ACTION OF HECKE AND RELATED OPERATORS ON M-SYMBOLS

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Abstract. Computing Hecke operators, Atkin-Lehner involutions and degeneracy maps on M-symbols using Heilbronn matrices and similar methods.

1. Introduction

In [2, pages 20–23], we showed how to compute the action of the Hecke operators T_p for $p \nmid N$ directly on the space $\mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]$ spanned by so-called M-symbols $(c:d)$. Here we review that result (whose proof needs some slight adjustment), and also present in a unified way a similar recipe for the computation of Atkin-Lehner involutions W_Q (where $Q \mid N$ and $\gcd(Q, N/Q) = 1$) as well as the degeneracy maps α_t (respectively β_t) from level N to level M (respectively, from level M to level N) where $M \mid N$ and $t \mid N/M$.

I worked out the formulae for the W_Q some time ago (around 1998?) but did not implement them then; Michael Müller (Essen) told me that he had done this and found it less efficient than the method of converting to modular symbols and back.

Regarding the degeneracy maps, I remember working out how to do this and may have implemented around 2005 at least the lowering maps. But the code seems to have disappeared, so with those I am starting more or less from scratch.

All the constructions depend on a basic idea, which we introduce in the following section together with some notation.

2. Preliminaries

For an integer matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$, we write (M) for the modular symbol $\{M(0), M(\infty)\} = \{b/d, a/c\}$, i.e. the path from the cusp $M(0)$ to the cusp $M(1)$ in the extended upper half plane $\mathbb{H}^*$, which is the image under M of the imaginary axis $\{0, \infty\}$. When $\det(M) = 1$, this is also represented by the M-symbol (or Manin symbol) $(c:d)$ of level N; the image of (M) in $H_1(X_0(N), \mathbb{Q})$ only depends on the left coset of M in $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$, and hence on $(c:d)$ as an element of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

We extend this notation to $2\times(k+1)$ integer matrices $M = \begin{pmatrix} x_0 & x_1 & \dots & x_k \\ y_0 & y_1 & \dots & y_k \end{pmatrix}$ of rank 2 with $k \geq 1$, with the restriction that all the 2×2 minors formed by adjacent column are nonsingular, so that each $x_i/y_i \in \mathbb{P}^1(\mathbb{Q})$ and $x_{i-1}/y_{i-1} \neq x_i : y_i$ for $1 \leq i \leq k$. We set $M(0) = x_k/y_k$ and $M(\infty) = x_0/y_0 \in \mathbb{P}^1(\mathbb{Q})$, and think of (M) as representing a k-step path from the cusp $M(0)$ to the cusp $M(\infty)$ via intermediate cusps x_j/y_j for $1 \leq j \leq k-1$. In modular symbol notation,

$$(M) = \{M(0), M(\infty)\} = \left\{\frac{x_k}{y_k}, \frac{x_0}{y_0}\right\} = \sum_{i=1}^{k} \left\{\frac{x_i}{y_i}, \frac{x_{i-1}}{y_{i-1}}\right\}.$$

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2.1. **Co-cyclic lattices and Λ-matrices.** A lattice $\Lambda \subseteq \mathbb{Z}^2$ is called *co-cyclic of index* $Q \geq 1$ if it has cyclic quotient $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}/Q\mathbb{Z}$. Since $\mathbb{Z}^2/\Lambda$ is cyclic, there exist coprime integers c, d such that

$$(x, y) \in \Lambda \iff cx + dy \equiv 0 \pmod{Q};$$

here (c, d) is well-defined as an element of $\mathbb{P}^1(\mathbb{Z}/Q\mathbb{Z})$, and there is a bijection between elements $(c : d) \in \mathbb{P}^1(\mathbb{Z}/Q\mathbb{Z})$ and co-cyclic lattices Λ of index Q. We write $\Lambda = \Lambda((c : d); Q)$, or simply $\Lambda = \Lambda(c : d)$ if Q is understood from the context.

