FORMAL MODULAR FORMS AND HECKE OPERATORS OVER NUMBER FIELDS

J. E. CREMONA

Abstract. We develop the foundations of an explicit theory of "formal" modular forms over arbitrary number fields K, including discussion of cusps, modular points, Hecke operators and generalized Atkin-Lehner operators. This description incorporates the classical theory for $K = \mathbb{Q}$, and also extends earlier work of the first author and his students for imaginary quadratic fields, and should be useful more generally in the computation of spaces of cusp forms for $\mathrm{GL}(2, K)$ via modular symbols and related methods.

1. Introduction

Let K be a number field with ring of integers $\mathcal{O}_K$. Spaces of cusp forms of weight 2 (and higher weights) for $\mathrm{GL}(2, K)$ have been computed for certain classes of fields K using methods based on modular symbols. When $K = \mathbb{Q}$ these computations are classical: see [6] and [17]. For some imaginary quadratic fields (with small class number) the methods have been developed by the first author and his students: see [4] and the theses of Bygott [2] and Lingham [14]. For some real quadratic fields, explicit computations have been carried out by students of Darmon, notably Dembélé [9] for $\mathbb{Q}(\sqrt{5})$.

In this paper we give a systematic treatment of those parts of the theory which are purely algebraic in nature, and apply to all number fields. In fact, for most of what follows we only need assume that $\mathcal{O}_K$ is a Dedekind domain; special properties enjoyed by the ring of integers of a number field, such as finiteness of the class group and of residue fields, will only be used where necessary. It is hoped that this framework will be useful in future work on explicit computation of cusp forms over general number fields.

1.1. **Outline of the paper.** In the introduction, we recall some basic concepts and notation introduced by the author and Aranés in [7]. In Section 2 we introduce lattices and "modular points" for a general number field K, with respect to the congruence subgroups $\Gamma_0(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})$, for an arbitrary level $\mathfrak{n}$. In Section 3, we introduce Hecke operators as operators on the free module generated by modular points, and define the grading of the Hecke algebra by the class group, and Hecke eigensystems. A key result is the determination of the extent to which such an eigensystem is determined by its restriction to the principal subalgebra of the Hecke algebra. This has important practical consequences, since in explicit computations, such as using modular symbols, computing principal Hecke operators can be much simpler than the general case; our result implies that *all* information about Hecke eigensystems may be recovered from the eigenvalues of principal operators. Formal modular forms are introduced in Section 4, first as functions on modular points, and then as functions on $\mathrm{GL}(2)$. This gives the link to earlier approaches, where automorphic forms are viewed as functions on adelic spaces, generalising the classical definition (in the case $K = \mathbb{Q}$) of modular forms as functions on the upper

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half-plane. Sections 5 and 6 make explicit the matrices defining principal Hecke operators Atkin-Lehner operators; these have been used in practice over imaginary quadratic fields, to determine Hecke eigensystems over arbitrary imaginary quadratic fields, as implemented in the author's `C++` package `bianchi-progs` [8].

1.2. **Notation and basic definitions.** Let K be an arbitrary number field, with ring of integers $\mathcal{O}_K$, and unit group $\mathcal{O}_K^\times$. Let $\mathrm{Mat}_2(K)$ and $\mathrm{Mat}_2(\mathcal{O}_K)$ denote the algebras of 2×2 matrices with entries in K or $\mathcal{O}_K$ respectively, and $\mathrm{GL}(2,K)$ and $\Gamma := \mathrm{GL}(2,\mathcal{O}_K)$ the corresponding multiplicative groups.

Nonzero ideals of $\mathcal{O}_K$ are denoted $\mathfrak{a},\mathfrak{b},\dots,\mathfrak{n}$ and prime ideals by $\mathfrak{p},\mathfrak{q}$. The norm of an ideal is $N(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a})$, assumed to be finite where used, satisfying $N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b})$ for all $\mathfrak{a},\mathfrak{b}$. We have $\#(\mathcal{O}_K/\mathfrak{a})^\times = \varphi(\mathfrak{a})$, where

$$\varphi(\mathfrak{a}) := N(\mathfrak{a})\prod_{\mathfrak{p}\mid\mathfrak{a}}\left(1-N(\mathfrak{p})^{-1}\right).$$

Associated to each nonzero integral ideal $\mathfrak{n}$ of $\mathcal{O}_K$, called the *level*, we have the standard congruence subgroups[1] $\Gamma_0(\mathfrak{n})$, $\Gamma_1(\mathfrak{n})$ and $\Gamma(\mathfrak{n})$ of Γ. We are mainly concerned with $\Gamma_0(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})$ here:

$$\Gamma_0(\mathfrak{n}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma \mid c\in\mathfrak{n}\right\};$$
$$\Gamma_1(\mathfrak{n}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma \mid c, d-1\in\mathfrak{n}\right\}.$$

We have $[\Gamma:\Gamma_0(\mathfrak{n})] = \psi(\mathfrak{n})$, where

$$\psi(\mathfrak{n}) := N(\mathfrak{n})\prod_{\mathfrak{p}\mid\mathfrak{n}}\left(1+N(\mathfrak{p})^{-1}\right),$$

and $[\Gamma_0(\mathfrak{n}):\Gamma_1(\mathfrak{n})] = \varphi(\mathfrak{n})$. Note that φ and ψ are multiplicative in the sense that $\varphi(\mathfrak{m}\mathfrak{n}) = \varphi(\mathfrak{m})\varphi(\mathfrak{n})$ when $\mathfrak{m},\mathfrak{n}$ are coprime, and similarly for ψ. This, and the index formulas, are proved exactly as in the case $K=\mathbb{Q}$.

The group of fractional ideals coprime to the level $\mathfrak{n}$ is denoted $\mathcal{J}_K^{\mathfrak{n}}$.

For a level $\mathfrak{n}$, divisors $\mathfrak{q}$ of $\mathfrak{n}$ such that $\mathfrak{q}$ and $\mathfrak{n}\mathfrak{q}^{-1}$ are coprime are called *exact divisors* of $\mathfrak{n}$; this relation is indicated by $\mathfrak{q}\mid\mid\mathfrak{n}$.

We will use several standard facts about finitely-generated modules over the Dedekind Domain $\mathcal{O}_K$. Specifically we are concerned with $\mathcal{O}_K$*-lattices* of rank 2, which will always be $\mathcal{O}_K$-submodules of $K\oplus K$ of full rank. Elements of $\mathcal{O}_K$-lattices are written as row vectors, with matrices acting on the right. For both the theoretical results we will need and an algorithmic approach to this theory as well as solutions to many problems which arise in practice, see [3, Chap. 1].

The group Γ may be characterized through its action on $\mathcal{O}_K$-lattices, as the set of all matrices $\gamma\in\mathrm{GL}(2,K)$ satisfying $(\mathcal{O}_K\oplus\mathcal{O}_K)\gamma = \mathcal{O}_K\oplus\mathcal{O}_K$, and the following similar characterization of $\Gamma_0(\mathfrak{n})$ is easily checked:

$$\begin{aligned}\Gamma_0(\mathfrak{n}) &= \{\gamma\in\Gamma \mid (\mathfrak{n}\oplus\mathcal{O}_K)\gamma = (\mathfrak{n}\oplus\mathcal{O}_K)\}\\ &= \{\gamma\in\mathrm{GL}(2,K)\mid (\mathcal{O}_K\oplus\mathcal{O}_K)\gamma = \mathcal{O}_K\oplus\mathcal{O}_K \quad\text{and}\quad (\mathfrak{n}\oplus\mathcal{O}_K)\gamma = (\mathfrak{n}\oplus\mathcal{O}_K)\}.\end{aligned}$$

Here, we could also replace $\mathfrak{n}\oplus\mathcal{O}_K$ by $\mathcal{O}_K\oplus\mathfrak{n}^{-1}$. Thus $\Gamma_0(\mathfrak{n})$ is the right stabilizer of the pair of lattices (L,L') where $L=\mathcal{O}_K\oplus\mathcal{O}_K$ and $L'=\mathcal{O}_K\oplus\mathfrak{n}^{-1}$, so that $L'\supseteq L$ and $L'/L\cong\mathcal{O}_K/\mathfrak{n}$ as $\mathcal{O}_K$-modules. The pair (L,L') is an example of a *modular point* for $\Gamma_0(\mathfrak{n})$ which will be studied in detail in Section 2 below.

[1] As we are using $\Gamma=\mathrm{GL}(2,\mathcal{O}_K)$ as our base group, and not $\mathrm{SL}(2,\mathcal{O}_K)$, we define $\Gamma_0(\mathfrak{n})$ and $\Gamma_0(\mathfrak{n})$ accordingly; the reasons for this will be explained later.

Alternatively we may consider the subalgebra $A_0(\mathfrak{n}) = \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{n} & \mathcal{O}_K \end{pmatrix}$ of $\mathrm{Mat}(2, \mathcal{O}_K)$; then $\Gamma_0(\mathfrak{n})$ is the intersection of this with Γ, consisting of those matrices γ such that $A_0(\mathfrak{n})\gamma = A_0(\mathfrak{n})$.

To characterize $\Gamma_1(\mathfrak{n})$ in the same way, we need to rigidify the lattice pair (L, L') by fixing an $\mathcal{O}_K$-module generator of the quotient L'/L (which is cyclic, being isomorphic to $\mathcal{O}_K/\mathfrak{n}$). To do this, let $n_0 \in \mathfrak{n}^{-1}$ generate $\mathfrak{n}^{-1}/\mathcal{O}_K$ (equivalently, such that $\mathfrak{n}^{-1}$ is generated by 1 and n_0) and set $\beta_0 = (0, n_0) \in L'$. Then $L' = L + \mathcal{O}_K\beta_0$, and $\Gamma_1(\mathfrak{n})$ is the subgroup of Γ fixing $\beta_0 \pmod{L}$:

$$\Gamma_1(\mathfrak{n}) = \{\gamma \in \mathrm{GL}(2, K) \mid (\mathcal{O}_K \oplus \mathcal{O}_K)\gamma = \mathcal{O}_K \oplus \mathcal{O}_K \quad \text{and} \quad \beta_0\gamma = \beta_0 \pmod{\mathcal{O}_K \oplus \mathcal{O}_K}\}$$

since for $x \in \mathcal{O}_K$ we have $n_0 x \in \mathcal{O}_K \iff x \in \mathfrak{n}$. This characterization is independent of the choice of β_0. When $\mathcal{O}_K = \mathbb{Z}$ and $\mathfrak{n} = N\mathbb{Z}$ one usually takes $n_0 = \frac{1}{N}$, but there is no canonical choice in general.

For $a, b \in \mathcal{O}_K$ (or K) we denote by $\langle a\rangle$ and $\langle a, b\rangle$ the (fractional) ideals $a\mathcal{O}_K$ and $a\mathcal{O}_K + b\mathcal{O}_K$. For an ideal $\mathfrak{a}$ we denote its class in the class group $\mathrm{Cl}(K)$ by $[\mathfrak{a}]$. When $\mathrm{Cl}(K)$ is finite (as in the number field case) we denote its order by $h(\mathcal{O}_K)$ or simply h.

Among the elementary properties of Dedekind Domains we will use are

(1) For any ideal $\mathfrak{a}$ and any nonzero $a \in \mathfrak{a}$ there exists $b \in \mathfrak{a}$ with $\mathfrak{a} = \langle a, b\rangle$;
(2) Every ideal class contains an ideal coprime to any given ideal.

1.3. **$(\mathfrak{a}, \mathfrak{b})$-matrices.** Certain matrices in $\mathrm{Mat}_2(\mathcal{O}_K)$ called $(\mathfrak{a}, \mathfrak{b})$-matrices were introduced in [7] and will play a useful important role in our theory. Over $\mathbb{Q}$, or more generally when the class number is 1, these matrices may not be visible, as they function like matrices in Hermite Normal Form. In [7], they were used to study and count the orbits of $\Gamma_0(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})$ on $\mathbb{P}^1(K)$.

$(\mathfrak{a}, \mathfrak{b})$-matrices are associated with any pair of ideals $\mathfrak{a}, \mathfrak{b}$ in inverse ideal classes, that $\mathfrak{a}\mathfrak{b}$ is principal, and $\mathfrak{a} \oplus \mathfrak{b} \cong \mathcal{O}_K \oplus \mathcal{O}_K$. An $(\mathfrak{a}, \mathfrak{b})$-matrix is any matrix $M \in \mathrm{Mat}_2(\mathcal{O}_K)$ such that

$$(\mathcal{O}_K \oplus \mathcal{O}_K)M = \mathfrak{a} \oplus \mathfrak{b}.$$

The set of all $(\mathfrak{a}, \mathfrak{b})$-matrices is denoted $X_{\mathfrak{a},\mathfrak{b}}$, and for all pairs of integral ideals $(\mathfrak{a}, \mathfrak{b})$ we set

$$\begin{aligned}\Delta(\mathfrak{a}, \mathfrak{b}) &= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, w \in \mathcal{O}_K, y \in \mathfrak{a}^{-1}\mathfrak{b}, z \in \mathfrak{a}\mathfrak{b}^{-1}, xw - yz \in \mathcal{O}_K^{\times} \right\} \\ &= \{\gamma \in \mathrm{GL}(2, K) \mid (\mathfrak{a} \oplus \mathfrak{b})\gamma = \mathfrak{a} \oplus \mathfrak{b}\},\end{aligned}$$

by Proposition 3 of [7].

In the case $\mathfrak{a} = \mathfrak{b} = \mathcal{O}_K$, an $(\mathfrak{a}, \mathfrak{b})$-matrix is simply an element of Γ. From [7] we recall the following facts about $(\mathfrak{a}, \mathfrak{b})$-matrices:

(1) $(\mathfrak{a}, \mathfrak{b})$-matrices exist whose first column is an arbitrary pair of generators of $\mathfrak{a}$. In particular, the lower left entry may be chosen to be any nonzero element of $\mathfrak{a}$, so may be taken to lie in any other ideal $\mathfrak{n}$ that we choose. An $(\mathfrak{a}, \mathfrak{b})$-matrix whose $(2, 1)$-entry lies in $\mathfrak{n}$ is called an $(\mathfrak{a}, \mathfrak{b})$-matrix of *level* $\mathfrak{n}$.
(2) $X_{\mathfrak{a},\mathfrak{b}} = \Gamma M = M\Delta((\mathfrak{a}, \mathfrak{b}))$ for all $M \in X_{\mathfrak{a},\mathfrak{b}}$.

1.4. **M-symbols and coset representatives for $\Gamma_0(\mathfrak{n})$.** As in Section 3 of [7], for an ideal $\mathfrak{n}$ of $\mathcal{O}_K$, an *M-symbol* (or *Manin symbol*) *of level* $\mathfrak{n}$ is an element $(c : d) \in \mathbb{P}^1(\mathcal{O}_K/\mathfrak{n})$. Here, $c, d \in \mathcal{O}_K$ are such that the ideal $\langle c, d\rangle$ is coprime to $\mathfrak{n}$, and

$$(c_1 : d_1) = (c_2 : d_2) \iff c_1 d_2 - c_2 d_1 \in \mathfrak{n}.$$

For brevity we will write $\mathbb{P}^1(\mathfrak{n})$ for $\mathbb{P}^1(\mathcal{O}_K/\mathfrak{n})$. The M-symbols of level $\mathfrak{n}$ are in bijection with the set of cosets of $\Gamma_0(\mathfrak{n})$ in Γ, since

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \Gamma_0(\mathfrak{n}) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \Gamma_0(\mathfrak{n}) \iff (c_1 : d_1) = (c_2 : d_2) \quad \text{in } \mathbb{P}^1(\mathfrak{n}).$$

In other words, the map $\Gamma \to \mathbb{P}^1(\mathfrak{n})$ is surjective, with fibres equal to the cosets of $\Gamma_0(\mathfrak{n})$. On Proposition 8 of [7] we show how to replace a pair c, d such that $\langle c, d\rangle + \mathfrak{n} = \mathcal{O}_K$ with an equivalent pair c', d' such that $\langle c', d'\rangle = \mathcal{O}_K$ and $(c : d) = (c' : d')$. Writing $1 = ad' - bc'$ with $a, b \in \mathcal{O}_K$, we obtain a section from $\mathbb{P}^1(\mathfrak{n})$ to Γ, the implementation of which is crucial when doing computations.

More generally, suppose that the ideals $\mathfrak{m}, \mathfrak{n}$ are coprime. Then M-symbols of level $\mathfrak{n}$ also give coset representatives of $\Gamma_0(\mathfrak{m}\mathfrak{n})$ in $\Gamma_0(\mathfrak{m})$. To see this, note that by the Chinese Remainder Theorem we may identify $\mathbb{P}^1(\mathfrak{m}\mathfrak{n})$ with $\mathbb{P}^1(\mathfrak{m}) \times \mathbb{P}^1(\mathfrak{n})$. Hence, for each $(c : d) \in \mathbb{P}^1(\mathfrak{n})$, we may find $c', d' \in \mathcal{O}_K$ which are coprime and such that

$$(c' : d') = (c : d) \quad \text{in } \mathbb{P}^1(\mathfrak{n}), \text{ and} \quad (c' : d') = (0 : 1) \quad \text{in } \mathbb{P}^1(\mathfrak{m}).$$

Then, lifting the M-symbol $(c' : d') \in \mathbb{P}^1(\mathfrak{m}\mathfrak{n})$ to Γ as before results in a matrix in $\Gamma_0(\mathfrak{m})$ representing the same $\Gamma_0(\mathfrak{n})$-coset as $(c : d)$.

The algorithmic procedures outlined here play an important part in the construction of explicit matrices giving the action of Hecke and related operators below.

2. Lattices and Modular points

One description of classical modular forms for $\mathrm{SL}(2, \mathbb{Z})$ is to view them as functions on classical lattices (discrete rank two $\mathbb{Z}$-submodules of $\mathbb{C}$) satisfying certain properties. This approach may be extended to modular forms for $\Gamma_0(N)$ or $\Gamma_1(N)$ by giving the lattices extra "level N" structure. For example, see [15, VIII §2] for the case of level 1 and [12, III §5] and [13, VII] for $\Gamma_0(N)$ and $\Gamma_1(N)$. This description of modular forms enables one to define Hecke operators via their action on lattices.

In his thesis [2, Chap. 7], Bygott began to extend this approach to number fields, the emphasis there being on the imaginary quadratic case. We follow and extend that approach here. The first step is to consider $\mathcal{O}_K$-lattices more general than $\mathcal{O}_K$-submodules of $K \oplus K$; then we introduce the level structure. We start with the case of $\Gamma_0(N)$-structure, and then consider $\Gamma_1(N)$.

Let $\widehat{K} = K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, where r_1 and r_2 are numbers of real and complex conjugate pairs of embeddings of K into its Archimedean completions, $\mathbb{R}$ or $\mathbb{C}$. The components of $\widehat{K}$ are denoted $K_{\infty,j}$ for $1 \le j \le r_1 + r_2$, and we have the diagonal embedding $K \hookrightarrow \widehat{K}$. For this section, we do everything component-wise, and the reader can think of $\widehat{K}$ as being a single completion of K, isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

Define $\mathcal{G} = \mathrm{GL}(2, \widehat{K})$; this acts on the right on the space of row vectors $\widehat{K}^2 = \widehat{K} \oplus \widehat{K}$.

Definition 1. An $\mathcal{O}_K$-lattice in $\widehat{K}^2$ is an $\mathcal{O}_K$-submodule L of $\widehat{K}^2$ satisfying the following conditions:

(1) $\widehat{K}L = \widehat{K}^2$;
(2) $F_1 \subseteq L \subseteq F_2$ where F_1, F_2 are free $\mathcal{O}_K$-modules of rank 2;
(3) L is finitely-generated and $\dim_K(KL) = 2$.

