Asymptotic Analysis of the Optimal Exercise Boundary in American Options, A Free Boundary Value Problem

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1 Introduction

1.1 What are options?

An *option* is a financial derivative, offering the buyer the right, but not the obligation, to buy (in the case of a call option) or sell (for a put option) an underlying asset (S) at a pre-agreed price (denoted strike price K) on or before a certain date (expiration time T). Option prices are split into two components - intrinsic value and extrinsic value (or time value). Intrinsic value can be considered the immediate profit gained if an option is exercised and the stock received is sold at the current market value. Extrinsic value is the premium paid for the time before expiry and approaches zero as the option approaches time T.

$$\text{Option Price} = \text{IV} + \text{TV} \tag{1}$$

A complete example is as follows. Suppose you purchase a call option on a stock currently worth $S = \$100$ with a strike value of $K = \$105$ for $T = 1$ expiration. Currently, $S < K$, and if you attempted to exercise the option, you would purchase shares at \$105 and sell it for \$100, incurring a profit of $-\$5$. However, there is no reason to incur a loss, so the intrinsic value is typically lower bound by 0, denoted by the mathematical equation (for call options)

$$\text{IV} = \max(S - K, 0) = [S - K]^+. \tag{2}$$

Suppose now the stock price S moves to \$110. Now $S > K$, and our option has an intrinsic value of \$5. According the above logic, at expiration T, if the option has intrinsic value, then it will be exercised. Otherwise, it expires worthless.

Options serve various purposes in the financial domain, from hedging and speculation to leveraging positions in the asset from which they are derived [2].

1.2 Pricing European options

Central to the pricing of options is the Black-Scholes model, a framework in financial mathematics that employs a partial differential equation (PDE) to estimate option prices [1]. The Black-Scholes PDE (for calls on a dividend-paying asset) is expressed as:

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC &= 0, \\ C(0, t) &= 0 \\ C(\infty, t) &= Se^{-\delta(T-t)} - Ke^{-r(T-t)} \\ C(S_T, T) &= [S_T - K]^+ \end{aligned} \tag{3}$$

where $C(t, S)$ represents the call option price as a function of the underlying asset price S and time t, K is the strike price, T is the expiration time, σ is the volatility of the stock's returns, r the risk-free interest rate, and δ is the dividend rate assuming a continuous-paying dividend. The solution to this equation yields the following Black-Scholes formula, a closed-form solution for the prices of European options.

$$
\begin{aligned}
C(S,t) &= Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \\
d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - \delta + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
\tag{4}
$$

1.3 European versus American Options and the Free Boundary Problem

The European options described above differ from American options primarily in their exercise terms; European options can only be exercised at expiration, whereas American options may be exercised at any time up to expiration. This allows American option holders to prevent their options from falling below intrinsic value (Figure 1). The additional flexibility embedded in American options complicates their pricing, as the Black-Scholes model, tailored for European options, does not accommodate early exercise features. Pricing American options is significantly more challenging than European options, involving no closed-form solution and no analytical solution, a topic expanded in the following section.

2 American Options as a Free Boundary Problem

2.1 Free Boundary Value Problems

Free boundary value problems (FBVPs) in partial differential equations (PDEs) are a class of mathematical problems where the solution domain (Ω) itself is unknown and must be determined as part of the solution. These problems are characterized by the presence of a *free boundary*, a surface or curve whose location is not specified a priori and is instead influenced by the solution of the PDE. Common examples include the Stefan problem, which models the

