An extension of James's compactness theorem
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1 Introduction

James's characterization of reflexivity states that a Banach space X is reflexive if and only if every $\Lambda \in X^*$ attains its norm at an element of B_X, the closed unit ball of X. The "only if" part is easy, as one can take a sequence $(x_n) \in B_X$ such that Λx_n approximates the supremum of Λ on B_X, and consider the weak limit of (x_n),which exists due to the reflexivity of X. However, the "if" part is nontrivial, and requires James's compactness theorem.

Theorem (James's compactness theorem)**.** *Let X be a Banach space and $K \subseteq X$ be bounded. If every $\Lambda \in X^*$ attains its supremum on K, then K is relatively weakly compact. That is, $\overline{K}$ is weakly compact.*

Indeed, taking $K = B_X$, then the formentioned "if" part follows, as B_X is weakly compact if and only if X is reflexive.
The goal of this paper is to prove the following extension of James's compactness theorem.

Theorem 1.1. *Let X, Y be Banach spaces and $F \subseteq B_{Y^*}$. Give Y the topology (Y, τ_F). If $T : X^* \to Y$ is bounded, linear, and (w^*, τ_F)-continuous, and*

$$\forall \Lambda \in X^*, \exists \Lambda' \in F \text{ such that } \|T\Lambda\| = |\Lambda' T\Lambda|,$$

then T is (w^*, w)-continuous.

Theorem 1.1 says that if F is a set of linear functionals that can attain the norm of elements in the image of T, then the $(w*, \tau_F)$-continuity can be enhanced to (w^*, w), which is powerful since τ_F might be much weaker than the weak topology. We will show that theorem 1.1 implies James's compactness theorem.

2 James's compactness theorem in separable Banach spaces

The pivotal results in this section are Corollary 2.2 and 2.4. The former helps us examine whether a sequence is weakly null, and the latter helps us examine whether a sequence is relatively compact. Merely using them, we can immediately deduce James's compactness theorem in separable Banach spaces. With more results, they can also be used to prove Theorem 1.1, which leads to the general case, but we shall do that in the next section.

Let $\ell_1^+ = \{(a_n) \in \ell_1 \mid a_n \geq 0 \ \forall n\}$ be the positive cone of ℓ_1. In this paper, it is important to look into sequences that in some sense behave like a "basis" for ℓ_1^+. Formally speaking, we have the following definitions.

Definition. *Let $(e_n) \subseteq X$ be normalized. We say (e_n) is an ℓ_1^+-sequence if there exists $c > 0$ such that*

$$\left\| \sum_{n=1}^{\infty} a_n e_n \right\| \geq c \sum_{n=1}^{\infty} a_n \text{ for all } (a_n) \in \ell_1^+.$$

In this case, (e_n) is called a $c - \ell_1^+$-sequence. In particular, if $c = 1$, then we say (e_n) generates ℓ_1^+ isometrically.

Definition. *Let $(e_n) \subseteq X$ be normalized. We say (e_n) generates ℓ_1^+ almost isometrically if for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $(e_n)_{n \geq n_\varepsilon}$ is an $1/(1+\varepsilon)$-sequence. That is,*

$$\left\| \sum_{n=n_\varepsilon}^{\infty} a_n e_n \right\| \geq \frac{1}{1+\varepsilon} \sum_{n=n_\varepsilon}^{\infty} a_n \text{ for all } (a_n) \in \ell_1^+.$$

Under these definitions, one sees that if $(x_n) \subseteq X$ is bounded, nonzero, and $x_n \not\rightharpoonup 0$, then $(x_n/\|x_n\|)$ has an ℓ_1^+-subsequence. Therefore, it is reasonable for us to investigate such sequences since we need many results involving weakly nullity.

2.1 Examining weakly null sequences

Lemma 2.1 is very technical and is only needed in the proof of Corollary 2.2.

Lemma 2.1. *Let X be a Banach space and $(e_n) \subseteq X$ be a normalized ℓ_1^+-sequence, then there exist scalars $a_n > 0$ such that*

$$(1 + a_n) \left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| \sum_{i=1}^{n+1} a_i e_i \right\| < 1 \; \forall n \in \mathbb{N}.$$

To prove Lemma 2.1, one uses induction to break the norm of the sum into products of $(1 + a_i)'s$, thus showing that the whole stuff can be bounded above by 1. Also, the succeeding a_i can always be found by the intermediate value theorem. □

To show that a sequence $x_n \in X$ is weakly null, originally one needs to check $\Lambda x_n \to 0$ for all $\Lambda \in X^*$. However, Corollary 2.3 tells us that we can restrict ourselves to a smaller class of linear functionals which has some good property.