Let Λ be a co-cyclic lattice of index Q. We say that an $2 \times (k + 1)$ matrix M as above is a Λ-*matrix* if each column lies in Λ and each consecutive pair of columns is an oriented basis for Λ, i.e. $x_{i-1}y_i - y_{i-1}x_i = +Q$ for $1 \leq i \leq k$. In particular, a 2×2 Λ-matrix is a matrix of determinant Q whose columns are a basis for Λ; one such for $\Lambda = \Lambda((c : d); Q)$ is

$$M_1 = \begin{pmatrix} Q/g & -du \\ 0 & g \end{pmatrix},$$

where $g = \gcd(c, Q) = cu + Qv$ with $u, v \in \mathbb{Z}$.

In the case $Q = 1$ (so $\Lambda = \mathbb{Z}^2$), we have

$$(M) = \sum_{i=1}^{k} \left\{ \frac{x_i}{y_i}, \frac{x_{i-1}}{y_{i-1}} \right\} = \sum_{i=1}^{k} (y_{i-1} : y_i),$$

where in the last step we used the fact that $x_{i-1}y_i - y_{i-1}x_i = 1$. A useful shorthand is to write the last expression as an "extended M-symbol": $(y_0 : y_1 : \cdots : y_k)$. Similarly the second row of any $\mathbb{Z}^2$-matrix, or of any Λ-matrix of index coprime to N, may be interpreted as an extended M-symbol of level N.

It is easy to see that every Λ-matrix M has the form $M = M_1M_2$ where M_2 is a $\mathbb{Z}^2$-matrix and M_1 is a 2×2 Λ-matrix. For, if M_1 is any 2×2 Λ-matrix, then $M_2 = M_1^{-1}M$ is a $\mathbb{Z}^2$-matrix of the same size as M.

In this notation, the basic continued fraction (CF) lemma ([2, (2.1.8), pp.14–15] may be expressed as follows; here the columns of M are the CF convergents to α, with alternate columns negated to give the correct sign in the determinant condition.

Lemma 2.1. *For all $\alpha \in \mathbb{P}^1(\mathbb{Q})$, there exists a $\mathbb{Z}^2$-matrix M such that*

$$(M) = \{M(0), M(\infty)\} = \{\alpha, \infty\}.$$

More generally, for all distinct $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ there is a $\mathbb{Z}^2$-matrix M such that $(M) = \{\alpha, \beta\}$.

Proof. For the first part, write the convergents of α as $x_0/y_0 = 1/0, x_1/y_1 = 0/1, \ldots, x_k/y_k = \alpha$ and let M be the matrix with x_j, y_j in column j.

For the general case, an efficient construction[1] is as follows. Let M_1 be a 2×2 matrix of positive determinant such that $(M_1) = \{\alpha, \beta\}$. Let the Hermite Normal Form (HNF) of M_1 be $UM_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $\det U = 1$. Construct a $\mathbb{Z}^2$-matrix M with $(M) = \{b/d, \infty\} = (UM_1)$, using the convergents of b/d as in the first part. Then $U^{-1}M$ is a $\mathbb{Z}^2$-matrix with $(U^{-1}M) = (M_1) = \{\alpha, \beta\}$. □

The following more general result and construction will be crucial. For a 2×2 matrix M we write $M' = \operatorname{adj} M = (\det M)M^{-1}$.

Proposition 2.2. *Let $\Lambda \subseteq \mathbb{Z}^2$ be a co-cyclic lattice of index Q, and let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be distinct. Then there exists a Λ-matrix M such that $(M) = \{\alpha, \beta\}$.*

[1] shown to me by Karim Belabas

Proof. Let M_1 be a 2×2 Λ-matrix, and define $\alpha' = M_1'(\alpha)$, $\beta' = M_1'(\beta)$. Let M_2 be a $\mathbb{Z}^2$-matrix with $(M_2) = \{\alpha', \beta'\}$, which exists by the lemma. Then $M = M_1M_2$ has the desired properties, since $M(0) = M_1M_2(0) = M_1(\alpha') = \alpha$, and similarly $M(\infty) = \beta$. □

For a cocyclic lattice Λ, write M_Λ for a Λ-matrix M with $(M) = \{0, \infty\}$, and write $M_{(c:d);Q} = M_{\Lambda((c:d);Q)}$.