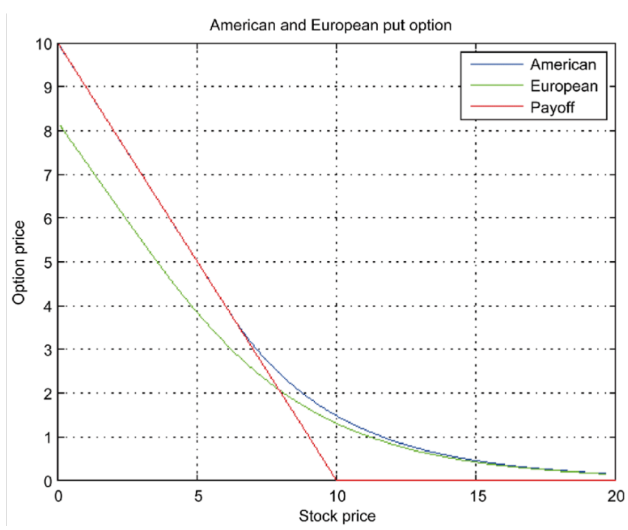


Figure 1: American option vs. European option value

phase change of materials (e.g., ice melting into water), and fluid dynamics problems, where the interface between two fluids evolves over time. Solving FBVPs often involves coupling PDEs with conditions on the moving boundary, typically requiring sophisticated numerical techniques and analytical insights to address both the PDE and the boundary's dynamics.

2.2 Formulating American Options as a FBVP

Figure 2: Sample Continuation and Stopping Region for an American Call Option

Mathematically, pricing an American option can be considered solving the PDE for $u = C(t, S)$ over the unknown domain

$$\Omega : (S, t) \in (0, ?) \times (0, ?). \tag{5}$$

For this paper, we simplify the scenario by only considering the pricing of options on continuous dividend-paying assets. Doing so allows us to define (and later prove) a continuous optimal exercise boundary [4]

$$S^*_{P|C}(\tau), \quad \tau = T - t \tag{6}$$

where if S exceeds S^*_C above or S^*_P below for calls and puts respectively, then the option should be exercised and a free boundary is hit. Thus, we can refine our domain to a continuation and stopping region as follows

$$\begin{aligned} \text{Continuation Region:} \quad & \{(S, t) \in (0, S^*(\tau)) \times (0, \tau)\} \\ \text{Stopping Region:} \quad & \{(S, t) \in (S^*(\tau), \infty) \times (\tau, T)\}, \end{aligned} \tag{7}$$

where the American option price follows the Black Scholes PDF in Equation **??**

A visualization of this for call options is shown in Figure 3.

2.3 An Economic Perspective of Early Exercise

For American *call options*, early exercise may be economically motivated if the dividends to be earned over the remaining life of the option outweigh the time premium that would be lost by exercising the option early. This decision is only relevant when dividends are expected, as represented by S_C, and does not apply to non-dividend-paying assets.

In the case of American *put options*, early exercise might be economically beneficial if the funds from exercising the option can be reinvested into a risk-free asset to earn interest over the remaining term. If this interest income is greater than the time premium lost, early exercise can be justified. This condition, denoted by S_P, is always applicable as long as the risk-free interest rate r is greater than zero, even on non-dividend-paying assets [4].

A mathematical case for both will be presented in the proceeding sections.

3 Asymptotic Analysis of S_C^*

In this section, we determine the functional form of S_C^* close to expiry ($\tau \to 0^+$) as well as with infinite time to expiry ($\tau \to \infty$), a type of option known as a *perpetual option*.

3.1 Properties of S_C^*

We first begin by establishing two properties of S_C^*.

1. For $S \geq S_C^*$, the option is exercised and within the stopping region. $C(S,t) = S - K$ (or the intrinsic value) is the lower bound for the option value.

2. S_C^* is a continuous increasing function of τ

We show property one as follows. Suppose $S >> K$ is true. Then, in our closed-form Black Scholes Equation 4,

$$N(d_1) \approx 1, \quad N(d_2) \approx 1, \tag{8}$$

since d_1 and d_2 are large. Substituting into the closed form equation, we get

$$C(S,t) = Se^{-\delta\tau} - Ke^{-r\tau}. \tag{9}$$

For sufficiently large S, we encounter a situation where the $e^{\delta\tau}$ term may allow the option to fall below the intrinsic value $S - K$. However, the American call option holder would not allow this to happen and would consequently exercise early.

