future ban reduces current consumption—users begin de-accumulating S in anticipation.

Mathematical Toolkit

Lagrangian & Karush-Kuhn-Tucker (KKT)

$\max f(x)$ s.t. inequality constraints $g_j(x) \le 0$ and equality constraints $h_k(x) = 0$:

$$\mathcal{L} = f - \sum_j \mu_j g_j - \sum_k \lambda_k h_k$$

KKT conditions (necessary for optimality):

1. **Stationarity:** $\nabla f = \sum_j \mu_j \nabla g_j + \sum_k \lambda_k \nabla h_k$ (gradient of objective is a linear combination of constraint gradients)
2. **Primal feasibility:** $g_j(x^*) \le 0$, $h_k(x^*) = 0$
3. **Dual feasibility:** $\mu_j \ge 0$ (shadow prices on inequalities are non-negative)
4. **Complementary slackness:** $\mu_j g_j(x^*) = 0$ — either the constraint binds ($g_j = 0$) or its multiplier is zero ($\mu_j = 0$), never both slack

Interpretation: λ_k is the marginal value of relaxing constraint h_k; μ_j is the marginal value of relaxing g_j. If $\mu_j = 0$, the inequality is slack and doesn't affect the optimum.
Always check corners. If interior FOC $\implies x < 0$, the corner $x = 0$ binds. Set $x = 0$, re-solve the reduced system, and verify $\mu \ge 0$.

Envelope Theorem

$V(\alpha) = \max_x f(x, \alpha)$ s.t. $g(x, \alpha) = 0$. Then:

$$\frac{dV}{d\alpha} = \frac{\partial \mathcal{L}}{\partial \alpha}\bigg|_{x=x^*}$$

Applications: $\partial V/\partial p_i = -\lambda x_i$ (Roy's identity). $\partial e/\partial p_i = x_i^H$ (Shephard). $\partial \pi/\partial p = q^S$ (Hotelling).
Why it matters: In multi-period models, the envelope theorem means the two-period FOC often collapses to the static FOC (the inventor's problem).

Implicit Function Theorem

From $F(x^*, \alpha) = 0$ with $F_x \neq 0$:

$$\frac{dx^*}{d\alpha} = -\frac{F_\alpha}{F_x}$$

Pattern: Sign $dx^*/d\alpha$ without solving explicitly. Use SOC ($F_x < 0$ at max) to sign denominator; economic logic signs numerator.
Application: From $u_f'(f)/p_f = u_m'(M)$ and knowing $u_f'' < 0$: if $T \uparrow$ raises food output f, then p_f must fall (since u_f' falls but RHS is fixed).

Homogeneous Functions

$f(\lambda x) = \lambda^k f(x)$ (HDk). **Euler:** $\sum x_i f_i = kf$.
Corollary: derivatives HD($k-1$). HD1 cost: $C(tw, q) = tC(w, q)$.
HD0 demand: $x^M(tp, tM) = x^M(p, M)$.
Application: HD1 production $\implies f = \sum x_i f_i$ (Euler) $\implies$ paying each factor its MP exhausts output $\implies$ zero profit under CRS.

Duality Table

Consumer	Producer
$\max U$ s.t. budget	$\min w \cdot x$ s.t. $f(x) \ge q$
Marshallian $x^M(p, M)$	Factor demand $x(w, q)$
Indirect utility $V(p, M)$	Cost $C(w, q)$
Hicksian $x^H(p, \bar{u})$	Supply $q^S(p)$
Expenditure $e(p, \bar{u})$	Profit $\pi(p, w)$
Slutsky equation	Marshall's Laws

Every consumer result has a producer dual. The Slutsky decomposition (substitution + income) parallels Marshall's decomposition (substitution + scale).

Elasticity Calculus

$\epsilon_{y,x} = \frac{\partial y}{\partial x}\frac{x}{y} = \frac{d \ln y}{d \ln x}$.
Chain: $\epsilon_{z,x} = \epsilon_{z,y} \cdot \epsilon_{y,x}$. **Product:**
$z = y_1 y_2 \implies \epsilon_{z,x} = \epsilon_{y_1,x} + \epsilon_{y_2,x}$.
Sum: $z = y_1 + y_2 \implies \epsilon_{z,x} = \frac{y_1}{z}\epsilon_{y_1,x} + \frac{y_2}{z}\epsilon_{y_2,x}$.
Application: Revenue $R = pq$. $\epsilon_{R,p} = 1 + \epsilon_{q,p}$. Revenue rises with p iff $|\epsilon_{q,p}| < 1$.