Corollary 2.3. *Let $(c : d) \in \mathbb{P}^1(\mathbb{Z}/Q\mathbb{Z})$, let $M = M_{(c:d);Q}$ have column entries x_i, y_i, and set $e_i = (cx_i + dy_i)/Q \in \mathbb{Z}$. Then*

$$(c : d) = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} (e_0 : e_1 : \cdots : e_k).$$

Proof. Let $M_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ have second row $(c\ d)$. Then $M_0M = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} M_1$, where M_1 is a $\mathbb{Z}^2$-matrix with entries e_i in the second row. Hence

$$(c : d) = (M_0) = (M_0M) = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} (M_1) = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \sum_{i=1}^{k} (e_0 : e_1 : \cdots : e_k).$$

□

3. Atkin-Lehner involutions W_Q

Let Q be a divisor of N coprime to $Q' = N/Q$. The Atkin-Lehner involution W_Q may be represented by any matrix of the form $W_Q = \begin{pmatrix} Qx & y \\ N & Q \end{pmatrix}$ where $Qx - Q'y = 1$, so $\det W_Q = Q$. For each M-symbol $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, we wish to evaluate $W_Q(c : d)$ as a sum of M-symbols.

Proposition 3.1. *Let Q be a divisor of N coprime to $Q' = N/Q$, and $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Let $M_0 \in \mathrm{SL}_2(\mathbb{Z})$ have second row $(c\ d)$. Let $M = M_{(c:d);Q}$. Then*

$$W_Q(c : d) = \sum_{i=1}^{k} (t_0 : t_1 : \cdots : t_k)$$

where

$$(t_0, \ldots, t_k) = \begin{pmatrix} Q' & 1 \end{pmatrix} M_0M.$$

More succinctly,

$$W_Q(c : d) = \begin{pmatrix} Q' & 1 \end{pmatrix} M_0M,$$

where the row vector on the right-hand side, whose entries are coprime, is interpreted as an extended M-symbol of level N.

Proof. We have

$$W_Q \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} = Q \begin{pmatrix} x & y \\ Q' & Q \end{pmatrix} = QA,$$

with $A \in \mathrm{SL}_2(\mathbb{Z})$. As in Corollary 2.3 we have $(c : d) = (M_0M) = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} (M_1)$. Since scalars act trivially on modular symbols, this gives

$$W_Q(c : d) = W_Q \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} (M_1) = QA(M_1) = (AM_1).$$

Finally, AM_1 is a $\mathbb{Z}^2$-matrix, with second row $\begin{pmatrix} Q' & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}^{-1} M_0M = \begin{pmatrix} Q' & 1 \end{pmatrix} M_0M$.

□

Note that computing the action of W_Q on $(c : d)$ involves computing the CF convergents of a single rational number of denominator $\gcd(c, Q)$; unfortunately this does depend on the symbol $(c : d)$.

4. Hecke operators T_p

Next we consider the action of the Hecke operator T_p on M-symbols of level N, where p is a prime not dividing N. This is more complicated than the previous cases since T_p is defined by the action of not just one matrix but $p + 1$ matrices of determinant p. Although normally one takes these to be $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & x \\ 0 & p \end{pmatrix}$ for $0 \le x \le p - 1$, the condition which the matrices are required to satisfy is that they represent the distinct right cosets in the double coset $\Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N)$, of which there are $p + 1$ (assuming $p \nmid N$), characterised by the property that the lattices spanned by their rows give all the $p+1$ sublattices of $\mathbb{Z}^2$ of index p, and that the lower left entry is divisible by N. This observation becomes important when generalising the formulae here to other number fields, where the representatives cannot always be taken to be upper triangular and independent of N.

