Problem Set 3 - Question 2

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Question

What is the relationship between the Slutsky equation and Marshall's Laws of Derived demand? Recall that Marshall's Laws are often analyzed with an equation (hereafter, "the ML equation") that, like Slutsky's, decomposes a price response into a "scale effect" and a "substitution effect." See Chapter 11 of Chicago Price Theory for a discussion of Marshall's Laws.

Marshall's Laws

Remember Marshall's Laws[2]:

1. "The demand for anything is likely to be more elastic, the more readily substitutes for the thing can be obtained ."
2. "The demand for anything is likely to be less elastic, the less important is the part played by the cost of that thing in the total cost of some other thing, in the production of which it is employed."
3. "The demand for anything is likely to be more elastic, the more elastic is the supply of co-operant agents of production."
4. "The demand for anything is likely to be more elastic, the more elastic is the demand for any further thing which it contributes to produce."

[2] Kennan 1998

Rewriting Marshall's First Two Laws

Note the most clearheaded prose...

1. Demand for an input is more elastic if there are readily available production substitutes
2. Demand for an input is less elastic when it plays a small role in the total cost of an output

The Slustky Equation and the ML Equation

Slutsky:

$$\epsilon_{ij}^{M} = \epsilon_{ij}^{H} - s_j \eta_i$$

ML (Under CRS):

$$\epsilon_{ij} = s_j \epsilon^{D} + s_j \sigma_{ij}$$

Part A

First discuss and analyze the relationship assuming the mapping from consumption amounts to utility (a.k.a., output) exhibits constant returns to scale.

Constant Returns to Scale

Whether you are producing Y or u, you can think of CRS as:

1. Prices $\mathbf{w}$ pin down optimal mix of inputs to produce 1 unit of Y or u

$$C(w_1, \ldots, w_N, Y) = YC(w_1, \ldots, w_N, 1)$$

2. Supply is perfectly elastic–need demand to pin down quantities.

Intuition Behind Differences

- Marshalls Law: quantity pinned down by the demand curve
- Slutsky: quantity pinned down by maintaining the expenditure of the optimizing agent

The key difference is that Marshall's Law relies on *equilibrium*.

To See Formally:

Start with totally differentiating X_i with respect to price w_j–from hint:

$$\frac{d}{dw_j}X_i = \frac{d}{dw_j}\frac{\partial C(u,\mathbf{w})}{\partial w_i} = \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial u}\frac{du}{dw_j} + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j}$$

What to make of $\frac{du}{dw_j}$?

- $\frac{du}{dw_j}$ is a total-derivative–a linear approximation allowing many factors to adjust
- Depending on the problem at hand, we hold various things fixed–e.g. in the Hicksian problem $du = 0$
- What are we holding fixed when thinking about Slutsky?

From Cost Function to Slutsky:

To get the Slutsky equation, utility can change but we hold the total expenditure fixed:

$$dc(u, \mathbf{w}) = 0$$

We can see what that implies for $\frac{du}{dw_j}$

$$0 = dc(u, \mathbf{w}) = \frac{\partial c}{\partial u} du + \sum_{i=1}^{N} \frac{\partial c}{\partial w_i} dw_i$$

We are only letting one price j adjust so $dw_i = 0$ for all $i \neq j$

$$\frac{du}{dw_j} = -\frac{\partial c / \partial w_j}{\partial c / \partial u}$$

From Cost Function to Slutsky:

Substituting in for $\frac{du}{dw_j}$:

$$\begin{aligned}
\frac{d}{dw_j}X_i &= -\frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial u}\frac{\partial c/\partial w_j}{\partial c/\partial u} + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j} \\
&= -X_j \frac{\partial H_i/\partial u}{\partial c/\partial u} + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j} \\
&= -X_j \frac{\partial X_i}{\partial c} + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j}
\end{aligned}$$

$\frac{\partial X_i}{\partial c}$ is the Marshallian income effect. To see that, recognize that $\frac{\partial H_i}{\partial u}$ is how much more x_i you consume if I gave you *mc* dollars and $\frac{\partial c}{\partial u}$ is exactly *mc*!