Corollary 2.2. *Let $(x_n) \subseteq X$ be bounded and $F \subseteq B_{X^*}$. If F has the nice property that*

$$\forall (a_n) \in \ell_1^+, \; \exists \Lambda \in F \text{ such that } \Lambda \left(\sum_{n=1}^{\infty} a_n x_n \right) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\|,$$

then as long as $\Lambda x_n \to 0$ for all $\Lambda \in F$, one has $x_n \rightharpoonup 0$.

Corollary 2.2 is proved by contradiction. If $x_n \not\rightharpoonup 0$, then upon normalization, we can extract an ℓ_1^+ subsequence (z_n), and find scalars $a_1, a_2, \ldots$ as in Lemma 3.2. Hence $u = \sum a_i z_i$ is well-defined.
However, for any $\Lambda \in F$, there is some $m \in \mathbb{N}$ such that $|\Lambda x_n|$ and a_n is small for all $n \geq m$, so by splitting the sum into the first m terms and the tail, it turns out that $|\Lambda u| < \|u\|$ for all $\Lambda \in F$, contradicting our assumption on F. □

2.2 Examining relative weak compactness

Recall that $J : X \to X^{**}$ is the canonical identification. First we have a characterization of relative weak compactness.

Observation. *Let X be a Banach space and $K \subseteq X$ be bounded, then the following are equivalent.*

1. *K is relatively weakly compact.*
2. *The weak* closure $\overline{J(K)}^{w^*}$ is contained in $J(X)$.*

The implication $(\Rightarrow)$ comes from the fact that J is (w, w^*) continuous, and the implication $(\Leftarrow)$ comes from the fact that the weak and weak* topologies of X^** coincide on $J(X)$. □

Therefore, we want to know what will happen if $\overline{J(K)}^{w^*}$ protrudes out of $J(X)$. Lemma 2.3 tells us that in this case, we can obtain two peculiar sequences, one from X and one from X^*, which can help derive contradiction in later proofs. The proof of Lemma 2.3 requires Goldstine's theorem.

Theorem (Goldstine). *Let X be a Banach space, then B_X via J is dense in $B_{X^{**}}$ under the weak* topology.*

Lemma 2.3. *Let X be a Banach space and $K \subseteq X$. View X, K as sets in X^{**} via J. Suppose that $\overline{K}^{w^*} \setminus X \neq \emptyset$, then there exists $(x_n) \subseteq K$ and a bounded $(\Lambda_n) \subseteq X^*$, $\delta > 0$ such that for all $n \in \mathbb{N}$, we have*

$$|\Lambda_n x_i| < \frac{1}{n} \ \forall i < n, \text{ and } |\Lambda_n x_i| \geq \delta \ \forall i \geq n.$$

To prove Lemma 2.3, first we choose $x^{**} \in \overline{K}^{w^*} \setminus X$, then by Hahn-Banach theorem, there exist $x^{***} \in X^{***}$ and $\delta > 0$ such that $|x^{***}(x^{**})| > \delta$ and $x^{***}|_X = 0$. By Goldstine's theorem, we can then construct (x_n) and (Λ_n) inductively. The point is to choose Λ_i close enough to x^{***} and x_i close enough to x^{**}. $\square$

After getting these two peculiar sequences, we may consider $|\Lambda_n x_i|$. If we fix i and let n run, then $|\Lambda_n x_i|$ will converge to 0, but it will not if we fix n and let i run instead. This is a strategy for us to derive contradiction in proofs. In fact, the following theorem, which shows the equivalence of different types of relative weak compactness, is an application of Lemma 2.3.

Theorem (Eberlein-Šmulian). *Let X be a Banach space and $K \subseteq X$, then K is relatively weakly compact if and only if K is relatively weakly sequentially compact.*

The "if" part is easy, and we need it in the proof of Theorem 2.5. On the other hand, the "only if" part can be proved by Lemma 2.3 using the aforementioned strategy. We will clearly depict this strategy in the proof of Corollary 2.4. $\square$

Corollary 2.4 shows how to check if a bounded set K is relatively weakly compact (which is equivalent to checking if $\overline{J(K)}^{w^*}$ protrudes out of $J(X)$). It transform this problem into examining the weakly compactness of a particular restriction map.

