An extension of James' compactness theorem by I. Gasparis

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James' compactness theorem

For any Banach space X, we write B_X for the closed unit ball of X.

James' compactness theorem

Let X be a Banach space, then the following are equivalent.

1. X is reflexive.
2. For all $\Lambda \in X^*$, there exists $x \in B_X$ such that $\Lambda x = \|\Lambda\|$.

Remark

- 2 states that $|\Lambda|$ attains maximum in the closed unit ball of X.
- We have seen 1 $\Rightarrow$ 2 in our homework.

James' compactness theorem

The implication 1 $\Rightarrow$ 2 is easy.

Proof.

Choose a sequence $(x_n) \in B_X$ such that

$$\Lambda x_n \to \sup_{x \in B_X} \Lambda x.$$

Since X is reflexive, B_X is weakly sequentially compact. Thus there is a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \rightharpoonup y$ for some $y \in B_X$. In particular, $\Lambda_{x_{n_i}} \to \Lambda y = \sup_{x \in B_X} \Lambda x$. □

This paper gives a result extending the implication 2 $\Rightarrow$ 1.

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The case where X is separable

In this section, we will prove an extension of James' compactness theorem under the special case where X is separable.

Corollaries 3.3 and 3.5 are crucial general results. The former examines weakly null sequences, and the latter examines relative weak compactness. (These definitions will come later)

Not only will we use them to prove the special case in this section, but we will also need them in the proof of the final main theorem.

Let $\ell_1^+ = \{(a_n) \in \ell_1 \mid a_n \geq 0 \ \forall n\}$ be the positive cone of ℓ_1. In this paper, sequences that in some sense behave like a "basis" for ℓ_1^+ play an important role.

ℓ_1^+-sequences

Definition

Let $(e_n) \subseteq X$ be normalized. We say (e_n) is an ℓ_1^+-sequence if there is $c > 0$ such that

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| \geq c \sum_{n=1}^{\infty} a_n \text{ for all } (a_n) \in \ell_1^+.$$

In this case, (e_n) is called a $c - \ell_1^+$-sequence. In particular, if $c = 1$, then we say (e_n) generates ℓ_1^+ isometrically.

Remark

By triangle inequality, the c above can only be between 0 and 1, and the inequality has to be tight if $c = 1$.

Sequences generating ℓ_1^+ almost isometrically

Definition

Let $(e_n) \subseteq X$ be normalized. We say (e_n) generates ℓ_1^+ almost isometrically if for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $(e_n)_{n \geq n_\varepsilon}$ is an $1/(1+\varepsilon)$-sequence. That is,

$$\left\| \sum_{n=n_\varepsilon}^{\infty} a_n e_n \right\| \geq \frac{1}{1+\varepsilon} \sum_{n=n_\varepsilon}^{\infty} a_n \text{ for all } (a_n) \in \ell_1^+.$$

We ask when it is possible to extract an ℓ_1^+ subsequence from a given sequence.

Extracting ℓ_1^+-subsequences

We say a sequence is weakly null if it weakly converges to 0.

Observation 1

If $(x_n) \subseteq X$ is bounded, nonzero, and $x_n \not\rightharpoonup 0$, then $(x_n/\|x_n\|)$ has an ℓ_1^+-subsequence.

Proof.

- By boundedness, $(e_n) = (x_n/\|x_n\|)$ also has $e_n \not\rightharpoonup 0$.
- By definition, $\exists \Lambda \in B_{X^*}$, $(v_n) \subseteq (e_n)$, $\delta > 0$ so that $|\Lambda v_n| > \delta \ \forall n \in \mathbb{N}$.
- We may assume there is a subsequence $(w_n) \subseteq (v_n)$ such that $\mathrm{Re}(\Lambda w_n) > (1/\sqrt{2})\delta$ for all $n \in \mathbb{N}$, then

$$\left\|\sum_{n=1}^{\infty} a_n w_n\right\| \geq \left|\Lambda\left(\sum_{n=1}^{\infty} a_n w_n\right)\right| = \left|\sum_{n=1}^{\infty} a_n \Lambda w_n\right| \geq \frac{\delta}{\sqrt{2}} \sum_{n=1}^{\infty} a_n \ \forall (a_n) \in \ell_1^+.$$

$\square$

An inequality about ℓ_1^+-sequences

Lemma 3.2 is quite technical, and is only needed in Corollary 3.3.

Lemma 3.2

Let X be a Banach space and $(e_n) \subseteq X$ be a normalized ℓ_1^+-sequence, then there exist scalars $a_n > 0$ such that for all $n \in \mathbb{N}$,

$$(1 + a_n) \left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| \sum_{i=1}^{n+1} a_i e_i \right\| < 1.$$

Proof of Lemma (3.2).

- Let (e_n) be a c-ℓ_1^+-sequence. We will construct (a_n) inductively.
- Choose $1 < a_1 < e^{-1/c}$, then $(1 + a_1)\|a_1 e_1\| = b_1(1 + b_1) < 1$.
- By IVT applied to $\phi_2 : [0, \infty) \to \mathbb{R}$, $t \mapsto \|a_1 e_1 + te_2\|$, we find $a_2 > 0$ such that $(1 + a_1)\|a_1 e_1\| = \|a_1 e_1 + a_2 e_2\| < 1$.

An inequality about ℓ_1^+-sequences

Proof of Lemma (3.2) (Cont.)

- Assume we already have $a_1, \ldots, a_n > 0$ such that

$$(1 + a_k)\left\|\sum_{i=1}^{k} a_i e_i\right\| = \left\|\sum_{i=1}^{k+1} a_i e_i\right\| < 1 \ \forall 1 \leq k \leq n-1,$$

then by induction hypothesis,

$$\begin{aligned}(1 + a_n)\left\|\sum_{i=1}^{n} a_i e_i\right\| &= (1 + a_n)(1 + a_{n-1})\left\|\sum_{i=1}^{n-1} a_i e_i\right\| = \ldots \\ &= a_1 \prod_{i=1}^{n}(1 + a_i) \leq a_1 e^{a_1 + \ldots + a_n} \leq a_1 e^{\|a_1 e_1 + \ldots + a_n e_n\|/c} \leq a_1 e^{1/c} < 1,\end{aligned}$$

and we can again find a_{n+1} by IVT. □

A way to examine weakly null sequences

Corollary 3.3

Let $(x_n) \subseteq X$ be bounded and $F \subseteq B_{X^*}$ be such that

$$\forall (a_n) \in \ell_1^+,\ \exists \Lambda \in F \text{ such that } \Lambda\left(\sum_{n=1}^{\infty} a_n x_n\right) = \left\|\sum_{n=1}^{\infty} a_n x_n\right\|.$$

Then, $x_n \rightharpoonup 0$ as long as $\Lambda x_n \to 0$ for all $\Lambda \in F$.

Remark

Originally, for $x_n \rightharpoonup 0$, we need to check $\Lambda x_n \to 0$ for all $\Lambda \in X^*$. Corollary 3.3 shows that it suffices to check for all $\Lambda \in F$ if F has such nice properties.

A way to examine weakly null sequences

Proof of Corollary (3.3).

- Suppose (x_n) is not weakly null, then by observation 1, there is an ℓ_1^+-subsequence of $(x_n/\|x_n\|)$.
- By boundedness and linearity, this subsequence also satisfies the hypothesis for (x_n).
  Therefore, we may assume (x_n) is a normalized ℓ_1^+-sequence, and try to derive a contradiction form this.

A way to examine weakly null sequences

Proof of Corollary (3.3) (Cont.)

- By Lemma (3.2), $\exists a_n > 0$ such that for $u_n = \sum_{i=1}^{n} a_i x_i$,

$$(1 + a_n)\|u_n\| = \|u_{n+1}\| < 1 \ \forall n \in \mathbb{N}.$$

In particular, $u = \sum_{i=1}^{\infty} a_i x_i \in X$ is well-defined and

$$\|u\| = \|u_n\|(1 + a_n)(1 + a_{n+1}) \ldots .$$

We want to show $|\Lambda u| < \|u\| \ \forall \Lambda \in F$, which is a contradiction.

- For all $\Lambda \in \mathcal{F}$, choose $0 < \delta < b_1/2$ and $m \in \mathbb{N}$ such that $|\Lambda x_n| < \delta$, $a_n < 1$ for all $n \geq m$.
Define $t_n = \|u\|/\|u_n\|$, then $t_n \searrow 1$ and $2\delta < b_1 \leq \|u_n\| = \|u\|/t_n$.

A way to examine weakly null sequences

Proof of Corollary (3.3) (Cont.)