Property two can be reasoned through multiple methods. The simplest method is the following line of thought. As the time to expiry, τ, increases, the time value portion of the option must also increase. Therefore, the required dividend earned to early exercise, which depends on S, must correspondingly increase to match the additional time value that would be sacrificed.

These two properties together will help us perform an asymptotic analysis of S_C^*.

3.2 S_C^* close to expiry ($\tau \to 0^+$)

To analyze the functional form of S_C^* close to expiry, we begin with a modified version of the Black Scholes PDE in equation 3.

$$\begin{aligned} -\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \\ C(0,\tau) = 0 \\ C(S_C^*(\tau),\tau) = S - K \\ C(S(0),0) = [S(0) - K]^+ \end{aligned} \tag{10}$$

Time t has been translated to an equivalent measure of τ in the $-\frac{\partial C}{\partial \tau}$ term. More importantly, using property one, we establish a new boundary condition at $S*_C(\tau)$ with a lower bound value of $S - K$.

Now suppose $S > K$ (if $S < K$ close to expiry, the option has zero intrinsic value and is worthless), rearrange the PDE to solve for $\frac{\partial C}{\partial \tau}$, and substitute in $\tau \to 0^+$.

$$\begin{aligned} \left.\frac{\partial C}{\partial \tau}\right|_{\tau=0^+} &= \frac{1}{2}\sigma^2 S^2 \left.\frac{\partial^2 C}{\partial S^2}\right|_{\tau=0^+} + (r-\delta)S \left.\frac{\partial C}{\partial S}\right|_{\tau=0^+} - r\left.C\right|_{\tau=0^+} \\ &= (r-\delta)S - r(S-K) \\ &= rK - \delta S \end{aligned} \tag{11}$$

This relationship is verified by the equations

$$\begin{aligned} \left.d_1\right|_{\tau=0^+} &= \left.\frac{\ln\left(\frac{S}{K}\right) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right|_{\tau=0^+} \approx \left.\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}}\right|_{\tau=0^+} = +\infty \\ \left.\frac{\partial^2 C}{\partial S^2}\right|_{\tau=0^+} &= \left.\frac{N'(d_1)}{S\sigma\sqrt{\tau}}\right|_{\tau=0^+} = \left.\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{\tau}}\right|_{\tau=0^+} \approx \left.\frac{e^{-d_1^2}}{\sqrt{\tau}}\right|_{\tau=0^+} = 0 \quad \text{Since } e^{\infty} \gg \sqrt{0^+} \\ \left.\frac{\partial C}{\partial S}\right|_{\tau=0^+} &= \left.N(d_1)\right|_{\tau=0^+} = N(+\infty) = 1 \end{aligned} \tag{12}$$

Now, at expiry, we observe that if

$$\frac{\partial C}{\partial \tau}(S, 0^+) < 0 \tag{13}$$

then the call option $C(S,\tau)$ would have fallen below the intrinsic value $S - K$ immediately prior to execution. However, we know that the option would already have been exercised, giving rise to a contradiction. Therefore, we must have

$$\frac{\partial C}{\partial \tau}(S, 0^+) \geq 0 \tag{14}$$

in order for the option to stay alive. Furthermore, we observe the sign change in $\frac{\partial C}{\partial \tau}(S, 0^+)$ at

$$S = \frac{rK}{\delta} \tag{15}$$

Knowing this, we now split our analysis into two cases, $\delta < r$ and $\delta \geq r$.