(It is easy to see that (2) and (3) are in fact equivalent.) So $E = KL$ is a 2-dimensional K-vector subspace of $\widehat{K}^2$, hence isomorphic to $K \oplus K$, but in general different lattices lie in different such subspaces. Each lattice L has $r_1 + r_2$ components, the jth component contained in $K^2_{\infty,j}$; for this section the reader may think of each lattice L as being contained in either $\mathbb{R} \oplus \mathbb{R}$ or $\mathbb{C} \oplus \mathbb{C}$.

The following elementary facts about lattices are stated without proof, as all follow easily from the theory of finitely-generated modules over Dedekind Domains.

Every lattice L is isomorphic as $\mathcal{O}_K$-module to $\mathcal{O}_K \oplus \mathfrak{a}$ for some ideal $\mathfrak{a}$, whose class in the class group $\mathrm{Cl}(K)$ is the (Steinitz) *class* of L, which we denote $[L]$. The structure theorem for $\mathcal{O}_K$-modules implies that two lattices are isomorphic as $\mathcal{O}_K$-modules if and only if they have the same class, since they are both torsion-free and of rank 2.

If L is a lattice and $U \in \mathcal{G}$, then LU is also a lattice, isomorphic to L and hence of the same class. Conversely, any $\mathcal{O}_K$-module isomorphism between lattices $L_1 \to L_2$ extends first to a K-linear isomorphism $KL_1 \to KL_2$, and then to a $\widehat{K}$-linear automorphism of $\widehat{K}^2$, so is represented by a matrix $U \in \mathcal{G}$. Hence

$$[L_1] = [L_2] \iff L_1 \cong L_2 \iff L_2 = L_1 U \qquad \text{for some } U \in \mathcal{G}.$$

One may also think of L and LU as being the same lattice, expressed in terms of different bases for $\widehat{K}^2$, with U being the change of basis matrix.

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$ (possibly fractional), then the module $\mathfrak{a}L$, consisting of all finite sums $\{\sum_i a_i x_i \mid a_i \in \mathfrak{a}, x_i \in L\}$, is also a lattice, of class $[\mathfrak{a}]^2[L]$. For example, if $[L]$ is a square, say $[L] = [\mathfrak{a}]^2$, then $L \cong \mathfrak{a} \oplus \mathfrak{a}$, so $\mathfrak{a}^{-1}L \cong \mathcal{O}_K \oplus \mathcal{O}_K$, a free lattice.

If L, L' are two lattices with $L' \supseteq L$ then L'/L is a finite torsion $\mathcal{O}_K$-module, hence is isomorphic to $\mathcal{O}_K/\mathfrak{a}_1 \oplus \mathcal{O}_K/\mathfrak{a}_2$ for unique ideals $\mathfrak{a}_1$, $\mathfrak{a}_2$ of $\mathcal{O}_K$ satisfying $\mathfrak{a}_1 \mid \mathfrak{a}_2$, called the *elementary divisors* of L'/L. The product $\mathfrak{a} = \mathfrak{a}_1\mathfrak{a}_2$ is called the *norm ideal* of L'/L or the *index ideal* of L in L', denoted $[L' : L]$; it determines the cardinality $\#(L'/L) = N(\mathfrak{a}) = \#(\mathcal{O}_K/\mathfrak{a})$, and $\mathfrak{a}_2$ is the annihilator of L'/L. The classes are related by $[L'] = [\mathfrak{a}][L]$. For any fractional ideal $\mathfrak{b}$ we have $\mathfrak{b}L'/\mathfrak{b}L \cong L'/L$. Moreover, there exist fractional ideals $\mathfrak{b}_1, \mathfrak{b}_2$ and a $\widehat{K}$-basis x, y of $\widehat{K}^2$ such that

$$L' = \mathfrak{b}_1 x \oplus \mathfrak{b}_2 y; \qquad L = \mathfrak{a}_1\mathfrak{b}_1 x \oplus \mathfrak{a}_2\mathfrak{b}_2 y;$$

we can write this alternatively as

$$L' = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U; \qquad L = (\mathfrak{a}_1\mathfrak{b}_1 \oplus \mathfrak{a}_2\mathfrak{b}_2)U$$

where $U \in \mathcal{G}$ is the matrix with rows x, y. By scaling x and y we may if we wish replace $\mathfrak{b}_1, \mathfrak{b}_2$ by integral ideals. Now we have $L'/L \cong \mathcal{O}_K/\mathfrak{a}_1 \oplus \mathcal{O}_K/\mathfrak{a}_2$.

2.1. **Standard Lattices.** We wish now to choose a standard lattice in each Steinitz class. One way to do this (as in Bygott's thesis [2]), which works for any field, is to take ideals $\mathfrak{p}_i$ representing the ideal classes, such as $\mathfrak{p}_1 = \mathcal{O}_K$ and $\mathfrak{p}_i$ prime for $i > 1$, and use the lattices $\mathfrak{p}_i \oplus \mathcal{O}_K$. When working with level $\mathfrak{n}$ structure, we may also wish to assume that the $\mathfrak{p}_i$ are coprime to $\mathfrak{n}$. However, there is a simpler choice available for lattices whose class is square, since a class of the form $[\mathfrak{q}]^2$ may be represented by the lattice $\mathfrak{q} \oplus \mathfrak{q} = \mathfrak{q}(\mathcal{O}_K \oplus \mathcal{O}_K)$. Again, we may wish to choose $\mathfrak{q}$ to be coprime to the current level $\mathfrak{n}$. When the class number is odd, we can use this second choice for all ideal classes, resulting in a simplification of the theory in several places.

In the general case, we combine these two approaches. Write $h = \#\,\mathrm{Cl}(K) = h_2 h_2'$ where $h_2 = \#\,\mathrm{Cl}[2] = [\mathrm{Cl}(K) : \mathrm{Cl}(K)^2]$. For $1 \le i \le h_2$ we fix ideals $\mathfrak{p}_i$ whose classes $[\mathfrak{p}_i]$ represent the cosets $\mathrm{Cl}(K)/\mathrm{Cl}(K)^2$, and for $1 \le j \le h_2'$ we fix $\mathfrak{q}_j$ whose classes $[\mathfrak{q}_j]^2$ comprise $\mathrm{Cl}(K)^2$. We always take $\mathfrak{p}_1 = \mathfrak{q}_1 = \mathcal{O}_K$; the other representatives may if convenient be taken to be primes, or coprime to a given level, or both. Then the set of all ideal classes is

$$\mathrm{Cl}(K) = \{c_{ij} = [\mathfrak{p}_i][\mathfrak{q}_j]^2 \mid 1 \le i \le h_2, 1 \le j \le h_2'\}.$$

When the class number is odd we have $h_2 = 1$ and then all representatives are of the form $[\mathfrak{q}_j]^2$; at the other extreme, if $\mathrm{Cl}(K)$ has exponent 2 then $h_2 = h$ and all are of the form $[\mathfrak{p}_i]$.

For each class $c = c_{ij} \in \mathrm{Cl}(K)$, we define the standard lattice in class c_{ij} to be

$$L_{ij} = \mathfrak{p}_i\mathfrak{q}_j \oplus \mathfrak{q}_j = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathcal{O}_K).$$

Since every lattice is isomorphic to L_{ij} for a unique pair (i, j), the set of lattices in class c_{ij} is $\{L_{ij}U \mid U \in \mathcal{G}\}$. The matrix U in these representations is of course not unique, as we now make precise.

Recall that $\Delta(\mathfrak{p}, \mathcal{O}_K) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathcal{O}_K, b \in \mathfrak{p}^{-1}, c \in \mathfrak{p}, ad - bc \in \mathcal{O}_K^\times\right\}$. For brevity, we set $\Gamma_\mathfrak{p} = \Delta(\mathfrak{p}, \mathcal{O}_K)$ and $\Gamma_i = \Gamma_{\mathfrak{p}_i}$. Note that if $\mathfrak{p} = \langle\pi\rangle$ were principal then $\Gamma_\mathfrak{p}$ would be simply the conjugate of Γ by $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. The following result follows from Proposition 3 of [7], after observing that for any lattice L, if $LU = L$ for some $U \in \mathcal{G}$, then the entries of U lie in K.

Proposition 2.1. *Let $U \in \mathcal{G}$. Then $L_{ij}U = L_{ij}$ if and only if $U \in \Gamma_i$ (independent of j); hence*

$$L_{ij}U = L_{ij}U' \iff U'U^{-1} \in \Gamma_i.$$

The set of lattices in class c_{ij} is in bijection with the coset space $\Gamma_i\backslash\mathcal{G}$. □

Hence when the class number is odd, each class of lattices is parameterised by the *same* space $\Gamma\backslash\mathcal{G}$, just as in the case of trivial class group.

2.2. Modular points for $\Gamma_0(\mathfrak{n})$.

Definition 2. A *modular point for* $\Gamma_0(\mathfrak{n})$ is a pair (L, L') of lattices with[2] $L' \supseteq L$ and $L'/L \cong \mathcal{O}_K/\mathfrak{n}$. We call L the *underlying lattice* of the pair, and $[L]$ its *class.* The set $\mathcal{M}_0(\mathfrak{n})$ of modular points for $\Gamma_0(\mathfrak{n})$ is the disjoint union of the sets $\mathcal{M}_0^{(c)}(\mathfrak{n})$ of modular points in each ideal class c.

Thus a modular point consists of an underlying lattice and a superlattice (not a sublattice) with relative index ideal $\mathfrak{n}$, such that the quotient is a cyclic $\mathcal{O}_K$-module. If $(L, L') \in \mathcal{M}_0(\mathfrak{n})$ then $L \subseteq L' \subseteq \mathfrak{n}^{-1}L$. Modular points certainly exist with any underlying lattice L, therefore, since $\mathfrak{n}^{-1}L/L \cong (\mathcal{O}_K/\mathfrak{n})^2$ has submodules isomorphic to $\mathcal{O}_K/\mathfrak{n}$.

If $P = (L, L') \in \mathcal{M}_0(\mathfrak{n})$ then $\mathfrak{a}P = (\mathfrak{a}L, \mathfrak{a}L') \in \mathcal{M}_0(\mathfrak{n})$ also for every ideal $\mathfrak{a}$, since $\mathfrak{a}L'/\mathfrak{a}L \cong L'/L$. Also, $PU = (LU, L'U) \in \mathcal{M}_0(\mathfrak{n})$ for all $U \in \mathcal{G}$.

In terms of the standard structure given earlier we have $\mathfrak{a}_1 = \mathcal{O}_K$ and $\mathfrak{a}_2 = \mathfrak{n}$. In other words, there exist fractional ideals $\mathfrak{b}_1, \mathfrak{b}_2$ and $U \in \mathcal{G}$ such that

$$L' = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U; \qquad L = (\mathfrak{b}_1 \oplus \mathfrak{n}\mathfrak{b}_2)U.$$

It is now convenient to change this notation, replacing $\mathfrak{b}_2$ by $\mathfrak{n}\mathfrak{b}_2$, so that now the modular point (L, L') has the form

$$L = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U; \qquad L' = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U.$$

The basic example of a modular point is the pair $(\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1})$ with underlying lattice $\mathcal{O}_K \oplus \mathcal{O}_K$. More generally, to each of our standard lattices L_{ij} we associate a *standard modular point* P_{ij}:

$$P_{ij} = (L_{ij}, L'_{ij}) = (\mathfrak{p}_i\mathfrak{q}_j \oplus \mathfrak{q}_j, \mathfrak{p}_i\mathfrak{q}_j \oplus \mathfrak{q}_j\mathfrak{n}^{-1})) = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathcal{O}_K, \mathfrak{p}_i \oplus \mathfrak{n}^{-1}).$$

The following proposition states that every modular point is the image of one of these under a suitable $U \in \mathcal{G}$.

[2]In [2], the pair (L, S) where $S = L'/L$ is called a modular point for $\Gamma_0(\mathfrak{n})$ rather than (L, L'), but this is only a cosmetic difference.

Definition 3. An *admissible basis matrix* for the modular point $P = (L, L') \in \mathcal{M}_0^{(c)}(\mathfrak{n})$ is a matrix $U \in \mathcal{G}$ such that $P = P_{ij}U$ where $c = c_{ij}$; that is, such that

$$L = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathcal{O}_K)U; \qquad L' = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathfrak{n}^{-1})U.$$

The rows of an admissible basis matrix are vectors $r_1, r_2 \in \widehat{K} \oplus \widehat{K}$, which we call an *admissible basis* for the modular point P; they are characterized by

$$L = \mathfrak{q}_j(\mathfrak{p}_i r_1 + \mathcal{O}_K r_2); \qquad L' = \mathfrak{q}_j(\mathfrak{p}_i r_1 + \mathfrak{n}^{-1} r_2).$$

We now show that admissible bases exist, and see to what extent they are unique.

Proposition 2.2. *Every modular point for $\Gamma_0(\mathfrak{n})$ has an admissible basis.*

Proof. First consider the case of a principal modular point (L, L'), where L is free. Write $L = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U_0$ and $L' = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U_0$ as above, where $\mathfrak{b}_1, \mathfrak{b}_2$ are integral ideals and $U_0 \in \mathcal{G}$. Since L is free, $\mathfrak{b}_1\mathfrak{b}_2$ is principal; let $V \in \mathrm{GL}(2, K)$ be a $(\mathfrak{b}_1, \mathfrak{b}_2)$-matrix of level $\mathfrak{n}$. Then $(\mathcal{O}_K \oplus \mathcal{O}_K)V = \mathfrak{b}_1 \oplus \mathfrak{b}_2$, so that putting $U = VU_0 \in \mathcal{G}$ we have $(\mathcal{O}_K \oplus \mathcal{O}_K)U = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U_0 = L$; moreover, since V has level $\mathfrak{n}$, Proposition 4 of [7] implies that $(\mathcal{O}_K \oplus \mathfrak{n}^{-1})V = \mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2$, and hence $(\mathcal{O}_K \oplus \mathfrak{n}^{-1})U = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U_0 = L'$.

Next we show how to construct an admissible basis matrix in the case where $L \cong L_{i1} = \mathfrak{p}_i \oplus \mathcal{O}_K$ with $i > 1$. Again we start from a representation of the form $L = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U_0$ and $L' = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U_0$ with $U_0 \in \mathcal{G}$. We claim the existence of $V \in \begin{pmatrix} \mathfrak{b}_2^{-1} & \mathfrak{b}_1^{-1} \\ \mathfrak{b}_1\mathfrak{n} & \mathfrak{b}_2 \end{pmatrix}$ with $\det V = 1$. Then V satisfies $(\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathcal{O}_K)V = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ and $(\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathfrak{n}^{-1})V = \mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2$. Finally, since $L \cong L_{i1}$, we have $\mathfrak{b}_1\mathfrak{b}_2 = t\mathfrak{p}_i$ for some $t \in K^*$, so if we set $U = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} VU_0$ then U is an admissible basis matrix for (L, L'):

$$\begin{aligned}(\mathfrak{p}_i \oplus \mathcal{O}_K)U &= (\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathcal{O}_K)VU_0 = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U_0 = L;\\ (\mathfrak{p}_i \oplus \mathfrak{n}^{-1})U &= (\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathfrak{n}^{-1})VU_0 = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U_0 = L'.\end{aligned}$$

For the existence of V we proceed as follows, generalizing [3, Prop. 1.3.12] where the special case $\mathfrak{n} = \mathcal{O}_K$ is proved. Choose an integral ideal $\mathfrak{a}$ coprime to $\mathfrak{b}_2$ in the inverse ideal class to $\mathfrak{n}\mathfrak{b}_1$. Then $\mathfrak{a}\mathfrak{n}\mathfrak{b}_1 = \langle z \rangle$ with $z \in \mathfrak{n}\mathfrak{b}_1$. Next choose x so that $x\mathfrak{b}_2$ is integral and coprime to $\mathfrak{a}\mathfrak{n}$. Then $x \in \mathfrak{b}_2^{-1}$, and since $\mathcal{O}_K = x\mathfrak{b}_2 + \mathfrak{a}\mathfrak{n} = x\mathfrak{b}_2 + z\mathfrak{b}_1^{-1}$, there exist $w \in \mathfrak{b}_2$ and $y \in \mathfrak{b}_1^{-1}$ such that $xw - yz = 1$. Now $V = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \begin{pmatrix} \mathfrak{b}_2^{-1} & \mathfrak{b}_1^{-1} \\ \mathfrak{n}\mathfrak{b}_1 & \mathfrak{b}_2 \end{pmatrix}$ with $\det V = 1$ as required.

Finally, if $P = (L, L')$ is a modular point with class c_{ij} and $j > 1$, we form $\mathfrak{q}_j^{-1}P = (\mathfrak{q}_j^{-1}L, \mathfrak{q}_j^{-1}L')$, which has class c_{i1}, and simply observe that any admissible basis for the latter is also an admissible basis for P. $\square$

Let

$$\Gamma_0^{\mathfrak{p}}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathcal{O}_K; b \in \mathfrak{p}^{-1}; c \in \mathfrak{n}\mathfrak{p}; ad - bc \in \mathcal{O}_K^\times \right\}.$$

It is easy to see that the group of matrices $\gamma \in \mathrm{GL}(2, K)$ which stabilize P_{ij} is $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$ from which the following is immediate.

Proposition 2.3. *Let P be a modular point for $\Gamma_0(\mathfrak{n})$ in class c_{ij}, and let U be an admissible basis matrix for P, so that $P = P_{ij}U$. Then the set of all admissible basis matrices for P is the coset $\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})U$.* $\square$

Again, the situation is simpler for square ideal classes, for which the right stabilizer of the standard modular point is the same as for principal modular points, namely $\Gamma_0(\mathfrak{n})$ itself. It can be shown that the more general groups $\Gamma_0^{\mathfrak{p}}(\mathfrak{n})$ defined above are conjugate to $\Gamma_0(\mathfrak{n})$ in $\mathrm{GL}(2,K)$ if and only if the ideal class $[\mathfrak{p}]$ is a square. More generally, if $[\mathfrak{p}_1]=[\mathfrak{p}_2][\mathfrak{a}]^2$ then $\Gamma_0^{\mathfrak{p}_2}(\mathfrak{n})=M\Gamma_0^{\mathfrak{p}_1}(\mathfrak{n})M^{-1}$ where $M\in\mathrm{GL}(2,K)$ satisfies $(\mathfrak{a}\mathfrak{p}_2\oplus\mathfrak{a})M=\mathfrak{p}_1\oplus\mathcal{O}_K$ and simultaneously $(\mathfrak{a}\mathfrak{p}_2\oplus\mathfrak{a}\mathfrak{n}^{-1})M=\mathfrak{p}_1\oplus\mathfrak{n}^{-1}$. Our choice of standard lattices and modular points has been made to minimise the number of different groups of the form $\Gamma_0^{\mathfrak{p}}(\mathfrak{n})$ which need to be considered.

Corollary 2.4. *For each ideal class* $c_{ij}=[\mathfrak{p}_i\mathfrak{q}_j^2]$,

$$\mathcal{M}_0^{(c_{ij})}(\mathfrak{n})=\{P_{ij}U\mid U\in\mathcal{G}\},$$

and the correspondence $P_{ij}U\leftrightarrow U$ *is a bijection*

$$\mathcal{M}_0^{(c_{ij})}(\mathfrak{n})\longleftrightarrow\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})\backslash\mathcal{G}.$$

2.3. **Modular points for $\Gamma_1(\mathfrak{n})$.** A modular point for $\Gamma_1(\mathfrak{n})$ consists of a modular point for $\Gamma_0(\mathfrak{n})$ with some extra structure.