The following estimations gives $\Lambda u < \|u\|$. The point is that for $n \geq m$, the contribution of $|\Lambda x_n|$ and a_n can be controlled.

$$
\begin{aligned}
|\Lambda u| &\leq \left|\Lambda\left(\sum_{n=1}^{m} a_n x_n\right)\right| + \left|\Lambda\left(\sum_{n=m+1}^{\infty} a_n x_n\right)\right| \leq \|u_m\| + \sum_{n=m+1}^{\infty} a_n |\Lambda x_n| \\
&\leq \|u_m\| + 2\delta \sum_{n=m+1}^{\infty} \frac{a_n}{1+a_n} \leq \|u_m\| + 2\delta \ln\left(\prod_{n=m+1}^{\infty} \frac{a_n}{1+a_n}\right) \\
&= \|u_m\| + 2\delta \ln \frac{\|u\|}{\|u_{m+1}\|} = \frac{\|u\|}{t_m} + 2\delta \ln(t_{m+1}) \leq \frac{\|u\|}{t_m} + 2\delta(t_m - 1) \\
&= \|u\| \left(\frac{1}{t_m} + \frac{2\delta}{\|u\|}(t_m - 1)\right) < \|u\|.
\end{aligned}
$$

□

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Examining relative weak compactness

Definition

Let X be a Banach space. We say $K \subseteq X$ is relatively compact if the closure of K is compact in X.

Observation 2

Let X be a Banach space and $K \subseteq X$ be bounded, then the following are equivalent.

1. K is relatively weakly compact. ($\overline{K}^w$ is weakly compact in X)
2. The weak* closure $\overline{J(K)}^{w^*}$ is contained in $J(X)$, where $J : X \to X^{**}$ is the canonical identification.

When we view a set A in a Banach space X as a set in X^{**} via $J : X \to X^{**}$, we may write A for $J(A)$ without confusion.

Examining relative weak compactness

Proof of Observation 2.

- For ① $\Rightarrow$ ②, note that $J : X \to X^{**}$ is (w, w^*)-continuous and $\overline{K}^w$ is weakly compact in X, so $J(\overline{K}^w)$ is weak*ly compact in X^{**}. Thus $J(\overline{K}^w)$ is closed in X^{**}, and since it contains $J(K)$, we get $\overline{J(K)}^{w^*} \subseteq J(\overline{K}^w) \subseteq J(X)$.
- For ② $\Rightarrow$ ①, first note that by Hahn-Banach theorem we know $\overline{J(K)}^{w^*}$ is bounded in X^{**}. Thus $\overline{J(K)}^{w^*}$ is weak*ly compact. Since $\overline{J(K)}^{w^*} \subseteq J(X)$, and the weak and weak* topologies of X^{**} coincide on $J(X)$, we see $J(\overline{K}^w) = \overline{J(K)}^w = \overline{J(K)}^{w^*}$ and they are weakly compact in $J(X)$. Hence $\overline{K}^w$ is weakly compact in X. □

What we will get if $\overline{K}^{w^*} \setminus X \neq \emptyset$

Lemma 3.4 is needed in Corollary 3.5, where we will prove the relative weak compactness of something by contradiction.

Lemma 3.4

Let X be a Banach space and $K \subseteq X$. View X, K as sets in X^{**} via J. Suppose that $\overline{K}^{w^*} \setminus X \neq \emptyset$, then $\exists (x_n) \subseteq K$ and bounded $(\Lambda_n) \subseteq X^*$, $\delta > 0$ such that for all $n \in \mathbb{N}$, we have

$$|\Lambda_n x_i| < \frac{1}{n} \ \forall i < n, \text{ and } |\Lambda_n x_i| \geq \delta \ \forall i \geq n.$$

We need the following theorem, which was proved in class.

Goldstine's theorem

Let X be a Banach space, then B_X as a subset of X^{**} via J is dense in $B_{X^{**}}$ under the weak* topology.

What we will get if $\overline{K}^{w^*} \setminus X \neq \emptyset$

Proof of Lemma (3.4) .

- Choose $x^{**} \in \overline{K}^{w^*} \setminus X$. By Hahn-Banach theorem, $\exists x^{***} \in X^{***}$ and $\delta > 0$ such that $|x^{***}(x^{**})| > \delta$ and $x^{***}|_X = 0$.
- By Goldstine's theorem applied to the closed ball in X^* with radius $\|x^{***}\|$, we find $\Lambda_1 \in X^*$, $\|\Lambda_1\| \leq \|x^{***}\|$ such that $|x^{**}\Lambda_1| > \delta$. $x^{**} \in \overline{K}^{w^*}$, so $\exists x_1 \in K$ such that $|\Lambda_1 x_1| > \delta$.

Proof of Lemma (3.4) (Cont.)

Suppose we have $x_1, \ldots, x_{n-1} \in K$, $\Lambda_1, \ldots, \Lambda_{n-1} \in X^*$ such that

1. $\|\Lambda_i\| \leq \|x^{***}\|$, and $|\Lambda_i(x_j)| < 1/i$ for all $j < i$.
2. $|\Lambda_i(x_j)| > \delta$ for all $j \geq i$.
3. $|x^{**}\Lambda_i| > \delta$ for all i.

Since $|x^{***}(x^{**})| > \delta$, $x^{***}|_X = 0$ and $x_1, \ldots, x_n \in X$, again by Goldstine's theorem we find $\Lambda_n \in X^*$, $\|\Lambda_n\| \leq \|x^{***}\|$ such that $|\Lambda_n(x_j)| < 1/n \ \forall j \leq n$ and $|x^{**}\Lambda_n| > \delta$.

Moreover, $x^{**} \in \overline{K}^{w^*}$, so $\exists x_n \in K$ such that $|\Lambda_i x_n| > \delta$ for all $i \leq n$. Thus we can indeed construct (x_n) and (Λ_n) inductively. □

Eberlein-Šmulian theorem

We introduce the Eberlein-Šmulian theorem, whose trivial side is needed later and nontrivial side is a consequence of Lemma 3.4.

Eberlein-Šmulian theorem

Let X be a Banach space and $K \subseteq X$, then K is relatively weakly compact if and only if K is relatively weakly sequentially compact.

Proof.

First we prove $(\Rightarrow)$.

- Let $(x_n) \subseteq \overline{K}^w$ and $S = \{x_n \mid n \in \mathbb{N}\}$. We may assume S is infinite.
- If (x_n) has no convergent subsequence, then S has no limit points, and hence is a closed subset of $\overline{K}^w$. Thus S is compact.
- However, for all $x_n \in S$, $\exists$ open $U_n \ni x_n$ such that $U_n \cap S = \{x_n\}$. By compactness, S can be covered by finitely many U_n's, which is a contradiction.

Proof.

Next we prove ($\Leftarrow$).

- By observation 2, it suffices to show that $\overline{K}^{w*} \subseteq X$ in X^{**}. If not, we choose $(x_n) \in K$, bounded $(\Lambda_n) \in X^*$, $\delta > 0$ by Lemma 3.4.
- Let Λ be a weak* cluster point of (Λ_n), and x be a weak* cluster point of (x_n). Since $\Lambda x_i = 0$ for all i, we have $\Lambda x = 0$. However, since $|\Lambda_i x| > \delta$ for all i, we have $|\Lambda x| \geq \delta$, which is a contradiction. □

Relative weak compactness obtained from the compactness of a restriction map

Corollary 3.5

Let X be a Banach space, $K \subseteq X$ be bounded, and view $X \subseteq X^{**}$ via J. Let $L = \overline{K}^{w^*}$ and define $R : X^* \to C(L)$, $\Lambda \mapsto \Lambda|_L$, where $C(L)$ is the space of functions continuous on L, equipped with the sup norm. If R is weakly compact, then K is relatively weakly compact.

Remark

Corollary 3.5 allow us to derive the relative compactness of K via the compactness of R, which is useful if we have more information about the latter.

Relative weak compactness obtained from the compactness of a restriction map

Proof of Corollary (3.5) .

- By observation 2, it suffices to show $L \subseteq X$. Suppose $L \not\subseteq X$.
- By Lemma 3.4 we get $(x_n) \in K$, $(\Lambda_n) \in X^*$, $\delta > 0$ such that

$$|\Lambda_n x_i| < 1/n \ \forall i < n \text{ and } |\Lambda_n x_i| > \delta \ \forall i \geq n.$$

- Let $f_n = R(\Lambda_n)$. Since R is weakly compact, (f_n) has a weakly convergent subsequence. Let f be a weak cluster point of (f_n), then since $\Lambda_n x_i < 1/n$ for large n, we get $f(x_i) = 0$ for all i.
- L is bounded and weak*-closed, so it is weak*-compact. By the Eberlein-Šmulian theorem, L is weak*-sequentially compact, so $(x_n) \in X^{**}$ has a weak* cluster point x^{**}. Then since $f_i(x^{**}) > \delta$ for all i, we get $|f(x^{**})| \geq \delta$, a contradiction. $\square$

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Theorem 3.6

Let X be a Banach space where $B_{X^*} \subseteq X^*$ is weak* sequentially compact, and $K \subseteq X$ be bounded. If every $\Lambda \in X^*$ attains its norm on K, then K is relatively weakly compact.