Corollary 2.4. *Let X be a Banach space, $K \subseteq X$ be bounded, and view $X \subseteq X^{**}$ via J. Let $L = \overline{K}^{w^*}$ and define the restriction map $R : X^* \to C(L)$, $\Lambda \mapsto \Lambda|_L$, where $C(L)$ is the space of functions continuous on L, equipped with the sup norm. If R is weakly compact, then K is relatively weakly compact.*

Corollary 2.4 is proved by contradiction. Suppose not, then we can obtain (x_n), (Λ_n), $\delta > 0$ as in Lemma 2.4. Let $f_n = R\Lambda_n$. On one hand, by the weak compactness of R, (f_n) has a weak cluster point f, so $|f(x_i)| = 0$ for all i. On the other hand, L is weak* sequentially compact by the Eberlein-Šmulian theorem, so (x_n) has a weak* cluster point x, and $|f_n(x)| \geq \delta$ for all n. Letting $i \to \infty$ we get $f(x) = 0$, but letting $n \to \infty$ we get $f(x) \neq 0$, which is a contradiction. $\square$

2.3 James's compactness theorem in separable Banach spaces

With corollary 2.2 and 2.4, we are ready to deduce James's compactness theorem in separable Banach spaces. Theorem 2.5 is actually stronger then that, because its assumption on X covers the case where X is separable, and if K is taken to be the closed unit ball of X, then K is relatively weakly compact if and only if X is reflexive.

Theorem 2.5. *Let X be a Banach space where $B_{X^*} \subseteq X^*$ is weak* sequentially compact, and $K \subseteq X$ be bounded. If every $\Lambda \in X^*$ attains its norm on K, then K is relatively weakly compact.*

To prove Theorem 2.5, Corollary 2.4 suggests that it suffices to show the weak compactness of the restriction map R. That is, given a bounded $(\Lambda_n) \in X^*$, which may be assumed to have a weak* limit Λ by our assumption on B_X, we need to find a weakly convergent subsequence of $(R\Lambda_n)$.
Letting $f_n = R\Lambda_n$ and $f = R\Lambda$, then it turns out that $(f_n - f)$ is bounded and K attains the norm of $\sum a_i f i$ for all $(a_n) \in \ell_1^+$. Moreover, $f_n(t) \to f(t)$ for all $t \in K$, so Corollary 2.2 suggests that $f_n \rightharpoonup f$. $\square$

3 The general extension of James's compactness theorem

In this section, we will prove Theorem 1.1, the general extension of James' compactness theorem. Although a bunch of lemmas and corollaries come before the main theorem, we can divide them into two parts.

The first part (3.1 to 3.3) deals with sequences generating ℓ_1^+ almost isometrically and the notion of "admissible collections", while the second part (3.4 to 3.7) deals with admissible collections with respect to symmetric sets of functionals.
To prove the main theorem, we will mainly use Corollaries 2.2, 2.4 and 3.7. We also need the notion of nets and some additional theorems, which will be treated as facts.

3.1 Sequences generating ℓ_1^+ almost isometrically, and admissibility

In Lemma 2.1 we established inequalities about ℓ_1^+ sequences. In Lemma 3.1, we do the same thing for sequences generating ℓ_1^+ almost isometrically. Again, this is technical and is only needed in Corollary 3.2.

Lemma 3.1. *Let X be a Banach space and (e_n) be a normalized sequence in X generating ℓ_1^+ almost isometrically. Then for all $0 < \delta_0 < 1$, there exist $m_0, n_0 \in \mathbb{N}$ and scalars $b_n > 0$ such that for $x_n = e_{m_0+n}$, we have*

1. $\|\sum_{i=1}^{n_0} b_i x_i\| = \delta_0$.
2. $(1 + b_n) \|\sum_{i=1}^{n} b_i x_i\| = \left\|\sum_{i=1}^{n+1} b_i x_i\right\| < 1 \quad \forall n \geq n_0$.
3. $(\prod_{i=n}^{\infty}(1 + b_i)) \|\sum_{i=1}^{n} b_i x_i\| = \|\sum_{i=1}^{\infty} b_i x_i\| \leq 1, \ \forall n \geq n_0$.

