Software Requirements Specification for ROC: software estimating the radius of convergence of a power series

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Revision History

Date	Version	Notes
12 October 2020	1.0	First submission

1 Reference Material

This section records information for easy reference.

1.1 Table of Units

A Table of Units is not applicable to ROC.

1.2 Table of Symbols

The table that follows summarizes the mathematical notation used in this document.

symbol	description
$z \in \mathbb{C}$	A member z of the complex numbers $\mathbb{C}$
$x \in \mathbb{R}$	A member x of the real numbers $\mathbb{R}$
$\left\{c_n\right\} \subset \mathbb{C}$	A sequence of complex numbers whose n^{th} term is $c_n \in \mathbb{C}$
$\left\{c_n\right\}_{n=0}^{N-1} \subset \mathbb{C}$	A finite sequence of N complex numbers whose n^{th} term is $c_n \in \mathbb{C}$
$\sum_{n=0}^{\infty} c_n(z - z_0)^n$	A power series. $c_n \in \mathbb{C}$ is the n^{th} coefficient. z^n is the n^{th} power of $z \in \mathbb{C}$. z_0 is the center point.
$\liminf\limits_{n\to\infty}$	Lower subsequential limit
$\limsup\limits_{n\to\infty}$	Upper subsequential limit
R_c	Radius of the circle of convergence
$\mathbb{R}^d$	A d-dimension real vector space
$\mathcal{D}$	An open subset of $\mathbb{R}^d$
$[a, b] \subset \mathbb{R}$	Interval of real numbers, $t \in [a, b]$ means $a \leq t \leq b$ and $t \in \mathbb{R}$
$(x_n)_n$	n^{th} Taylor coefficient of $t \in \mathbb{R} \mapsto x \in \mathbb{R}^d$ evaluated at $t_n \in \mathbb{R}$

1.3 Abbreviations and Acronyms

symbol	description
A	Assumption
BM	Bergsma and Mooij Bergsma and Mooij (2016)
CC	Chang and Corliss Chang and Corliss (1982)
C++	The programming language
$\mathbb{C}$	Complex numbers
DD	Data Definition
FORTRAN 77	The programming language
GD	General Definition
GS	Goal Statement
IM	Instance Model
IVP	Initial Value Problem
JZ	Jorba and Zou Jorba and Zou (2005)
LC	Likely Change
$\mathbb{N}$	The non-negative integers
ODE	Ordinary Differential Equation
$\mathbb{R}$	Real numbers
R	Requirement
ROC	Radius of Convergence software developed for this project
SRS	Software Requirements Specification
T	Theoretical Model
TC	Taylor coefficient
TS	Taylor series

2 Introduction

Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n(z - z_0)^n \tag{1}$$

is called a *power series*. The number $c_n \in \mathbb{C}$ is the n^{th} coefficient in the power series. The symbol z^n denotes the n^{th} power of the complex number z. This power series is *centered* at $z_0 \in \mathbb{C}$.

In general, a power series will converge or diverge, depending on the magnitude of $z - z_0$. With every power series, there is associated a circle of convergence such that (1) converges if z is in the interior of the circle of convergence or diverges if z is in the exterior of the circle of convergence. The convergence/divergence behavior of (1) on the circle of convergence can not be described so simply. By convention, the entire complex plane is the interior of a circle of infinite radius, and a point is the interior of a circle of zero radius.

This project is concerned with estimating the radius R_c of the circle of convergence.

2.1 Purpose of Document

The purpose of this document is to facilitate communication between the stakeholders and developers during the software development of project ROC by communicating and reflecting its software requirements. The scientific and business problem ROC solves is described in Section 4.1, the "Problem Description".

2.2 Scope of Requirements

In the late 1800's, several authors resolved problems concerning the characterization and analysis of singularities for power series. CC discuss this history Chang and Corliss (1982).

The CC approach to series analysis is the topic of the following quote.

> Our approach to series analysis was motivated by the observation that series for solutions to ODEs follow a few very definite patterns which are characterized by the location of primary singularities. In general, the coefficients of a power series follow no patterns, so few theorems about truncated series can be proved. However, series which are real-valued on the real axis can have poles, logarithmic branch points, and essential singularities only on the real axis or in conjugate pairs. Further, the effects of all secondary singularities disappear if sufficiently long series are used. (Chang and Corliss, 1982, p. 122)

A primary singularity of (1) is the closest singularity to the series expansion point z_0 in the complex plane. All other singularities are secondary singularities.

CC proposed a method of four parts to estimate the radius R_c of the circle of convergence as well as the order and location of primary singularities. The top-hump analysis applies to the power series of entire functions. The 3-term analysis applies to the power series of functions exhibiting a single primary singularity. The 6-term analysis applies to the power series of functions exhibiting a conjugate pair of primary singularities. Whenever these three analysis fail to resolve the R_c, singularity order, and singularity location parameters for the series, the CC method does a top-line analysis. Each CC sub-method works by fitting R_c, singularity order, and singularity location parameters of a known model to the given sequence.