The recipe given in [2, pages 20–23] constructs a larger finite set of matrices R_p, all of determinant p, and called Heilbronn matrices (or in Stein's book "Cremona-Heilbronn matrices" since we do not actually prove in [2] that they are the same as the traditional Heilbronn matrices). The formula is then that

$$T_p(c : d) = \sum_{R \in R_p} (c : d)R.$$

The exposition given in [2] has a flaw: it apparently shows how to express each of the $p + 1$ terms $T(c : d)$ as a sum of M-symbols, but the representative integers c, d used in each term are different. It is then not clear that each term is well-defined, although the sum of all the $p + 1$ subexpressions is well-defined. To correct that argument, it suffices to lift $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ to a matrix $M_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that $M_0 \equiv I \pmod{p}$. This is possible when $p \nmid N$, using the Chinese Remainder Theorem together with the surjectivity of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/Np\mathbb{Z})$. The matrix M_0 then satisfies

$$MM_0M^{-1} \in \mathrm{SL}_2(\mathbb{Z})$$

for all integral matrices M of determinant p. In [2], we used a different lift M_0 for each matrix M of determinant p.

Using the notions developed here, we can express the conclusion in the following more concise way. Recall that M_Λ denotes a Λ-matrix with $(M_\Lambda) = \{0, \infty\}$.

Proposition 4.1. *Let p be a prime not dividing N. Then for all $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ we have*

$$T_p(c : d) = \sum_{[\mathbb{Z}^2:\Lambda]=p} (c : d)M_\Lambda,$$

where the sum is over the $p + 1$ lattices $\Lambda \subset \mathbb{Z}^2$ of index p. When $p \mid N$, the above formula is still valid if we omit the term with $\Lambda = \Lambda_{(c:d);p}$.

Proof. First suppose that $p \nmid N$. Let M_p be a 2×2 matrix with determinant p, representing a double coset as above, with the lower left entry of M_p divisible by N; set $M'_p = \operatorname{adj} M_p = pM_p^{-1}$. Since $\det M'_p = p$ and $M_pM'_p \equiv 0 \pmod{p}$ it follows that M'_p is a Λ-matrix for some lattice Λ of index p; moreover, we obtain all $p + 1$ such lattices as M_p runs over a set of double coset representatives. Construct a Λ-matrix $M_\Lambda = M'_pM$ as in Proposition 2.2 using a $\mathbb{Z}^2$-matrix M with $(M) = (M_p)$.

Now $(c:d) = (M_0)$ and $(M_\Lambda) = (I)$ imply

$$M_p(c:d) = (M_pM_0) = (M_pM_0M_\Lambda) = (c:d)M_\Lambda.$$

The last equality requires some explanation. By the remarks made before the statement of the proposition, we may assume that $M_pM_0 = M_1M_p$ where $M_1 \in \mathrm{SL}_2(\mathbb{Z})$. Then $(M_pM_0M_\Lambda) = (M_1M_pM_p'M) = (M_1M)$, since we may discard the scalar factor p. So $M_p(c:d)$ is given as an extended M-symbol by the second row of the $\mathbb{Z}^2$-matrix M_1M, and p times this row is the second row of $M_pM_0M_\Lambda$. However, since $M_p \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N$, and extended M-symbols are invariant under multiplication by units modulo N, this is the same as the extended M-symbol given by the second row of M_0M_Λ, which is $(c:d)M_\Lambda$.

Adding over all double cosets gives the result.

When $p \mid N$, the operator T_p is sometimes denoted U_p, and some modification of the above argument is necessary. The definition of T_p in this case omits the matrix $M_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. Also, we can no longer assume that $M_pM_0M_p^{-1} \in \mathrm{SL}_2(\mathbb{Z})$; however there is a permutation σ (depending on $(c:d)$ modulo p) of the set of $p+1$ matrices M_p so that $M_pM_0\sigma(M_p)^{-1} \in \mathrm{SL}_2(\mathbb{Z})$. One can easily check that $\sigma(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})$ is the matrix whose associated lattice is $\Lambda_{(c:d);p}$, so this lattice must be omitted from the sum. Finally, in the last part of the proof, we use the fact that the bottom row of each M_p is now simply $(0,p)$ and the factor of p cancels, so that the second rows of M_0M_Λ and M_1M are equal. □

Note that the matrices M_Λ required here only depend on p, and not on either the symbol $(c:d)$ or the level N, except that when $p \mod N$ the lattice to be omitted does depend on $(c:d)$; so they may be precomputed. The 2×2 blocks constituting the M_Λ are the (Cremona-)Heilbronn matrices of [2].