To prove lemma 3.1, we take a $1/(1+\varepsilon)$-ℓ_1^+ subsequence (x_n) of (e_n) by definition. Choosing n_0 large enough such that b_i is small for all $i \leq n_0$, we can apply an argument similar to that in Lemma 2.1. That is, we can find the succeeding b_i using intermediate value theorem and use induction to break the norm into products, thus showing that the whole thing can be bounded above by 1. □

Although Corollary 3.2 shows something different from Corollary 2.2, we shall see that the idea and arguments are similar in essence.

Corollary 3.2. *Let X be a Banach space and (e_n) be a normalized sequence in X generating ℓ_1^+ almost isometrically. Let $F \subseteq B_{X^*}$ be such that $\sum_n a_n e_n$ attains its norm at some element of F for all $(a_n) \in \ell_1^+$. Then for all $0 < \delta < 1$, there exists $\Lambda \in F$ such that*

$$\limsup_n |\Lambda e_n| \geq \delta.$$

Moreover, if every subsequence of (e_n) admits a τ_F-cluster point which attains its norm at some element of F, then there exists $\Lambda \in F$ such that

$$\limsup_n |\Lambda e_n| = 1.$$

For the fisrt part, one sees that the converse statement leads to conditions and estimations similar to those in Corollary 2.2. That is, for some $u = \sum b_i x_i$ with (x_i) being a suitable subsequence of (e_i), we can control the contribution of $|\Lambda x_i|$ and the coefficients b_i given by Lemma 3.1, thus getting $|\Lambda u| < \|u\|$ for all $\Lambda \in F$, which is a contradiction.
For the "moreover" part, we can again prove by contradiction. The point is to apply the first part repeatedly and then pick a subsequence diagonally.. □

In Corallary 3.2, we discussed a set F of functionals having a good property about attaining norms with respect to a normalized sequence generating ℓ_1^+ almost isometrically. This motivates the following definition.

Definition. *Let X be a Banach space and $F \subseteq B_{X^*}$. For $K \subseteq X$, we say K is F-admissible if*

1. *K is bounded.*
2. *K is τ_F-compact.*
3. *For all $(x_n) \in K$ and $(a_n) \in \ell_1$, $\sum_n a_n x_n$ attains its norm at some $\Lambda \in F$.*

Lemma 3.3 is needed in Lemma 3.4 and 3.5. It shows that if we have an ordinary sequence (x_n) in K at first, and if we can combine the x_i's using a fixed number of bounded scalars such that the resulting (u_n) is normalized and generates ℓ_1^+ almost isometrically, then (u_n) will satisfy the "Moreover" part in Corollary 3.2.

Lemma 3.3. *Let X be a Banach space, $F \subseteq B_{X^*}$, and $K \subseteq X$ be F-admissible. Let $(x_n) \subseteq K$ be a sequence, (I_n) be a sequence of finite subsets of $\mathbb{N}$ with the same cardinality, (λ_n) be a bounded sequence of scalars, and $u_n = \sum_{i \in I_n} \lambda_i x_i$ for all $n \in \mathbb{N}$. Assume that (u_n) is normalized and generates ℓ_1^+ almost isometrically, then there exists $\Lambda \in F$ such that* $\limsup_n |\Lambda u_n| = 1$.

To prove Lemma 3.3, by taking convergent subsequence of λ_i and cluster points of x_i by the compactness of K, we can obtain a norm-attaining cluster point of u_n. □

3.2 Admissibility with respect to a symmetric set of functionals

Having established Lemma 3.4, our goal now is to prove Corollary 3.7, whose argument requires sequences that are equivalent to the ℓ_1-basis. Let us first give the following definitions.

Definition. *Let X, Y be Banach spaces and $\{x_n\}$, $\{y_n\}$ be bases of X and Y, respectively. We say $\{x_n\}$, $\{y_n\}$ are equivalent if there exists an isomorphism $T : X \to Y$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.*

Definition. *Let X be a Banach space, $(x_n) \in X$. We say (x_n) is an ℓ_1-sequence if*

1. *$\{x_n\}$ is a basis of $\overline{span\{x_1, x_2, \ldots\}}$.*
2. *$\{x_n\}$ is equivalent to the usual basis of ℓ_1.*

From the above definitions, we have the following observation.