3.2.1 $\delta < r$

Consider some small time interval Δt before expiry. If $\delta < r$, then over that small interval we have

$$\delta S \Delta t < rK\Delta t \tag{16}$$

which implies that the dividend earned from holding the asset over the time interval is less than the potential interest earned from investing in a risk-free asset. Therefore, the American call option holder would not exercise the option early and we can draw the following conclusions

$$\begin{aligned} S < \frac{rK}{\delta} &\rightarrow \frac{\partial C}{\partial \tau}(S, 0^+) > 0 \\ S > \frac{rK}{\delta} &\rightarrow \frac{\partial C}{\partial \tau}(S, 0^+) < 0. \end{aligned} \tag{17}$$

Hence, we also know that the point of sign change is also our optimal exercise price.

$$\boldsymbol{S_C^*(0^+) = \frac{rK}{\delta}} \tag{18}$$

Furthermore, as an interesting aside, according to this equation, if $\delta \rightarrow 0^+$, then $S_C^*(0^+) \rightarrow +\infty$. Since we know that $S_C^*(\tau)$ is an increasing function, then $S_C^*(\tau) \rightarrow +\infty$ for all τ. This implies that there is no optimal exercise value when dividends do not exist, which matches our economic intuition.

3.2.2 $\delta \geq r$

We take a slightly different approach for $\delta \geq r$. Suppose $S_C^*(0^+) > K$ so the American option remains alive. This implies the relationship $S*_C > S > K$. Then, if we consider the same small time interval Δt, for $\delta \geq r$ and $S > K$, we have

$$\delta S \Delta t \geq rK\Delta t \tag{19}$$

where the dividend earned from holding the asset over the time interval is greater than the potential interest earned from investing in a risk-free asset. If so, then the American option would already have been exercised, bringing about a contradiction. Therefore, we must have

$$S_C^*(0^+) \leq K. \tag{20}$$

Since $S_C^*(0^+) \leq K$, $\S_C^*(\tau) \geq K$ by definition of optimal exercise, and $S_C^*(\tau)$ is increasing, we must have

$$\boldsymbol{S_C^*(0^+) = K} \tag{21}$$

3.2.3 Combined Equation

We can then combine equations 18 and 21 into one equation as follows

$$\boldsymbol{S_C^*(0^+) = \begin{cases} \frac{rK}{\delta}, & \text{if } \delta < r \\ K, & \text{if } \delta \geq r \end{cases} = K \max\left(1, \frac{r}{\delta}\right)} \tag{22}$$

3.3 S^*_C infinite time to expiry ($\tau \to \infty$)

The case of an option with infinite time to expiry is known as a *perpetual option*. Perpetual options tend to have no true financial meaning as they are not traded anywhere on the markets, but their concept makes for interesting mathematical analysis.

At infinite time to expiry, the concept of time value decay does not exist, or $\frac{\partial C}{\partial \tau} = 0$. Therefore, we can reduce the Black Scholes PDE in Equation 3 to a homogenous second-order ordinary differential equation (ODE)

$$\begin{aligned} \frac{1}{2}\sigma^2 S^2 \left.\frac{\partial^2 C_\infty}{\partial S^2}\right|_{\tau=\infty} + (r-\delta)S \left.\frac{\partial C_\infty}{\partial S}\right|_{\tau=\infty} - r\, C_\infty|_{\tau=\infty} = 0 \\ C_\infty(0) = 0 \\ C_\infty\left(S^*_{C,\infty}\right) = S^*_{C,\infty} - K \end{aligned} \tag{23}$$

where $C_\infty(S) = C(S,\infty)$, and $S^*_{C,\infty} = \lim_{\tau\to\infty} S^*(\tau)$.

Given this ODE, we take the ansatz

$$f(S) = S^\lambda, \quad f'(S) = \lambda S^{\lambda-1}, \quad f\text{"}(S) = \lambda(\lambda-1)S^{\lambda-2} \tag{24}$$

and subtitute it in our ODE to get

$$\begin{aligned} \frac{1}{2}\sigma^2 S^2 \lambda(\lambda-1)S^{\lambda-2} + (r-\delta)S\lambda S^{\lambda-1} - rS^\lambda = 0 \\ \iff S^\lambda \left[\frac{1}{2}\sigma^2 S^2 \lambda^2 + (r-\delta-\frac{1}{2}\sigma^2)\lambda - r\right] = 0. \end{aligned} \tag{25}$$