Remark

- B_{X^*} is weak* compact by the Banach-Alaoglu theorem. Moreover, if X is separable, then B_{X^*} is metrizable, so B_{X^*} is weak* sequentially compact.
- Taking $K = B_X$, we have K is weakly compact iff X is reflexive.

James' compactness theorem in separable spaces

Proof of Theorem (3.6) .

- Define $L = \overline{K}^{w^*}$ and $R : X^* \to C(L)$ as before. We may assume that L is contained in $B_{X^{**}}$. By corollary 3.5, it suffices to show that R is weakly compact.
  Given a bounded $(\Lambda_n) \in X^*$, we will find a weakly convergent subsequence of $(R\Lambda_n)$.
- Since B_{X^*} is weak* sequentially compact, we may assume that Λ_n weak*ly converges to some Λ.

James' compactness theorem in separable spaces

Proof of Theorem (3.6) (Cont.)

- Let $f = R\Lambda$ and $f_n = R\Lambda_n$, then $(f_n - f)$ is bounded in $C(L)$ and $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in K$.
- For all $(a_n) \in \ell_1^+$, by definition we have

$$\left\|\sum_{n=1}^{\infty} a_n(f_n - f)\right\| = \sup_{x^{**}\in L}\left|\sum_{n=1}^{\infty} a_n x^{**}(\Lambda_n - \Lambda)\right| = \sup_{x\in K}\left|\sum_{n=1}^{\infty} a_n(\Lambda_n - \Lambda)x\right|.$$

  By assumption, the right hand side is attained at some $x \in K$.
- $(f_n - f) \in C(L) \subseteq X^{***}$ is bounded, and $K \subseteq X^{**} \subseteq X^{****}$ attains the norm of $\sum a_n(f_n - f)$ for all $(a_n) \in \ell_1^+$. Moreover, we have shown that $\lim_{n\to\infty} f_n(t) = f(t)$ for all $t \in K$.
  By Corollary 3.3, $f_n \rightharpoonup f$. Hence $(R\Lambda_n)$ has a weakly convergent subsequence. □

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Proof of the main theorem

In this section, we will prove the general extension of James' compactness theorem.

Although a bunch of lemmas and corollaries come before the main theorem, we can divide them into two parts.

The first part (4.1 to 4.4) deals with sequences generating ℓ_1^+ almost isometrically and admissible collections, while the second part (4.5 to 4.8) deals with admissible collections with respect to symmetric sets of functionals. (Again, these definition will come later)

To prove the main theorem, we will mainly use Corollaries 3.3, 3.5 and 4.8. We also need the notion of nets. Some additional theorems are required, and we will treat them as facts.

Some estimations analogous to Lemma 3.2

In Lemma 3.2 we established inequalities about ℓ_1^+ sequences. In Lemma 4.1, we do the same thing for sequences generating ℓ_1^+ almost isometrically. Again, this is technical and is only needed in Corollary 4.2.

Lemma 4.1

Let X be a Banach space and (e_n) be a normalized sequence in X generating ℓ_1^+ almost isometrically. Then for all $0 < \delta_0 < 1$, there exist $m_0, n_0 \in \mathbb{N}$ and scalars $b_n > 0$ such that for $x_n = e_{m_0+n}$, we have

1. $\left\|\sum_{i=1}^{n_0} b_i x_i\right\| = \delta_0$.
2. $(1 + b_n)\left\|\sum_{i=1}^{n} b_i x_i\right\| = \left\|\sum_{i=1}^{n+1} b_i x_i\right\| < 1 \quad \forall n \geq n_0$.
3. $\left(\prod_{i=n}^{\infty}(1 + b_i)\right)\left\|\sum_{i=1}^{n} b_i x_i\right\| = \left\|\sum_{i=1}^{\infty} b_i x_i\right\| \leq 1, \ \forall n \geq n_0$.

Some estimations analogous to Lemma 3.2

Proof of Lemma (4.1) .

- We first choose $0 < \varepsilon < 1 - \delta_0$ such that $\delta_0 e^{(1+\varepsilon)(1+2\varepsilon-\delta_0)} < 1$.
- Since (e_n) generates ℓ_1^+ almost isometrically, there exists $m_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i \geq m_0} a_i e_i \right\| \geq (1+\varepsilon)^{-1} \sum_{i \geq m_0} a_i, \ \forall (a_i)_{i \geq m_0} \in \ell_1^+.$$

  Set $x_n = e_{m_0+n}$ for all $n \in \mathbb{N}$.
- Next we choose $n_0 \in \mathbb{N}$ such that

$$1 + \varepsilon < n_0 \varepsilon$$

  and set $v_0 = \frac{1}{n_0} \sum_{i=1}^{n_0} x_i$. Note that $\|v_0\| \geq (1+\varepsilon)^{-1}$.

Some estimations analogous to Lemma 3.2

Proof of Lemma (4.1) (Cont.)

- Define
$$b_i = \frac{\delta_0}{n_0\|v_0\|} \; \forall i \le n_0, \; u_{n_0} = \sum_{i=1}^{n_0} b_i x_i,$$
then $\|u_{n_0}\| = \delta_0$ and $0 < b_i < \varepsilon$ for all $i \le n_0$. In particular, 1 is satisfied. We shall inductively construct (b_n) for $n > n_0$.
- Since
$$(1 + b_{n_0})\|u_{n_0}\| \le (1 + \varepsilon)\delta_0 < e^{\varepsilon}\delta_0 < \delta_0 e^{1-\delta_0} < 1,$$
applying the IVT in a manner similar to that in the proof of Lemma 3.2, we find $b_{n_0+1} > 0$ such that
$$(1 + b_{n_0})\|u_{n_0}\| = \|u_{n_0} + b_{n_0+1}x_{n_0+1}\|.$$

Some estimations analogous to Lemma 3.2

Proof of Lemma (4.1) (Cont.)

- Inductively, assume that $n > n_0$ and that we have constructed scalars $b_{n_0}, \ldots, b_n > 0$ satisfying

$$(1 + b_k)\left\|\sum_{i=1}^{k} b_i x_i\right\| = \left\|\sum_{i=1}^{k+1} b_i x_i\right\| < 1, \ n_0 \le k \le n-1.$$

By our construction, we have the following estimations.

$$(1 + b_n)\left\|\sum_{i=1}^{n} b_i x_i\right\| = \left[\prod_{i=n_0}^{n}(1 + b_i)\right]\left\|\sum_{i=1}^{n_0} b_i x_i\right\| \le \delta_0 e^{\sum_{i=n_0}^{n} b_i}$$
$$\le \delta_0 e^{(1+\varepsilon)\|\sum_{i=n_0}^{n} b_i x_i\|}.$$

Some estimations analogous to Lemma 3.2

Proof of Lemma (4.1) (Cont.)

- Now it remains to show $\| \sum_{i=n_0}^{n} b_i x_i \| \leq 1 + 2\varepsilon - \delta_0$.
  Were this true, then

$$(1 + b_n) \left\| \sum_{i=1}^{n} b_i x_i \right\| \leq \delta_0 e^{(1+\varepsilon)\| \sum_{i=n_0}^{n} b_i x_i \|} < 1,$$

and once again the IVT gives some $b_{n+1} > 0$ such that

$$(1 + b_n) \left\| \sum_{i=1}^{n} b_i x_i \right\| = \left\| \sum_{i=1}^{n+1} b_i x_i \right\| < 1,$$

meaning that we have inductively constructed scalars $b_n > 0$ satisfying 1 and 2. It follows that $\sum_n b_n$ is a convergent series and hence 3 is an immediate consequence of 2.

Proof of Lemma (4.1) (Cont.)

- To show $\|\sum_{i=n_0}^{n} b_i x_i\| \leq 1 + 2\varepsilon - \delta_0$, we observe that

$$\left\|\sum_{i=1}^{n_0-1} b_i x_i\right\| + \left\|\sum_{i=n_0}^{n} b_i x_i\right\| \leq \sum_{i=1}^{n} b_i \leq (1+\varepsilon)\left\|\sum_{i=1}^{n} b_i x_i\right\| \leq 1+\varepsilon$$

and that

$$\left\|\sum_{i=1}^{n_0-1} b_i x_i\right\| \geq \left\|\sum_{i=1}^{n_0} b_i x_i\right\| - b_{n_0} \geq \delta_0 - \varepsilon.$$

Therefore,

$$\left\|\sum_{i=n_0}^{n} b_i x_i\right\| \leq 1 + \varepsilon - (\delta_0 - \varepsilon) = 1 + 2\varepsilon - \delta_0. \qquad \square$$

Set of functionals attaining the norm

Corollary 4.2

Let X be a Banach space and (e_n) be a normalized sequence in X generating ℓ_1^+ almost isometrically. Let $F \subseteq B_{X^*}$ be such that $\sum_n a_n e_n$ attains its norm at some element of F for all $(a_n) \in \ell_1^+$. Then for all $0 < \delta < 1$, there exists $\Lambda \in F$ such that

$$\limsup_n |\Lambda e_n| \geq \delta.$$

Moreover, if every subsequence of (e_n) admits a τ_F-cluster point which attains its norm at some element of F, then there exists $\Lambda \in F$ such that

$$\limsup_n |\Lambda e_n| = 1.$$

Despite proving something different from Corollary 3.3, the idea and arguments are similar in essence.