Definition 4. A *modular point for* $\Gamma_1(\mathfrak{n})$ is a triple $P=(L,L',\beta)$ where (L,L') is a $\Gamma_0(\mathfrak{n})$-modular point (so L, L' are lattices with $L'\supseteq L$ and $L'/L\cong\mathcal{O}_K/\mathfrak{n}$), and $\beta\in L'$ generates L'/L as $\mathcal{O}_K$-module (that is, $L'=L+\mathcal{O}_K\beta$). We call L the *underlying lattice* of P, $[L]$ its *class*, and (L,L') the underlying $\Gamma_0(\mathfrak{n})$-modular point.

We identify (L,L',β_1) and (L,L',β_2) when $\beta_1-\beta_2\in L$; in particular, $(L,L',\beta)=(L,L',u\beta)$ for $u\in\mathcal{O}_K$ coprime to $\mathfrak{n}$. The condition that $\beta \pmod{L}$ generates L'/L is equivalent to $r\beta\in L\iff r\in\mathfrak{n}$, for $r\in\mathcal{O}_K$.

The set $\mathcal{M}_1(\mathfrak{n})$ of modular points for $\Gamma_1(\mathfrak{n})$ is the disjoint union of the sets $\mathcal{M}_1^{(c)}(\mathfrak{n})$ of modular points in each ideal class c.

If $P=(L,L',\beta)\in\mathcal{M}_1(\mathfrak{n})$, then also $PU=(LU,L'U,\beta U)\in\mathcal{M}(\mathfrak{n})$ for all $U\in\mathcal{G}$. Further transformations on modular points are defined below.

Together with each of the standard lattices L_{ij}, we have already associated a standard $\Gamma_0(\mathfrak{n})$-modular point P_{ij}. We now lift each of these to a standard $\Gamma_1(\mathfrak{n})$-modular point. As in Section 1, we fix $n_0\in\mathfrak{n}^{-1}$ such that $\mathfrak{n}^{-1}=\langle 1,n_0\rangle$, or equivalently such that $n_0 \pmod{\mathcal{O}_K}$ generates $\mathfrak{n}^{-1}/\mathcal{O}_K$ as $\mathcal{O}_K$-module, and set $\beta_0=(0,n_0)\in\mathcal{O}_K\oplus\mathfrak{n}^{-1}$. Then the standard modular points are as follows. For each j, let $\mathfrak{a}_j$ be an ideal coprime to $\mathfrak{n}$ in the inverse class to $\mathfrak{q}_j$, so that $\mathfrak{a}_j\mathfrak{q}_j=\langle z_j\rangle$ with $z_j\in\mathcal{O}_K$. Then multiplication by z_j induces an $\mathcal{O}_K$-module isomorphism $\mathfrak{n}^{-1}/\mathcal{O}_K\to\mathfrak{q}_j\mathfrak{n}^{-1}/\mathfrak{q}_j$. Set $n_j=n_0z_j\in\mathfrak{q}_j\mathfrak{n}^{-1}$ and $\beta_j=(0,n_j)$; then n_j generates the cyclic module $\mathfrak{q}_j\mathfrak{n}^{-1}/\mathfrak{q}_j$. We define

$$\tilde{P}_{ij}=(L_{ij},L'_{ij},\beta_j)=(\mathfrak{q}_j(\mathfrak{p}_i\oplus\mathcal{O}_K),\mathfrak{q}_j(\mathfrak{p}_i\oplus\mathfrak{n}^{-1}),\beta_j).$$

Although this does depend on various choices made, we regard these as fixed once and for all. Also, since β_j depends only on j and not on i, it only depends on the class modulo squares. In particular, when the class group has exponent 2 we have $\beta=\beta_0$ in all cases[3].

Any other generator of L'_{ij}/L_{ij} has the form $\beta=u\beta_j \pmod{L_{ij}}$ where $u\in\mathcal{O}_K$ is coprime to $\mathfrak{n}$. If we let $\gamma_u\in\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$ be any matrix with $(2,2)$-entry u, then

$$(L_{ij},L'_{ij},\beta)=(L_{ij},L'_{ij},\beta_j)\gamma_u=\tilde{P}_{ij}\gamma_u$$

since γ_u stabilises (L_{ij},L'_{ij}) by Proposition 2.3. Moreover,

$$\tilde{P}_{ij}\gamma_u=\tilde{P}_{ij}\iff u\equiv 1 \pmod{\mathfrak{n}}\iff\gamma_u\in\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n}),$$

[3]This is one place where the situation is not simpler for fields of odd class number.

where

$$\Gamma_1^{\mathfrak{p}}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{\mathfrak{p}}(\mathfrak{n}) \mid d-1 \in \mathfrak{n} \right\}.$$

Hence the set of all lifts of P_{ij} to a $\Gamma_1(\mathfrak{n})$-modular point is the set of $P_{ij}\gamma$, as γ runs through $\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})\backslash\Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$. This has cardinality $\varphi(\mathfrak{n})$: the index $[\Gamma_0^{\mathfrak{p}}(\mathfrak{n}) : \Gamma_1^{\mathfrak{p}}(\mathfrak{n})] = \varphi(\mathfrak{n})$ always, since $\Gamma_1^{\mathfrak{p}}(\mathfrak{n})$ is the kernel of the surjective homomorphism $\epsilon : \Gamma_0^{\mathfrak{p}}(\mathfrak{n}) \to (\mathcal{O}_K/\mathfrak{n})^\times$ which maps a matrix to its $(2,2)$-entry.

More generally, to describe the set of all $\Gamma_1(\mathfrak{n})$ modular points with an arbitrary underlying $\Gamma_0(\mathfrak{n})$ modular point we first extend the notion of admissible basis.

Definition 5. An *admissible basis matrix* for the modular point $\tilde{P} = (L, L', \beta) \in \mathcal{M}_1^{(c)}(\mathfrak{n})$ is a matrix $U \in \mathcal{G}$ such that $\tilde{P} = \tilde{P}_{ij}U$ where $c = c_{ij}$; that is, such that

$$L = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathcal{O}_K)U; \qquad L' = \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathfrak{n}^{-1})U; \qquad \beta = \beta_j U.$$

The rows of an admissible basis matrix are an *admissible basis* for the modular point.

Proposition 2.5. *Every modular point $\tilde{P}$ for $\Gamma_1(\mathfrak{n})$ has an admissible basis matrix U; the set of all admissible basis matrices for $\tilde{P}$ is the coset $\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})U$ where $c = c_{ij}$ is the class of $\tilde{P}$; the set of $\Gamma_1(\mathfrak{n})$-modular points with the same underlying $\Gamma_0(\mathfrak{n})$-modular point as $\tilde{P}$ is $\{\tilde{P}U^{-1}\gamma U \mid \gamma \in \Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})\}$.*

Hence

$$\mathcal{M}_1^{(c)}(\mathfrak{n}) = \{\tilde{P}_{ij}U \mid U \in \mathcal{G}\},$$

and the correspondence $\tilde{P}_{ij}U \leftrightarrow U$ is a bijection

$$\mathcal{M}_1^{(c)}(\mathfrak{n}) \longleftrightarrow \Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})\backslash\mathcal{G}.$$

Proof. Let $\tilde{P}$ be a $\Gamma_1(\mathfrak{n})$-modular point with underlying $\Gamma_0(\mathfrak{n})$-modular point P. Let U be an admissible basis matrix for P. Then $\tilde{P}U^{-1} = (\mathfrak{q}_j(\mathfrak{p}_i \oplus \mathcal{O}_K), \mathfrak{q}_j(\mathfrak{p}_i \oplus \mathfrak{n}^{-1}), \beta)$ for some β. As seen above, there exists $\gamma \in \Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})$ with $\beta = \beta_j\gamma$. Then $\tilde{P} = \tilde{P}_{ij}\gamma U$, so γU is an admissible basis matrix for $\tilde{P}$. The other parts are clear. □

3. The Hecke algebra

Let $\mathcal{M} = \mathcal{M}_0(\mathfrak{n})$ or $\mathcal{M}_1(\mathfrak{n})$ be the set of modular points for either $\Gamma_0(\mathfrak{n})$ or $\Gamma_1(\mathfrak{n})$, and let $\mathbb{Q}\mathcal{M}$ be the $\mathbb{Q}$-vector space with the elements of $\mathcal{M}$ as basis; $\mathbb{Q}\mathcal{M}$ is the direct sum of spaces $\mathbb{Q}\mathcal{M}_c$, as c runs over the ideal classes. We will define an algebra $\mathbb{T}$ of commuting linear operators called Hecke operators, which act on $\mathbb{Q}\mathcal{M}$, and prove their formal properties. We also define Atkin-Lehner operators on $\mathbb{Q}\mathcal{M}_0(\mathfrak{n})$. Later we will consider the complexification $\mathbb{C}\mathcal{M}$, for example when considering eigenvectors for the Hecke action, but we start with rational scalars, since Hecke operators respect the rational structure.

Next we will discuss the notion of an “eigenvalue system” for $\mathbb{T}$. Then we define “formal modular forms” for $\Gamma_0(\mathfrak{n})$ or $\Gamma_1(\mathfrak{n})$ to be functions on $\mathcal{M}$ satisfying certain transformation properties; in general these functions are complex vector-valued, i.e. they are functions $\mathcal{M} \to \mathbb{C}^d$ for some $d \geq 1$. Putting these together we can define the action of the Hecke algebra $\mathbb{T}$ on the set of formal modular forms. Then, using the admissible bases of the previous section we can view formal modular forms as functions on $\mathcal{G}$ which are left-invariant under $\Gamma_0(\mathfrak{n})$ or $\Gamma_1(\mathfrak{n})$ and which satisfy certain other transformation properties, bringing the theory closer to more traditional approaches.

3.1. **Formal Hecke operators.** We will only sketch the proofs of the main results here, since they are almost identical to those for the classical theory; full details were given in Bygott's thesis [2], though our notation differs slightly from there.

First we make a general observation. If (L, L') is a $\Gamma_0(\mathfrak{n})$-modular point and $M \supseteq L$ is a lattice with index $[M : L] = \mathfrak{a}$, we can attempt to construct a new modular point (M, M') by setting $M' = M + L'$. This is certainly valid when $\mathfrak{a}$ is coprime to $\mathfrak{n}$, since then $L' \cap M = L$, so that $M'/M \cong L'/L' \cap M = L'/L \cong \mathcal{O}_K/\mathfrak{n}$, but not in general. In this situation when we talk of the modular point (M, M') we will always mean the one with $M' = M + L'$, assuming that this is a valid modular point. Similarly for $\Gamma_1(\mathfrak{n})$-modular points.

Each of the following operators on $\mathbb{Q}\mathcal{M}_1(\mathfrak{n})$ will be defined by specifying the image of each modular point in $\mathcal{M}_1(\mathfrak{n})$ as a $\mathbb{Q}$-linear combination of modular points, and extending by $\mathbb{Q}$-linearity:

- an operator $T_{\mathfrak{a},\mathfrak{a}}$ for each fractional ideal $\mathfrak{a}$ coprime to $\mathfrak{n}$;
- an operator $T_{\mathfrak{a}}$ for each integral ideal $\mathfrak{a}$;
- an operator $[d]$ for each $d \in K^\times$ coprime to $\mathfrak{n}$.

In each case, we also obtain an operator on $\mathbb{Q}\mathcal{M}_0(\mathfrak{n})$ (which is trivial in the case of the $[d]$), by ignoring the action on the third component of modular points. Hence, in these definition, we may restrict our attention to $\mathcal{M} = \mathcal{M}_1(\mathfrak{n})$.

We will also define

- an Atkin-Lehner operator $W_{\mathfrak{q}}$ for each $\mathfrak{q} \mid \mathfrak{n}$ coprime to $\mathfrak{n}\mathfrak{q}^{-1}$,

on $\mathbb{Q}\mathcal{M}_0(\mathfrak{n})$, as linear extensions of maps $\mathcal{M}_0(\mathfrak{n}) \to \mathcal{M}_0(\mathfrak{n})$.

Definition 6 (The operators $T_{\mathfrak{a}}$). For each integral ideal $\mathfrak{a}$, the operator $T_{\mathfrak{a}}$ is defined by

$$T_{\mathfrak{a}}(L, L', \beta) = N(\mathfrak{a})^{-1} \sum_{\substack{M \supseteq L \\ [M:L]=\mathfrak{a} \\ (M,M',\beta)\in\mathcal{M}}} (M, M', \beta).$$

Here the sum is over all (M, M', β) such that M is a super-lattice of L with $[M : L] = \mathfrak{a}$ and $M' = M+L'$, in which case it is automatic that β generates M'/M. This sum is certainly finite, since $L \subseteq M \subseteq \mathfrak{a}^{-1}L$ and $\mathfrak{a}^{-1}L/L$ is finite, so has only finitely many submodules.

Definition 7 (The operators $T_{\mathfrak{a},\mathfrak{a}}$). Here, the definition for $\mathcal{M}_0(\mathfrak{n})$ is simpler. We define $T_{\mathfrak{a},\mathfrak{a}} : \mathbb{Q}\mathcal{M}_0(\mathfrak{n}) \to \mathbb{Q}\mathcal{M}_0(\mathfrak{n})$ for all fractional ideals $\mathfrak{n}$, by setting

$$T_{\mathfrak{a},\mathfrak{a}}(L, L') = N(\mathfrak{a})^{-2}(\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L')$$

and extending by linearity.

To extend this definition to $\mathbb{Q}\mathcal{M}_1(\mathfrak{n})$, it is necessary to restrict to fractional ideals $\mathfrak{a}$ which are coprime to $\mathfrak{n}$. Note that for each $(L, L', \beta) \in \mathcal{M}_1$, we may assume that $\beta \in L' \cap \mathfrak{a}^{-1}L'$, on replacing β by $u\beta$ where $u \in \mathcal{O}_K$ satisfies $u \equiv 1 \pmod{\mathfrak{n}}$ and $u \in \mathfrak{a}^{-1}$; such a u exists since $\mathcal{O}_K \cap \mathfrak{a}^{-1}$ and $\mathfrak{n}$ are coprime.

Lemma 3.1. *Let* $(L, L', \beta) \in \mathcal{M}_1(\mathfrak{n})$. *If* $\beta \in \mathfrak{a}^{-1}L'$, *then also* $(\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L', \beta) \in \mathcal{M}_1(\mathfrak{n})$.

Proof. To check that $(\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L', \beta) \in \mathcal{M}_1(\mathfrak{n})$, we must show that $\mathfrak{a}^{-1}L' = \mathcal{O}_K\beta + \mathfrak{a}^{-1}L$, so that β generates $\mathfrak{a}^{-1}L'/\mathfrak{a}^{-1}L$.

We first claim that $L' = (\mathcal{O}_K \cap \mathfrak{a})L' + L$. Since $\mathfrak{n}L' \subseteq L \subseteq L'$, and $\mathfrak{a}$ and $\mathfrak{n}$ are coprime, we have

$$L' = (\mathcal{O}_K \cap \mathfrak{a})L' + \mathfrak{n}L' \subseteq (\mathcal{O}_K \cap \mathfrak{a})L' + L \subseteq L' + L = L'.$$

Next, given that $L' = \mathcal{O}_K\beta + L$ and $\mathfrak{a}\beta \subseteq L'$, we have $L' = (\mathcal{O}_K + \mathfrak{a})\beta + L$, and hence

$$L' = (\mathcal{O}_K \cap \mathfrak{a})(\mathcal{O}_K + \mathfrak{a})\beta + ((\mathcal{O}_K \cap \mathfrak{a}) + \mathcal{O}_K)L = \mathfrak{a}\beta + L.$$

This implies that $\mathfrak{a}^{-1}L' = \mathcal{O}_K\beta + \mathfrak{a}^{-1}L$, as required. □

Assuming the condition $\beta \in L' \cap \mathfrak{a}^{-1}L'$, setting

$$T_{\mathfrak{a},\mathfrak{a}}(L, L', \beta) = N(\mathfrak{a})^{-2}(\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L', \beta)$$

gives a well-defined operator $T_{\mathfrak{a},\mathfrak{a}} : \mathbb{Q}\mathcal{M}_1(\mathfrak{n}) \to \mathbb{Q}\mathcal{M}_1(\mathfrak{n})$ for all fractional ideals coprime to $\mathfrak{n}$, *i.e.*, for $\mathfrak{a} \in \mathcal{J}_K^{\mathfrak{n}}$. From the definition, it is clear that $T_{\mathcal{O}_K,\mathcal{O}_K}$ is the identity, and that for $\mathfrak{a}, \mathfrak{b} \in \mathcal{J}_K^{\mathfrak{n}}$ we have

$$T_{\mathfrak{ab},\mathfrak{ab}} = T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}}.$$

Definition 8 (The operators $[d]$). Let $d \in \mathcal{O}_K$ be coprime to $\mathfrak{n}$. The operator $[d]$ is defined by

$$[d](L, L', \beta) = (L, L', d\beta).$$

This is valid since multiplication by d induces an automorphism of L'/L. Since $d\beta \pmod{L}$ only depends on $d \pmod{\mathfrak{n}}$ we may regard $[d]$ as defined for $d \in (\mathcal{O}_K/\mathfrak{n})^\times$, and hence extend it to all $d \in K^\times$ coprime to $\mathfrak{n}$.

An alternative for the definition is to set $[d](L, L', \beta) = (L, L', \beta)\gamma_d$ where $\gamma_d \in \Gamma_0(\mathfrak{n})$ satisfies $\epsilon(\gamma_d) = d$. Hence $[d]$ permutes the $\Gamma_1(\mathfrak{n})$-modular points above each $\Gamma_0(\mathfrak{n})$-modular point, and acts trivially on the latter.

Definition 9 (Atkin-Lehner operators $W_{\mathfrak{q}}$). For each exact divisor $\mathfrak{q}$ of the level $\mathfrak{n}$, we define $W_{\mathfrak{q}} : \mathbb{Q}\mathcal{M}_0(\mathfrak{n}) \to \mathbb{Q}\mathcal{M}_0(\mathfrak{n})$ as follows. For $(L, L') \in \mathcal{M}_0(\mathfrak{n})$, set

$$W_{\mathfrak{q}}(L, L') = N(\mathfrak{q})^{-1}(L + \mathfrak{q}'L', \mathfrak{q}^{-1}L + L').$$

Remark. We have $(L, L') = (L_1 \cap L_2, L_1 + L_2)$ where $L_1 = L + \mathfrak{q}L'$ and $L_2 = L + \mathfrak{q}'L'$. Hence $L_1/L \cong L'/L_2 \cong \mathcal{O}_K/\mathfrak{q}$ and $L_2/L \cong L'/L_1 \cong \mathcal{O}_K/\mathfrak{q}'$, and $W_{\mathfrak{q}}(L, L') = N(\mathfrak{q})^{-1}(L_2, \mathfrak{q}^{-1}L_1)$.

With respect to bases, if $L = (\mathfrak{b}_1 \oplus \mathfrak{b}_2)U$ and $L' = (\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U$, then

$$(L + \mathfrak{q}'L', \mathfrak{q}^{-1}L + L') = ((\mathfrak{b}_1 \oplus \mathfrak{q}^{-1}\mathfrak{b}_2)U, (\mathfrak{q}^{-1}\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2)U);$$

this is again in $\mathcal{M}_0(\mathfrak{n})$ since $(\mathfrak{b}_1 \oplus \mathfrak{q}^{-1}\mathfrak{b}_2)/(\mathfrak{q}^{-1}\mathfrak{b}_1 \oplus \mathfrak{n}^{-1}\mathfrak{b}_2) \cong \mathcal{O}_K/\mathfrak{n}$.