Then determining the roots of λ through the quadratic formula to solve this equation we get

$$\lambda_\pm = \frac{-(r-\delta-\frac{1}{2}\sigma^2) \pm \sqrt{(r-\delta-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{26}$$

Since we have two real solutions, we know the general solution of the ODE takes the form

$$f(S) = c_1 S^{\lambda_+} + c_2 S^{\lambda_-}. \tag{27}$$

Substituting in boundary condition one (from Equation 23), $C_\infty(0) = 0$, we must have $c_2 = 0$ otherwise $f(S)$ is undefined at the boundary condition. Boundary condition two, $C_\infty\left(S^*_{C,\infty}\right) = S^*_{C,\infty} - K$, gives us

$$f(S^*_{C,\infty}) = c_1 S^{\lambda_+} = S^*_{C,\infty} - K \iff c_1 = \frac{S^*_{C,\infty} - K}{{S^*_{C,\infty}}^{\lambda_+}} \tag{28}$$

We now rewrite our equation in the form

$$f(S) = (S^*_{C,\infty} - K)\left(\frac{S}{S^*_{C,\infty}}\right)^{\lambda_+}. \tag{29}$$

To find the true value of C_∞, we need to find the value of $S^*_{C,\infty}$ that maximizes $f(S)$, or

$$C_\infty(S) = \max_{\{S^*_{C,\infty}\}} \left\{ (S^*_{C,\infty} - K) \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+} \right\}. \tag{30}$$

Through basic calculus, we find

$$\begin{aligned}
\frac{dC}{dS^*_{C,\infty}} &= (S^*_{C,\infty} - K) \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+} (-\lambda_+) \left(\frac{1}{S^*_{C,\infty}} \right) + \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+} = 0, \\
&\iff (S^*_{C,\infty} - K) \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+} (\lambda_+) \left(\frac{1}{S^*_{C,\infty}} \right) = \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+}, \\
&\iff 1 = \lambda_+ - \frac{\lambda_+ - K}{S^*_{C,\infty}}, \\
&\iff S^*_{C,\infty} = \frac{\lambda_+ + K}{\lambda_+ - 1}.
\end{aligned} \tag{31}$$

and substitute the results of equations 31 and 26 into Equation 29 to get the following set of solutions

$$\begin{aligned}
\boldsymbol{C_\infty(S; K, \delta)} &= \boldsymbol{\frac{S^*_{C,\infty}}{\lambda_+} \left(\frac{S}{S^*_{C,\infty}} \right)^{\lambda_+}}, \\
\boldsymbol{S^*_{C,\infty}} &= \boldsymbol{\frac{\lambda_+ K}{\lambda_+ - 1}}, \\
\boldsymbol{\lambda_+} &= \boldsymbol{\frac{-(r - \delta - \frac{1}{2}\sigma^2) + \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}}.
\end{aligned} \tag{32}$$

4 Asymptotic Analysis of S^*_P

In this section, we use our results from Section 3 and the well established theory of put-call symmetry to compute the asymptotes of S^*_P.

4.1 Put-Call Symmetry Relationship

In financial theory, call and put options can be considered as means to exchange two assets, cash and stock, where cash grows at the risk-free rate r and stocks "grow" at the dividend rate δ. Purchasing a call option is equivalent to exchanging \$$K$ of cash for 1 stock worth \$$S$. Inversely, a put option exchanges 1 stock worth \$$S$ for \$$K$ cash. Intuitively, if we exchange the roles of cash within the put option, we obtain the following symmetry

$$C(S, \tau; K, r, \delta) = P(K, \tau; S, \delta, r). \tag{33}$$

Using the same line of reasoning, we establish put-call symmetry for the optimal exercise boundary. Let $S^*_C(\tau; r, \delta)$ and $S^*_P(S, \tau; r, \delta)$ be the optimal exercise boundary for calls and puts respectively. When $S = S_C^*(\tau; r, \delta)$, we would exchange \$$K$ cash for 1 stock worth \$$S_C^*$. Similarly, when $S = S_P^*(\tau; r, \delta)$, we would exchange 1 stock worth \$$S_P^*$ for \$$K$ cash. Interchanging the roles of stock and cash in boundary for puts, we get $S_P^*(\tau; \delta, r)$. Now, exchanging \$$K$ cash to stock via a call, and back to cash via the interchanged put provides the following symmetry relation