From Cost Function to Marshall's:

Under Marshall's Law, we have to consider movement along the demand curve. Let r be the marginal cost of producing one unity of utility, then:

$$u = D(r)$$

$$du = D'(r)dr$$

We can rewrite r under CRS:

$$r = \frac{\partial C}{\partial u}$$

$$\frac{du}{dw_j} = \frac{D'(r)dr}{dw_j}$$

What is $\frac{dr}{dw_j}$? The change in marginal cost from a change in the price of good j holding everything else fixed:

$$\frac{dr}{dw_j} = \frac{\partial^2 C(u, \mathbf{w})}{\partial u \partial w_j}$$

From Cost Function to Marshall's:

Can we rewrite the following?

$$\frac{\partial^2 C(u, \mathbf{w})}{\partial u \partial w_j}$$

Another way to interpret it is how Hicksian demand for X_j changes as you increase u (another way is to ask how does marginal cost increase if you raise w_j).
With CRS, increasing u by 1 requires increasing X_i by how much it takes to produce one u:

$$\frac{\partial^2 C(u, \mathbf{w})}{\partial u \partial w_j} = \frac{X_j}{u}$$

From Cost Function to Marshall's:

$$\begin{aligned}
\frac{d}{dw_j}X_i &= \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial u}\frac{du}{dw_j} + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j} \\
&= \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial u}\frac{\partial^2 C(u,\mathbf{w})}{\partial u \partial w_j}D'(r) + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j} \\
&= \frac{X_i X_j}{u^2}D'(r) + \frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j}
\end{aligned}$$

Multiplying and dividing to convert to elasticities, we get

$$\beta_{ij} = s_j \epsilon^D + s_j \sigma_{ij}$$

where

$$\sigma_{ij} = \frac{c}{X_i X_j}\frac{\partial^2 C(u,\mathbf{w})}{\partial w_i \partial w_j}$$

Part B

Cite an application where the Slutsky equation would be your preferred analytical tool. Cite another where the ML equation would be preferred.

Part B

- Key difference is that for Marshall's Law we are in equilibrium
- Differences clearest on firm side: Slutsky imagines firms have a fixed budget, Marshall's lets quantities be determined by demand (under CRS).
- When does considering equilibrium matter especially? The more elastic demand, the larger the differences in scale effects.
- Example: What is the effect of an increase in lumber prices on labor in construction?
- Marshall's Law: clear way to compare short and long run scale effects using different housing demand elasticities
- Slutsky: scale effect comes from firm's budget contracting–no reason for it to differ from short to long run.

Part C

Where in your derivation does the partial elasticity of substitution (in output) between two quantities appear? If all pairs of quantities had the same elasticity of substitution, what form would the utility function take? The cost function?

Partial Elasticity of Substitution

- The Partial Elasticity of Substitution is the elasticity form of the substitution effect:

$$\sigma_{ij} = \frac{c}{X_i X_j} \frac{\partial^2 C(u, \mathbf{w})}{\partial w_i \partial w_j}$$

- How the ratio of i to j changes as you change the ratio of prices

Constant Elasticity of Substitution

- If all elasticities of substitution are the same, we have CES utility:

$$\left(\sum_{i=1}^{N} X_i^{\rho}\right)^{1/\rho}$$

  where $\sigma = \frac{1}{1-\rho}$
- There is some general taste for variety governed by σ but each good plays the same role. Implies formulas for aggregating X_i and p_i.

Part D

What does this approach say about inferior inputs?