As a special case, with $\mathfrak{q} = \mathfrak{n}$ we obtain the *Fricke operator* $W_{\mathfrak{n}}$ which maps $(L, L') \mapsto N(\mathfrak{n})^{-1}(L', \mathfrak{n}^{-1}L)$.

3.1.1. *Relations between the operators.*

Proposition 3.2. *The operators $T_{\mathfrak{a}}$ and $T_{\mathfrak{a},\mathfrak{a}}$ satisfy the following identities:*

(1) $T_{\mathcal{O}_K} = T_{\mathcal{O}_K,\mathcal{O}_K} =$*identity.*

(2) *For all $\mathfrak{a}, \mathfrak{b}$ coprime to $\mathfrak{n}$,*

$$T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}} = T_{\mathfrak{ab},\mathfrak{ab}} = T_{\mathfrak{b},\mathfrak{b}}T_{\mathfrak{a},\mathfrak{a}}.$$

(3) *For all $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a}$ coprime to $\mathfrak{n}$,*

$$T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}} = T_{\mathfrak{b}}T_{\mathfrak{a},\mathfrak{a}}.$$

(4) *If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime then*

$$T_{\mathfrak{a}}T_{\mathfrak{b}} = T_{\mathfrak{ab}} = T_{\mathfrak{b}}T_{\mathfrak{a}},$$

and in particular $T_{\mathfrak{a}}$ and $T_{\mathfrak{b}}$ commute.

(5) *If $\mathfrak{p}$ is a prime dividing $\mathfrak{n}$ then for all $n \geq 1$,*

$$T_{\mathfrak{p}^n} = (T_{\mathfrak{p}})^n.$$

(6) *If $\mathfrak{p}$ is a prime not dividing $\mathfrak{n}$ then for all $n \geq 1$,*

$$T_{\mathfrak{p}^n} T_{\mathfrak{p}} = T_{\mathfrak{p}^{n+1}} + N(\mathfrak{p}) T_{\mathfrak{p}^{n-1}} T_{\mathfrak{p},\mathfrak{p}}.$$

Proof. In all cases it suffices to compare the action on $\Gamma_0(\mathfrak{n})$-modular points since it is clear that the condition on β is satisfied.

(1) and (2) are clear. For (3), both sides map (L, L') to $N(\mathfrak{a}^2\mathfrak{b})^{-1}$ times the sum of the modular points $(M, M') = (\mathfrak{a}^{-1}M_1, \mathfrak{a}^{-1}M_1')$ such that $[M : \mathfrak{a}^{-1}L] = [\mathfrak{a}M : L] = [M_1 : L] = \mathfrak{b}$.

For (4), first observe that each tower $L \subseteq M \subseteq N$ with $[M : L] = \mathfrak{a}$ and $[N : M] = \mathfrak{b}$ such that $(M, M') = (M, L' + M) \in \mathcal{M}$ and $(N, N') = (N, M' + N) \in \mathcal{M}$ gives $L \subseteq N$ with $[N : L] = \mathfrak{ab}$ and $(N, L' + N) \in \mathcal{M}$. Conversely, every extension $L \subseteq N$ with $[N : L] = \mathfrak{ab}$ and $(N, N') = (N, L' + N) \in \mathcal{M}$ arises uniquely this way for a suitable intermediate M, which automatically satisfies $(M, L' + M) \in \mathcal{M}$.

For (5) we show by induction that $T_{\mathfrak{p}^{n+1}} = T_{\mathfrak{p}} T_{\mathfrak{p}^n}$ for $n \geq 1$. The only part which is not quite obvious is that to each extension $L \subseteq N$ with $[N : L] = \mathfrak{p}^{n+1}$ with $(N + L')/N \cong \mathcal{O}_K/\mathfrak{n}$, there is a unique intermediate M with $[M : L] = \mathfrak{p}$ and $(M + L')/L \cong \mathcal{O}_K/\mathfrak{n}$. The fact that $(N + L')/N$ has exact annihilator $\mathfrak{n}$ and $\mathfrak{p} \mid \mathfrak{n}$ implies that $L \not\subseteq \mathfrak{p}N$, and the unique such M is $M = L + \mathfrak{p}N$.

For (6), each side when applied to $(L, L') \in \mathcal{M}$ is a linear combination of (M, M') where $[M : L] = \mathfrak{p}^{n+1}$. If the multiplicity of (M, M') in $T_{\mathfrak{p}^n} T_{\mathfrak{p}}(L, L')$ is a and in $T_{\mathfrak{p}^{n-1}} T_{\mathfrak{p},\mathfrak{p}}(L, L')$ is c then the identity follows from $a = 1 + N(\mathfrak{p})c$ since the multiplicity in $T_{\mathfrak{p}^{n+1}}(L, L')$ is 1 by definition. One checks that either $M \supseteq \mathfrak{p}^{-1}L$, in which case $c = 1$ and $a = N(\mathfrak{p}) + 1$; or $M \not\supseteq \mathfrak{p}^{-1}L$, in which case $c = 0$ and $a = 1$. $\square$

Corollary 3.3. *The $\mathbb{Q}$-algebra $\mathbb{T}$ of formal Hecke operators on $\mathbb{Q}\mathcal{M}$ generated by $T_{\mathfrak{a}}$ (for all integral ideals $\mathfrak{a}$) and $T_{\mathfrak{a},\mathfrak{a}}$ (for all fractional ideals $\mathfrak{a}$ coprime to $\mathfrak{n}$) is commutative and generated (as a $\mathbb{Q}$-algebra) by $T_{\mathfrak{p}}$ (for all primes $\mathfrak{p}$) and $T_{\mathfrak{p},\mathfrak{p}}$ (for all primes $\mathfrak{p} \nmid \mathfrak{n}$). As a module, $\mathbb{T}$ is generated by the operators of the form $T_{\mathfrak{a},\mathfrak{a}} T_{\mathfrak{b}}$.* $\square$

The multiplicative relations between these operators may be summarised in terms of a formal Dirichlet series:

Corollary 3.4. *We have the formal identity between the formal Hecke operators:*

$$\sum_{\mathfrak{a}} T_{\mathfrak{a}} N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}\mid\mathfrak{n}} (1 - T_{\mathfrak{p}} N(\mathfrak{p})^{-s})^{-1} \prod_{\mathfrak{p}\nmid\mathfrak{n}} (1 - T_{\mathfrak{p}} N(\mathfrak{p})^{-s} + T_{\mathfrak{p},\mathfrak{p}} N(\mathfrak{p})^{1-2s})^{-1}.$$

Here the sum on the left is over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$, and the product on the right is over all prime ideals $\mathfrak{p}$. $\square$

If we were to define $T_{\mathfrak{a},\mathfrak{a}} = 0$ for $\mathfrak{a}$ not coprime to $\mathfrak{n}$, and in particular for $\mathfrak{a} = \mathfrak{p}$ with $\mathfrak{p} \mid \mathfrak{n}$, then we would not have to separate the two kinds of primes in the above product, and also Proposition 3.2(6) would apply to all primes.

Properties (1) and (2) imply that the $T_{\mathfrak{a},\mathfrak{a}}$ are invertible and form a subgroup of the group of invertible elements of $\mathbb{T}$, namely the image of the group homomorphism $\mathfrak{a} \mapsto T_{\mathfrak{a},\mathfrak{a}}$.

3.1.2. *Atkin-Lehner relations.* A short calculation shows that

$$W_{\mathfrak{q}}^2 = T_{\mathfrak{q},\mathfrak{q}},$$

since both map (L, L') to $N(\mathfrak{q})^{-2}(\mathfrak{q}^{-1}L, \mathfrak{q}^{-1}L')$. More generally, suppose that for $i = 1, 2$ we have divisors $\mathfrak{q}_i \mid \mathfrak{n}$ coprime to $\mathfrak{n}\mathfrak{q}_i^{-1}$; then

$$W_{\mathfrak{q}_1} W_{\mathfrak{q}_2} = T_{\mathfrak{q},\mathfrak{q}} W_{\mathfrak{q}_3},$$

where $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$ (the greatest common divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$) and $\mathfrak{q}_3 = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}^{-2}$.

3.2. Alternative normalisations.

Scaling. It is possible to define the operators $T_{\mathfrak{a}}$ and $T_{\mathfrak{a},\mathfrak{a}}$ without the scaling factors $N(\mathfrak{a})^{-1}$ and $N(\mathfrak{a})^{-2}$ respectively. The effect on the formal Dirichlet series $\sum T_{\mathfrak{a}} N(\mathfrak{a})^{-s}$ would be to shift the variable s by 1. This is the normalization used by Bygott in [2]. Similarly for $W_{\mathfrak{q}}$.

Using sublattices. In our definition of Hecke operators (see Definitions 6 and 7 above), we map each lattice to one or more superlattices; this follows the treatment (for $K=\mathbb{Q}$) by Lang [13] and Koblitz [12]. One can alternatively use sublattices instead, as in Serre [15] (for $K=\mathbb{Q}$ and level 1). The latter approach is slightly more convenient for computation, as the operators can be described explicitly in terms of 2 matrices with entries in $\mathcal{O}_K$, as we show later in Sections 5 and 6. We briefly describe this second approach here.

We define a new operator $\tilde{T}_{\mathfrak{a}}$, for ideals $\mathfrak{a}$ coprime to $\mathfrak{n}$ only, as follows. First observe that whenever $L \subseteq M$ with $[M:L]=\mathfrak{a}$, then $L \subseteq M \subseteq \mathfrak{a}^{-1}L$ with also $[\mathfrak{a}^{-1}L : M]=\mathfrak{a}$, and hence $\mathfrak{a}L \subseteq \mathfrak{a}M \subseteq L$ where $[L:\mathfrak{a}M]=\mathfrak{a}$. Thus to each superlattice $M \supseteq L$ with index $\mathfrak{a}$ there corresponds a sublattice $\mathfrak{a}M \subseteq L$ also with index $\mathfrak{a}$. This correspondence extends to modular points in either $\mathcal{M}_0(\mathfrak{n})$ or $\mathcal{M}_1(\mathfrak{n})$.

Now we set

$$\tilde{T}_{\mathfrak{a}}(L,L',\beta) = N(\mathfrak{a}) \sum_{\substack{M \subseteq L \\ [L:M]=\mathfrak{a} \\ (M,M',\beta') \in \mathcal{M}}} (M,M',\beta').$$

Here, $M' = M + \mathfrak{a}L'$, and the term (M,M',β') is only included when $M'/M \cong \mathcal{O}_K/\mathfrak{n}$; also, $\beta' = u\beta \in \mathfrak{a}L' \subseteq M'$ where $u \in \mathfrak{a}$ and $u \equiv 1 \pmod{\mathfrak{n}}$. A simple calculation using the correspondence between superlattices and sublattices shows that

$$\tilde{T}_{\mathfrak{a}} = T_{\mathfrak{a},\mathfrak{a}}^{-1} T_{\mathfrak{a}}.$$

For $\mathfrak{a}$ coprime to the level $\mathfrak{n}$, set $\tilde{T}_{\mathfrak{a},\mathfrak{a}} = T_{\mathfrak{a}^{-1},\mathfrak{a}^{-1}}$, so that $\tilde{T}_{\mathfrak{a},\mathfrak{a}}((L,L',\beta)) = N(\mathfrak{a})^2(\mathfrak{a}L,\mathfrak{a}L',\beta')$ with β' defined as above.

These new operators satisfy all the relations listed above in Proposition 3.2 (omitting (5)).

Similarly, in the definition of Atkin-Lehner operators above in Definition 9 we map the modular point $(L,L') \in \mathcal{M}_0(\mathfrak{n})$ to $(L+\mathfrak{q}'L', \mathfrak{q}^{-1}L+L')$, where the lattice $L_2 = L+\mathfrak{q}'L'$ satisfies $L \subseteq L_2 \subseteq L'$. (Here, as above, $\mathfrak{n}=\mathfrak{q}\mathfrak{q}'$ with $\mathfrak{q}$, $\mathfrak{q}'$ coprime.) For the alternative normalization we define

$$\tilde{W}_{\mathfrak{q}}(L,L') = N(\mathfrak{q})(\mathfrak{q}L+\mathfrak{n}L', L+\mathfrak{q}L'),$$

noting that $\mathfrak{q}L+\mathfrak{n}L' \subseteq L$. We now have

$$\tilde{W}_{\mathfrak{q}}^2 = \tilde{T}_{\mathfrak{q},\mathfrak{q}}.$$

3.3. Grading the Hecke algebra.

Each of the operators T we have just defined has a well-defined *ideal class* $c=[T]$ such that $T(\mathcal{M}_{c'}) \subseteq \mathcal{M}_{c'c}$ for each $c' \in \mathrm{Cl}(K)$. From the definitions, we see that

- $[T_{\mathfrak{a}}] = [\mathfrak{a}]$;
- $[T_{\mathfrak{a},\mathfrak{a}}] = [\mathfrak{a}]^2$;
- $[W_{\mathfrak{q}}] = [\mathfrak{q}]$;
- $[\tilde{T}_{\mathfrak{a}}] = [\mathfrak{a}]^{-1}$;
- $[\tilde{T}_{\mathfrak{a},\mathfrak{a}}] = [\mathfrak{a}]^{-2}$;
- $[\tilde{W}_{\mathfrak{q}}] = [\mathfrak{q}]^{-1}$.

In explicit computations it is simplest to work with *principal* operators, whose class is trivial. The basic principal operators have the form $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}$ (or $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}W_{\mathfrak{q}}$) where $\mathfrak{a}^2\mathfrak{b}$ (or $\mathfrak{a}^2\mathfrak{b}\mathfrak{q}$) is principal; they map each $\mathcal{M}_c$ to itself. In Sections 5 and 6 we will describe the action of principal operators explicitly in terms of matrices in $M_2(K)$, as is done in the classical setting.

Note that $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}$ (respectively, $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}W_{\mathfrak{q}}$) is principal if and only if $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{b}}$ (respectively, $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{b}}\tilde{W}_{\mathfrak{q}}$) is.

Each element of $\mathbb{T}$ is (non-uniquely) a linear combination of the basic operators. Since in each of the relations listed in Proposition 3.2 all terms have the same class, it follows that we have a well-defined decomposition

$$\mathbb{T} = \bigoplus_{c\in \mathrm{Cl}(K)} \mathbb{T}_c$$

where $\mathbb{T}_c$ is spanned by all basic operators with class c. By an operator of class c, we now mean any element of $\mathbb{T}_c$; thus, every Hecke operator is uniquely expressible as a sum of operators, one in each graded component $\mathbb{T}_c$, and operators $T \in \mathbb{T}_c$ are characterized by the property that

$$T(\mathbb{Z}\mathcal{M}_{c'}) \subseteq \mathbb{Z}\mathcal{M}_{cc'} \qquad \forall c' \in \mathrm{Cl}(K).$$

This decomposition of the algebra $\mathbb{T}$ is a grading by the Abelian group $\mathrm{Cl}(K)$, on account of the following.

Proposition 3.5.

$$\mathbb{T}_{c_1}\mathbb{T}_{c_1} \subseteq \mathbb{T}_{c_1c_2}.$$

Proof. The statement follows immediately from the preceding characterization of $\mathbb{T}_c$, or simply from the definition of the class of a basic operator. □

Corollary 3.6. *The component $\mathbb{T}_1$, where 1 denotes the trivial ideal class, is a subalgebra of $\mathbb{T}$. More generally, if $H \le \mathrm{Cl}(K)$ is any subgroup then $\mathbb{T}_H := \oplus_{c\in H}\mathbb{T}_c$ is a subalgebra of $\mathbb{T}$.* □

3.4. **Eigenvalue systems.** If we have a vector space on which $\mathbb{T}$ acts linearly, such as $\mathbb{C}\mathcal{M}$, then we may consider simultaneous eigenvectors for all $T \in \mathbb{T}$; to such an eigenvector we can associate a system of eigenvalues, one for each $T \in \mathbb{T}$, and this determines an algebra homomorphism $\mathbb{T} \to \mathbb{C}$. In this section we study such eigenvalue systems from a purely formal point of view without considering any particular $\mathbb{T}$-action.

Definition 10. An *eigenvalue system of level* $\mathfrak{n}$ is an algebra homomorphism

$$\lambda : \mathbb{T} \to \mathbb{C}.$$

To specify an eigenvalue system we must give the values $\lambda(\mathfrak{a}) = \lambda(T_{\mathfrak{a}})$ for all ideals $\mathfrak{a}$ and $\lambda(\mathfrak{a},\mathfrak{a}) = \lambda(T_{\mathfrak{a},\mathfrak{a}})$ for all ideals $\mathfrak{a}$ coprime to $\mathfrak{n}$, where these values are arbitrary complex numbers satisfying the multiplicative relations (numbered as in Proposition 3.2)

(1) $\lambda(\mathcal{O}_K) = \lambda(\mathcal{O}_K, \mathcal{O}_K) = 1$.
(2) For all $\mathfrak{a}, \mathfrak{b}$,
$$\lambda(\mathfrak{a},\mathfrak{a})\lambda(\mathfrak{b},\mathfrak{b}) = \lambda(\mathfrak{a}\mathfrak{b},\mathfrak{a}\mathfrak{b}).$$
(3) (trivial) $\lambda(\mathfrak{a},\mathfrak{a})\lambda(\mathfrak{b}) = \lambda(\mathfrak{b})\lambda(\mathfrak{a},\mathfrak{a})$.
(4) If $\mathfrak{a}$ and $\mathfrak{b}$ are coprime then
$$\lambda(\mathfrak{a})\lambda(\mathfrak{b}) = \lambda(\mathfrak{a}\mathfrak{b}).$$
(5) If $\mathfrak{p}$ is a prime dividing $\mathfrak{n}$ then for all $n \ge 1$,
$$\lambda(\mathfrak{p}^n) = \lambda(\mathfrak{p})^n.$$
(6) If $\mathfrak{p}$ is a prime not dividing $\mathfrak{n}$ then for all $n \ge 1$,
$$\lambda(\mathfrak{p}^n)\lambda(\mathfrak{p}) = \lambda(\mathfrak{p}^{n+1}) + N(\mathfrak{p})\lambda(\mathfrak{p}^{n-1})\lambda(\mathfrak{p},\mathfrak{p}).$$

In particular, restriction of λ to the operators $T_{\mathfrak{a},\mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{J}_K^{\mathfrak{n}}$ gives a group homomorphism $\chi : \mathcal{J}_K^{\mathfrak{n}} \to \mathbb{C}^*$ via $\chi(\mathfrak{a}) = \lambda(\mathfrak{a},\mathfrak{a}) = \lambda(T_{\mathfrak{a},\mathfrak{a}})$. It is for this reason that we defined $T_{\mathfrak{a},\mathfrak{a}}$ for fractional ideals, since they form a group rather than just a monoid, which forces the values $\lambda(\mathfrak{a},\mathfrak{a})$ to be nonzero.

We call χ the *character* of the eigenvalue system λ. For convenience we may set $\chi(\mathfrak{a}) = \lambda(\mathfrak{a},\mathfrak{a}) = 0$ when $\mathfrak{a}$ is not coprime to $\mathfrak{n}$. Then these multiplicative relations may be expressed via the formal Dirichlet series and Euler product

$$\sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} (1 - \lambda(\mathfrak{p}) N(\mathfrak{p})^{-s} + \chi(\mathfrak{p}) N(\mathfrak{p})^{1-2s})^{-1}.$$

If λ is an eigenvalue system, and ψ is any another character of $\mathcal{J}_K^{\mathfrak{n}}$, then we can define the *twist* $\lambda \otimes \psi$ of λ by ψ to be the eigenvalue system defined by

$$(\lambda \otimes \psi)(\mathfrak{a}) = \psi(\mathfrak{a})\lambda(\mathfrak{a}); \qquad (\lambda \otimes \psi)(\mathfrak{a},\mathfrak{a}) = \psi(\mathfrak{a})^2 \lambda(\mathfrak{a},\mathfrak{a}).$$

It would be possible to restrict attention to characters of $\mathcal{J}_K^{\mathfrak{n}}$ with additional properties, such as ray class characters modulo $\mathfrak{n}$, which are trivial on $\mathcal{P}_K^{\mathfrak{n}}$ (the principal ideals $\langle a \rangle$ with $a \equiv 1 \pmod{\mathfrak{n}}$), and hence may be viewed as characters of the ray class group $\mathcal{J}_K^{\mathfrak{n}}/\mathcal{P}_K^{\mathfrak{n}}$; or unramified characters, which are trivial on all principal ideals, and hence may be viewed as characters of the ideal class group.