Observation. *Let (e_n) be a sequence in a Banach space which is isometrically equivalent to the ℓ_1-basis. Given $d \in \mathbb{N}$ and $m_1 < \ldots < m_d \in \mathbb{N}$, $(\lambda_1 \ldots, \lambda_{d+1}) \in (\mathbb{C}^\times)^{d+1}$ with $|\lambda_1| + \ldots + |\lambda_{d+1}| = 1$, if we define*

$$u_n = \sum_{i=1}^{d} \lambda_i e_{m_i} + \lambda_{d+1} e_n$$

for all $n > m_d$, then $(u_n)_{n > m_d}$ generates ℓ_1^+ isometrically.

We will also use the notion of symmetric sets in Banach spaces, which is defined as follows.

Definition. *Let X be a Banach space and $A \subseteq X$. We say A is symmetric if $zA \subseteq A$ for all $z \in \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.*

All the following lemmas are about admissible sets with respect to a symmetric collection of functionals in the closed unit ball. Lemma 3.4 is used to derive contradiction in the arguments of Corollary 3.7.

Lemma 3.4. *Let X be a Banach space, $F \subseteq B_{X^*}$ be symmetric, and $K \subseteq X$ be F-admissible. Then K contains no sequence isometrically equivalent to the ℓ_1-basis.*

Lemma 3.4 is proved by contradiction. Suppose $(x_n) \subseteq K$ is isometrically equivalent to the ℓ_1-basis, then by the above observation, the symmetry of F, and Lemma 3.3, we can inductively find a sequence of infinite subsets $\mathbb{N} \supsetneq M_1 \supsetneq M_2 \supsetneq \dots$ and $(\Lambda_n) \subseteq F$ such that for all $n \in \mathbb{N}$, we have $m_n := \min M_{n-1} \notin M_n$ and $\lim_{k\in M_n} \Lambda_n \left(\sum_{i=1}^n \frac{1}{2n} x_{m_i} - \frac{1}{2}x_k\right) = 1$. Let $x_0 \in K$ be a τ_F-cluster point of (x_{m_n}). Then we can show that $(\frac{1}{2}x_{m_n} - \frac{1}{2}x_0)_{n=1}^\infty$ is normalized and indeed generates ℓ_1^+ isometrically. However, by the F-admissibility of K and the symmetry of F, for every $(a_n) \in \ell_1^+$ with $\sum_n a_n = 1$, there exists $\Lambda \in F$ such that $\sum_n a_n \Lambda(\frac{1}{2}x_{m_n} - \frac{1}{2}x_0) = 1$, and hence $\Lambda(x_{m_n} - x_0) = 2$ for all $n \in \mathbb{N}$. This contradicts the fact that x_0 is a τ_F-cluster point of x_{m_n}. □

We will use Lemma 3.5 to construct an important structure (tree) in Lemma 3.6. For Lemma 3.5, we first define a notation. For any set S, we write $[S]$ for the collection of infinite subsets of S.

Lemma 3.5. *Let X be a Banach space, $F \subseteq B_{X^*}$ be symmetric, and $K \subseteq X$ be F-admissible. Assume (e_n) is a normalized sequence in K generating ℓ_1 almost isometrically. Given $r \in \mathbb{N}$, finite $\Delta \subseteq \mathbb{T}$, and a collection $(N_i)_{i=1}^r$ of pairwise disjoint members of $[\mathbb{N}]$, there is a collection $(P_i)_{i=1}^r$ with $P_i \in [N_i]$ satisfying the following property:*

For every choice $\theta_1, \dots, \theta_r$ of members of Δ, there is $\Lambda \in F$ so that

$$\lim_{k\in P_i} \Lambda e_k = \theta_i, \ i = 1, \dots, r.$$

To prove Lemma 3.5, let $\theta_1, \dots, \theta_r \in \Delta$ be fixed. It suffices to find $M_i \in [N_i]$ for $1 \le i \le r$ and $\Lambda_0 \in F$ so that

