Radius of Convergence

John M. Ernsthausen

Joint work with Ned Nedialkov

McMaster University
Canada

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Problem statement

Given sequence $\{c_n\}$ and *expansion point* $z_0 \in \mathbb{C}$ or $\mathbb{R}$ construct power series (PS)

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

What is an infinite sum?

When does an infinite sum converge?

But computers cannot sum an infinite sum!

- INPUTS: A finite sequence of real numbers, an expansion point, a scale
- OUTPUTS: Radius of convergence and Order of singularity

What is an infinite sum?

Given any sequence $\{c_n\}$ of complex numbers. Associate sequence $\{s_n\}$ of finite sumes

$$s_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} c_k$$

If $\{s_n\} \to s$ as $n \to \infty$ we say $\{s_n\}$ converges to s and write

$$\sum_{n=0}^{\infty} c_n = s$$

When does it converge?

Root test Given a series $\sum_{n=0}^{\infty} c_n$. Set $\alpha \stackrel{\text{def}}{=} \limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Then

(a) if $\alpha < 1$, then $\sum_{n=0}^{\infty} c_n$ converges;

(b) if $\alpha > 1$, then $\sum_{n=0}^{\infty} c_n$ diverges;

(c) if $\alpha = 1$, then this test gives no information.

Radius of convergence (RC) Given any sequence $\{c_n\}$, construct $\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Set $a_n = c_n(z-z_0)^n$. Root test $\sum_{n=0}^{\infty} a_n$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z-z_0| \limsup_{n\to\infty} \sqrt[n]{|c_n|} \stackrel{\text{def}}{=} \frac{|z-z_0|}{R_c}.$$

Obtain

(a) if $|z-z_0| < R_c$, then the power series converges;

(b) if $|z-z_0| > R_c$, then the power series diverges;

(c) if $|z-z_0| = R_c$, then this test gives no information.

The RC tests of CC

Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

The coefficients of a general PS follow no patterns

PS of real valued functions follow a few very definite patterns characterized by the location of primary singularities

PS which are real valued on the real axis can have poles, logarithmic branch points, and essential singularities only on the real axis or in conjugate pairs

The effects of secondary singularities disappear if sufficiently long PS are used

CC tests fit your (finite) sequence to their model

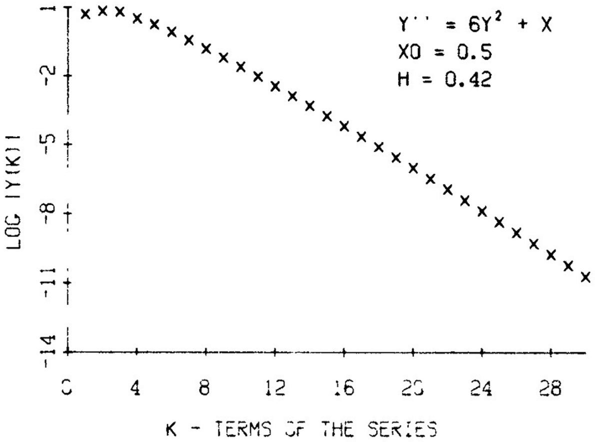

Figure 1: One necessarily real (primary) pole

This graph from Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

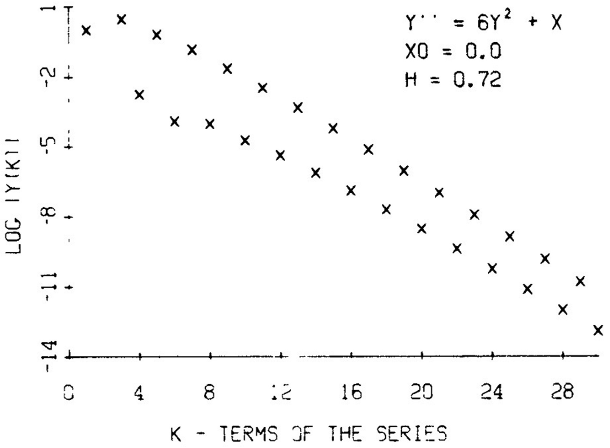

Figure 2: One necessarily real (primary) pole with a secondary pole nearby

This graph from Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

One pole: Three term analysis

Data fitting

Model the given (finite) real series (scale h) by a pole on the real axis which is radius a away and has order of singularity s

$$v(z) \stackrel{\text{def}}{=} (a - z)^{-s} \qquad s \neq 0, -1, -2, \ldots$$

Any function with only one primary pole or logarithmic singularity has the form $f(v(z))$ where f is analytic in some region

On circle centered at zero v has Taylor coefficients

$$k(v)_{k+1} = (v)_k \, (k + s - 1) \, \frac{h}{a}$$

Automatic differentiation formula for u^p with $p = -a$ and $u = a - z$. All derivatives of u zero except the first one which is -1

One pole: Three term analysis (continued)

Data fitting

Derive this result from k and $k-1$ recursion relations.