Set of functionals attaining the norm

Proof of Corollary (4.2) (Cont.)

- For the first part, suppose not, then $\exists 0 < \delta < 1$ such that

$$\limsup_n |\Lambda e_n| < \delta, \ \forall \Lambda \in F.$$

Choose $0 < \delta_0 < 1$ and $\varepsilon_0 > 0$ such that

$$\delta_0 > \delta(1 + \varepsilon_0).$$

- By Lemma 4.1, we can find $n_0 \in \mathbb{N}$, scalars $b_n > 0$ and a subsequence (x_n) of (e_n) such that
  1. $\left\|\sum_{i=1}^{n_0} b_i x_i\right\| = \delta_0$.
  2. $(1 + b_n)\left\|\sum_{i=1}^{n} b_i x_i\right\| = \left\|\sum_{i=1}^{n+1} b_i x_i\right\| < 1 \ \forall\ n \geq n_0$.

Proof of Corollary (4.2) (Cont.)

- Set $u_n = \sum_{i=1}^{n} b_i x_i$ for $n \in \mathbb{N}$, and $u = \sum_n b_n x_n$, then

$$(1 + b_n)\|u_n\| = \|u_{n+1}\| \text{ and } \prod_{i=n}^{\infty}(1 + b_i)\|u_n\| = \|u_{n+1}\| \ \forall\ n \geq n_0.$$

  In particular, $\|u_n\| < \|u_{n+1}\| < \|u\|\ \forall n \geq n_0$.
- Set $t_n = \|u\|/\|u_n\|$ for $n \geq n_0$, then we obtain a sequence of scalars $(t_n)_{n \geq n_0}$, which strictly decreases to 1.
- Choose $\Lambda \in F$, $m > n_0$ so that $|\Lambda x_n| < \delta$ and $b_n < \varepsilon_0$ for all $n > m$. Using estimations similar to those in Corollary 3.3, we will get $|\Lambda u| < \|u\|\ \forall \Lambda \in F$, contradicting our hypothesis that u attains its norm at some element of F.

Set of functionals attaining the norm

Proof of Corollary (4.2) (Cont.)

- For the moreover part, suppose $\limsup_n |\Lambda e_n| < 1 \;\; \forall \Lambda \in F$, then

$$|\Lambda x| < 1 \text{ for all } \Lambda \in F \text{ and } \tau_F\text{-cluster point } x \text{ of } (e_n).$$

- Successive applications of the first part of this corollary now yield a nested sequence $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ of infinite subsets of $\mathbb{N}$, and a sequence $(\Lambda_n) \in F$ so that

$$|\Lambda_n e_i| > 1 - \frac{1}{n}, \;\; \text{for all } i \in M_n, \;\; n \in \mathbb{N}.$$

We choose integers $m_1 < m_2 < m_3 < \ldots$ with $m_n \in M_n \; \forall n \in \mathbb{N}$. Our hypothesis now yields a τ_F-cluster point x_0 of (e_{m_n}) which attains its norm at some element of F.

Set of functionals attaining the norm

Proof of Corollary (4.2) (Cont.)

- Since

$$|\Lambda x| < 1 \text{ for all } \Lambda \in F \text{ and } \tau_F\text{-cluster point } x \text{ of } (e_n),$$

we have $\|x_0\| < 1$. However, since

$$|\Lambda e_i| > 1 - \frac{1}{n}, \quad \text{for all } i \in M_n, \ n \in \mathbb{N},$$

we have $\|x_0\| \geq 1$, which is a contradiction. Therefore, there exists $\Lambda \in F$ such that

$$\limsup_n |\Lambda e_n| = 1. \qquad \square$$

F-admissibility

In Corallary 4.2, we discussed a set F of functionals having a good property about attaining norms with respect to a normalized sequence generating ℓ_1^+ almost isometrically. This motivates the following definition.

Definition (4.3)

Let X be a Banach space and $F \subseteq B_{X^*}$. For $K \subseteq X$, we say K is F-admissible if

1. K is bounded.
2. K is τ_F-compact.
3. $\forall (x_n) \in K$ and $(a_n) \in \ell_1$, $\sum_n a_n x_n$ attains its norm at some $\Lambda \in F$.

A good subsequence from an F-admissible set

Lemma 4.4

Let X be a Banach space, $F \subseteq B_{X^*}$, and $K \subseteq X$ be F-admissible.

Let $(x_n) \subseteq K$ be a sequence. Let (I_n) be a sequence of finite subsets of $\mathbb{N}$ with the same cardinality, (λ_n) be a bounded sequence of scalars, and set $u_n = \sum_{i \in I_n} \lambda_i x_i$ for all $n \in \mathbb{N}$.

Assume that (u_n) is normalized and generates ℓ_1^+ almost isometrically, then there exists $\Lambda \in F$ such that $\limsup_n |\Lambda u_n| = 1$.

Remark

Originally we only have an ordinary sequence (x_n) in K.
If we can combine the x_i's using a fixed number of bounded scalars such that the resulting (u_n) is normalized and generates ℓ_1^+ almost isometrically, (u_n) will satisfy the "Moreover" part in Corollary 4.2.

A good subsequence from an F-admissible set

Proof of Lemma (4.4) .

- Since K is F-admissible, for all $(a_n) \in \ell_1$, $\sum_n a_n u_n$ attains its norm at some element of F. So by Corollary 4.2, it suffices to show that every subsequence of (u_n) admits a τ_F-cluster point which attains its norm at some element of F.
- A subsequence of (u_n) also satisfies the hypotheses on (u_n), so it is enough to show that (u_n) admits a τ_F-cluster point which attains its norm at some element of F.
- Let $d = |I_n|$ and write $m(n, i)$ for the i-th element of I_n for all $n \in \mathbb{N}$ and $i \leq d$.
- Since (λ_n) is bounded, by passing to a subsequence, we may assume that $\lim_n \lambda_{m(n,i)} = \mu_i \in \mathbb{C}$ for all $i \leq d$.

A good subsequence from an F-admissible set

Proof of Lemma (4.4) (Cont.)

- By the F-admissibility of K, we know K^d is compact in the product topology induced by τ_F.
- Define $\vec{x_n} = (x_{m(n,1)}, \ldots, x_{m(n,d)})$ for all $n \in \mathbb{N}$. By compactness, $(\vec{x_n})$ has a cluster point $(z_1, \ldots, z_d) \in K^d$.
- $u_n = \sum_i \lambda_i x_i = \sum_{i=1}^d \lambda_{m(n,i)} x_{m(n,i)}$, so (u_n) has a τ_F-cluster point $\sum_{i=1}^d \mu_i z_i$, which attains its norm at some element of F since it is in K and K is F-admissible. □

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Equivalence to the ℓ_1-basis

Having established Lemma 4.4, our goal now is to prove Corollary 4.8, whose argument requires sequences that are equivalent to the ℓ_1-basis.

Definition

Let X, Y be Banach spaces and $\{x_n\}$, $\{y_n\}$ be bases of X and Y, respectively. We say $\{x_n\}$, $\{y_n\}$ are equivalent if there exists an isomorphism $T : X \to Y$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

Definition

Let X be a Banach space, $(x_n) \in X$. We say (x_n) is an ℓ_1-sequence if

1. $\{x_n\}$ is a basis of $\overline{\operatorname{span}\{x_1, x_2, \ldots\}}$.
2. $\{x_n\}$ is equivalent to the usual basis of ℓ_1.

Combining elements in the ℓ_1-basis while preserving the isometric generation of ℓ_1^+

Observation 3

Let (e_n) be a sequence in a Banach space isometrically equivalent to the ℓ_1-basis. Given $d \in \mathbb{N}$, $m_1 < \ldots < m_d \in \mathbb{N}$, $(\lambda_1 \ldots, \lambda_{d+1}) \in (\mathbb{C}^\times)^{d+1}$ with $|\lambda_1| + \ldots + |\lambda_{d+1}| = 1$, if we define

$$u_n = \sum_{i=1}^{d} \lambda_i e_{m_i} + \lambda_{d+1} e_n$$

for all $n > m_d$, then $(u_n)_{n>m_d}$ generates ℓ_1^+ isometrically.