$$\lim_{k\in M_i} \Lambda_0 e_k = \theta_i, \ i = 1, \dots, r.$$

We first choose a sequence $I_1 < I_2 < I_3 < \dots$ of successive finite subsets of $\mathbb{N}$ the form $I_j = \{l_{j1} < \dots < l_{jr}\}$, where $l_{ji} \in N_i, \ 1 \le i \le r, \ j \in \mathbb{N}$. Then define $v_k = \sum_{i=1}^r \bar{\theta}_i e_{l_{ki}}$ and $u_k = \frac{v_k}{\|v_k\|}, \ k \in \mathbb{N}$. We may assume $\lim_k \|v_k\| = r$ and that (u_k) generates ℓ_1 almost isometrically. By Lemma 3.3, there exists $\Lambda_1 \in F$ such that $\limsup_k |\Lambda_1 u_k| = 1$. We may assume $\lim_k |\Lambda_1 u_k| = 1$ and $\lim_k \Lambda_1 e_{l_{ki}} = a_i \in \mathbb{C}, \ i = 1, \dots, r$. Then, $\left|\sum_{i=1}^r a_i \bar{\theta}_i\right| = r$, and hence there exists $z \in \mathbb{T}$ such that $a_i\bar{\theta}_i = z, \ i = 1, \dots, r$. Finally, set $M_i = \{l_{ki} \mid k \in \mathbb{N}\}, \ i = 1, \dots, r$ and $\Lambda_0 = \bar{z}\Lambda_1 \in F$. Then

$$\lim_{k\in M_i} \Lambda_0 e_k = \theta_i, \ i = 1, \dots, r,$$

as disired. □

Definition. *Let X be a Banach space and $\Gamma \subseteq X$. We say Γ generates $\ell_1(|\Gamma|)$ isometrically if for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in \Gamma$, $a_1, \dots, a_n \in \mathbb{C}$, we have*

$$\left\|\sum_{i=1}^n a_i x_i\right\| = \sum_{i=1}^n |a_i|.$$

Lemma 3.6. *Let X be a Banach space, $F \subseteq B_{X^*}$ be symmetric, and $K \subseteq X$ be F-admissible. Assume that (e_n) is a normalized sequence in K generating ℓ_1 almost isometrically, then there exists a subset of K, whose cardinality is equal to the continum with elements being the τ_F-cluster points of (e_n), that generates $\ell_1(\mathfrak{c})$ isometrically.*

Before we prove Lemma 3.6, let us first define the dyadic tree and a partial ordering on it.

Definition. *Let $\mathcal{T}$ denote the dyadic tree $\bigcup_{n=1}^\infty \{0,1\}^n$, and define a partial ordering $\le'$ on $\mathcal{T}$ by*

$$(a_1, \dots, a_n) \le' (b_1, \dots, b_m) \iff n \le m \text{ and } a_i = b_i, \ i = 1, \dots, n.$$

A branch of $\mathcal{T}$ is a maximal well-ordered subset of $(\mathcal{T}, \le')$.

Now, let us prove Lemma 3.6. Let (ε_n) be a scalar sequence strictly decreasing to 0 and choose an increasing sequence $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \ldots$ of finite subsets of $\mathbb{T}$ such that Δ_n is an ε_n-net for $\mathbb{T}$ for all $n \in \mathbb{N}$. Namely, $d(x, \Delta_n) < \varepsilon_n$ for all $n \in \mathbb{N},\ x \in \mathbb{T}$ where $d(x, \Delta_n) := \inf\{d(x,t) \mid t \in \Delta_n\}$. Successive applications of Lemma 3.5 yield a binary tree $(M_\alpha)_{\alpha \in \mathcal{T}}$ of infinite subsets of $\mathbb{N}$ with the following property:

For each $n \in \mathbb{N}$ and all choices $(\zeta_\alpha)_{\alpha \in \{0,1\}^n}$ of elements from Δ_n, there exists $\Lambda \in F$ so that