5. Downward degeneracy operators α_t

The "downward" degeneracy map α_t is defined for $t \mid N$ and maps symbols of level N to symbols of level M for all $M \mid (N/t)$, i.e., for all $M \mid N$ with $t \mid (N/M)$. Since

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & bt \\ (N/t)c & d \end{pmatrix},$$

so that $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(N)\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \Gamma_0(N/t) \cap \Gamma^0(t) \subseteq \Gamma_0(M)$, the map $z \mapsto tz$ on the upper half plane induces a well-defined map

$$\alpha_t \colon \Gamma_0(N)\backslash\mathbb{H}^* \to \Gamma_0(M)\backslash\mathbb{H}^*.$$

Proposition 5.1. *Let $M \mid N$ and $t \mid N/M$. For each $(c:d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ we have*

$$\alpha_t(c:d) = (e_0 : e_1 : \cdots : e_k),$$

where the right-hand side is the extended M-symbol at level M defined by

$$(e_0, e_1, \ldots, e_k) = t^{-1}\begin{pmatrix} c & d \end{pmatrix} M_\Lambda$$

with $\Lambda = \Lambda_{(c:d);t}$.

Proof. Applying Corollary 2.3 with $Q = t$ we find that the action of α_t on M-symbols is given by

$$\alpha_t(c:d) = (M)$$

where $M = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} M_\Lambda$, which gives the result. □

We can write this result more succinctly as

$$\alpha_t(c:d) = (c:d)M_{(c:d);t}$$

with the understanding that before being interpreted as an extended M-symbol of level M, the vector on the right, whose entries are all divisible by t, should have these factors divided out.

Note that computing this action involves computing the CF convergents of a single rational number of denominator $\gcd(c,t)$.

5.1. **Special cases.** We make the preceding formulae more explicit, first for α_1 and then for α_p in the case $N = pM$ with p prime, where α_1 and α_p are the only degeneracy maps.

α_1 is always very simple, as it is the natural reduction map $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{P}^1(\mathbb{Z}/M\mathbb{Z})$:

$$\alpha_1(c:d) = (c:d),$$

where the symbols on the left and right are interpreted as M-symbols of levels N, M respectively.

The formula for α_p is, as in the general case,

$$\alpha_p(c:d) = p^{-1}(c:d)M_\Lambda$$

where $\Lambda = \Lambda_{(c:d);p}$. However we can express this alternatively as follows, using the observation that the matrix $\begin{pmatrix} p & 1 \\ 1 & 0 \end{pmatrix}$ defining α_p is precisely the matrix "missing" from the definition of the Hecke operator T_p at level N when $p \mid N$. In case $p \parallel N$ (that is, $p \nmid M$) this observation implies the equality

$$\alpha_p = T_p\alpha_1 - \alpha_1 T_p$$

of maps from level N to level M, where the first T_p is acting at level M and the second at level N. One interesting consequence is that any element of the modular symbol (or modular form) space at level N which is both an eigenvector for T_p and in the kernel of α_1 is automatically also in the kernel of α_p and hence new at p.

More generally, if $N = p^k N_0$ with $p \nmid N_0$ then

$$\alpha_1^{k-1}\alpha_p = T_p\alpha_1^k - \alpha_1^k T_p$$

as maps from level N to level N_0. But note that (for $k \geq 2$) this does not mean that an eigenvector for T_p which is in the kernel of α_1 is also in the kernel of α_p.

6. Upward degeneracy operators β_t

Again let $M \mid N$ and $t \mid (N/M)$. The second, "upward", family of degeneracy maps β_t map modular symbols at level M to the modular symbol space at level N. We follow [3] for the definition and the proof that the map is well-defined.