Combining elements in the ℓ_1 basis while preserving the isometric generation of ℓ_1^+

Proof of Observation 3.

For all $(a_n) \in \ell_1^+$, we have

$$\left\|\sum_{n>m_d} a_n u_n\right\| = \left\|\sum_{n>m_d} a_n(\lambda_1 e_{m_1} + \ldots + \lambda_d e_{m_d} + \lambda_{d+1} e_n)\right\|$$

$$= \left\|(\lambda_1 e_{m_1} + \ldots + \lambda_d e_{m_d}) \sum_{n>m_d} a_n + \lambda_{d+1} \sum_{n>m_d} a_n e_n\right\|$$

$$= (|\lambda_1| + \ldots + |\lambda_d|) \sum_{n>m_d} a_n + |\lambda_{d+1}| \sum_{n>m_d} a_n = \sum_{n>m_d} a_n. \qquad \square$$

Symmetry

We will also use the notion of symmetric sets in Banach spaces.

Definition

- We define $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.
- Let X be a Banach space and $A \subseteq X$. We say A is symmetric if $zA \subseteq A$ for all $z \in \mathbb{T}$.

All the following lemmas are about admissible sets with respect to a symmetric collection of functionals in the closed unit ball.

No sequence isometrically equivalent to the ℓ_1-basis

Lemma 4.5

Let X be a Banach space, $F \subseteq B_{X^*}$ symmetric, $K \subseteq X$ F-admissible, then K contains no seq. isometrically equivalent to the ℓ_1-basis.

Proof of Lemma (4.5) .

Suppose $(x_n) \subseteq K$ is isometrically equivalent to the ℓ_1-basis.

- By Observation 3, $(\frac{1}{2}x_1 - \frac{1}{2}x_n)_{n\geq 2}$ generates ℓ_1^+ isometrically.
- By Lemma 4.4 where $I_n = (1, n)$ and $(\lambda_1, \lambda_n) = (\frac{1}{2}, -\frac{1}{2})$, $\exists \Lambda_1' \in F$ such that $\limsup_n |\Lambda_1'(\frac{1}{2}x_1 - \frac{1}{2}x_n)| = 1$.
- Let $m_1 = 1$. By the symmetry of F, we may find an infinite subset M_1 of $\mathbb{N} \setminus \{m_1\}$ and $\Lambda_1 \in F$ such that

$$\lim_{n\in M_1} \Lambda_1 \left(\frac{1}{2}x_{m_1} - \frac{1}{2}x_n \right) = 1.$$

Proof of Lemma (4.5) (Cont.)

- Let $m_2 = \min M_1$. By Observation 3, $(\frac{1}{4}x_{m_1} + \frac{1}{4}x_{m_2} - \frac{1}{2}x_n)_{n \in M_1 \setminus \{m_2\}}$ generates ℓ_1^+ isometrically.
- By Lemma 4.4 and the symmetry of F, we can find an infinite subset M_2 of $M_1 \setminus \{m_2\}$ and $\Lambda_2 \in F$ such that

$$\lim_{n \in M_2} \Lambda_2 \left(\frac{1}{4}x_{m_1} + \frac{1}{4}x_{m_2} - \frac{1}{2}x_n \right) = 1.$$

- Inductively, we get $\mathbb{N} \supsetneq M_1 \supsetneq M_2 \supsetneq \dots$ and $(\Lambda_n) \subseteq F$ such that for all $n \in \mathbb{N}$, we have $m_n := \min M_{n-1} \notin M_n$ and

$$\lim_{k \in M_n} \Lambda_n \left(\sum_{i=1}^{n} \frac{1}{2n} x_{m_i} - \frac{1}{2}x_k \right) = 1.$$

No sequence isometrically equivalent to the ℓ_1-basis

Proof of Lemma (4.5) (Cont.)

- K is τ_F-compact, so (x_{m_n}) has a τ_F-cluster point $x_0 \in K$. Since

$$1 = \Lambda_n\left(\sum_{i=1}^{n} \frac{1}{2n} x_{m_i} - \frac{1}{2} x_0\right) = \frac{1}{n}\left(\sum_{i=1}^{n} \Lambda_n\left(\frac{1}{2} x_{m_i} - \frac{1}{2} x_0\right)\right),$$

  we get $\Lambda_n(\frac{1}{2} x_{m_i} - \frac{1}{2} x_0) = 1$ for all $n \in \mathbb{N}$, $i = 1, \ldots, n$.

- Since

$$\lim_{k \in M_n} \Lambda_n\left(\sum_{i=1}^{n} \frac{1}{2n} x_{m_i} - \frac{1}{2} x_k\right) = 1 \text{ and } \left\|\sum_{i=1}^{n} \frac{1}{2n} x_{m_i} - \frac{1}{2} x_k\right\| = 1,$$

  we have $\|\Lambda_n\| = 1$ for all $n \in \mathbb{N}$, so $(\frac{1}{2} x_{m_n} - \frac{1}{2} x_0)_{n=1}^{\infty}$ is normalized and indeed generates ℓ_1^+ isometrically.

Proof of Lemma (4.5) (Cont.)

- By the F-admissibility of K, $\sum_n a_n(\frac{1}{2}x_{m_n} - \frac{1}{2}x_0)$ attains its norm at some element of F for all $(a_n) \in \ell_1^+$ with $\sum_n a_n = 1$.
- In particular, since F is symmetric, there exists $\Lambda \in F$ such that $\sum_n a_n\Lambda(\frac{1}{2}x_{m_n} - \frac{1}{2}x_0) = 1$, and hence $\Lambda(x_{m_n} - x_0) = 2$ for all $n \in \mathbb{N}$. This contradicts the fact that x_0 is a τ_F-cluster point of x_{m_n}. □

We will use Lemma 4.5 to derive contradiction in the arguments of Corollary 4.8.

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

For any set S, we write $[S]$ for the collection of infinite subsets of S.

Lemma 4.6

Let X be a Banach space, $F \subseteq B_{X^*}$ symmetric, $K \subseteq X$ F-admissible. Assume (e_n) is a normalized sequence in K generating ℓ_1 almost isometrically. Given $r \in \mathbb{N}$, finite $\Delta \subseteq \mathbb{T}$, and a collection $(N_i)_{i=1}^r$ of pairwise disjoint members of $[\mathbb{N}]$, there is a collection $(P_i)_{i=1}^r$ with $P_i \in [N_i]$ satisfying the following property:

For every choice $\theta_1, \ldots, \theta_r$ of members of Δ, there is $\Lambda \in F$ so that

$$\lim_{k \in P_i} \Lambda e_k = \theta_i, \ i = 1, \ldots, r.$$

We will use Lemma 4.6 to construct an important structure (tree) in Lemma 4.7.

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) .

- Let $\theta_1, \ldots, \theta_r \in \Delta$ be fixed. It suffices to find $M_i \in [N_i]$ for $1 \leq i \leq r$ and $\Lambda_0 \in F$ such that

$$\lim_{k \in M_i} \Lambda_0 e_k = \theta_i, \; i = 1, \ldots, r.$$

  This is because Δ^r is finite, so by repeating this process a finite number of times we shall arrive at the desired choices of $P_1, \ldots, P_r$.

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) (Cont.)

- We first choose a sequence

$$I_1 < I_2 < I_3 < \dots$$

  of successive finite subsets of $\mathbb{N}$ the form $I_j = \{l_{j1} < \dots < l_{jr}\}$, where $l_{ji} \in N_i,\ 1 \leq i \leq r,\ j \in \mathbb{N}$.
- Define

$$v_k = \sum_{i=1}^{r} \bar{\theta}_i e_{l_{ki}} \text{ and } u_k = \frac{v_k}{||v_k||},\ k \in \mathbb{N}.$$

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) (Cont.)

- By the fact that (e_n) is a normalized sequence in K generating ℓ_1 almost isometrically, we may assume that

$$\lim_k \|v_k\| = r$$

and that

$$(u_k) \text{ generates } \ell_1 \text{ almost isometrically.}$$

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) (Cont.)

- By our construction, (u_k) fulfills the hypothesis in Lemma 4.4, so there exists $\Lambda_1 \in F$ such that

$$\limsup_k |\Lambda_1 u_k| = 1.$$

Moreover, we may assume that

$$\lim_k |\Lambda_1 u_k| = 1$$

and

$$\lim_k \Lambda_1 e_{l_{ki}} = a_i \in \mathbb{C}, \; i = 1, \ldots, r.$$

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) (Cont.)