$$\lim_{k \in M_\alpha} \Lambda e_k = \zeta_\alpha,\ \alpha \in \{0,1\}^n.$$

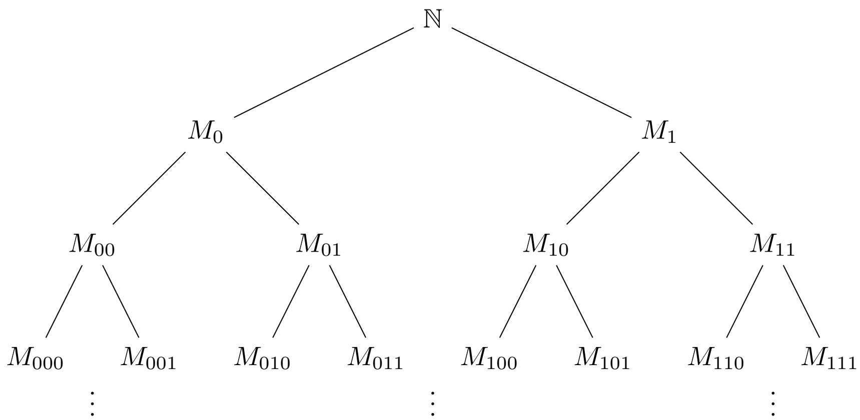

figure: binary tree $(M_\alpha)_{\alpha \in \mathcal{T}}$

Let $\mathfrak{b} = \{\beta_1 < \beta_2 < \beta_3 < \ldots\}$ be a branch of $(\mathcal{T}, \leq')$ and $N_\mathfrak{b} \in [\mathbb{N}]$ be almost contained in each M_{β_j}, namely $|N_\mathfrak{b} \setminus M_{\beta_j}| < \infty$ for all $j \in \mathbb{N}$. Let $e_\mathfrak{b} \in K$ be a τ_F-cluster point of $(e_n)_{n \in N_\mathfrak{b}}$, and $\mathcal{B} = \{\mathfrak{b}' : \text{branch of } (\mathcal{T}, \leq')\}$.
We claim that $(e_\mathfrak{b})_{\mathfrak{b} \in \mathcal{B}}$ generates $\ell_1(\mathfrak{c})$ isometrically. Let $m \in \mathbb{N}$, and let $\mathfrak{b}^1, \ldots, \mathfrak{b}^m$ be distinct branch of $(\mathcal{T}, \leq')$. Write $\mathfrak{b}^i = \{\beta_1^i < \beta_2^i < \beta_3^i < \ldots\},\ i = 1, \ldots, m$. Given $p \in \mathbb{N}$, we choose $n_0 > p$ so that $\beta_{n_0}^i \neq \beta_{n_0}^j$ whenever $i \neq j$ in $\{1, \ldots, m\}$. Let $(a_i)_{i=1}^m$ be scalars and write $a_i = |a_i| z_i$, where $z_i \in \mathbb{T}$. For each $i = 1, \ldots, m$, we can find $\theta_i \in \Delta_{n_0}$ so that $|\theta_i - \bar{z}_i| < \varepsilon_{n_0}$. By our construction there exists $\Lambda \in F$ so that

$$\lim_{k \in M_{\beta_{n_0}^i}} \Lambda e_k = \theta_i,\ i = 1, \ldots, m, \text{ whence } \Lambda e_{\mathfrak{b}^i} = \lim_{k \in N_{\mathfrak{b}^i}} \Lambda e_k = \theta_i,\ i = 1, \ldots, m.$$

Then $\|e_{\mathfrak{b}^i}\| = 1,\ i = 1, \ldots, m$. Finally, we can show that $\|\sum_{i=1}^m a_i e_{\mathfrak{b}^i}\| \geq (1 - \varepsilon_p) \sum_{i=1}^m |a_i|$ for all $p \in \mathbb{N}$. Taking $p \to \infty$, then we get $\|\sum_{i=1}^m a_i e_{\mathfrak{b}^i}\| \geq \sum_{i=1}^m |a_i|$. Therefore, $(e_\mathfrak{b})_{\mathfrak{b} \in \mathcal{B}}$ generates $\ell_1(\mathfrak{c})$ isometrically. □

We arrive at the main result of this subsection, which is needed in Theorem 1.1. In short, it shows that a closed subspace isomorphic to ℓ_1 cannot have its closed unit ball contained in a specific admissible set.

Corollary 3.7. *Let X be a Banach space, $F \subseteq B_{X^*}$ be symmetric, and $K \subseteq X$ be F-admissible. Suppose that Y is a closed linear subspace of X isomorphic to ℓ_1, then $B_Y \setminus K \neq \emptyset$.*

To prove Corollary 3.7, we need James's distortion theorem.

Theorem (James's distortion theorem)**.** *Let X be a Banach space. Then there exists a closed linear subspace Y of X isomorphic to ℓ_1 if and only if for every null sequence $(\varepsilon_n) \in (0,1)$, there exists a normalized sequence $(e_n) \in X$ such that*

$$\sum_{n=k}^\infty |a_n| \geq \left\| \sum_{n=k}^\infty a_n e_n \right\| \geq \frac{1}{1 + \varepsilon_k} \sum_{n=k}^\infty |a_n|,\ \forall (a_n) \in \ell_1,\ k \in \mathbb{N}.$$

Now, let us prove Corollary 3.7 by contradiction. Suppose not, then $B_Y \subseteq K$. James' distortion theorem now yields a normalized sequence (e_n) in K generating ℓ_1 almost isometrically. We deduce from Lemma 3.6 that K contains a normalized sequence isometrically equivalent to the ℓ_1-basis, contradicting Lemma 3.5. □

3.3 The general extension of James's compactness theorem

Let us again state theorem 1.1, which is the main result of this paper.

