Derivation of a General Formula for element n in a Fibonacci Sequence

In this technique, we will try to solve the problem in a more algorithmically effective way. We will derive a General Formula for an element n in the Fibonacci Sequence

For the Fibonacci sequence, we have the following recurrence.

$$f(n) = f(n-1) + f(n-2) \quad \text{for every } n \geq 2 \quad \text{—— } (1)$$

$f(0), f(1), f(2) \ldots$ are no other than the Fibonacci numbers

(1): Is therefore a situation of homogeneous linear recurrence.

We look at solutions of the type $f(n) = x^n$

The reason we expect such solution to work out is because for every $n \geq 2$ the equation

$$x^n = x^{n-1} + x^{n-2} \quad (\text{when replaced in } (1))$$

can then be simplified further.

let's multiply by (x^{2-n})

$$x^2 = x^1 + 1$$

This is a simple quadratic equation

$$x^2 - x - 1 = 0$$

Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$x_1 = \frac{1 + \sqrt{5}}{2} \qquad x_2 = \frac{1 - \sqrt{5}}{2}$$

It is easy to check that

$f(n) = x_1^n$ and $f(n) = x_2^n$ satisfy ①, but we also need to include our base cases of

$$f(0) = 1$$
$$f(1) = 1$$

Let us take s, t two arbitrary real numbers

$$f(n) = s x_1 + t x_2$$

$$f(n) = s\left(\frac{1+\sqrt{5}}{2}\right) + t\left(\frac{1-\sqrt{5}}{2}\right)$$

Replacing our base cases:

i) $f(0) = 1$

$$1 = s x_1^0 + t x_2^0$$

$$1 = s + t \quad \text{———} \quad (*)$$

ii) $f(1) = 1$

$$1 = s x_1 + t x_2$$

$$1 = s \cdot \left(\frac{1+\sqrt{5}}{2}\right) + t\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{———} \quad (**)$$

We then solve $(*)$ and $(**)$ simultaneously and obtain

$$s = \frac{1}{2} + \frac{1}{2\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)$$

$$t = 1 - s$$

$$= -\frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)$$

$$f(n) = s x_1^n + t x_2^n$$

$$= \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

$$\boxed{f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}$$

This is the general formula for an element n in the Fibonacci sequence