Clearly, if λ has character χ then $\lambda \otimes \psi$ has character $\chi\psi^2$. In particular, the twist of λ by a quadratic character ψ has the same character as λ. In this situation, it is possible that $\lambda \otimes \psi = \lambda$, in which case $\lambda(\mathfrak{a}) = 0$ for all $\mathfrak{a}$ such that $\psi(\mathfrak{a}) = -1$.

3.5. **Restriction to principal operators.** In practical situations, where we are computing the action of Hecke operators explicitly on some finite-dimensional vector space, it is much simpler to deal with principal operators, i.e. those in $\mathbb{T}_1$, as these may be expressed in terms of 2×2 matrices in $\mathcal{M}_2(K)$ (see Section 5 below). This leads us to consider the question, to what extent is an eigenvalue system λ determined by its restriction to $\mathbb{T}_1$? We will give a complete theoretical answer to this question, and then make some remarks about how to handle the situation in practice, if we can only compute principal Hecke operators but wish to determine the entire eigenvalue system.

The first step is to consider the extent to which the character χ of an eigenvalue system λ (with character) is determined by the restriction of λ to $\mathbb{T}_1$. Since $\chi(\mathfrak{a}) = \lambda(T_{\mathfrak{a},\mathfrak{a}})$ and $T_{\mathfrak{a},\mathfrak{a}} \in \mathbb{T}_1$ if and only if $\mathfrak{a}^2$ is principal, the restriction of λ to $\mathbb{T}_1$ only determines the restriction of χ to the subgroup H of $\mathcal{J}_K^{\mathfrak{n}}$ consisting of ideals whose square is principal. In general the number of characters with the same restriction to H is the index of H, namely h/h_2 where $h_2 = \#\,\mathrm{Cl}[2]$. More explicitly, it is not hard to see that the characters which are trivial on elements of order 2 are precisely those which are squares.

Hence, if λ' with character χ' has the same restriction to $\mathbb{T}_1$, then $\chi' = \chi\psi^2$ for some character ψ. Since ψ is trivial on principal ideals it is unramified and we may view it as a character of the ideal class group. Now $\lambda \otimes \psi$ has both the same restriction to $\mathbb{T}_1$ and the same character as λ'. Note that we can multiply ψ here by any unramified quadratic character without affecting the character of $\lambda \otimes \psi$. Replacing λ by $\lambda \otimes \psi$, we may now assume that λ and λ' have the same character, and for any unramified quadratic character ψ the same is true for $\lambda' \otimes \psi$ in place of λ'. Finally, we show that for a suitable quadratic ψ we in fact have $\lambda = \lambda' \otimes \psi$.

Certainly λ and λ' now agree on all $T_{\mathfrak{a},\mathfrak{a}}$ since these values are given by χ. Moreover, for all ideals $\mathfrak{a}$ whose class is a square we have $\lambda(\mathfrak{a}) = \lambda'(\mathfrak{a})$: for if we take $\mathfrak{b}$ to be an ideal coprime to $\mathfrak{n}$ such that $\mathfrak{a}\mathfrak{b}^2$ is principal, then

$$\lambda'(\mathfrak{a})\chi(\mathfrak{b}) = \lambda'(\mathfrak{a})\lambda'(\mathfrak{b},\mathfrak{b}) = \lambda'(T_{\mathfrak{a}} T_{\mathfrak{b},\mathfrak{b}}) = \lambda(T_{\mathfrak{a}} T_{\mathfrak{b},\mathfrak{b}}) = \lambda(\mathfrak{a})\chi(\mathfrak{b}),$$

since $T_{\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}} \in \mathbb{T}_1$, so $\lambda'(\mathfrak{a}) = \lambda(\mathfrak{a}) = \lambda(T_{\mathfrak{a}}T_{\mathfrak{b},\mathfrak{b}})/\chi(\mathfrak{b})$. Thus the restrictions of λ and λ' to $\mathbb{T}_{H_2}$ are equal, where H_2 is the subgroup of squares in the class group.

The restriction of λ to $\mathbb{T}_c$ for any square class c cannot be identically zero, since these classes contain the operators $T_{\mathfrak{a},\mathfrak{a}}$ on which $\lambda(T_{\mathfrak{a},\mathfrak{a}}) = \chi(\mathfrak{a}) \neq 0$. However, it is possible that for a non-square class c we may have $\lambda(T) = 0$ for all $T \in \mathbb{T}_c$. If the restrictions of λ to two classes c_1 and c_2 are both not identically zero then also its restriction to class $\mathbb{T}_{c_1c_2}$ is nonzero, so the set of classes c such that λ is not identically zero on c forms a subgroup H_0 of $\mathrm{Cl}(K)$ containing the subgroup of squares H_2.

We now compare the values of λ and λ' on ideals $\mathfrak{a}$ in non-square classes. We always have

$$\lambda(\mathfrak{a})^2 = \lambda(T_{\mathfrak{a}}^2) = \lambda'(T_{\mathfrak{a}}^2) = \lambda'(\mathfrak{a})^2,$$

so $\lambda'(\mathfrak{a}) = \pm\lambda(\mathfrak{a})$. We show that these signs are the values $\psi(\mathfrak{a})$ of a quadratic character ψ.

There is a potential difficulty caused by the possibility that $\lambda(\mathfrak{a}) = 0$; otherwise we could just define $\psi(\mathfrak{a}) = \lambda(\mathfrak{a})/\lambda'(\mathfrak{a})$ and prove that ψ is a character. However, the set of classes on which λ' is not identically zero are the same as those for λ, namely the classes in H_0. For each $c \in H_0$, let $\mathfrak{a} \in c$ be such that $\lambda(\mathfrak{a}) \neq 0$, and define $\psi(\mathfrak{a}) = \lambda(\mathfrak{a})/\lambda'(\mathfrak{a}) = \pm 1$. Now we have

- $\psi(\mathfrak{a}) = 1$ if $c \in H_2$.
- $\psi(\mathfrak{a}) = \psi(\mathfrak{a}')$ for all $\mathfrak{a}' \in c$ with $\lambda(\mathfrak{a}') \neq 0$. For then $T_{\mathfrak{a}}T_{\mathfrak{a}'} \in \mathbb{T}_{c^2}$; hence $\lambda(\mathfrak{a})\lambda(\mathfrak{a}') = \lambda(T_{\mathfrak{a}})\lambda(T_{\mathfrak{a}'}) = \lambda(T_{\mathfrak{a}}T_{\mathfrak{a}'}) = \lambda'(T_{\mathfrak{a}}T_{\mathfrak{a}'}) = \lambda'(\mathfrak{a})\lambda'(\mathfrak{a}')$, so $\psi(\mathfrak{a}') = \lambda(\mathfrak{a}')/\lambda'(\mathfrak{a}') = \lambda'(\mathfrak{a})/\lambda(\mathfrak{a}) = \psi(\mathfrak{a})$. (Here it is useful to remember that λ is completely multiplicative as a function on Hecke operators, but not as a function on ideals, since in general $T_{\mathfrak{a}}T_{\mathfrak{a}'} \neq T_{\mathfrak{a}\mathfrak{a}'}$; this is why we use $T_{\mathfrak{a}}T_{\mathfrak{a}'}$ here rather than $T_{\mathfrak{a}\mathfrak{a}'}$.)
- For $i = 1, 2$ let $\mathfrak{a}_i \in c_i$; without loss we may assume that $\mathfrak{a}_1, \mathfrak{a}_2$ are coprime, so that $\lambda(\mathfrak{a}_1\mathfrak{a}_2) = \lambda(\mathfrak{a}_1)\lambda(\mathfrak{a}_2)$ and similarly for λ'; then

$$\psi(\mathfrak{a}_1)\psi(\mathfrak{a}_2) = \frac{\lambda(\mathfrak{a}_1)}{\lambda'(\mathfrak{a}_1)}\frac{\lambda(\mathfrak{a}_2)}{\lambda'(\mathfrak{a}_2)} = \frac{\lambda(\mathfrak{a}_1\mathfrak{a}_2)}{\lambda'(\mathfrak{a}_1\mathfrak{a}_2)} = \psi(\mathfrak{a}_1\mathfrak{a}_2).$$

Hence ψ induces a well-defined quadratic character on H_0, trivial on H_2, such that $\lambda' = \lambda \otimes \psi$.

We have only defined ψ as a character of H_0, which may be a proper subgroup of $\mathrm{Cl}(K)$. In that case, ψ has more than one (in fact precisely $[\mathrm{Cl}(K) : H_0]$) extensions to $\mathrm{Cl}(K)$ and hence to an unramified character. For each such extension, the value of $\lambda \otimes \psi$ is the same. In other words, $\lambda = \lambda \otimes \psi'$ for all quadratic characters ψ' whose restrictions to H_0 are trivial; the existence of non-trivial characters ψ' of $\mathrm{Cl}(K)/H_0$, or "inner twists" of λ, when $[\mathrm{Cl}(K) : H_0] > 1$, causes λ to be identically zero on all ideals in classes not in $\ker(\psi')$.

In summary we have shown that when λ and λ' have the same restriction to the subalgebra $\mathbb{T}_1$ of principal operators, then we may first twist λ by an unramified character so that its character matches that of λ', and then twist again by a quadratic unramified character to give λ' precisely without affecting the character of the system further. The overall twist required is unique except in case λ has inner twists, that is, quadratic twists which fix λ. In the latter case, the restriction of λ is identically zero on operators in classes forming the complement of a subgroup.

Proposition 3.7. *Let λ be an eigenvalue system of character χ. Then the eigenvalue systems μ whose restriction to $\mathbb{T}_1$ is the same as that of λ are precisely the twists $\lambda \otimes \psi$ as ψ runs through unramified characters, that is, characters of the class group* $\mathrm{Cl}(K)$*. These twists have the same character χ as λ if and only if ψ is quadratic.*

We single out two special cases which are in a sense at opposite extremes.

3.5.1. *Special case: odd class number.* When $\mathrm{Cl}(K)$ has odd order, the above situation simplifies somewhat, since all ideal classes are squares, the class group has no quadratic characters, and $H_2 = H_0 = \mathrm{Cl}(K)$ in the earlier notation. Now, if two eigenvalue systems have the same restriction to $\mathbb{T}_1$ and the same character then they are equal. More generally, if λ and λ' have the same restriction to $\mathbb{T}_1$ then $\lambda' = \lambda \otimes \psi$ for a unique unramified character ψ.

Concrete examples for this situation appear in Lingham's thesis [14] for the fields $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$, both of class number 3. There, the interest was in eigenvalue systems with trivial character, so all $T_{\mathfrak{a},\mathfrak{a}}$ act trivially, and the eigenvalue systems are uniquely determined by their principal eigenvalues. To compute the eigenvalue of $T_{\mathfrak{p}}$ in general in this case, one can choose $\mathfrak{a}$ coprime to $\mathfrak{n}$ such that $\mathfrak{a}^2\mathfrak{p}$ is principal and compute the operator $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}}$, which is principal, and can be described using a finite set of matrices in $\mathrm{Mat}_2(K)$; then $\lambda(\mathfrak{p}) = \lambda(T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{p}})$. See the examples at the end of Section 5.

3.5.2. *Special case: elementary Abelian* 2*-group.* In this case $\mathrm{Cl}(K)$ has exponent 2 and every square class is principal. Hence the restriction of λ to $\mathbb{T}_1$ completely determines its character, and determines λ itself up to an unramified quadratic twist. Each restricted eigenvalue system extends to (in general) h complete eigenvalue systems, forming a set of h unramified quadratic twists; in the presence of inner twists the number is a proper divisor of h, and the eigenvalues for operators in some classes is identically zero.

Examples for this situation appear in Bygott's thesis [2] for the field $\mathbb{Q}(\sqrt{-5})$, of class number 2. There, eigenvalues systems with any unramified character were considered, that is, both those with trivial character (called "plusforms" in [2]) and those with the unique unramified quadratic character (called "minusforms"). In general the eigenvalue systems come in pairs which are unramified quadratic twists of each other and have the same character, though there are examples of inner twist. For example, at level $\mathfrak{n} = \langle 8 \rangle$ there are two eigenvalue systems denoted f_{15} and f_{16} in [2], each of which is its own quadratic twist so has all eigenvalues for non-principal Hecke operators equal to 0; of these, f_{16} has trivial character, while f_{15} has nontrivial character.

Although we have not yet considered the question of rationality of the eigenvalues, it is interesting to note several cases in [2] where the eigenvalues of principal operators are rational integers while those of non-principal operators lie in a quadratic field. Both real and imaginary quadratic fields can occur, but only real fields for forms with trivial character, since the principal Hecke operators are all Hermitian in the context of [2] where the class number is 2.

3.5.3. *Special case: unramified character.* Consider eigenvalue systems whose character χ is unramified and so may be regarded as a character of the ideal class group. The restriction of such a system to $\mathbb{T}_1$ does not determine its character uniquely, only up multiplication by the square of a character, giving h/h_2 possible characters. Fixing the character, there are still h_2 possibilities for the eigenvalue system in general (fewer in the presence of inner twists).

4. Formal modular forms

We will define *formal modular forms* for $\Gamma_0(\mathfrak{n})$ or $\Gamma_1(\mathfrak{n})$ to be functions on modular points satisfying certain transformation properties with respect to the right action of the centre $\mathcal{Z}$ and the compact subgroup $\mathcal{K}$ of $\mathcal{G} = \mathcal{G}$, which is the product of the maximal compact subgroups in each component $\mathrm{GL}(2, K_{\infty,j})$. We will also define the action of the Hecke algebra on such functions.

From Corollary 2.4 and Proposition 2.5, it follows that there is a bijection between functions on modular points and collections of h functions on $\mathcal{G}$, indexed by the ideal classes c_{ij}, each left-invariant by $\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})$ for the appropriate index i. Hence each formal modular form induces a collection of functions on $\mathcal{G}$ which transform in a prescribed way under the action of $\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})$ on the left and $\mathcal{ZK}$ on the right. These in turn induce functions on the space $\mathcal{H}_2^r \times \mathcal{H}_3^s$, the product of r copies of the upper half-plane $\mathcal{H}_2$ and s copies of its three-dimensional analogue the upper half-space $\mathcal{H}_3$. (Recall that r, s are the numbers of real and complex places of the number field K.) To continue the development of the theory one would impose extra analytic conditions on these functions (such as holomorphicity for real components and harmonicity for complex components), in order to define modular, or automorphic, forms over K.

Some motivation. We will consider functions which are vector-valued. In essence, this is necessary in order to allow for a more general definition of the *weight* of a modular form. Classically, the weight of a modular form is an integer k; when interpreted in the language of automorphic representations, what happens is that the relevant representations of $\mathrm{GL}(2,\mathbb{R})$ are classified according to the induced representation on the compact subgroup $\mathrm{SO}(2)$. Since $\mathrm{SO}(2)$ is Abelian, its irreducible representations are all one-dimensional, indexed by integers k corresponding to the weight of the form. When considering fields with complex embeddings, however, the role of $\mathrm{GL}(2,\mathbb{R})$ is replaced by $\mathrm{GL}(2,\mathbb{C})$ and that of $\mathrm{SO}(2)$ by $\mathrm{SU}(2)$, which is compact but not Abelian. Its irreducible representations are not one-dimensional: there is one in dimension n for every $n \geq 1$, afforded by the action of 2×2 matrices on binary forms of degree $n-1$. The associated modular forms take values in $\mathbb{C}^n$.

Recall that $\widehat{K} = K \otimes_{\mathbb{Q}} \mathbb{R}$, and all our lattices are $\mathcal{O}_K$-submodules of $\widehat{K}^2 = \widehat{K} \oplus \widehat{K}$. Define $\mathcal{Z}$ to be the centre of $\mathcal{G}$, (the subgroup of scalar matrices), $\mathcal{K} = \mathrm{U}(2) \cap \mathcal{G}$ and $\mathcal{K}_1 = \mathcal{K} \cap \mathrm{SL}(2, \widehat{K})$. Note that the complex components of $\mathcal{ZK}$ and $\mathcal{ZK}_1$ are equal, but at a real component the latter has index 2 in the former. We also set

$$(\mathcal{ZK})_+ = \{g \in \mathcal{ZK} \mid \det(g) \in \mathbb{R}_+^\times\}.$$

At a real component,

$$(\mathcal{ZK})_{+,j} = \mathcal{ZK}_{1,j} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in \mathbb{R},\ \text{not both zero} \right\} \cong \mathbb{C}^\times,$$

the isomorphism being via $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \leftrightarrow x+yi$. Now $K_{\infty,j} = \mathbb{R}$ and one can identify the jth component $\mathbb{R}^2$ of $\widehat{K}^2$ with $\mathbb{C}$. Multiplication by $z = x+yi \in \mathbb{C}^\times$ is the same as right multiplication by the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in (\mathcal{ZK})_{+,j} \cong \mathbb{C}^\times$. To obtain classical modular forms of weight k, one considers functions on lattices (viewed as embedded in $\mathbb{R}^2 = \mathbb{C}$) which are homogeneous of degree $-k$. Specifying the weight can then be thought of as specifying an action of $(\mathcal{ZK})_{+,j}$ via the quasicharacter $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mapsto (x+yi)^k$. Similarly for the various real embeddings of a totally real field where one considers integer weight vectors, with possibly a different weight for each embedding.

At a complex component,

$$(\mathcal{ZK})_{+,j} = \mathbb{R}^\times \mathcal{K}_{1,j} = \left\{ \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \mid u, v \in \mathbb{C},\ \text{not both zero} \right\} \cong \mathbb{H}^\times,$$

where $\mathbb{H}$ is the algebra of quaternions, the isomorphism being via $\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \leftrightarrow u+vj$. Now we may identify the component $\mathbb{C}^2$ of $\widehat{K}^2$ with $\mathbb{H}$ via $(u,v) \leftrightarrow u+vj$, and then right scalar multiplication by $u+vj \in \mathbb{H}^\times$ corresponds to right multiplication by $\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in (\mathcal{ZK})_+ \cong \mathbb{H}^\times$, since $j(u+vj) = -\overline{v} + \overline{u}j$. The key difference here is that $\mathbb{H}^\times$ is non-commutative, and has irreducible representations of dimension n for every $n \geq 1$. Thus, instead of restricting weights to integers k we allow more general representations ρ of $(\mathcal{ZK})_{+,j}$, or of $(\mathcal{ZK})_j$, of arbitrary (finite) dimension, and call this representation the weight of the modular form.

Observe that the determinant map induces isomorphisms$(\mathcal{ZK})_j/(\mathcal{ZK})_{+,j} \cong \mathbb{R}^\times/\mathbb{R}^\times_+$ of order 2 in the real case, and in the complex case $(\mathcal{ZK})_j/(\mathcal{ZK})_{+,j} \cong \mathbb{C}^\times/\mathbb{R}^\times_+$, the circle group. The sequence

$$1 \to (\mathcal{ZK})_{+,j} \to (\mathcal{ZK})_j \to K^\times_{\infty,j}/\mathbb{R}^\times_+ \to 1$$

splits in each case, a complement being the subgroup $\{\begin{pmatrix} \pm1 & 0 \\ 0 & 1 \end{pmatrix}\}$, respectively $\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\}$. An alternative complement in the complex case is the central subgroup $\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\}$.