- It follows that

$$\left|\sum_{i=1}^{r} a_i\bar{\theta}_i\right| = r.$$

- However

$$|a_i\bar{\theta}_i| \leq 1,\ i = 1, \ldots, r,$$

so there exists $z \in \mathbb{T}$ such that

$$a_i\bar{\theta}_i = z,\ i = 1, \ldots, r.$$

A finite subset of $\mathbb{T}$ as limits of the range of some Λ

Proof of Lemma (4.6) (Cont.)

- Finally, set

$$M_i = \{l_{ki} \mid k \in \mathbb{N}\},\ i = 1, \ldots, r \text{ and } \Lambda_0 = \bar{z}\Lambda_1 \in F.$$

Then

$$\lim_{k \in M_i} \Lambda_0 e_k = \theta_i,\ i = 1, \ldots, r. \qquad \square$$

Generating ℓ_1 from a subset of an admissible set

Definition

Let X be a Banach space and $\Gamma \subseteq X$. We say Γ generates $\ell_1(|\Gamma|)$ isometrically if for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in \Gamma$, $a_1, \dots, a_n \in \mathbb{C}$, we have

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| = \sum_{i=1}^{n} |a_i|.$$

Lemma 4.7

Let X be a Banach space, $F \subseteq B_{X^*}$ symmetric, $K \subseteq X$ F-admissible. Assume that (e_n) is a normalized sequence in K generating ℓ_1 almost isometrically. Then there exists a subset of K, whose cardinality is equal to the continuum with elements being τ_F-cluster points of (e_n), that generates $\ell_1(\mathfrak{c})$ isometrically.

Before we prove Lemma 4.7, let us first define the dyadic tree and a partial ordering on it.

Definition

Let $\mathcal{T}$ denote the dyadic tree $\bigcup_{n=1}^{\infty}\{0,1\}^n$.
Define a partial ordering $\leq'$ on $\mathcal{T}$ by

$$(a_1,\ldots,a_n) \leq' (b_1,\ldots,b_m) \iff n \leq m \text{ and } a_i = b_i,\ i = 1,\ldots,n.$$

A branch of $\mathcal{T}$ is a maximal well-ordered subset of $(\mathcal{T}, \leq')$.

Generating ℓ_1 from a subset of an admissible set

Proof of Lemma (4.7) .

- Let (ε_n) be a scalar sequence strictly decreasing to 0 and choose an increasing sequence

$$\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \dots$$

of finite subsets of $\mathbb{T}$ such that

$$\Delta_n \text{ is an } \varepsilon_n\text{-net for } \mathbb{T}, \forall n \in \mathbb{N}.$$

Namely,

$$d(x, \Delta_n) < \varepsilon_n, \ \forall n \in \mathbb{N}, \ x \in \mathbb{T}$$

where $d(x, \Delta_n) := \inf\{d(x,t) \mid t \in \Delta_n\}$.

Generating ℓ_1 from a subset of an admissible set

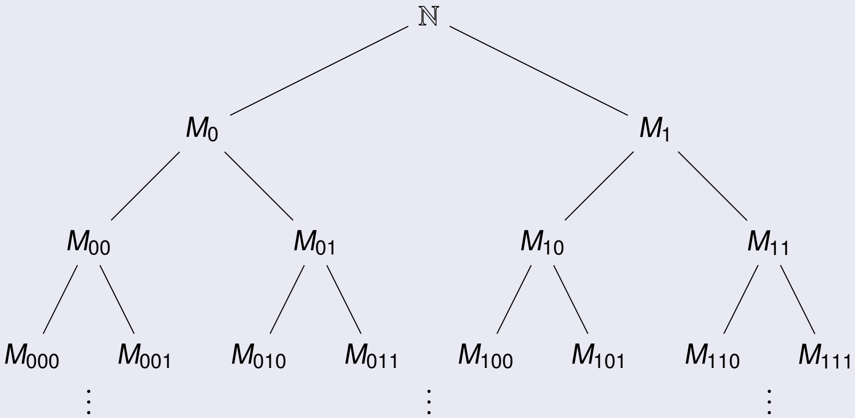
Proof of Lemma (4.7) (Cont.)

- Successive applications of Lemma 4.6 yield a binary tree $(M_\alpha)_{\alpha \in \mathcal{T}}$ of infinite subsets of $\mathbb{N}$ with the following property:

  For each $n \in \mathbb{N}$ and all choices $(\zeta_\alpha)_{\alpha \in \{0,1\}^n}$ of elements from Δ_n, there exists $\Lambda \in F$ so that

$$\lim_{k \in M_\alpha} \Lambda e_k = \zeta_\alpha, \ \alpha \in \{0,1\}^n.$$

Proof of Lemma (4.7) (Cont.)



Proof of Lemma (4.7) (Cont.)

- Let $\mathfrak{b} = \{\beta_1 < \beta_2 < \beta_3 < \dots\}$ be a branch of $(\mathcal{T}, \leq')$. We observe that $M_{\beta_j} \subseteq M_{\beta'_j}$ if and only if $\beta_{j'} \leq' \beta_j$.
- Let $N_\mathfrak{b} \in [\mathbb{N}]$ be almost contained in each M_{β_j}, namely $|N_\mathfrak{b} \setminus M_{\beta_j}| < \infty$, $\forall j \in \mathbb{N}$. Note that such $N_\mathfrak{b}$ exists, for example

$$N_\mathfrak{b} = \{n_k \mid k \in \mathbb{N}\}, \text{ where } n_k = \min\left(M_{\beta_k} \setminus \{n_j \mid j = 1, \dots, k-1\}\right).$$

- Let $e_\mathfrak{b} \in K$ be a τ_F-cluster point of $(e_n)_{n \in N_\mathfrak{b}}$.
- Let $\mathcal{B} = \{\mathfrak{b}' : \text{branch of } (\mathcal{T}, \leq')\}$.

Proof of Lemma (4.7) (Cont.)

- We claim that $(e_{\mathfrak{b}})_{\mathfrak{b}\in\mathcal{B}}$ generates $\ell_1(\mathfrak{c})$ isometrically.
- Let $m \in \mathbb{N}$, and let $\mathfrak{b}^1, \ldots, \mathfrak{b}^m$ be distinct branch of $(\mathcal{T}, \leq')$. Write $\mathfrak{b}^i = \{\beta_1^i < \beta_2^i < \beta_3^i < \ldots\},\ i = 1, \ldots, m$.
- Given $p \in \mathbb{N}$, we choose $n_0 > p$ so that

$$\beta_{n_0}^i \neq \beta_{n_0}^j \text{ whenever } i \neq j \text{ in } \{1, \ldots, m\}.$$

- Let $(a_i)_{i=1}^m$ be scalars and write

$$a_i = |a_i| z_i, \text{ where } z_i \in \mathbb{T}.$$

Generating ℓ_1 from a subset of an admissible set

Proof of Lemma (4.7) (Cont.)

- For each $i = 1, \ldots, m$, we can find $\theta_i \in \Delta_{n_0}$ so that

$$|\theta_i - \bar{z}_i| < \varepsilon_{n_0}$$

  by the fact that Δ_{n_0} is an ε_{n_0}-net for $\mathbb{T}$.
- By our construction, there exists $\Lambda \in F$ so that

$$\lim_{k \in M_{\beta^i_{n_0}}} \Lambda e_k = \theta_i, \; i = 1, \ldots, m.$$

  So

$$\lim_{k \in N_{\mathfrak{b}^i}} \Lambda e_k = \theta_i, \; i = 1, \ldots, m.$$

Proof of Lemma (4.7) (Cont.)

- Since $e_{\mathfrak{b}^i}$ is a τ_F-cluster point of $(e_n)_{n\in N_{\mathfrak{b}}^i}$, we get

$$\Lambda e_{\mathfrak{b}^i} = \theta_i,\ i = 1, \dots, m.$$

  This implies

$$\|e_{\mathfrak{b}^i}\| = \|e_{\mathfrak{b}^i}^{**}\| \geq |\Lambda e_{\mathfrak{b}^i}| = |\theta_i| = 1.$$

- However K is F-admissible means there exists $\Lambda' \in F$ such that

$$\|e_{\mathfrak{b}^i}\| = |\Lambda' e_{\mathfrak{b}^i}| \leq \|\Lambda'\| \leq 1.$$

  Hence we get

$$\|e_{\mathfrak{b}^i}\| = 1,\ i = 1, \dots, m.$$

Generating ℓ_1 from a subset of an admissible set

Proof of Lemma (4.7) (Cont.)

- Finally we have the estimates

$$
\begin{aligned}
&\left\|\sum_{i=1}^{m} a_i e_{\mathfrak{b}^i}\right\| \geq \left|\sum_{i=1}^{m} a_i \Lambda e_{\mathfrak{b}^i}\right| = \left|\sum_{i=1}^{m} a_i \theta_i\right| \\
&= \left|\sum_{i=1}^{m} a_i \bar{z}_i - \sum_{i=1}^{m} a_i(\bar{z}_i - \theta_i)\right| \geq \sum_{i=1}^{m} |a_i| - \sum_{i=1}^{m} |a_i|\varepsilon_{n_0} \\
&\geq (1-\varepsilon_p) \sum_{i=1}^{m} |a_i|.
\end{aligned}
$$

Generating ℓ_1 from a subset of an admissible set

Proof of Lemma (4.7) (Cont.)

