ECMA 31000: Problem Set 4

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Question 1 (Partial Identification) Suppose the entire population of 22-25 year olds is surveyed on their preference for rock music. For each individual i, let $Y_i = 1$ if i likes rock music and $Y_i = 0$ if not. Not all individuals respond. For these individuals, Y_i is unobserved. Let X_i be the age of individual i, in years. Suppose we have the following data about the joint distribution of taste for rock music and age:

	Age in Years			
Likes Rock?	22	23	24	25
$Y = 1$	0.05	0.10	0.05	0.15
$Y = 0$	0.10	0.10	0.10	0
Y unobserved	0.05	0.10	0.15	0.05

We wish to learn $\mathrm{P}(Y = 1)$.

a) Using the data alone, what can you conclude about $\mathrm{P}(Y = 1)$?

b) You make the assumption that preferences for rock are independent of age. Is this assumption consistent with the data? If it is, what can you conclude about $\mathrm{P}(Y = 1)$ now?

c) You make the assumption that older individuals are more likely to have a taste for rock music. That is, you assume $\mathrm{P}(Y = 1|X = x)$ is non-decreasing in x. Is this assumption consistent with the data? If so, what can you conclude about $\mathrm{P}(Y = 1)$ now?

d) A third of the 24 year olds who did not respond initially now respond. They report that they do not like rock. How does your answer to parts a) and c) change?

e) Now suppose all 24 year olds who did not respond initially now report they don't like rock. How does your answer to parts a) and c) change?

Question 2 Suppose we have an iid sample $\{X_i\}_{i=1}^n$ drawn from a distribution represented by the pdf

$$f_\theta(x) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\theta > 0$.

a) Find a Method of Moments estimator and the Maximum Likelihood estimator of θ.

b) Show that both estimators are consistent.

c) Derive the asymptotic distribution of your Method of Moments estimator.

d) Derive the asymptotic distribution of the maximum likelihood estimator.

e) Compute the information matrix $I(\theta)$ defined in Lecture 9. Compare your answer to the solution in d). Compare the asymptotic variances of the MLE and MoM estimator.

Question 3 a) Let $\{X_i\}_{i\geq 1}$ be an iid sequence with $\mathrm{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 > 0$. Calculate MSE of the (biased) estimate of σ^2 given by

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2.$$

Decompose your result into $Var\left(S_n^2\right) + Bias\left(S_n^2\right)^2$. Use your result to deduce the MSE of $\tilde{S}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$ stated in class.

b) Consider the family of estimators $\left\{k\tilde{S}_n^2 : k \in \mathbb{R}\right\}$ of σ^2. Calculate the value k^* which minimizes the MSE of $k\tilde{S}_n^2$, and show that for any distribution F with $0 < Var(F) < \infty$,

$$k^* \leq \frac{n^2 - n}{n^2 - n + 2}.$$

Show, in particular, that for the normal distribution,

$$k^* = \frac{n-1}{n+1}.$$

Hint: Use the optimal value of k derived in Lecture 8.

Question 4 Suppose you observe an iid sample $\{X_i\}_{i=1}^{n}$, where $X_i \sim U[0, \theta]$.

a) Compute $\mathrm{E}\left(X^k\right)$, and derive the method of moments estimator $(\hat{\theta}_k)$ of θ based on this calculation.

b) Show that $\hat{\theta}_{MM}^k \overset{a.s.}{\to} \theta$. Is $\hat{\theta}_{MM}^k$ an unbiased estimator of θ?

c) Find the asymptotic distribution of $\hat{\theta}_{MM}^k$. How does it vary with k?

d) Compute the Maximum Likelihood Estimator $\left(\hat{\theta}_{ML}\right)$ of θ. How does it relate to the method of moments estimator based on computing $\mathrm{E}\left(X^k\right)$? (Think about large k).

e) Show that $\hat{\theta}_{ML}$ is a consistent estimator of θ and compute its asymptotic distribution.

f) Is $\hat{\theta}_{ML}$ unbiased? If not, construct an unbiased estimate of θ based on $\hat{\theta}_{ML}$. Call it $\tilde{\theta}$.

g) Compute the MSE of $\hat{\theta}_{ML}$ and $\tilde{\theta}$. Which is lower?

h) Use your answer to part g) to conclude that $\hat{\theta}_{ML} \overset{a.s.}{\to} \theta$.

Question 5 (Not quite a sample average) Let $\{X_i\}_{i=1}^n$ be an iid sample, with $X_i \sim U[0, \theta]$ for some $\theta > 0$. Consider the following estimator of θ:

$$\hat{\theta}_n = \left(\Pi_{i=1}^n X_i\right)^{\frac{1}{n}}.$$

a) Is $\hat{\theta}_n$ a consistent estimator of θ? If not, find a function f such that $\tilde{\theta}_n = f\left(\hat{\theta}_n\right)$ is a consistent estimator of θ.

b) Find the asymptotic distribution of $\tilde{\theta}_n$.

Question 6 Suppose you observe a random sample of $\{Y_i, X_i\}_{i=1}^n$ where

$$Y_i = \begin{cases} 1 & \text{if individual } i \text{ is employed,} \\ 0 & \text{if individual } i \text{ is unemployed.} \end{cases}$$

is a scalar outcome indicating employment status and X_i is a $(K \times 1)$ vector of observable characteristics. You model the probability of employment conditional on X as follows:

$$\mathrm{P}\left(Y = 1 | X = x\right) = F\left(x'\beta\right),$$

for some known distribution function F and unknown parameter β. Write down the conditional log likelihood of the observed sample, as a function of β.