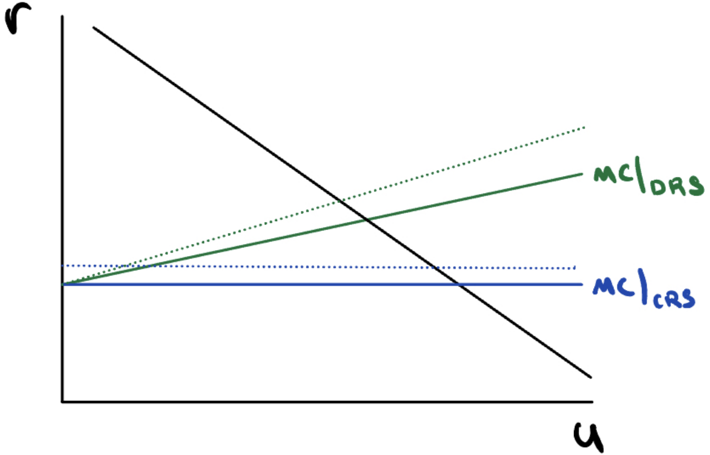
Part D

To this point we have constant returns to scale. No inferior goods under CRS because shares are constant as output increases.

Part E

Now drop the constant returns assumption and show the revised Slutsky and ML equations. [Hint 1: revisit part (a) with a supply-demand diagram and think about how diminishing returns changes it. Hint 2: Define "returns to scale" as the ratio of average cost to marginal cost and use that as a summary statistic.]

Supply and Demand



Scale Effect without CRS

- Under CRS–no need to consider demand when computing change in marginal cost from a change in an input's price
- Without CRS–slope of demand curve will impact marginal cost

$$r = \frac{\partial c(D(r), \mathbf{w})}{\partial u}$$

Computing Change in Marginal Cost

$$r = \frac{\partial c(D(r), \mathbf{w})}{\partial u}$$

$$\frac{dr}{dw_j} = \frac{\partial^2 c(D(r), \mathbf{w})}{\partial u^2} D'(r) \frac{dr}{dw_j} + \frac{\partial^2 c(D(r), \mathbf{w})}{\partial u \partial w_j}$$

$$\implies \frac{dr}{dw_j} = \left[1 - \frac{\partial^2 c(D(r), \mathbf{w})}{\partial u^2} D'(r)\right]^{-1} \frac{\partial^2 c(D(r), \mathbf{w})}{\partial u \partial w_j}$$

Let $\theta = \left[1 - \frac{\partial^2 c(D(r), \mathbf{w})}{\partial u^2} D'(r)\right]^{-1}$

Computing Change in Marginal Cost

$$u = D(r)$$

$$\frac{du}{dw_j} = D'(r)\frac{dr}{dw_j}$$

$$\frac{du}{dw_j} = D'(r)\theta\frac{\partial^2 c(D(r), \mathbf{w})}{\partial u \partial w_j}$$

Now combining with our cost function equation:

$$\frac{d}{dw_j}X_i = \frac{\partial^2 C(u, \mathbf{w})}{\partial w_i \partial u}\frac{\partial^2 c(u, \mathbf{w})}{\partial u \partial w_j}D'(r)\theta + \frac{\partial^2 C(u, \mathbf{w})}{\partial w_i \partial w_j}$$

Marshall's Law without CRS

Taking the prior expression and multiplying and dividing to get elasticity format:

$$\beta_{ij} = \left(\frac{\theta}{RTS} \eta_i \eta_j \epsilon^D + \sigma_{ij} \right) s_j$$

where:

$$RTS = \frac{r}{c/u} \quad \eta_i = \frac{u}{x_i} \frac{\partial^2 C(u, \mathbf{w})}{\partial w_i \partial u}$$

RTS is the ratio of marginal cost to average cost and η_i are the output elasticities of factor demand.

When is the Scale Effect Positive?

$$\beta_{ij} = \left(\frac{\theta}{RTS} \eta_i \eta_j \epsilon^D + \sigma_{ij} \right) s_j$$

One of the input elasticities must be negative. So either i is a normal input and the price occurred to an inferior input or vice-versa.
We have no change to the Slutsky equation because we did not use CRS in it's derivation–so again the differences between the two operate through equilibrium and the scale effect.