$$K = \frac{S_C^*(\tau; r, \delta) S_P^*(\tau; \delta, r)}{K} \iff S_P^*(\tau; \delta, r) = \frac{K^2}{S_C^*(\tau; r, \delta)} \tag{34}$$

4.2 Properties of S_P^*

Like S_C^*, S_P^* has two crucial properties.

1. For $S \leq S_P^*$, the option is exercised and within the stopping region. $P(S, \tau) = K - S$ (or the intrinsic value) is the lower bound for the option value.

2. S_P^* is a continuous decreasing function of τ

Property one can be shown through the same logic as Section 3.1 using the closed form BSM for put options. The put-call symmetry established in equation 34 shows an inverse relationship with the optimal exercise boundary of puts and calls, thereby proving property two.

4.3 S_P^* close to expiry ($\tau \to 0^+$)

Finding the precise functional value of S_P^* actually poses a significant mathematical challenge. However, using put-call symmetry, we can easily determine a lower bound using our results from Section 3 as follows

$$\lim_{\boldsymbol{\tau \to)^+}} \boldsymbol{S_P^*(\tau; r, \delta)} = \frac{K^2}{\lim_{\tau \to)^+} S_C^*(\tau; \delta, r)} = \frac{K^2}{K \max(1, \frac{\delta}{r})} = \boldsymbol{K \min(1, \frac{r}{\delta})} \tag{35}$$

We can note that when $\delta \to 0^+$, $S_P^*(0^+) \neq 0$ and with $S_P^*(\tau)$ decreasing, $S_P^*(\tau) \neq 0$. Therefore, even with no dividends, an optimal early exercise price for a put option exists. Similarly, when $r = 0$, $S_P^*(0^+) = 0$ and we know that $S_P^*(\tau) = 0$. When the interest rate is not positive, there exists no optimal early exercise price. These two conditions follow our economic intuition for S_P^*.

4.4 S_P^* infinite time to expiry ($\tau \to \infty$)

The process for S_P^* infinite time to expiry is exactly identical to S_C^* besides beginning with the put version of the BSM PDE. The corresponding solutions are as follows

$$
\begin{aligned}
\boldsymbol{P_\infty(S;K,\delta)} &= \frac{\boldsymbol{S^*_{P,\infty}}}{\boldsymbol{\lambda_-}}\left(\frac{\boldsymbol{S}}{\boldsymbol{S^*_{P,\infty}}}\right)^{\boldsymbol{\lambda_-}}, \\
\boldsymbol{S^*_{P,\infty}} &= \frac{\boldsymbol{\lambda_- K}}{\boldsymbol{\lambda_- - 1}}, \\
\boldsymbol{\lambda_+} &= \frac{\boldsymbol{-(r-\delta-\frac{1}{2}\sigma^2) - \sqrt{(r-\delta-\frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}}{\boldsymbol{\sigma^2}}.
\end{aligned}
\tag{36}
$$

5 Numerical Simulations of S^*_C close to expiry

5.1 The Binomial Pricing Model

We utilize the Cox, Ross, Rubenstein Model (named after the creators), better known as the binomial pricing model, to simulate the price of an option [3]. The binomial pricing model offers a simple, brute-force manner of computing the price of an option throughout time. The pricing model, modified for dividends, works as such.