In the real case, $(\mathcal{ZK})_{+,j}$ is Abelian and hence equal to its centre, isomorphic to $\mathbb{C}^\times$; the larger subgroup $(\mathcal{ZK})_j$ has smaller centre $\mathcal{Z}_j \cong \mathbb{R}^\times$. However in the complex case, $(\mathcal{ZK})_{+,j}$ is non-Abelian with centre $\mathbb{R}^\times$, while $(\mathcal{ZK})_j$ has larger centre $\mathcal{Z}_j \cong \mathbb{C}^\times$.

4.1. **Definition of formal modular forms.** We now define formal modular forms as functions on modular points which transform under the right action of $\mathcal{ZK}$ via a given representation of that subgroup. In view of the motivational discussion above it might seem more natural to use instead representations of the smaller subgroup $(\mathcal{ZK})_+$, but we will see later the advantage of the approach taken.

Fix an irreducible complex r-dimensional representation $\rho : \mathcal{ZK} \to \mathrm{GL}(r,\mathbb{C})$. Since ρ is irreducible, its restriction to the centre $\mathcal{Z}$ of $\mathcal{ZK}$ consists of scalar matrices. For $x \in \widehat{K}^\times$, write $\rho(x) = \rho(xI_2)$ (where I_2 is the 2×2 identity matrix); we consider $\rho(x)$ to be a scalar where convenient.

To the representation ρ, and each integral ideal $\mathfrak{n}$, we associate the following space of functions on the set of modular points $\mathcal{M}_1(\mathfrak{n})$ with values in $\mathbb{C}^r$ where $r = \dim\rho$:

Definition 11. A *formal modular form of weight ρ for* $\Gamma_1(\mathfrak{n})$ is a function

$$F : \mathcal{M}_1(\mathfrak{n}) \to \mathbb{C}^r,$$

where $r = \dim\rho$, satisfying

$$F(P\zeta\kappa) = F(P)\rho(\zeta\kappa) \qquad (\forall \zeta \in \widehat{K}^\times, \kappa \in \mathcal{K}).$$

(Elements of $\mathbb{C}^r$ are viewed as row vectors.) For fixed $\mathfrak{n}$ and ρ, these form a complex vector space, denoted $\mathcal{S}^{\mathcal{M}}_\rho(\mathfrak{n})$.

Each formal modular form consists of a collection of h functions $F_{ij} : \mathcal{M}_1^{(c_{ij})}(\mathfrak{n}) \to \mathbb{C}^r$, one for each ideal class c_{ij}, each satisfying the same transformation condition.

4.2. **Hecke action on formal modular forms.** Let $\mathcal{M} = \mathcal{M}_1(\mathfrak{n})$. Functions $F : \mathcal{M} \to \mathbb{C}^r$ may be extended by linearity to functions $\mathbb{C}\mathcal{M} \to \mathbb{C}^r$. Now we can define an action of the Hecke algebra $\mathbb{T}$ on such functions by setting

$$F\,|T = F \circ T$$

for $T \in \mathbb{T}$. Explicitly,

$$F\,|T_{\mathfrak{a}}\,((L, L', \beta)) = N(\mathfrak{a})^{-1} \sum_{\substack{[M:L]=\mathfrak{a} \\ (M,M',\beta)\in\mathcal{M}}} F((M, M', \beta)),$$

and for $\mathfrak{a}$ coprime to $\mathfrak{n}$,

$$F\,|T_{\mathfrak{a},\mathfrak{a}}\,((L, L', \beta)) = N(\mathfrak{a})^{-2} F((\mathfrak{a}^{-1}L, \mathfrak{a}^{-1}L', \beta)).$$

Also, for $d \in \mathcal{O}_K$ coprime to $\mathfrak{n}$,

$$F\,|[d]\,((L, L', \beta)) = F((L, L', d\beta)).$$

It is immediate that these actions preserve the space $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n})$, since the left Hecke action on $\mathcal{M}$ commutes with the right action by $\mathcal{G}$. The operators $[d]$ define an action of the finite Abelian group $(\mathcal{O}_K/\mathfrak{n})^\times$ on this space, which therefore decomposes as a direct sum

$$\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}) = \bigoplus_\chi \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$$

where χ runs over the group of Dirichlet characters modulo $\mathfrak{n}$, *i.e.*, homomorphisms $(\mathcal{O}_K/\mathfrak{n})^\times \to \mathbb{C}^\times$, and

$$\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi) = \left\{ F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}) \mid F\,|[d] = \chi(d)F \right\}.$$

Since each $[d]$ commutes with the operators in $\mathbb{T}$ it is again immediate that $\mathbb{T}$ acts on each subspace $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$. If χ_0 denotes the principal character modulo $\mathfrak{n}$ then $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi_0)$ may be identified as the space of functions $\mathcal{M}_0(\mathfrak{n}) \to \mathbb{C}^r$ satisfying the same transformation formula with respect to the action of $\mathcal{ZK}$, which we naturally call a formal modular form of weight ρ for $\Gamma_0(\mathfrak{n})$.

On $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$, the action of $T_{\mathfrak{a},\mathfrak{a}}$ for principal ideals $\mathfrak{a} = \langle d \rangle$ is completely determined in terms of ρ and χ. For $d \in \mathcal{O}_K$ coprime to $\mathfrak{n}$, note that

$$[d](L, L', \beta) = (L, L', d\beta) = N(d)^2 T_{d,d}(dL, dL', d\beta) = N(d)^2 T_{d,d}(L, L', \beta)(dI_2)$$

where I_2 denotes the 2×2 identity matrix. Hence, recalling that $\rho(d)$ is scalar,

$$F\,|[d] = N(d)^2 \rho(d)(F\,|T_{d,d}),$$

so for $F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$ we have

$$F\,|T_{d,d} = N(d)^{-2}\rho(d)^{-1}\chi(d)F.$$

In particular, $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$ consists of eigenforms for each $T_{d,d}$. The analogous classical formula for forms of weight k (see [12, III, Prop. 34]) would read $T_{d,d}F = d^{k-2}\chi(d)F$, since then $\rho(d) = d^{-k}$ and $N(d) = d$.

As a special case of the above formula, if $d \in \mathcal{O}_K^\times$ then $T_{d,d}$ is the identity, so we must then have $\rho(d) = \chi(d)$. Thus $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi) = 0$ unless the restrictions of χ and ρ to $\mathcal{O}_K^\times$ agree in this way; just as in the classical case where there are no forms of odd weight k and character χ unless $\chi(-1) = -1$.

It follows that each $T_{\mathfrak{a},\mathfrak{a}}$ acts on $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$ via matrices whose hth powers are (non-zero) scalars, so we can diagonalise the action further with respect to these operators; in effect we have an action of the class group on each $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \chi)$ and can split these further into eigenspaces for that action. To this end we define, for each quasicharacter $\lambda : \mathcal{J}_K^{\mathfrak{n}} \to \mathbb{C}^\times$,

$$\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}, \lambda) = \left\{ F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}) \mid F\,|T_{\mathfrak{a},\mathfrak{a}} = \lambda(\mathfrak{a})F \right\}.$$

Now $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda) \subseteq \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\chi)$, where

$$\chi(d) = N(d)^2\rho(d)\lambda(\langle d\rangle). \tag{1}$$

Fixing χ, the space $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\chi)$ is the direct sum of those $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ for which λ is related to χ by (1); these λ differ by quasicharacters of $\mathcal{J}_K^{\mathfrak{n}}$ which are trivial on principal ideals, *i.e.,* by characters of the ideal class group. Thus we have, as expected, split up each $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\chi)$ further using characters of the class group.

In particular, consider the case where χ is trivial, so that $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\chi)$ consists of forms for $\Gamma_0(\mathfrak{n})$. For such a form F we have

$$F\,|T_{d,d} = N(d)^{-2}\rho(d)^{-1}F$$

for all $d \in \mathcal{O}_K$ coprime to $\mathfrak{n}$. This space is the direct sum of subspaces $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ as λ runs over the quasicharacters whose values on principal ideals $\langle d\rangle$ are given by $\lambda(\langle d\rangle) = N(d)^{-2}\rho(d)^{-1}$.

If we further have that $\rho(d) = N(d)^{-2}$, as in the classical situation for forms of weight 2, then all $T_{d,d}$ act trivially, the λ are trivial on principal ideals, and we actually do have a well-defined action of the class group on the space of such forms since $T_{\mathfrak{a},\mathfrak{a}}$ then only depends on the ideal class $[\mathfrak{a}]$.

We summarise this discussion in the following.

Proposition 4.1. *The space $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n})$ of formal modular forms for $\Gamma_1(\mathfrak{n})$ is a direct sum*

$$\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}) = \bigoplus_\lambda \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$$

where λ runs over the group of quasicharacters of $\mathcal{J}_K^{\mathfrak{n}}$, and

$$\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda) = \left\{F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n}) \mid F\,|T_{\mathfrak{a},\mathfrak{a}} = \lambda(\mathfrak{a})F\right\}.$$

Forms in $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ are also eigenforms for the operators $[d]$ for $d \in \mathcal{O}_K$ coprime to $\mathfrak{n}$, with eigenvalues $\chi(d)$ where χ is defined in terms of λ by (1). The space of forms for $\Gamma_0(\mathfrak{n})$ consists of the direct sum of those $\mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ for which the restriction of λ to principal ideals is given by $\lambda(\langle d\rangle) = N(d)^{-2}\rho(d)^{-1}$.

Recall that each $F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n})$ consists of a collection of h functions $F_{ij} : \mathcal{M}_1^{(c_{ij})}(\mathfrak{n}) \to \mathbb{C}^r$ indexed by the ideal classes c_{ij}. If $F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ then the F_{ij} for fixed i and varying j are linked:

Proposition 4.2. *Let $F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$. For each class c_{ij} and ideal $\mathfrak{a} \in \mathcal{J}_K^{\mathfrak{n}}$, let j' be the unique index such that $[\mathfrak{q}_{j'}\mathfrak{a}]^2 = [\mathfrak{q}_j]^2$. Then*

$$F_{ij} = \lambda(\mathfrak{a})^{-1}F_{ij'} \circ T_{\mathfrak{a},\mathfrak{a}}.$$

In particular,

$$F_{ij} = \lambda(\mathfrak{q}_j)^{-1}F_{i1} \circ T_{\mathfrak{q}_j,\mathfrak{q}_j}.$$

Proof. Let $P \in \mathcal{M}_1^{(c_{ij'})}(\mathfrak{n})$; then $T_{\mathfrak{a},\mathfrak{a}}(P) \in \mathcal{M}_1^{(c_{ij})}(\mathfrak{n})$, so

$$F(T_{\mathfrak{a},\mathfrak{a}}(P)) = F\,|T_{\mathfrak{a},\mathfrak{a}}\,(P) = \lambda(\mathfrak{a})F(P)$$

implies

$$F_{ij}(T_{\mathfrak{a},\mathfrak{a}}(P)) = \lambda(\mathfrak{a})F_{ij'}(P).$$

□

Thus a function $F \in \mathcal{S}_\rho^{\mathcal{M}}(\mathfrak{n},\lambda)$ is completely determined by the h_2 components $F_{i1} : \mathcal{M}_1^{(c_{i1})}(\mathfrak{n}) \to \mathbb{C}^r$. When the class number is odd there is only one such component. In any case, if F is also an eigenform for the full Hecke algebra, it is easy to see that F is completely determined by the "principal" component F_{11} once the eigenvalue system (or at least, the eigenvalue of $T_{\mathfrak{p}_i}$ for each i) is known.

4.3. **Modular forms as functions on** GL(2)**.** We now pass from formal modular forms, which are functions defined on modular points $\mathcal{M}_1(\mathfrak{n})$, to modular forms as functions on $\mathcal{G} = \mathrm{GL}(2, \widehat{K})$; this is done using admissible bases and the bijections in Corollary 2.4 and Proposition 2.5. We have $\mathcal{G} = \mathrm{GL}(2,\mathbb{R})^r \times \mathrm{GL}(2,\mathbb{C})^s$, with the j'th component $\mathcal{G}_j$ being either $\mathrm{GL}(2,\mathbb{R})$ or $\mathrm{GL}(2,\mathbb{C})$, according to whether the j'th completion is real or complex.

Each formal modular form $F \in \mathcal{S}^{\mathcal{M}}_{\rho}(\mathfrak{n})$ with $\dim(\rho) = r$ restricts to h functions $F_c : \mathcal{M}_1^{(c)}(\mathfrak{n}) \to \mathbb{C}^r$ indexed by the classes $c = c_{ij} \in \mathrm{Cl}(K)$. We define a corresponding set of h functions

$$\phi = (\phi_{ij}) : \mathcal{G} \to \mathbb{C}^r$$

as follows, using the standard modular points $\tilde{P}_{ij} = (L_{ij}, L'_{ij}, \beta_j)$:

$$\phi_{ij}(U) = F(\tilde{P}_{ij}U) = F(L_{ij}U, L'_{ij}U, \beta_j U).$$

These functions satisfy

$$\phi_{ij}(\zeta U \kappa) = \phi_{ij}(U)\rho(\zeta\kappa); \qquad (\forall \zeta \in \widehat{K}^{\times}, \kappa \in \mathcal{K}) \tag{2}$$

$$\phi_{ij}(\gamma U) = \phi_{ij}(U) \qquad (\forall \gamma \in \Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})), \tag{3}$$

the latter since $\tilde{P}_{ij}\gamma = \tilde{P}_{ij}$ for $\gamma \in \Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})$ by Proposition 2.5. In case $F \in \mathcal{S}^{\mathcal{M}}_{\rho}(\mathfrak{n}, \lambda)$ we have more generally

$$\phi_{ij}(\gamma U) = \chi(\epsilon(\gamma))\phi_{ij}(U) \qquad (\forall \gamma \in \Gamma_0^{\mathfrak{p}_i}(\mathfrak{n})), \tag{4}$$

where $\chi(d) = N(d)^2\rho(d)\lambda(\langle d\rangle)$ and $\epsilon : \Gamma_0^{\pi_i}(\mathfrak{n}) \to (\mathcal{O}_K/\mathfrak{n})^{\times}$ maps a matrix to its $(2,2)$-entry.

Conversely, given a set of functions $\phi = (\phi_{ij})$ satisfying conditions (1) and (2), we can recover the formal modular form of weight ρ by setting $F(\tilde{P}_{ij}U) = \phi_{ij}(U)$. This will lie in $\mathcal{S}^{\mathcal{M}}_{\rho}(\mathfrak{n}, \lambda)$ if (3) also holds.

We denote collections of functions on $\mathcal{G}$ satisfying these conditions by $\mathcal{S}^{\mathcal{G}}_{\rho}(\mathfrak{n})$ and $\mathcal{S}^{\mathcal{G}}_{\rho}(\mathfrak{n}, \lambda)$ respectively.

It is unfortunate but apparently unavoidable in general that the component functions ϕ_{ij} on $\mathcal{G}$ are invariant under different groups $\Gamma_1^{\mathfrak{p}_i}(\mathfrak{n})$. When the class number is odd, the situation is simpler since then all classes have the form $c = c_{ij}$ with $i = 1$, and all component functions ϕ_{1j} are invariant under $\Gamma_1(\mathfrak{n})$ itself.

5. Matrices for principal Hecke operators

Classically, Hecke operators may be defined in terms of certain matrices in $\mathrm{Mat}(2,\mathbb{Z})$. This description is particularly useful for explicit computations. Here, such a description (using matrices in $\mathrm{Mat}(2, \mathcal{O}_K)$) is only possible for principal operators, essentially because only these map modular points to modular points whose underlying lattice has the same Steinitz class. On the other hand, we saw in Section 3 that the eigenvalue system for a formal Hecke eigenform is essentially determined by the eigenvalues of principal operators. Hence, in practice, it suffices to know how to compute the action of $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}$ on $\mathcal{M}_1^{(1)}(\mathfrak{n})$, or $T_{\mathfrak{a},\mathfrak{a}}T_{\mathfrak{b}}W_{\mathfrak{q}}$ on $\mathcal{M}_0^{(1)}(\mathfrak{n})$, when $\mathfrak{a}^2\mathfrak{b}$, or $\mathfrak{a}^2\mathfrak{b}\mathfrak{q}$ respectively, is principal. It will be more convenient to use the alternative normalization introduced in subsection 3.2: the operator $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{b}}\tilde{W}_{\mathfrak{q}}$ maps a lattice L to a formal sum of sublattices of index $\mathfrak{a}^2\mathfrak{b}\mathfrak{q}$.

In this section we discuss the operators $\tilde{T}_{\mathfrak{a},\mathfrak{a}}$ and $\tilde{T}_{\mathfrak{b}}$, leaving the Atkin-Lehner operators to the following section.

5.0.1. Consider the operator $\tilde{T}_{\mathfrak{b}}$ where $\mathfrak{b}$ is principal. Considering first the action on principal lattices, and $L = \mathcal{O}_K \oplus \mathcal{O}_K$ in particular, we wish to determine the sublattices $M \subset \mathcal{O}_K \oplus \mathcal{O}_K$ with $[\mathcal{O}_K \oplus \mathcal{O}_K : M] = \mathfrak{b}$.

For each such M, there is a basis x, y for $\mathcal{O}_K \oplus \mathcal{O}_K$, and uniquely determined ideals $\mathfrak{b}_1, \mathfrak{b}_2$ with $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2^2$, such that $M = \mathfrak{b}_1\mathfrak{b}_2 x \oplus \mathfrak{b}_2 y = \mathfrak{b}_2(\mathfrak{b}_1 x + \mathcal{O}_K y)$. If $\gamma \in \Gamma$ is the matrix with rows x, y then we have $M = (\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathfrak{b}_2)\gamma$. Writing $\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathfrak{b}_2 = (\mathcal{O}_K \oplus \mathcal{O}_K)B$, with B a fixed $(\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{b}_2)$-matrix, we have $M = (\mathcal{O}_K \oplus \mathcal{O}_K)B\gamma$. As γ varies in Γ, this gives all sublattices M with index $\mathfrak{b}$ and quotient $\mathcal{O}_K \oplus \mathcal{O}_K/M \cong \mathcal{O}_K/\mathfrak{b}_1\mathfrak{b}_2 \oplus \mathcal{O}_K/\mathfrak{b}_2$. Moreover, $\gamma_1, \gamma_2 \in \Gamma$ give the same lattice M if and only if

$$\Gamma_0(\mathfrak{b}_1)\gamma_1 = \Gamma_0(\mathfrak{b}_1)\gamma_2$$

since $\Gamma \cap \Delta(\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{b}_2) = \Gamma_0(\mathfrak{b}_1)$. Hence, to obtain each such M exactly once, we let γ run through a complete set of $\psi(\mathfrak{b}_1)$ right cosets of $\Gamma_0(\mathfrak{b}_1)$ in Γ. Thus, for each factorization $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2^2$, we have $\psi(\mathfrak{b}_1)$ matrices BC where B is a fixed $(\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{b}_2)$-matrix, and C runs through coset representatives of $\Gamma_0(\mathfrak{b}_1)$ in Γ. These cosets are in bijection with $\mathbb{P}^1(\mathfrak{b}_1)$, and we may construct a set of coset representatives by lifting M-symbols from $\mathbb{P}^1(\mathfrak{b}_1)$ to Γ, as explained in subsection 1.4. The operator $\tilde{T}_{\mathfrak{b}}$ is then given by the formal sum of all these, as $\mathfrak{b}_2$ runs over the ideals whose square divides $\mathfrak{b}$.