Top-line analysis always applies to power series. It resolves situations where secondary singularities are less distinguishable from primary singularities. However it is less accurate, and it does not have a convergence analysis.

The scope of this ROC project is limited to top-line analysis.

2.3 Characteristics of Intended Reader

This document assumes the intended reader has familiarity with basic real analysis and complex analysis. Sequences and power series as well as the ratio and root tests will be discussed in this document. However, our exposition will only cover the concepts needed for our purposes. For proofs and for a complete exposition of all background materials, the interested reader should consult a beginning level graduate text such as Rudin (1976) for proofs.

2.4 Organization of Document

This document is built on the template recommendations in Smith and Lai (2005); Smith et al. (2007) that seeks to standardize communication tools for software development. The suggested order for reading this SRS document is: Goal Statement (Subsection 4.1.3), Instance Models (Subsection 4.2.5), Requirements (Section 5), Introduction (Section 2), and Specific System Description (Section 4).

3 General System Description

This section provides general information about the system. It identifies the interfaces between the system and its environment, describes the user characteristics, and lists the system constraints.

3.1 System Context

The following equation depicts ROC in symbols:

$$\{c_n\}_{n=0}^{N-1} \to R_c \tag{2}$$

3.2 User Characteristics

One intended user of ROC is a user of MAPLE or MATLAB. While neither MATLAB nor MAPLE currently implements an estimate for R_c, presumably the companies that develop MATLAB and MAPLE will want to provide such a facility when robust, reliable, and accurate computational tools are available.

After generating a real-valued Taylor series (TS) approximate solution of order p to the ordinary differential equation (ODE) initial-value problem (IVP)

$$y'(t) = f\big(t, y(t)\big), \quad y(t_0) = y_0 \in \mathcal{D} \subset \mathbb{R}^d, \quad t \in [t_0, t_{\text{end}}] \subset \mathbb{R}, \tag{3}$$

the TS methods defined in Jorba and Zou Jorba and Zou (2005) (JZ), Bergsma and Mooij Bergsma and Mooij (2016) (BM), and Chang and Corliss Chang and Corliss (1982) (CC) explicitly require an estimate for the TS radius of convergence.

At each (t_n, x_n), $n \geq 0$, the TS method for the numerical solution of (3) computes Taylor coefficients (TCs) $(x_n)_i$ at t_n to construct the TS approximate solution

$$T(t) = x_n + \sum_{i=1}^{p} (x_n)_i (t - t_n)^i \quad \text{on} \quad [t_n, t_{n+1}]. \tag{4}$$

Another type of intended user is any developers like BM, JZ, or CC of a TS method seeking the accuracy assurance from knowing the domain $[t_n, t_{n+1}]$ is in the circle of convergence.

3.3 System Constraints

The method developed in this project is expected to be independent of system constraints. However most TS methods are developed in C++ or FORTRAN 77, the goto languages of scientific computing. Certainly a scripting language would not be sufficient for large systems.

4 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

4.1 Problem Description

ROC is intended to estimate the radius of the circle of convergence of a power series.

4.1.1 Terminology and Definitions

A *sequence* is a function f whose domain is the non-negative integers $\mathbb{N}$ and range is in E, that is, a sequence is the mapping $n \in \mathbb{N} \mapsto f(n) = c_n \in E$. It is customary to denote the sequence f by the symbol $\{c_n\}$ or by $c_0, c_1, c_2, \ldots$. The values of f, that is, the elements c_n

are called the *terms* of the sequence. If A is a subset of E and if $c_n \in A$ for all $n \in \mathbb{N}$, the $\{c_n\}$ is said to be a *sequence in* A. The terms of a sequence need not be distinct. Typically, E is the complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$.

Given a sequence $\{c_n\}$, we use the notation

$$\sum_{n=p}^{q} c_n \qquad \text{with} \qquad p \leq q \tag{5}$$

to denote the sum $c_p + c_{p+1} + \cdots + c_q$. With the sequence $\{c_n\}$, we associate a sequence $\{s_n\}$, where

$$s_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} c_k. \tag{6}$$

For the sequence $\{s_n\}$, we may use the symbolic expression $c_0 + c_1 + c_2 + \ldots$ or

$$\sum_{n=0}^{\infty} c_n. \tag{7}$$

The symbol (10) is called an *infinite series* or just *series*. The terms s_n are called *partial sums* of the series, they are just numbers. If $\{s_n\} \to s$ as $n \to \infty$, then we say $\{s_n\}$ converges to s, the series converges, and write

$$\sum_{n=0}^{\infty} c_n = s. \tag{8}$$

The number s is the limit of a sequence of sums called the *sum of the series*. If $\{s_n\}$ diverges, then the series is said to diverge.