Let $\{g_1, \dots, g_n\}$ be left coset representatives for $\Gamma_0(N/t) \cap \Gamma^0(t)$ in $\Gamma_0(M)$. The index here is independent of t and equal to $[\Gamma_0(M) : \Gamma_0(N)]$. Then for all $(c:d) \in \mathbb{P}^1(\mathbb{Z}/M\mathbb{Z})$ we have

$$\beta_t(c:d) = \sum_i \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} g_i(c:d),$$

the right-hand side being in the modular symbol space of level N. Writing $M_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ and taking $M_0 \in \mathrm{SL}_2(\mathbb{Z})$ to be any lift of $(c:d)$ this can also be written

$$\beta_t(c:d) = \sum_i (M_t g_i M_0).$$

To evaluate the ith term, we let $\Lambda_i = \Lambda_{(u:v);t}$ be the cocyclic lattice of index t defined by the top row $(u\ v)$ of g_iM_0. Then $g_iM_0M_{\Lambda_i}$ is a Λ_i-matrix with all the entries in the top row divisible by t, with $(g_iM_0) = (g_iM_0M_{\Lambda_i})$. The effect of pre-multiplying by M_t is the same as dividing all the top row entries by t, so the ith term is the extended M-symbol of level N defined by the bottom row of $g_iM_0M_{\Lambda_i}$. Hence

$$\beta_t(c:d) = \sum_i (0\ 1)(g_iM_0M_{\Lambda_i}).$$

Note that we cannot simplify this to $\sum_i (c:d)M_{\Lambda_i}$, since g_i is in $\Gamma_0(M)$, not $\Gamma_0(N)$.

6.1. **Special cases.** For β_1 we have $\beta_1(c:d) = \sum(g_iM_0)$ where as usual M_0 is a lift of $(c:d)$ to a coset representative for $\Gamma_0(M)$ in $\mathrm{SL}_2(\mathbb{Z})$, and g_i runs through coset representatives of $\Gamma_0(N)$ in $\Gamma_0(M)$. Hence $\beta_1(c:d)$ is the formal sum of all lifts of $(c:d)$ from $\mathbb{P}^1(\mathbb{Z}/M\mathbb{Z})$ to $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

When $N = Mp$ with p prime we can compute these lifts easily. If $p \mid M$ then the index $[\Gamma_0(N) : \Gamma_0(M)] = p$ and the lifts of $(c:d) \in \mathbb{P}^1(\mathbb{Z}/M\mathbb{Z})$ to $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ are

$$\begin{array}{lll} (c+Mj:d) & \text{for } j \mod p & \text{if } p \nmid d; \text{ or} \\ (c:d+Mj) & \text{for } j \mod p & \text{if } p \nmid c. \end{array}$$

If $p \nmid M$ then the index is $p+1$, we may assume that $p \nmid d$ (replace d by $d+p$ otherwise), and the extra lift is

$$(cp+M:dp).$$

Now we consider β_p, still with $N = Mp$. The g_i are now coset representatives of $\Gamma_0(M)\cap\Gamma^0(p)$ in $\Gamma_0(M)$ of which there are again either p (when $p \mid M$) or $p+1$. It is easy to see (or see [1, Lemma 5]) that in the first case coset representatives are $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ for $j \mod p$, while in the second case the additional representative is any unimodular matrix of the form $\begin{pmatrix} pa & 1 \\ Mc & 1 \end{pmatrix}$. In the first case, after pre-multiplying by $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ we recover the p matrices used to define the Hecke operator T_p at level N; in the second, the additional term is just W_p (at level N). Hence we have the following equality of operators at level N:

$$\beta_p\alpha_1 = \begin{cases} T_p & \text{if } p \mid M; \\ T_p + W_p & \text{if } p \nmid M. \end{cases}$$

Thus, to evaluate β_p on an M-symbol at level M, we may take an arbitrary lift to level N and then apply T_p (if $p \mid M$) or $T_p + W_p$ (if $p \nmid M$). Multiplying by β_1 on the right and recalling that $\alpha_t\beta_t = [\Gamma_0(N) : \Gamma_0(M)]$, we obtain

$$\beta_p = \begin{cases} \frac{1}{p}T_p\beta_1 & \text{if } p \mid M; \\ \frac{1}{p+1}(T_p + W_p)\beta_1 & \text{if } p \nmid M. \end{cases}$$

Remark 6.1. *One consequence of the expression for $\beta_p\alpha_1$ is that for an element of the modular symbol space at level N which is in the kernel of α_1, which includes p-new elements, and which is an eigenvector for all the Hecke operators, the eigenvalue of T_p is 0 when $p^2 \mid N$ and minus the W_p-eigenvalue when $p \,||\, N$.*

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