In the special case where $\mathfrak{b}$ is square-free and principal, say with generator β, we may take $B = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$; then, $\tilde{T}_{\mathfrak{b}}$ is given by the formal sum of matrices BC where C runs over coset representatives of $\Gamma_0(\mathfrak{b})$ in Γ, the number of which is $\psi(\mathfrak{b}) = \prod_{\mathfrak{p}|\mathfrak{b}}(1 + N(\mathfrak{p}))$.

In general, the number of sublattices of index $\mathfrak{b} = \prod_{\mathfrak{p}} \mathfrak{p}^e$ is given (after a straightforward calculation) by $\eta(\mathfrak{b})$, where

$$\eta(\mathfrak{b}) = \prod_{\mathfrak{p}|\mathfrak{b}}(1 + N(\mathfrak{p}) + N(\mathfrak{p})^2 + \cdots + N(\mathfrak{p})^e) = \prod_{\mathfrak{p}|\mathfrak{b}} \left(\frac{N(\mathfrak{p})^{e+1} - 1}{N(\mathfrak{p}) - 1} \right).$$

5.0.2. The operator $\tilde{T}_{\mathfrak{a},\mathfrak{a}}$, where $\mathfrak{a}$ is integral with $\mathfrak{a}^2$ principal, is represented by an $(\mathfrak{a}, \mathfrak{a})$-matrix.

5.0.3. For the general case of $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{b}}$ where $\mathfrak{a}^2\mathfrak{b}$ is principal, we simply replace the $(\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{b}_2)$-matrix B in the above discussion of $\tilde{T}_{\mathfrak{b}}$ with an $(\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{a}\mathfrak{b}_2)$-matrix.

5.1. **Action on modular points.** In the previous subsection, we saw how to construct, for all principal ideals $\mathfrak{b}$, a set of $\eta(\mathfrak{b})$ "Hecke matrices" g_i such that the sublattices of $\mathcal{O}_K \oplus \mathcal{O}_K$ of index $\mathfrak{b}$ are precisely the $(\mathcal{O}_K \oplus \mathcal{O}_K)g_i$ for $1 \le i \le \eta(\mathfrak{b})$.

When L is an arbitrary free lattice we have $L = (\mathcal{O}_K \oplus \mathcal{O}_K)U$ for some $U \in \mathcal{G}$; then the sublattices $M \subseteq L$ with $[L : M] = \mathfrak{b}$ are given by $M_i = (\mathcal{O}_K \oplus \mathcal{O}_K)g_iU$ for $1 \le i \le \eta(\mathfrak{b})$, since $MU^{-1} \subseteq \mathcal{O}_K \oplus \mathcal{O}_K$, with the same index. Here, the matrix U is only determined up to left multiplication by a matrix $\gamma \in \Gamma$ such that $L\gamma = L$. Replacing M_i by $(\mathcal{O}_K \oplus \mathcal{O}_K)g_i\gamma U$ for each i gives the same sublattices, though possibly permuted. Hence the Hecke action on free lattices is well-defined by

$$\tilde{T}_{\mathfrak{b}} : L = (\mathcal{O}_K \oplus \mathcal{O}_K)U \mapsto \sum_i (\mathcal{O}_K \oplus \mathcal{O}_K)g_iU.$$

Extending this to an action on principal modular points, we must take into account the level structure. Now we restrict to principal ideals $\mathfrak{b}$ which are coprime to the level $\mathfrak{n}$. In the construction of the g_i we must ensure that each matrix has its $(2,1)$-entry in $\mathfrak{n}$. To do this, we first use $(\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{b}_2)$-matrices B of level $\mathfrak{n}$ (as defined in subsection 1.3). Secondly, the coset representatives for $\Gamma_0(\mathfrak{b}_1)$ in Γ we use must lie in $\Gamma_0(\mathfrak{n})$. These may be obtained by lifting from $\mathbb{P}^1(\mathfrak{b}_1)$ to Γ via $\mathbb{P}^1(\mathfrak{b}_1\mathfrak{n})$, as explained in subsection 1.4.

When the Hecke matrices g_i have been constructed in this way we say that they are "Hecke matrices of level $\mathfrak{n}$". These now give a well-defined action on $\Gamma_0(\mathfrak{n})$-modular points, via

$$\tilde{T}_{\mathfrak{b}} : (L, L') = (\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1})U \mapsto \sum_i (\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1}) g_i U.$$

Similarly, for the operator $\tilde{T}_{\mathfrak{a},\mathfrak{a}}$ when $\mathfrak{a}$ is coprime to $\mathfrak{n}$ and $\mathfrak{a}^2$ is principal, we must use an $(\mathfrak{a},\mathfrak{a})$-matrix of level $\mathfrak{n}$, and in the construction of the matrices for $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{b}}$, the $(\mathfrak{a}\mathfrak{b}_1\mathfrak{b}_2, \mathfrak{a}\mathfrak{b}_2)$-matrices we use must be of level $\mathfrak{n}$.

To complete this section, we make the constructions above completely explicit is some cases of particular use in computations, namely

- $\tilde{T}_{\mathfrak{p}}$ with $\mathfrak{p}$ a principal prime;
- $\tilde{T}_{\mathfrak{p}^2}$ with $\mathfrak{p}$ prime, with $\mathfrak{p}^2$ principal;
- $\tilde{T}_{\mathfrak{p}\mathfrak{q}}$ with $\mathfrak{p}$, $\mathfrak{q}$ distinct primes with $\mathfrak{p}\mathfrak{q}$ principal;

together with extended version of these where the ideals concerned are not principal, but whose ideal class is square, so that we obtain principal operators by composing with $\tilde{T}_{\mathfrak{a},\mathfrak{a}}$ for suitable $\mathfrak{a}$. Here we assume that the level $\mathfrak{n}$ is fixed, and all these ideals ($\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{a}$) are coprime to $\mathfrak{n}$.

Principal primes. Taking $\mathfrak{b} = \mathfrak{p}$ to be a principal prime ideal, say $\mathfrak{p} = \langle \pi \rangle$. Then there is only one factorisation of the form $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2^2$, namely $(\mathfrak{b}_1, \mathfrak{b}_2) = (\mathfrak{p}, \mathcal{O}_K)$, so we may take $B = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$. As coset representatives for $\Gamma_0(\mathfrak{p})$ we take $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ for $x \pmod{\mathfrak{p}}$. Then the matrices giving the sublattices of index $\mathfrak{p}$ are

$$\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} 0 & \pi \\ 1 & x \end{pmatrix} \qquad (\text{for } x \in \mathcal{O}_K \pmod{\mathfrak{p}}).$$

As the last $N(\mathfrak{p})$ of these do not have level $\mathfrak{n}$, we adjust them by multiplying each on the left by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to obtain the familiar set of Hecke matrices for a principal prime:

$$\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & x \\ 0 & \pi \end{pmatrix} \quad \text{for } x \in \mathcal{O}_K \mod \mathfrak{p}.$$

These have level $\mathfrak{n}$ for every $\mathfrak{n}$.

Primes with square class. Now suppose that the ideal class $[\mathfrak{p}]$ is a square, with $\mathfrak{a}^2\mathfrak{p}$ principal and $\mathfrak{a}$ coprime to $\mathfrak{n}$. Let B be an $(\mathfrak{a}\mathfrak{p}, \mathfrak{a})$-matrix of level $\mathfrak{n}$, and let $\nu \in \mathfrak{n} \setminus \mathfrak{p}$ (noting that $\mathfrak{p} \nmid \mathfrak{n} \implies \mathfrak{p} \not\supseteq \mathfrak{n}$). Then a suitable set of $N(\mathfrak{p}) + 1$ matrices representing $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{p}}$ consists of

$$B, \quad \text{and} \quad B\begin{pmatrix} 1 & x \\ \nu & 1 + x\nu \end{pmatrix} \quad \text{for } x \pmod{\mathfrak{p}},$$

since it is easy to see that the matrices $\begin{pmatrix} 1 & x \\ \nu & 1 + x\nu \end{pmatrix}$ for $x \pmod{\mathfrak{p}}$ together with the identity matrix represent all the cosets of $\Gamma_0(\mathfrak{p})$, and lie in $\Gamma_0(\mathfrak{n})$.

Principal prime squares. Let $\mathfrak{b} = \mathfrak{p}^2 = \langle\beta\rangle$ where $\mathfrak{p}$ is a prime whose class has order 2. There are two factorizations $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2^2$ to be considered, namely $(\mathfrak{b}_1, \mathfrak{b}_2) = (\mathcal{O}_K, \mathfrak{p}_2)$ and $(\mathfrak{b}_1, \mathfrak{b}_2) = (\mathfrak{p}^2, \mathcal{O}_K)$.

Taking $\mathfrak{b}_2 = \mathfrak{p}$ gives the lattice $M = \mathfrak{p} \oplus \mathfrak{p} = (\mathcal{O}_K \oplus \mathcal{O}_K)B$, where B is a $(\mathfrak{p}, \mathfrak{p})$ matrix. Hence, for the first Hecke matrix, B_1 can be any $(\mathfrak{p}, \mathfrak{p})$ matrix of level $\mathfrak{n}$.

When $\mathfrak{b}_1 = \mathfrak{p}^2$ and $\mathfrak{b}_2 = \mathcal{O}_K$, we set $B_2 = \begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}$, and use the matrices B_2C as C runs through lifts to $\Gamma_0(\mathfrak{n})$ of all $(c:d) \in \mathbb{P}^1(\mathfrak{p}^2)$.

To make this more explicit, fix $\nu \in \mathfrak{n}\backslash\mathfrak{p}$, and first take as coset representatives for $\Gamma_0(\mathfrak{p}^2)$ matrices $\begin{pmatrix}0 & 1\\ 1 & x\end{pmatrix}$ for $x \in \mathcal{O}_K \pmod{\mathfrak{p}^2}$, together with matrices $\begin{pmatrix}1 & 0\\ y\nu & 1\end{pmatrix}$ for $y \in \mathfrak{p} \pmod{\mathfrak{p}^2}$. The additional $N(\mathfrak{p})^2 + N(\mathfrak{p})$ Hecke matrices are then:

$$\begin{pmatrix}1 & x\\ 0 & \beta\end{pmatrix} \qquad \text{for } x \in \mathcal{O}_K \mod \mathfrak{p}^2;$$
$$\begin{pmatrix}\beta & 0\\ y\nu & 1\end{pmatrix} \qquad \text{for } y \in \mathfrak{p} \mod \mathfrak{p}^2;$$

here, for the first set, we have again swapped the rows to obtain matrices of level $\mathfrak{n}$.

A similar set of Hecke matrices were used in Bygott's thesis [2, Prop. 123]; he specified that $\nu \equiv 1 \pmod{\mathfrak{p}}$, which is stronger than necessary.

$\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}(\mathfrak{p}^2)$ *for arbitrary primes, with* $\mathfrak{a}\mathfrak{p}$ *principal.* For any prime $\mathfrak{p}$ not dividing $\mathfrak{n}$, we may take $\mathfrak{a}$ to be an ideal coprime to $\mathfrak{n}$ the inverse class to $\mathfrak{p}$, so that the operator $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}(\mathfrak{p}^2)$ is principal. As before, this operator is a formal sum of $N(\mathfrak{p})^2 + N(\mathfrak{p}) + 1$ matrices. Taking B_1 to be $(\mathfrak{a}\mathfrak{p}, \mathfrak{a}\mathfrak{p})$-matrix of level $\mathfrak{n}$ and B_2 to be an $(\mathfrak{a}\mathfrak{p}^2, \mathfrak{a})$-matrix of level $\mathfrak{n}$, we use B_1 and B_2C as C again runs through lifts to $\Gamma_0(\mathfrak{n})$ of all $(c:d) \in \mathbb{P}^1(\mathfrak{p}^2)$.

Principal products of two primes. Let $\mathfrak{b} = \mathfrak{p}\mathfrak{q} = \langle\beta\rangle$ where $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals in inverse ideal classes. There is only one factorization of $\mathfrak{b}$ to be considered, namely $(\mathfrak{b}_1, \mathfrak{b}_2) = (\mathfrak{p}\mathfrak{q}, \mathcal{O}_K)$. Using similar arguments as before, we find the following set of $\eta(\mathfrak{p}\mathfrak{q}) = (N(\mathfrak{p}) + 1)(N(\mathfrak{q}) + 1)$ Hecke matrices of level $\mathfrak{n}$, taking $\nu \in \mathfrak{n} \setminus (\mathfrak{p} \cup \mathfrak{q})$ (always assuming that $\mathfrak{p}\mathfrak{q}$ is coprime to $\mathfrak{n}$):

$$\begin{pmatrix}1 & x\\ 0 & \beta\end{pmatrix} \qquad \text{for } x \mod \mathfrak{p}\mathfrak{q};$$
$$\begin{pmatrix}\beta & 0\\ y\nu & 1\end{pmatrix} \qquad \text{for } y \in \mathfrak{p} \cup \mathfrak{q} \mod \mathfrak{p}\mathfrak{q};$$

together with both a $(\mathfrak{p}, \mathfrak{q})$-matrix of level $\mathfrak{n}$ and a $(\mathfrak{q}, \mathfrak{p})$-matrix of level $\mathfrak{n}$. There are $N(\mathfrak{p})N(\mathfrak{q})$ matrices of the first kind and $N(\mathfrak{p}) + N(\mathfrak{q}) - 1$ of the second kind. Again, these are similar to Hecke matrices used by Bygott in [2, Prop. 124].

Products of two primes whose product has square class. More generally, suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are distinct primes such that the class of $\mathfrak{p}\mathfrak{q}$ is a square, with $\mathfrak{a}^2\mathfrak{p}\mathfrak{q}$ principal (all ideals coprime to $\mathfrak{n}$). Then the principal operator $\tilde{T}_{\mathfrak{a},\mathfrak{a}}\tilde{T}_{\mathfrak{p}\mathfrak{q}}$ is again a sum of $(N(\mathfrak{p})+1)(N(\mathfrak{q})+1)$ matrices, namely BC where B is a fixed $(\mathfrak{a}\mathfrak{p}\mathfrak{q}, \mathfrak{a})$-matrix of level $\mathfrak{n}$ and C runs through lifts to $\Gamma_0(\mathfrak{n})$ of all $(c:d) \in \mathbb{P}^1(\mathfrak{p}\mathfrak{q})$.

6. Matrices for Atkin-Lehner operators

The material in this section extends the special cases treated in Bygott's thesis ([2, Section 1.4]) and Lingham's thesis ([14, Section 5.3]), and first appeared in the thesis of Aranés ([1, §2.3.1]). Bygott used an overgroup of $\Gamma_0(\mathfrak{n})$, containing $\Gamma_0(\mathfrak{n})$ as a normal subgroup of finite index, constructed from ideals whose class has order 2 in the class group. In our notation, this overgroup consists of all $(\mathfrak{a},\mathfrak{a})$-matrices where $\mathfrak{a}$ is an integral ideal whose square is principal. When the class number is odd, a method of generalizing the classical construction of Atkin-Lehner operators was given by Lingham.

Here we treat the following cases, where $\mathfrak{n}$ is the level and $\mathfrak{q}$ a divisor of $\mathfrak{n}$ coprime to $\mathfrak{n}\mathfrak{q}^{-1}$:

- the Atkin-Lehner operator $W_{\mathfrak{q}}$, when $\mathfrak{q}$ is principal;
- the operator $T_{\mathfrak{m},\mathfrak{m}}W_{\mathfrak{q}}$ with $\mathfrak{m}$ coprime to $\mathfrak{n}$, when $\mathfrak{m}^2\mathfrak{q}$ is principal;
- The operator $T_{\mathfrak{p}}W_{\mathfrak{q}}$ with $\mathfrak{p}$ a prime not dividing $\mathfrak{n}$, when $\mathfrak{p}\mathfrak{q}$ is principal.

The first two of these can each be defined using a single matrix in $\mathrm{Mat}(2,\mathcal{O}_K)$, whose determinant generates $\mathfrak{q}$ or $\mathfrak{m}^2\mathfrak{q}$ respectively, and we deal with these first.

For the rest of the section, we fix the level $\mathfrak{n}$, and denote by $\mathfrak{q}$ an exact divisor of $\mathfrak{n}$, so that $\mathfrak{q}$ is coprime to $\mathfrak{q}'=\mathfrak{n}\mathfrak{q}^{-1}$.

As in the previous section, it is more convenient to use the operators $\tilde{W}_{\mathfrak{q}}$, $\tilde{T}_{\mathfrak{m},\mathfrak{m}}\tilde{W}(\mathfrak{q})$, and $\tilde{T}_{\mathfrak{p}}\tilde{W}(\mathfrak{q})$, since these are realised by integral matrices.

6.1. **$W_{\mathfrak{q}}$ with $\mathfrak{q}$ principal.** By definition, the operator $\tilde{W}_{\mathfrak{q}}$ maps the standard modular point $(\mathcal{O}_K\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{n}^{-1}\mathcal{O}_K)$ to $(\mathfrak{q}\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{q}\mathfrak{n}^{-1})$.

Definition 12. Let $\mathfrak{q}$ be a principal ideal such that $\mathfrak{q}\mid\mathfrak{n}$ and $\mathfrak{q}$ is coprime to $\mathfrak{n}\mathfrak{q}^{-1}$. A *$W_{\mathfrak{q}}$-matrix of level* $\mathfrak{n}$ is a matrix $M=\begin{pmatrix}x & y\\ z & w\end{pmatrix}$ such that

$$x,w\in\mathfrak{q};\qquad y\in\mathcal{O}_K;\qquad z\in\mathfrak{n};\qquad \langle\det M\rangle=\mathfrak{q}.$$

More concisely,

$$M\in\begin{pmatrix}\mathfrak{q} & \mathcal{O}_K\\ \mathfrak{n} & \mathfrak{q}\end{pmatrix}\qquad\text{with}\quad\langle\det M\rangle=\mathfrak{q}.$$

$W_{\mathfrak{q}}$-matrices are exactly the matrices M such that

$$\begin{aligned}(\mathcal{O}_K\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{n}^{-1})M&=(\mathfrak{q}\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{q}\mathfrak{n}^{-1});\\ (\mathfrak{q}\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{q}\mathfrak{n}^{-1})M&=\mathfrak{q}(\mathcal{O}_K\oplus\mathcal{O}_K,\mathcal{O}_K\oplus\mathfrak{n}^{-1}).\end{aligned}$$

Equivalently,

$$\begin{aligned}\begin{pmatrix}\mathcal{O}_K & \mathcal{O}_K\\ \mathfrak{n} & \mathcal{O}_K\end{pmatrix}M&=\begin{pmatrix}\mathfrak{q} & \mathcal{O}_K\\ \mathfrak{n} & \mathfrak{q}\end{pmatrix};\\ \begin{pmatrix}\mathfrak{q} & \mathcal{O}_K\\ \mathfrak{n} & \mathfrak{q}\end{pmatrix}M&=\mathfrak{q}\begin{pmatrix}\mathcal{O}_K & \mathcal{O}_K\\ \mathfrak{n} & \mathcal{O}_K\end{pmatrix}.\end{aligned}$$

$W_{\mathfrak{q}}$-matrices exist, and may be constructed as follows:

Proposition 6.1 (Existence of $W_{\mathfrak{q}}$-matrices)**.** *$W_{\mathfrak{q}}$-matrices exists for every principal exact divisor $\mathfrak{q}\mid\mid\mathfrak{n}$.*

Proof. let $\mathfrak{q}=\langle g\rangle$, let $\mathfrak{a}\in[\mathfrak{n}]^{-1}$ be an integral ideal coprime to $\mathfrak{n}$, with $\mathfrak{a}\mathfrak{n}=\langle z\rangle$, and let $\mathfrak{b}\in[\mathfrak{q}]^{-1}$ be an integral ideal coprime to $\mathfrak{a}\mathfrak{q}'$, with $\mathfrak{b}\mathfrak{q}=\langle x\rangle$. Then $\mathfrak{b}\mathfrak{q}$ and $\mathfrak{a}\mathfrak{q}'$ are coprime, so $\mathfrak{q}=\mathfrak{b}\mathfrak{q}^2+\mathfrak{a}\mathfrak{n}=\langle xg\rangle+\langle z\rangle$, so we can write $g=gxw-zy$ with $y,w\in\mathcal{O}_K$. Now $M=\begin{pmatrix}x & y\\ z & gw\end{pmatrix}$ has the desired properties, with determinant g. □

For further properties of $W_{\mathfrak{q}}$-matrices, see the next subsection, taking $\mathfrak{m}=\mathcal{O}_K$.