Theorem 1.1. *Let X, Y be Banach spaces and $F \subseteq B_{Y^*}$. Give Y the topology (Y, τ_F). If $T : X^* \to Y$ is bounded, linear, and (w^*, τ_F)-continuous, and*

$$\forall \Lambda \in X^*, \exists \Lambda' \in F \text{ such that } \|T\Lambda\| = |\Lambda' T\Lambda|,$$

then T is (w^*, w)-continuous.

To prove Theorem 1.1, we first claim that for any bounded sequence $(\Lambda_n) \subseteq X^*$, $(T\Lambda_n)$ has no ℓ_1-subsequence. This can be proved by contradiction. Suppose that $(T\Lambda_n)$ is equivalent to the ℓ_1-basis, then we may pull it back, and thus (Λ_n) is also an ℓ_1-basis. Restricting T to the spaces spanned by (Λ_n) and $(T\Lambda_n)$, the Banach inverse mapping theorem suggests that B_{Y^*} is a subset of an F-admissible set, contradicting Corollary 3.7.

Definition. *Let X be a Banach space. A sequence $(x_n) \in X$ is weakly Cauchy if (Λx_n) is Cauchy sequence for all $\Lambda \in X^*$.*

Theorem (Rosenthal's ℓ_1-theorem)**.** *Let X be a Banach space and $(x_n) \in X$ be a bounded sequence, then (x_n) has an ℓ_1-subsequence or a weakly Cauchy subsequence.*

Next we show the weak compactness of T. For any bounded sequence $(\Lambda_n) \subseteq X^*$, by Rosenthal's ℓ_1-theorem and the previous claim, we see that $(T\Lambda_n)$ has a weakly Cauchy subsequence. By Corollary 2.2, this subsequence weakly converges to $T\Lambda_0$, a cluster point provided by the continuity of T.

Theorem (Krein-Šmulian)**.** *Let X be a Banach space and $A \subseteq X^*$ be convex. If $A \cap \{\Lambda \in X^* \mid \|\Lambda\| \le r\}$ is weak* closed for all $r > 0$, then A is weak* closed.*

Theorem (Mazur)**.** *Let X be a Banach space, $A \subseteq X$ be convex, then the weak closure and the norm closure of A are equal.*

With the above theorems, we are ready to show that for any norm-closed convex set $V \subseteq Y$, $T^{-1}V$ is weak* closed in X^*. It suffices to prove that $U^* \cap T^{-1}V$ is closed for all closed balls U^* centered at the origin, as $T^{-1}V$ is convex. Note that the image of any converging net converges to some element in $V \cap TU^*$ by weakly compactness shown in the last step, and thus the image of the limit $T\Lambda$ does as well.
We are finally in a position to finish the proof. If a net $(\Lambda_\beta)_{\beta \in M}$ is weak* converging to 0 but its image is not weakly converging to 0, then we can find a $\Lambda' \in Y^*$ such that $|\Lambda' T\Lambda_\beta|$ is bounded away from 0. However, since $V_0 := \{y \in Y \mid \mathrm{Re}(\Lambda' y) > \delta\}$ is norm-closed and convex, from the last claim we see $T^{-1}V_0$ is weak* closed and contains $(\Lambda_\beta)_{\beta \in M}$ by properly choosing $\delta > 0$. Hence $0 \in V_0$, contradicting the choice of V_0. □

Lastly, we show that Theorem 1.1 implies James's compactness theorem. Corollary 3.8 is actually stronger than that, because if K is taken to be the closed unit ball of X, then K is relatively weakly compact if and only if X is reflexive.

Corollary 3.8. *Let X be a Banach space and $K \subseteq X$ be bounded. If $|\Lambda|$ attains its maximum on K for all $\Lambda \in X^*$, then K is relatively weakly compact.*

To prove corollary 3.8, let us first identify X canonically in X^{**}. Let $L = \overline{K}^{w^*}$ and $R : X^* \to C(L), \Lambda \mapsto \Lambda|_L$. Observe that R is (w^*, τ_F)-continuous and that it satisfies the requirements of Theorem 1.1, so it is also (w^*, w)-continuous and hence weakly compact. Corollary 2.4 then suggests that K is relatively weakly compact. □