$RC \stackrel{\text{def}}{=} a$. Then

$$\frac{h}{RC} = \frac{h}{a} = k\frac{(v)_{k+1}}{(v)_k} - (k-1)\frac{(v)_k}{(v)_{k-1}}$$

For Order s (Use estimated RC from above)

$$\text{order} \stackrel{\text{def}}{=} s = k\frac{(v)_{k+1}}{(v)_k}\frac{RC}{h} - k + 1$$

If the series has one primary pole or logarithmic singularity, this test will detect it

Compute two estimates. Terms $N-2$, $N-1$, N. If the two estimates of h/RC do not agree (Relative backward error), then series does not have one primary real pole or logarithmic singularity

Complex conjugate pair of poles: Six term analysis

Data fitting

Model the given (finite) real series (scale h) by a Complex conjugate pair of poles which is radius a away and has order of singularity s

$b \in \mathbb{R}$ and $\cos\theta \stackrel{\text{def}}{=} b/a$ (Picture would help here)

Model

$$w(z) \stackrel{\text{def}}{=} \left(z^2 - 2bz + a^2\right)^{-s} \qquad s \neq 0, -1, -2, \ldots$$

On circle centered at zero w has Taylor coefficients

$$k(w)_{k+1} = 2(w)_k\,(k+s-1)\,\frac{h}{a}\cos\theta - (w)_{k-1}\,(k+2s-2)\left(\frac{h}{a}\right)^2$$

Reduced derivative formula for u^p with $p = -a$ and $u = z^2 - 2bz + a^2$.

Six term analysis (continued)

Data fitting

Unknowns: $\beta_1 = (h/a)\cos\theta$, $\beta_2 = s\beta_1$, $\beta_3 = (h/a)^2$, $\beta_4 = s\beta_3$

Equations:

$$\begin{aligned} k(w)_{k+1} &= (k-1)\,(w)_k\beta_1 + (w)_k\beta_2 \\ &\quad - (k-2)\,(w)_{k-1}\beta_3 - 2(w)_{k-1}\beta_4 \end{aligned}$$

for $k = N-1, N-2, N-3, N-4$

Linear system. Feasible solution unless $\beta_3 < 0$ or $|\cos\theta| > 1$ or large residual where residual is backward error checked against data from $N-5$ equation

Two values for s may disagree (a lot). Optimization problem minimizing norm of β. Enforce $\beta_3 \geq 0$ and $|\cos\theta| \leq 1$ in box constraints.

Top line analysis: The ratio test

Data fitting

Best linear fit $y(k) = mk + b$ in the 2-norm to the points

$$\left\{(n, \log_{10}|c_n|), (n+1, \log_{10}|c_{n+1}|), \ldots, (N, \log_{10}|c_N|)\right\}$$

for some sufficiently large n so that convergence guaranteed

$\sum_{n=0}^{\infty} c_n$ converges then $c_n \to 0$ as $n \to \infty$ The model parameter m will be negative

Compute the ratio in the ratio test with linear least squares best fit model. Then

$$\log_{10}\left|\frac{y(k+1)}{y(k)}\right| = \log_{10}|y(k+1)| - \log_{10}|y(k)| = m$$

which is independent of k.

Top line analysis (continued)

Data fitting

Have convergence analysis for top-line analysis

To find Order of singularity μ and the best possible RC

- Integrate sequence 3 times. Set $\mu = 4$
- Search for the sequence which opens downward.
- Differentiate the result up to 7 times reducing μ by one for each differentiation
- If cannot find such a sequence, conservatively use the final estimate RC but μ unknown

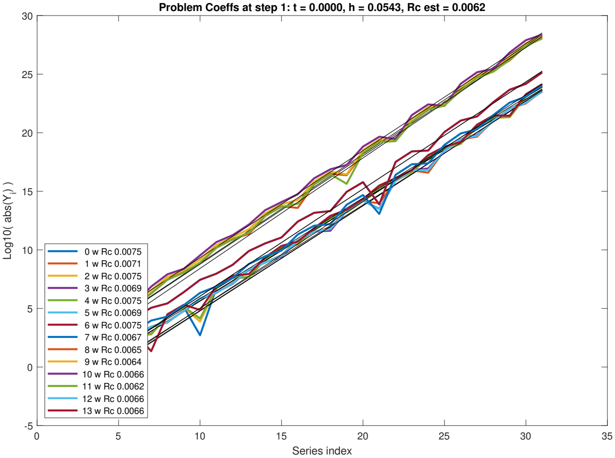

Figure 3: Top-line analysis shows divergent TS and rejected step

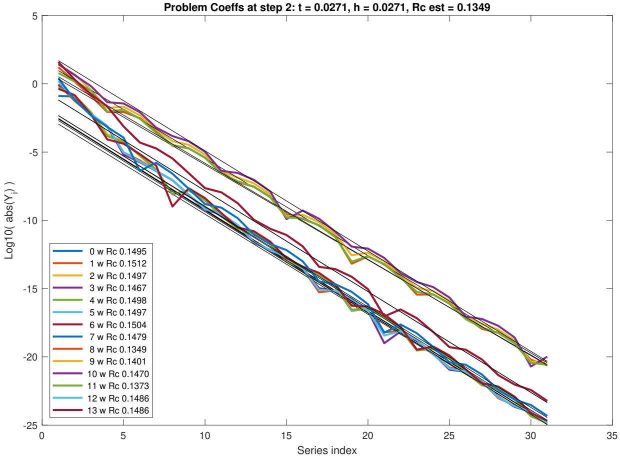

Figure 4: Top-line analysis indicates convergent TS and stepsize