

(Q5 Proof of Hint) Let $A = \{\omega : X_n(\omega) \rightarrow X(\omega)\}$ and $B = \{\omega : X(\omega) = Y(\omega)\}$. Then $P(A) = 1$ since $X_n \xrightarrow{a.s.} X$. Since for any events A, B , $P(A \cap B) = P(A) + P(B) - P(A \cup B)$, we have

$$\begin{aligned} P(\{\omega : X_n(\omega) \rightarrow X(\omega) \text{ and } X(\omega) = Y(\omega)\}) &= P(A) + P(B) - P(A \cup B) \\ &= 1 + P(B) - 1 \\ &= P(\{\omega : X(\omega) = Y(\omega)\}). \end{aligned}$$

Since $X_n \not\xrightarrow{a.s.} Y$, the LHS probability is less than 1, which means $P(X = Y) < 1$. For the second part, note that $P(X \neq Y) = P(|X - Y| > 0) > 0$ and define

$$A_n = \left\{ |X - Y| > \frac{1}{n} \right\}.$$

Note that

$$\{|X - Y| > 0\} = \bigcup_{n=1}^{\infty} A_n$$

and suppose that $P(A_n) = 0$ for all n . Then

$$P(\{|X - Y| > 0\}) = P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = 0,$$

which is a contradiction. So $P\left(|X - Y| > \frac{1}{n}\right) > 0$ for some n sufficiently large. Choose $0 < c \leq \frac{1}{n}$ and set $\delta = \frac{1}{2} \cdot P\left(|X - Y| > \frac{1}{n}\right)$. This concludes the proof.