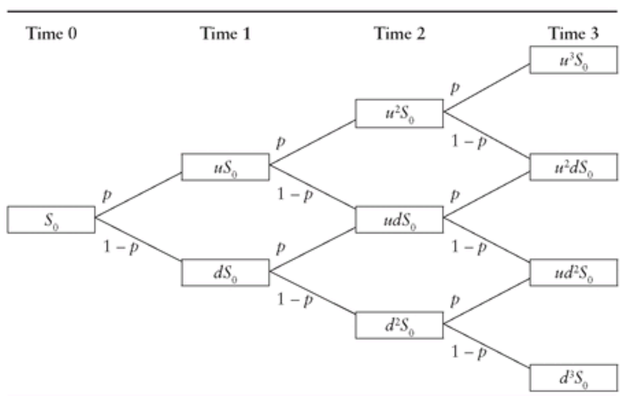


Figure 3: The behavior of S over time in the Binomial Pricing Model

We begin with the price of a stock at time 0, S_0. After some small time interval dt, the price of the stock increases by a factor of u with probability p, or decreases by a factor of d with probability $1-p$. The equations for u, d, and p are as follows

$$
\begin{aligned}
u &= e^{\sigma\sqrt{t}} \\
d &= 1/u \\
p &= \frac{e^{(r-\delta)}dt - u}{u-d}.
\end{aligned}
\tag{37}
$$

The values for u and d are chosen assuming the stock follows a log-normal distribution through time averaging around a risk-free increase in value, $S(t) = S_0e^{rt}$. The $\sqrt{t}$ term results from considering how the standard deviations of returns scale with respect to time. p is known as the risk-neutral probability of a stock increasing in a small time frame, and its formulation results from following the no-arbitrage theory, a topic outside the purview of this paper.

After computing all possible values of S_t, we begin at the end, time T, and compute the value of an option according to the boundary condition,

$$C = \max(S - K, 0) \tag{38}$$

Then, we can recursively backtrack and compute the value of an option for every possible instance of S by the following relationship

$$C_{n-1} = (pC_{n,H} + (1-p)C_{n,T}) * e^{-rdt}, \tag{39}$$

where $C_{n,H}$ represents the option price if the underlying asset increased from time $n-1$ to n and $C_{n,T}$ represents the price if the asset decreased. The e^{-rdt} term discounts the value of the option appropriately.

In the case of American options, we have the additional lower bound condition of the intrinsic value, $S - K$, established in Section 3.1. Therefore, while recursively computing option prices, we find where the computed option price C_{n-1} drops below the intrinsic value $S-K$. The price S corresponding to that drop is the point of our optimal exercise boundary.

The full code used to simulate our exercise boundary is provided in Appendix A.

5.2 Simulation of $S_C^*(\tau)$

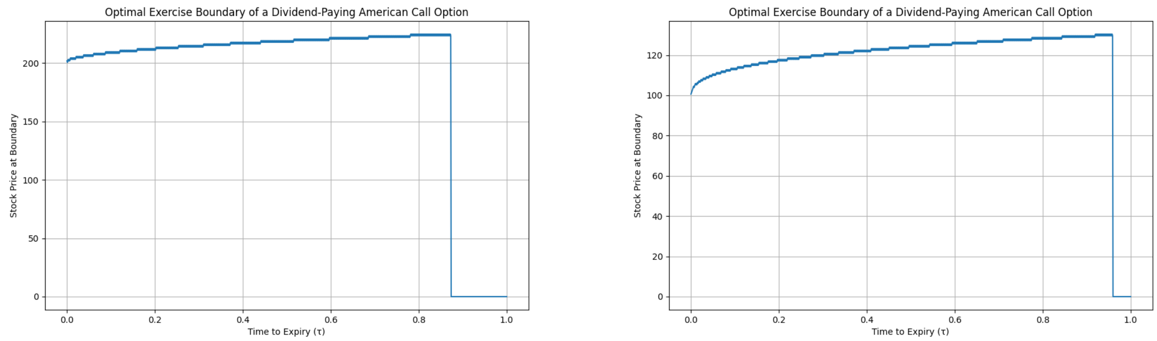
We simulate the call option with the parameters in Table 1. The optimal exercise boundary for the $\delta < r$ case is shown in Figure 4a and $\delta \geq r$ in Figure 4b. In the $\delta < r$ case, since $\frac{r}{\delta} = 2$, we expect $S_C^*(0+)$ to approach $2K = 200$ according to our formulation in Section 3, which we see to be the case. Conversely, in the $\delta \geq r$ case, we expect $S_C^*(0+)$ to approach $K = 100$. The sudden drop to 0 is due to the binomial model not containing large enough discrete values of S_t to determine the optimal exercise boundary, resulting in meaningless values.