Given a sequence $\{e_n\}$, consider a sequence $\{n_k\}$ of non-negative integers such that $n_0 < n_1 < n_2 < \cdots$. Then the sequence $\{e_{n_k}\}$ is called a *subsequence* of $\{e_n\}$. If $\{e_{n_k}\}$ converges, its limit is called a *subsequential limit* of $\{e_n\}$.

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x in the extended real number system such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits plus possibly $+\infty$ and $-\infty$. Define $s^* \stackrel{\text{def}}{=} \sup E$ and $s_* \stackrel{\text{def}}{=} \inf E$. The numbers s^* and s_* are called the *upper limit* and *lower limit* of $\{s_n\}$. We use the notation

$$\liminf_{n\to\infty} s_n = s_* \qquad \text{and} \qquad \limsup_{n\to\infty} s_n = s^*. \tag{9}$$

4.1.2 Physical System Description

This subsection doesn't apply to ROC.

4.1.3 Goal Statements

Given the inputs, the goal statements are:

GS1: Implement CC top line analysis to estimate R_c.

GS2: Use top line analysis to estimate R_c for three problems from real analysis where R_c is known. Report on the accuracy of the result.

GS3: Use top line analysis to estimate the order of a singularity as described in CC (Chang and Corliss, 1982, pp. 127–128).

GS4: Use top line analysis to estimate R_c while solving ODEIVP by the TS method.

4.2 Solution Characteristics Specification

This section characterizes the attributes of an acceptable solution. Both analysts and stakeholders should agree on these attributes so that the solution can be accepted when the project is complete.

4.2.1 Assumptions

Given a tolerance TOL, consider the sequence $\{c_n\}$ and its power series $\sum_{n=0}^{\infty} c_n$ under the following assumptions:

A1: We know an integer N such that, for all $m \geq n \geq N$, $|\sum_{k=n}^{m} c_k| <$ TOL.

A2: The sequence $\{c_n\}$ is a subset of $\mathbb{R}$.

A3: The software ROC will estimate the radius of convergence from a finite number of terms in the power series. It will not compute R_c exactly.

A4: A Linear Least Squares fit to the input sequence approximates the Least Squares residual within TOL.

4.2.2 Theoretical Models

For the sequence $\{s_n\}$, we may use the symbolic expression $c_0 + c_1 + c_2 + \ldots$ or

$$\sum_{n=0}^{\infty} c_n. \tag{10}$$

A series (10) converges if an only if, for every $\epsilon > 0$, there is an integer N such that

$$|\sum_{k=n}^{m} c_k| < \epsilon \tag{11}$$

if $m \geq n \geq N$. It follows that, if (10) converges, then $\lim_{n\to\infty} c_n = 0$.

Indeed, ROC accepts a sufficiently large integer N and a finite sequence $\{c_0, c_1, \ldots, c_{N-1}\}$ with each $c_n \in \mathbb{C}$ as inputs. ROC returns as output an estimate for the radius of the circle of convergence, $R_c \in \mathbb{R}$.

[TM] Root test.

[TM] Ratio test.

[GD] Compare these tests.

Remark.

[GD] Definition of R_c.

[IM] Derivation of slope equation.

[Likely change] Error estimate. Sum the tail using linear model as truth.

This section focuses on the general equations and laws that ROC is based on.

4.2.3 General Definitions

This section collects the laws and equations that will be used in building the instance models.

Detailed derivation of simplified rate of change of temperature

4.2.4 Data Definitions

This section collects and defines all the data needed to build the instance models. The dimension of each quantity is also given.

4.2.5 Instance Models

This section transforms the problem defined in Section 4.1 into one which is expressed in mathematical terms. It uses concrete symbols defined in Section 4.2.4 to replace the abstract symbols in the models identified in Sections 4.2.2 and 4.2.3.

Derivation of ...

4.2.6 Input Data Constraints

Properly scaled sequence.

4.2.7 Properties of a Correct Solution

A correct solution must exhibit .

5 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

Table 1: Output Variables

Var	Physical Constraints
T_W	$T_{\text{init}} \leq T_W \leq T_C$ (by A??)

5.1 Functional Requirements

The method developed in this project is expected to be independent of system constraints. However most TS methods are developed in C++ or FORTRAN 77, the goto languages of scientific computing. Certainly a scripting language would not be sufficient for large systems.

R1:

R2:

R3:

R4:

R5:

5.2 Nonfunctional Requirements

Section 3.2 This section summarizes the knowledge/skills expected of the user. Measuring usability, which is often a required non-function requirement, requires knowledge of a typical user.

6 Likely Changes

LC1:

7 Unlikely Changes

LC2:

8 Traceability Matrices and Graphs

9 Values of Auxiliary Constants

ROC does not have symbolic parameters at this time.

References

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