- The above estimate holds for every $p \in \mathbb{N}$, and we know that $\lim_n \varepsilon_n = 0$, so we conclude that

$$\left\| \sum_{i=1}^{m} a_i e_{\mathfrak{b}^i} \right\| \geq \sum_{i=1}^{m} |a_i|$$

for every $m \in \mathbb{N}$ and all choices of scalars $(a_i)_{i=1}^{m}$. Therefore, $(e_{\mathfrak{b}})_{\mathfrak{b} \in \mathcal{B}}$ generates $\ell_1(\mathfrak{c})$ isometrically. □

Closed unit ball of a subspace isomorphic to ℓ_1

We arrive at the main result of this subsection, which is needed in Theorem 1.1.

In short, it shows that a closed subspace isomorphic to ℓ_1 cannot have its closed unit ball contained in a specific admissible set.

Corollary 4.8

Let X be a Banach space, $F \subseteq B_{X^*}$ symmetric, $K \subseteq X$ F-admissible. Suppose that Y is a closed linear subspace of X isomorphic to ℓ_1. Then $B_Y \setminus K \neq \emptyset$.

Closed unit ball of a subspace isomorphic to ℓ_1

To prove Corollary 4.8, we need James' distortion theorem, whose proof is too long to be included here, so we will state the theorem without proving it.

James' distortion theorem

Let X be a Banach space. Then there exists a closed linear subspace Y of X isomorphic to ℓ_1 if and only if for every null sequence $(\varepsilon_n) \in (0,1)$, there exists a normalized sequence $(e_n) \in X$ such that

$$\sum_{n=k}^{\infty} |a_n| \geq \left\| \sum_{n=k}^{\infty} a_n e_n \right\| \geq \frac{1}{1+\varepsilon_k} \sum_{n=k}^{\infty} |a_n|, \ \forall (a_n) \in \ell_1, \ k \in \mathbb{N}.$$

Likewise, later on several other theorems are needed in the proof of Theorem 1.1, and we will state them without proof by then.

Proof of Corollary (4.8) .

- Suppose not, then $B_Y \subseteq K$. James' distortion theorem now yields a normalized sequence (e_n) in K generating ℓ_1 almost isometrically. We deduce from Lemma 4.7 that K contains a normalized sequence isometrically equivalent to the ℓ_1-basis, contradicting Lemma 4.5. □

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Directed sets and nets

Definition

A directed set $(D, \geq)$ consists of a nonempty set D and a relation $\geq$ on D satisfying

1. $\forall d \in D$, we have $d \geq d$.
2. $\forall d, d', d'' \in D$, if $d \geq d'$ and $d' \geq d''$, then $d \geq d''$.
3. $\forall d, d' \in D$, $\exists d'' \in D$ such that $d'' \geq d$ and that $d'' \geq d'$.

Definition

Let X be a set. A net in X consists of a directed set $(D, \geq)$ and a map $x_\bullet : (D, \geq) \to X, \alpha \mapsto x_\alpha$. We write the net as $(x_\alpha)_{\alpha \in D}$.

Remark

A sequence is a net with $D = \mathbb{N}$.

The convergence of nets

Nets generalize the notion of sequences. We can define the convergence of nets, which turns out to be able to characterize closed sets and continuity.

Definition

Let $(x_\alpha)_{\alpha\in D}$ be a net in X and $S \subseteq X$. We say $(x_\alpha)_{\alpha\in D}$ lies in S eventually if there is some $\delta \in D$ such that $x_\alpha \in S$ for all $\alpha \geq \delta$.

Definition

Let X be a topolgical space, $(x_\alpha)_{\alpha\in D}$ be a net in X and $x \in X$. We say $(x_\alpha)_{\alpha\in D}$ converges to x if $(x_\alpha)_{\alpha\in D}$ lies in U eventually for all open neighborhood U of x.

Characterization of closed sets by nets

Proposition

Let X be a topological space and $A \subseteq X$, then the set of limit points of A is $\{x \in X \mid \exists$ a net $(x_\alpha)_{\alpha \in D}$ in $A \setminus \{x\}$ converging to $x\}$.

Proof.

Let x be a limit point of A. Define the directed set $(D, \geq)$ by

- $D = \{$open neighborhoods of x in $X\}$.
- $U \geq V \iff U \subseteq V$.

For all $U \in D$, choose $x_U \in U \cap (A \setminus \{x\})$, then $(x_U)_{U \in D}$ is a net in $A \setminus \{x\}$ that lies in V eventually for all open neighborhood V of x.
Thus $\{$limit points of $A\} \subseteq \{x \in X \mid \exists (x_\alpha)_{\alpha \in D} \in A \setminus \{x\}$ converging to $x\}$, and equality holds since the converse containment is clear. $\square$

Characterization of closed sets by nets

Corollary

Let X be a topological space and $A \subseteq X$, then A is closed if and only if the limits of nets in A are still in A.

Remark

If X is first countable (every point has a countable neighborhood basis), then the above can be characterized by sequences instead of nets.
However, the weak topology on an infinite dimensional space is not first countable, so later on we need nets to examine closed sets.

Characterization of continuity by nets

Likewise, continuity may not be equivalent to sequential continuity in general topological spaces, so we need nets.

Proposition

Let $f : X \to Y$ be a map between topological spaces and $x_0 \in X$, then the following are equivalent.

1. f is continuous at x_0.
2. $\forall$ net $(x_\alpha)_{\alpha \in D} \in X$, $f(x_\alpha)$ converges to $f(x_0)$ if x_α converges to x_0.

Proof.

First we prove 1 $\Rightarrow$ 2.

- $\forall$ open neighborhood V of $f(x_0)$, $\exists$ open neighborhood U of x_0 such that $f(x) \in V$ for all $x \in U$.
- $(x_\alpha)_{\alpha \in D}$ lies in U eventually, so $(f(x_\alpha))_{\alpha \in D}$ lies in V eventually.

Characterization of continuity by nets

Proof (Cont.)

Next we prove 2 $\Rightarrow$ 1 by contradiction.

- If f is not continuous at x_0, then $\exists$ open neighborhood V of $f(x_0)$ such that for $K = f^{-1}(V)$, we have $x_0 \in K \setminus K^o$.
- Define the directed set $(D, \geq)$ by

$$D = \{\text{open neighborhood of } x_0\}, \text{ and } U \geq U' \iff U \subseteq U'.$$

- For all $A \in D$, choose $x_A \in A \setminus K$, then $(x_A)_{A \in D}$ is a net which lies in U eventually for all $U \in D$, so it converges to x_0.
- By assumption, $(f(x_A))$ also converges to $f(x_0)$. However, since $x_A \notin K$, we have $f(x_A) \notin V$, so $(f(x_A))$ cannot lie in V eventually, which is a contradiction. $\square$

Final maps and subnets

We define the notion of subnets, which preserves some features of a sunsequence.

Definition

Let $(D, \geq), (D', \geq')$ be directed sets. A map $h : D' \to D$ is final if $\forall \delta \in D$, $\exists \delta' \in D'$ such that for all $\alpha \in D'$ with $\alpha \geq' \delta'$, we have $h(\alpha) \geq \delta$.

Definition

Let $x_\bullet : D \to X$ be a net and $h : D' \to D$ be a final map, then $x_{h(\bullet)} : D' \to X$ is a subnet of $x_\bullet$.

Proposition

If $x_{h(\alpha)}$ is a subnet of x_α and $x_\alpha \to x$, then $x_{h(\alpha)} \to x$.

Final maps and subnets

Proposition

If $(x_\alpha)_{\alpha \in D}$ is a bounded net in $\mathbb{C}$ that does not converge to 0, then there exists a subnet $(x_\beta)_{\beta \in M}$ and $\delta > 0$ such that $|x_\beta| > \delta$ for all $\beta \in M$.

Proof.

- There exists $\delta > 0$ such that $\forall \alpha \in D$, $\exists \alpha' \geq \alpha$ so that $|x_{\alpha'}| > \delta$.
- Let $\preceq$ be a well-order of D and o be the first element in this order. Choose $t_o \geq o$ with $|x_{\alpha_o}| > \delta$.
- Suppose that we have chosen α_j for all $j \preceq i$ and $j \neq i$ such that $|x_{\alpha_j}| > \delta$. If we have defined $\alpha_j \geq i$ for some $j \preceq i$, then skip this i. Otherwise, choose α_i such that $|x_{\alpha_i}| > \delta$.
- Transfinite inductively, we defined a directed set $(M, \geq)$, whose elements are the chosen α_i's, such that $(x_\beta)_{\beta \in M}$ is the required subnet. $\square$

Table of contents

An extension of James' compactness theorem

Theorem 1.1

Let X, Y be Banach spaces and $F \subseteq B_{Y^*}$. Give Y the topology (Y, τ_F). If $T : X^* \to Y$ is bounded, linear, and (w^*, τ_F)-continuous, and

$$\forall \Lambda \in X^*, \exists \Lambda' \in F \text{ such that } \|T\Lambda\| = |\Lambda' T\Lambda|,$$

then T is (w^*, w)-continuous.