6.2. $T_{\mathfrak{m},\mathfrak{m}}W_{\mathfrak{q}}$ **with** $\mathfrak{m}^2\mathfrak{q}$ **principal.** With $\mathfrak{q}$ still an exact divisor of $\mathfrak{n}$, we now only assume that the ideal class $[\mathfrak{q}]$ is a square, and let $\mathfrak{m}$ be an ideal coprime to $\mathfrak{q}'$ such that $\mathfrak{q}\mathfrak{m}^2$ is principal.

Definition 13. A $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix of level $\mathfrak{n}$ is a matrix $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that

$$x, w \in \mathfrak{m}\mathfrak{q}; \qquad y \in \mathfrak{m}; \qquad z \in \mathfrak{m}\mathfrak{n}; \qquad \langle \det M \rangle = \mathfrak{q}\mathfrak{m}^2.$$

More concisely,

$$M \in \begin{pmatrix} \mathfrak{m}\mathfrak{q} & \mathfrak{m} \\ \mathfrak{m}\mathfrak{n} & \mathfrak{m}\mathfrak{q} \end{pmatrix} \qquad \text{with} \quad \langle \det M \rangle = \mathfrak{q}\mathfrak{m}^2.$$

$W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices are exactly the matrices M such that

$$(\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1})M = \mathfrak{m}(\mathfrak{q} \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{q}\mathfrak{n}^{-1});$$
$$(\mathfrak{q} \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{q}\mathfrak{n}^{-1})M = \mathfrak{m}\mathfrak{q}(\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1}),$$

or alternatively,

$$\begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{n} & \mathcal{O}_K \end{pmatrix} M = \begin{pmatrix} \mathfrak{m}\mathfrak{q} & \mathfrak{m} \\ \mathfrak{m}\mathfrak{n} & \mathfrak{m}\mathfrak{q} \end{pmatrix} = \mathfrak{m} \begin{pmatrix} \mathfrak{q} & \mathcal{O}_K \\ \mathfrak{n} & \mathfrak{q} \end{pmatrix};$$
$$\begin{pmatrix} \mathfrak{q} & \mathcal{O}_K \\ \mathfrak{n} & \mathfrak{q} \end{pmatrix} M = \begin{pmatrix} \mathfrak{m}\mathfrak{q} & \mathfrak{m}\mathfrak{q} \\ \mathfrak{m}\mathfrak{n}\mathfrak{q} & \mathfrak{m}\mathfrak{q} \end{pmatrix} = \mathfrak{m}\mathfrak{q} \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{n} & \mathcal{O}_K \end{pmatrix}.$$

When $\mathfrak{q} = \mathfrak{m} = \langle 1 \rangle$, these matrices are simply elements of $\Gamma_0(\mathfrak{n})$. When $\mathfrak{n} = \mathfrak{q} = \langle 1 \rangle$, these matrices, for all $\mathfrak{m}$ such that $\mathfrak{m}^2$ is principal, generate the normalizer of Γ in $\mathrm{GL}(2, K)$ (modulo scalars), denoted Δ in [2]. When $\mathfrak{q} = \langle 1 \rangle$ and $\mathfrak{m}^2$ is a principal ideal coprime to $\mathfrak{n}$, then $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices of level $\mathfrak{n}$ give matrix representations of the Hecke operators $\tilde{T}_{\mathfrak{m},\mathfrak{m}}$, as in the previous section, while if $\mathfrak{q} = \mathcal{O}_K$ then a $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix is just a $W_{\mathfrak{q}}$-matrix.

Note that if we replace $\mathfrak{m}$ by an alternative ideal $\mathfrak{m}'$ in the same class, then $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices and $W_{\mathfrak{q}}^{\mathfrak{m}'}$-matrices only differ by a scalar factor. If we identify two such matrices when they differ by a scalar factor, then $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices are associated with pairs $(\mathfrak{q}, [\mathfrak{m}])$ such that $[\mathfrak{q}\mathfrak{m}^2] = 0$, and the number of such pairs is finite for each level $\mathfrak{n}$.

A $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix may also be described as a $(\mathfrak{m}\mathfrak{q}, \mathfrak{m})$-matrix of level $\mathfrak{n}$, whose adjugate is also an $(\mathfrak{m}\mathfrak{q}, \mathfrak{m})$-matrix. In case $\mathfrak{q} = \langle 1 \rangle$, a $W_{\langle 1 \rangle}^{\mathfrak{m}}$-matrix is just an $(\mathfrak{m}, \mathfrak{m})$-matrix of level $\mathfrak{n}$ (where $\mathfrak{m}^2$ is principal).

$W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices may be constructed in a similar way to $W_{\mathfrak{q}}$-matrices:

Proposition 6.2 (Existence of $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices)**.** *$W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices exists for every exact divisor $\mathfrak{q} \mid\mid \mathfrak{n}$ and $\mathfrak{m}$ such that $\mathfrak{q}\mathfrak{m}^2$ is principal.*

Proof. Let $\mathfrak{q}\mathfrak{m}^2 = \langle g \rangle$, let $\mathfrak{a} \in [\mathfrak{m}\mathfrak{n}]^{-1}$ be an integral ideal coprime to $\mathfrak{m}\mathfrak{n}$, with $\mathfrak{a}\mathfrak{m}\mathfrak{n} = \langle z \rangle$, and let $\mathfrak{b} \in [\mathfrak{m}\mathfrak{q}]^{-1}$ be an integral ideal coprime to $\mathfrak{a}\mathfrak{q}'$, with $\mathfrak{b}\mathfrak{m}\mathfrak{q} = \langle x \rangle$.

Then

$$\langle gx, z \rangle = \mathfrak{q}\mathfrak{m}^2\mathfrak{b}\mathfrak{m}\mathfrak{q} + \mathfrak{a}\mathfrak{m}\mathfrak{n} = \mathfrak{m}\mathfrak{q}(\mathfrak{b}\mathfrak{m}^2\mathfrak{q} + \mathfrak{a}\mathfrak{q}') = \mathfrak{m}\mathfrak{q},$$

since each of $\mathfrak{b}$, $\mathfrak{m}$, $\mathfrak{q}$ is coprime to each of $\mathfrak{a}$, $\mathfrak{q}'$. Hence $g \in \mathfrak{m}^2\mathfrak{q} = \mathfrak{m}\langle gx, z \rangle$, so $g = gxw - zy$ with $y, w \in \mathfrak{m}$. Now $\begin{pmatrix} x & y \\ z & gw \end{pmatrix}$ is an $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix. □

Proposition 6.3 (Uniqueness of $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices)**.** *Let M, M_1, and M_2 be $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices of level $\mathfrak{n}$ (with the same $\mathfrak{q}$ and $\mathfrak{m}$). Then*

(1) $M_1 M_2 \in \Gamma_0(\mathfrak{n})$ *(up to a scalar factor).*
(2) $M_1 M_2^{-1} \in \Gamma_0(\mathfrak{n})$ *and* $M_1^{-1} M_2 \in \Gamma_0(\mathfrak{n})$.

(3) *The set of all* $W_{\mathfrak{q}}^{\mathfrak{m}}$*-matrices of level* $\mathfrak{n}$ *equals the left coset* $M\Gamma_0(\mathfrak{n})$ *and also the right coset* $\Gamma_0(\mathfrak{n})M$.

(4) *If* $\mathfrak{m}_1$ *is another ideal such that* $\mathfrak{q}\mathfrak{m}_1^2$ *is principal, say* $\mathfrak{m}_1 = \mathfrak{m}\mathfrak{b}$ *with* $\mathfrak{b}^2$ *principal, then the set of all* $W_{\mathfrak{q}}^{\mathfrak{m}_1}$*-matrices is*

$$\{MB \mid B \quad \text{is a } (\mathfrak{b},\mathfrak{b})\text{-matrix of level } \mathfrak{n}\}.$$

Proof. Part (1) is a special case of the more general result in Proposition 6.4 below (recalling that $\Gamma_0(\mathfrak{n})$ is the set of $W_{\langle 1\rangle}^{\langle 1\rangle}$-matrices). Part (2) follows from this, since up to scalars we have $M_i^{-1} = \operatorname{adj} M_i$, another $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix. This already shows that every $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix is in both $M\Gamma_0(\mathfrak{n})$ and $\Gamma_0(\mathfrak{n})M$; that these both consist only of $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices also follows from Proposition 6.4. Part (4) is a similar easy calculation, using the fact that when $\mathfrak{b}^2$ is principal, a $(\mathfrak{b},\mathfrak{b})$ matrix of level $\mathfrak{n}$ is the same as a $W_{\langle 1\rangle}^{\mathfrak{b}}$-matrix. □

Proposition 6.4 (Products of $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices)**.** *For* $i = 1, 2$ *let* M_i *be a* $W_{\mathfrak{q}_i}^{\mathfrak{m}_i}$*-matrix of level* $\mathfrak{n}$*, where* $\mathfrak{q}_i \mid\mid \mathfrak{n}$ *and* $\mathfrak{m}_i^2\mathfrak{q}_i$ *is principal. Then* $M_3 = M_1M_2$ *is (up to a scalar factor) a* $W_{\mathfrak{q}_3}^{\mathfrak{m}_3}$*-matrix of level* $\mathfrak{n}$*, where* $\mathfrak{m}_3 = \mathfrak{a}\mathfrak{m}_1\mathfrak{m}_2$*,* $\mathfrak{a} = \mathfrak{q}_1 + \mathfrak{q}_2$ *and* $\mathfrak{q}_3 = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{a}^{-2}$.

Proof. If we write $\mathfrak{q}_1 = \mathfrak{a}\mathfrak{q}_3'$ and $\mathfrak{q}_2 = \mathfrak{a}\mathfrak{q}_3''$ then $\mathfrak{q}_3 = \mathfrak{q}_3'\mathfrak{q}_3''$ and $\mathfrak{q}_3$ is another divisor of $\mathfrak{n}$ coprime to $\mathfrak{n}\mathfrak{q}_3^{-1}$. We also clearly have $\mathfrak{q}_3\mathfrak{m}_3^2 = \mathfrak{q}_3(\mathfrak{a}\mathfrak{m}_1\mathfrak{m}_2)^2 = (\mathfrak{q}_1\mathfrak{m}_1^2)(\mathfrak{q}_2\mathfrak{m}_2^2)$, which is principal, showing that M_3 has the right determinant to be a $W_{\mathfrak{q}_3}^{\mathfrak{m}_3}$-matrix of level $\mathfrak{n}$. Finally,

$$\begin{aligned} M_3 \in \begin{pmatrix} \mathfrak{m}_1\mathfrak{q}_1 & \mathfrak{m}_1 \\ \mathfrak{m}_1\mathfrak{n} & \mathfrak{m}_1\mathfrak{q}_1 \end{pmatrix} \begin{pmatrix} \mathfrak{m}_2\mathfrak{q}_2 & \mathfrak{m}_2 \\ \mathfrak{m}_2\mathfrak{n} & \mathfrak{m}_2\mathfrak{q}_2 \end{pmatrix} &\subseteq \begin{pmatrix} \mathfrak{m}_1\mathfrak{m}_2(\mathfrak{q}_1\mathfrak{q}_2 + \mathfrak{n}) & \mathfrak{m}_1\mathfrak{m}_2(\mathfrak{q}_1 + \mathfrak{q}_2) \\ \mathfrak{m}_1\mathfrak{m}_2\mathfrak{n}(\mathfrak{q}_1 + \mathfrak{q}_2) & \mathfrak{m}_1\mathfrak{m}_2(\mathfrak{q}_1\mathfrak{q}_2 + \mathfrak{n}) \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{m}_3\mathfrak{q}_3 & \mathfrak{m}_3 \\ \mathfrak{m}_3\mathfrak{n} & \mathfrak{m}_3\mathfrak{q}_3 \end{pmatrix} \end{aligned}$$

since $\mathfrak{q}_1 + \mathfrak{q}_2 = \mathfrak{a}$ and $\mathfrak{q}_1\mathfrak{q}_2 + \mathfrak{n} = \mathfrak{a}\mathfrak{q}_3$. □

Corollary 6.5. *For a fixed level* $\mathfrak{n}$*, the set of all* $W_{\mathfrak{q}}^{\mathfrak{m}}$*-matrices of level* $\mathfrak{n}$*, as* $\mathfrak{q}$ *ranges over all exact principal divisors of* $\mathfrak{n}$*, form a group* $\widetilde{\Gamma_0(\mathfrak{n})}$ *under multiplication modulo scalar matrices. This group contains* $\Gamma_0(\mathfrak{n})$ *(modulo scalars) as a normal subgroup of finite index, such that the quotient is an elementary* 2*-group.* □

Since $\Gamma_0(\mathfrak{n})$-modular points are right invariant under multiplication by $\Gamma_0(\mathfrak{n})$, and using Proposition 6.3, we have a well defined action on the set $\mathcal{M}_0^{(1)}(\mathfrak{n})$ of all principal $\Gamma_0(\mathfrak{n})$-modular points by mapping

$$P = (\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1})U \mapsto (\mathcal{O}_K \oplus \mathcal{O}_K, \mathcal{O}_K \oplus \mathfrak{n}^{-1})MU;$$

this is well defined, as the image is unchanged if we either replace U by another admissible basis matrix γU or replace M by another $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrix γM for any $\gamma \in \Gamma_0(\mathfrak{n})$.

As already mentioned, in the special case $\mathfrak{q} = \langle 1\rangle$, we recover the action of $\tilde{T}_{\mathfrak{m},\mathfrak{m}}$ for $[\mathfrak{m}] \in \mathrm{Cl}[2]$. When $\mathfrak{m} = \langle 1\rangle$ and $\mathfrak{q}$ is principal, we obtain the Atkin-Lehner operator $W_{\mathfrak{q}}$.

6.3. $T_{\mathfrak{a}}W_{\mathfrak{q}}$ **with** $\mathfrak{a}\mathfrak{q}$ **principal.** In applications, it is convenient to be able to apply the operator $\tilde{T}_{\mathfrak{a}}\tilde{W}_{\mathfrak{q}}$ at level $\mathfrak{n}$, where $\mathfrak{q} \mid\mid \mathfrak{n}$ and $\mathfrak{a}$ is coprime to $\mathfrak{n}$, with $\mathfrak{q}\mathfrak{a}$ principal. The case where $\mathfrak{a}$ is prime is most important, but here we handle the general case with no more effort. Now, $\tilde{T}_{\mathfrak{a}}\tilde{W}_{\mathfrak{q}}$ will be the formal sum of $\psi(\mathfrak{a})$ matrices M with entries in $\mathcal{O}_K$, satisfying the following properties, where $\langle g\rangle = \mathfrak{a}\mathfrak{q}$ and L runs through all $\psi(\mathfrak{a})$ lattices of index $\mathfrak{a}$:

(1) $\det(M) = g$;

(2) $M \in \begin{pmatrix} \mathfrak{q} & \mathcal{O}_K \\ \mathfrak{n} & \mathfrak{q} \end{pmatrix}$;
(3) $(\mathcal{O}_K \oplus \mathcal{O}_K)M \subseteq L$.

We construct these as follows. Let $\mathfrak{b} \in [\mathfrak{q}']^{-1}$ be coprime to $\mathfrak{a}\mathfrak{q}$, so $\mathfrak{b}\mathfrak{q}' = \langle h \rangle$ with $h \in \mathcal{O}_K$ coprime to $\mathfrak{a}$. For each $(c_0 : d_0) \in \mathbb{P}^1(\mathfrak{a})$, use the Chinese remainder Theorem to find $(c : d) \in \mathbb{P}^1(\mathfrak{a}\mathfrak{n}\mathfrak{b})$ such that

$$(c : d) = \begin{cases} (c_0 : d_0) & \text{in } \mathbb{P}^1(\mathfrak{a}); \\ (1 : 0) & \text{in } \mathbb{P}^1(\mathfrak{q}); \\ (0 : 1) & \text{in } \mathbb{P}^1(\mathfrak{b}\mathfrak{q}') \end{cases}$$

and lift to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{b}\mathfrak{q}')$. Then set $M = \begin{pmatrix} d & c/h \\ bgh & ag \end{pmatrix}$. Note that $c/h \in \mathcal{O}_K$, since $c \in \mathfrak{b}\mathfrak{q}' = \langle h \rangle$. Properties (1) and (2) for M are immediate, using $d, g \in \mathfrak{q}$ and $h \in \mathfrak{b}$. As for (3), the entries in the second row of M lie in $\mathfrak{a}$, while those in the top row lie in $L = \{(x, y) \in \mathcal{O}_K \oplus \mathcal{O}_K \mid c_0 x \equiv d_0 hy \pmod{\mathfrak{a}}\}$, and L ranges over all the lattices of index $\mathfrak{a}$ as $(c_0 : d_0)$ runs over $\mathbb{P}^1(\mathfrak{a})$, since h is coprime to $\mathfrak{a}$.

6.4. **Further remarks.** Elements of the quotient group $\widetilde{\Gamma_0(\mathfrak{n})}$ induce involution operators on any space on which Γ acts, since they preserve $\Gamma_0(\mathfrak{n})$-invariant subspaces. These involutions generalize both classical Atkin-Lehner involutions, and also include the involutions coming from elements of order 2 in the class group as utilized in [2]. The size of the group of involutions depends on both the structure of the ideal class group and also the ideal classes of the divisors of $\mathfrak{n}$. At one extreme, if the ideal class group is an elementary Abelian 2-group, then no non-trivial ideal class is a square. In this case the only $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices are those for which $\mathfrak{q}$ is principal, but $\mathfrak{m}$ is arbitrary. On the other hand, if the class number is odd (as in Lingham's thesis [14]) then every ideal class is a square but only principal ideals have principal squares; in this case we have $W_{\mathfrak{q}}^{\mathfrak{m}}$-matrices for all $\mathfrak{q} \mid\mid \mathfrak{n}$ but $\mathfrak{m}$ (or rather, its ideal class) is uniquely determined by $\mathfrak{q}$. (In [14], the choice was fixed as $\mathfrak{m} = \mathfrak{q}^n$ where $2n+1$ is the class number.) This is closest to the classical situation.

The analogue of the classical Fricke involution at level $\mathfrak{n}$ is $W_{\mathfrak{n}}$, which only acts on principal modular points when $\mathfrak{n}$ is principal, when it is given by a $W_{\mathfrak{n}}^{\langle 1 \rangle}$-matrix of level $\mathfrak{n}$. If $\mathfrak{n}$ has square ideal class, with $\mathfrak{n}\mathfrak{m}^2$ principal, then $W_{\mathfrak{n}}\tilde{T}_{\mathfrak{m},\mathfrak{m}}$ acts via a $W_{\mathfrak{n}}^{\mathfrak{m}}$-matrix.

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Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK.
Email address: J.E.Cremona@warwick.ac.uk