Hamilton-Jacobi-Bellman (HJB)

The HJB equation characterizes the value function $V(x)$ in continuous-time dynamic programming. With discount rate r, flow payoff $u(x, a)$, and state law of motion $\dot{x} = g(x, a)$:

$$rV(x) = \max_a \{u(x, a) + V'(x) g(x, a)\}$$

Interpretation: The *flow return* on holding value V (i.e. rV, what you'd earn investing V at rate r) must equal the flow payoff u plus the capital gain $V'\dot{x}$ from the state changing. At the optimum, you can't do better by switching actions.
With Poisson shocks (arrival rate λ, new state x'):

$$rV(x) = \max_a \{u(x, a) + V'(x) g(x, a) + \lambda[\mathbb{E}V(x') - V(x)]\}$$

Steady state ($\dot{V} = 0$, $\dot{x} = 0$): HJB reduces to algebra. Subtract value functions ($V_1 - V_0$) to eliminate common terms.
Application: In search models, V_e (employed) and V_u (unemployed) satisfy two HJB equations. The reservation wage solves $V_e(\bar{w}) = V_u$.
Euler equation (discrete time): $p_t = \mathbb{E}\Big[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\Big]$. Asset price = expected discounted payoff weighted by the stochastic discount factor (SDF).

Problem-Solving Patterns

Applied Problem Template

1. **Identify the economic question:** What prices or quantities do we want to predict?
2. **Write the model:** Objective, constraints, market clearing. Count eqs vs. unknowns
3. **Solve:** FOCs $\to$ demand/supply $\to$ equilibrium. Check corners
4. **Comparative statics:** Use IFT to sign $dx^*/d\alpha$ without re-solving
5. **Equilibrium effects:** Trace through indirect channels. Compare PE vs. GE

Worked Example: Tax on Ride-Sharing

Setup: Per-trip tax t on Uber. Riders have demand $Q^D(p)$, drivers have supply $Q^S(p - t)$.

PE: Consumer price $\uparrow$, driver price $\downarrow$. Split by ϵ_D vs. ϵ_S. DWL $\propto t^2$.

GE: Higher commuting cost $\implies$ rent gradient steepens, CBD rents rise. Landlords bear part via capitalization.

Corner Solutions & Regime Switching

- Interior FOC $\implies x < 0$? Set $x = 0$, re-solve the reduced system
- With multiple preventions: agents sort into regimes. Analyze each separately. Verify boundary conditions at switching thresholds
- Fixed costs create **switching thresholds:** consumer switches when indifferent between two options. Find the price ratio at indifference

Application (opioids): Rx vs. illicit opioids with different fixed costs ($f_R < f_I$). Consumer switches from Rx to illicit when:

$$p_R \ge \left(\frac{y - f_R}{y - f_I}\right)^{1/k} p_I \quad \text{(Cobb-Douglas, rational)}$$

The rational consumer switches *before* Rx is strictly dominated—anticipating the budget shift.

Using σ to Classify Results

Many results depend on σ relative to other elasticities:

- $\sigma > |\epsilon_D|$: substitution dominates scale $\implies$ factor share *rises* with its own price
- $\sigma < |\epsilon_D|$: scale dominates $\implies$ factor share falls
- $\sigma = 1$ (CD): shares are constant, simplifies most problems
- $\sigma = 0$ (Leontief): no substitution, only scale effects

When stuck: try $\sigma = 1$ first, solve, then ask what changes for $\sigma \neq 1$.

Steady-State Tricks

In dynamic models, steady state ($\dot{V} = 0$) turns PDEs into algebra.

- HJB $\to$ algebraic equation for V (or ΔV)
- Subtract value functions ($V_1 - V_0$) to eliminate common terms
- Poisson arrival rates (γ, λ) enter as discount-rate adjustments: effective rate becomes $r + \gamma + \lambda$

Nash Bargaining Shortcut

Surplus split by bargaining power $(\theta, 1 - \theta)$:
$B = \theta V_{\text{buyer}} + (1 - \theta) V_{\text{seller}}$. The bid is a weighted average of continuation values.