Parameter	$\delta < r$ **Value**	$\delta \geq r$ **Value**
S (Initial Stock Price)	100	100
K (Strike Price)	100	100
T (Time to Maturity)	1 year	1 year
r (Risk-free Rate)	0.05	0.025
δ (Dividend Yield)	0.025	0.05
σ (Volatility)	0.20	0.20
N (Number of Simulations)	1000	1000

Table 1: Financial Parameters for the Options model

6 Conclusion

We performed asymptotic analysis of the optimal exercise boundary for puts and calls as they approached expiration as well as with infinite time to analysis. Functional forms of S_C^*



(a) (b)

Figure 4: Plots for Simulated $S_C^*(\tau)$. For (a), $\delta < r$ where $\frac{r}{d} = 2$, so we expect $S_C^*(0^+) = 2K = 200$. For (b), $\delta \geq r$, so we expect $S_C^*(0^+) = K = 100$. Drops to 0 are due to solving limitations of the binomial model.

and S_P^* were determined. Additionally, we performed a numerical simulation of american call options to verify the case of $\tau \to 0^+$. Overall, solving free boundary value problems are mathematically challenging and require numerical simulations. Additionally, determining properties of the free boundary requires sufficient domain knowledge of the problem itself.

A Code for Simulations

The original code can be found at https://github.com/JohnDLee/ACO_Analysis/tree/main.

```
import numpy as np
import matplotlib.pyplot as plt

def american_call_dividend_binomial(S, K, T, r, q, sigma, N):
    dt = T / N
    df = np.exp(-r * dt)
    growth_factor = np.exp((r - q) * dt)
    u = np.exp(sigma * np.sqrt(dt))
    d = 1 / u
    p = (growth_factor - d) / (u - d)

    # Stock price tree initialization
    stock_price = np.zeros((N + 1, N + 1))
    for i in range(N + 1):
        for j in range(i + 1):
            stock_price[j, i] = S * (u ** (i - j)) * (d ** j)

    # Option value tree
    option_value = np.zeros_like(stock_price)
    option_value[:, N] = np.maximum(stock_price[:, N] - K, 0)
```

```
    # Backward induction
    # compute exercise boundary
    exercise_boundary = np.zeros(N)
    for i in range(N - 1, -1, -1):
        for j in range(i + 1):
            hold = (p * option_value[j, i + 1] + (1 - p) * option_value[j + 1,
            exercise = stock_price[j, i] - K
            option_value[j, i] = max(hold, exercise)

            # if we have an exercise greater than hold, we set it as the bounda
            if exercise > hold:
                exercise_boundary[i] = stock_price[j, i]


    return stock_price, option_value, exercise_boundary


# Parameters
S = 100
K = 100
T = 1
r = 0.05
q = 0.025
sigma = 0.20
N = 1000

stock_price, option_value, boundary = american_call_dividend_binomial(S, K, T,

# Time to expiry
tau = np.linspace(T, 0, N)
fig = plt.figure(figsize=(10, 6))
plt.plot(tau, boundary)
plt.title('Optimal Exercise Boundary of a Dividend-Paying American Call Option'
plt.xlabel('Time to Expiry (  )')
plt.ylabel('Stock Price at Boundary')
plt.grid(True)
#plt.show()
fig.savefig('images/r_d.png')
```

References

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[3] John C. Cox, Stephen A. Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263, 1979.

[4] Yue Kwok. *American Options*. 01 2008.