Remark

The weak topology on Y may be much finer than τ_F. However, if F satisfies some sort of "norm attaining" property as above, then the (w^*, τ_F)-continuity of T extends to the (w^*, w)-continuity.

An extension of James' compactness theorem

Proof of Theorem 1.1.

Claim 1: For a bounded $(\Lambda_n) \subseteq X^*$, $(T\Lambda_n)$ has no ℓ_1-subsequence.

- Suppose that $(\Lambda_n) \in X^*$ is bounded and $(T\Lambda_n)$ is equivalent to the ℓ_1-basis.
- Let $X' = \overline{\operatorname{span}\{\Lambda_1, \Lambda_2, \dots\}}$, $Y' = \overline{\operatorname{span}\{T\Lambda_1, T\Lambda_2, \dots\}}$. By definition, $(T\Lambda_n)$ is a basis of Y', and by linearity, (Λ_n) is also a basis of X'.
- Let (e_n) be the standard basis of ℓ_1. By definition, there exists $f : Y' \xrightarrow{\sim} \ell_1$ such that $f(T\Lambda_n) = e_n$.
- Define $g : X' \to \ell_1$, $\Lambda_n \mapsto f(T\Lambda_n) = e_n$. Since (Λ_n) and $(T\Lambda_n)$ are bases, g is bijective. Moreover, by the linearity of T and f, g is also linear. Hence g is an isomorphism, and (Λ_n) is equivalent to the ℓ_1-basis.

Proof of Theorem 1.1 (Cont.)

Claim 1: For a bounded $(\Lambda_n) \subseteq X^*$, $(T\Lambda_n)$ has no ℓ_1-subsequence.

- Since (Λ_n) and $(T\Lambda_n)$ are both equivalent to the ℓ_1-basis, $S : X' \to Y'$, $x_i \mapsto T(x_i)$ is a bounded linear bijection. By the Banach inverse mapping theorem, there is a closed ball $B^* \subseteq X^*$, centered at the origin, such that $B_{Y'} \subseteq TB^*$.
- $K = TB^*$ is τ_F-compact since T is (w^*, τ_F)-continuous, and K is bounded since T is. Together with our assumption, we see K is F-admissible.
- By Corollary 4.8, $B_{Y'} \setminus K \neq \emptyset$, which is a contradiction.

An extension of James' compactness theorem

Definition

Let X be a Banach space. A sequence $(x_n) \in X$ is weakly Cauchy if (Λx_n) is a Cauchy sequence for all $\Lambda \in X^*$.

Rosenthal's ℓ_1-theorem

Let X be a Banach space and $(x_n) \in X$ be a bounded sequence, then (x_n) has an ℓ_1-subsequence or a weakly Cauchy subsequence.

Proof of Theorem 1.1 (Cont.)

Claim 2: T is weakly compact.

- Let $(\Lambda_n) \subseteq X^*$ be bounded. By Rosenthal's ℓ_1-theorem and Claim 1, $(T\Lambda_n)$ has a weakly Cauchy subsequence $(T\Lambda_{k_n})$.
- By Banach-Alaoglu theorem, $\exists$ weak* cluster point Λ_0 of (Λ_{k_n}).
- T is (w^*, τ_F)-continuous, so $T\Lambda_0$ is a τ_F-cluster point of $(T\Lambda_n)$.
- Since $\sum_{n\in\mathbb{N}} a_n(T\Lambda_{k_n} - T\Lambda_0) \in R(T)$ for all $(a_n) \in \ell_1^+$, it attains its norm at some element of F, by assumption.
  Also, $\Lambda'(T\Lambda_{k_n} - T\Lambda_0) \to 0$ for all $\Lambda' \in F$ by weak Cauchiness.
- Corollary 3.3 suggests $T\Lambda_{k_n} \rightharpoonup T\Lambda_0$. Thus T is weakly compact.

An extension of James' compactness theorem

Krein-Šmulian theorem

Let X be a Banach space and $A \subseteq X^*$ be convex. If

$$A \cap \{\Lambda \in X^* \mid \|\Lambda\| \leq r\}$$

is weak* closed for all $r > 0$, then A is weak* closed.

Mazur's theorem

Let X be a Banach space and $A \subseteq X$ be convex, then the weak closure of A is equal to the norm closure of A.

Proof of Theorem 1.1 (Cont.)

Claim 3: If $V \subseteq Y$ is norm-closed and convex, then $T^{-1}V$ is weak* closed in X^*.

- T is linear, so $T^{-1}V$ is convex. By Krein-Šmulian theorem, it suffices so show that $U^* \cap T^{-1}V$ is closed for all closed ball U^* centered at the origin.
- Let $(\Lambda_\alpha)_{\alpha \in D}$ be a net in $U^* \cap T^{-1}V$ with a weak* limit $\Lambda \in X^*$. By the characterization of closed sets using nets, it suffices to show that $T\Lambda \in V$.

Proof of Theorem 1.1 (Cont.)

Claim 3: If $V \subseteq Y$ is norm-closed and convex, then $T^{-1}V$ is weak* closed in X^*.

- TU^* is weakly compact since T is, so we may assume $(T\Lambda_\alpha)_{\alpha\in D}$ weakly converges to some $y_0 \in TU^*$. By Mazur's theorem, we get $y_0 \in V \cap TU^*$.
- T preserves the convergence of nets, so $(T\Lambda_\alpha)_{\alpha\in D}$ converges to $T\Lambda$ under τ_F.
- $y_0, T\Lambda \in R(T)$, and F separates points in $R(T)$ by assumption. Thus $T\Lambda = y_0$ and hence $T\Lambda \in V$.

Proof of Theorem 1.1 (Cont.)

Claim 4: T is (w^*, w)-continuous.

- If suffices to show if $(\Lambda_\alpha)_{\alpha\in D} \in X^*$ is a net w* converging to 0, then $(T\Lambda_\alpha)_{\alpha\in D} \in X^*$ also weakly converges to 0.
- If not, then $\exists\Lambda' \in Y^*$, $\delta > 0$, and a subset $(T\Lambda_\beta)_{\beta\in M}$ such that $|\Lambda' T\Lambda_\beta| > \delta$ for all $\beta \in M$.
- Let $V_0 = \{y \in Y \mid \mathrm{Re}(\Lambda' y) \geq \delta/2\}$. By replacing Λ' by $\theta\Lambda'$ for some suitable $\theta \in \mathbb{T}$ and passing to a subnet, we may assume that $T\Lambda_\beta \in V_0$ for all $\beta \in M$.
- V_0 is norm-closed and convex in Y. By Claim 3, $T^{-1}V_0$ is weak* closed in X^* and it contains $(\Lambda_\beta)_{\beta\in M}$. However, Λ_β converges to 0 weak*ly, so $0 \in V_0$, which is a contradiction. □

James' compactness theorem

Corollary 4.9

Let X be a Banach space and $K \subseteq X$ be bounded. If $|\Lambda|$ attains its maximum on K for all $\Lambda \in X^*$, then K is relatively compact.

Remark

In particular, if we take K to be the closed unit ball, then we get the nontrivial implication in James' compactness theorem, since K is relatively compact iff X is reflexive.

Proof of Corollary 4.9.

- Identify $X \hookrightarrow X^{**}$. Let $L = \overline{K}^{w^*}$, $R : X^* \to C(L)$, $\Lambda \mapsto \Lambda|_L$.
- Take $F = \{\delta_x \mid x \in K\}$, where $\delta_x \in C(L)^*$, $\delta_x(f) = f(x)$.
- By definition, w^* is the coarsest topology on X^* making all functionals in $J(X)$ continuous, and τ_F is the coarsest topology on $C(L)$ making all functionals in $J(K)$ continuous. Therefore, R is (w^*, τ_F)-continuous.
- By our assumption on K, $\forall \Lambda \in X^*$, $\exists x \in K$ such that

$$\|R\Lambda\| = \sup_{y \in L} |\Lambda y| = \sup_{y \in K} |\Lambda y| = |\delta_x R\Lambda|.$$

By Theorem 1.1, R is (w^*, w)-continuous, so is weakly compact. By Corollary 3.5, K is relatively weakly compact. $\square$

Thank you for your attention

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