Radius of Convergence

John M. Ernsthausen

Joint work with Ned Nedialkov

McMaster University
Canada

Nedialkov Group Presentation
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Problem

Given the first $k+1$ coefficients of the power series (PS) expansion of an $f : \mathbb{R} \to \mathbb{R}$

$$f(t) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{1}$$

estimate the radius of convergence (RC) of this series

We are interested in estimating the RC of Taylor series solutions of ODEs and DAEs

- Needed for reliable stepsize control

The RC tests of CC

Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

The coefficients of a general PS follow no patterns
PS of a real valued function any or ODE solution? follow a few very definite patterns

- characterized by the location of primary (closest to z_0) singularity
- effects of secondary (second closest) singularities diminish as number of coefficients increases

CC idea

For sufficiently large k, c_k behave like the coefficients of

- single-pole model

$$v(z) \stackrel{\text{def}}{=} (a-z)^{-s} \qquad s \neq 0, -1, -2, \ldots \tag{2}$$

- or conjugate-pair model

$$w(z) \stackrel{\text{def}}{=} \left(z^2 - 2bz + a^2\right)^{-s} \qquad s \neq 0, -1, -2, \ldots \tag{3}$$

We know the coefficients of (2) and (3)
Find which model the c_k fit, i.e. find a, s in (2) and a, b, s if (3)
We know the RC in (2) and (3)
Use it as an approximation of RC of (1).
Given f analytic (has PS expansion) at $x_0 = a$.
a is an *pole of f of order m* means f can be written as PS at a of the form $f(x) = (a-x)^m * H(x)$ where $H(x) = \sum_0^\infty c_k(a-x)^k$ is entire
a is an *essential singularity* means $f(x) = \sum_{-\infty}^{\infty} c_k(a-x)^k$
a is a *logarithmic singularity*

Example

Single pole

Example

Conjugate pair of poles

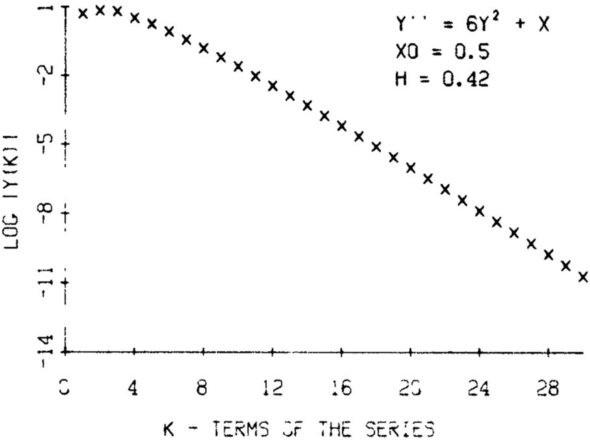

Figure 1: One necessarily real (primary) pole

This graph from Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

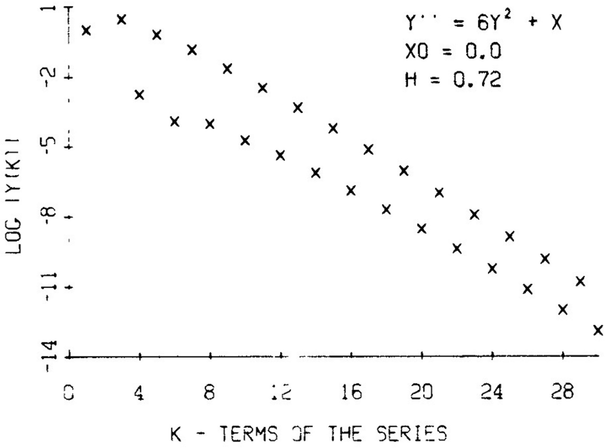

Figure 2: One necessarily real (primary) pole with a secondary pole nearby

This graph from Corliss and Chang, Solving Ordinary Differential Equations Using Taylor Series, ACM TOMS, 1982, pp. 114 - 144

One pole: Three term analysis

Data fitting

Model the given (finite) real series (scale h) by a pole on the real axis which is radius a away and has order of singularity s

$$v(z) \stackrel{\text{def}}{=} (a - z)^{-s} \qquad s \neq 0, -1, -2, \ldots$$

Any function with only one primary pole or logarithmic singularity has the form $f(v(z))$ where f is analytic in some region

On circle centered at zero v has Taylor coefficients

$$k(v)_{k+1} = (v)_k\,(k + s - 1)\,\frac{h}{a}$$

Automatic differentiation formula for u^p with $p = -a$ and $u = a - z$. All derivatives of u zero except the first one which is -1

One pole: Three term analysis (continued)

Data fitting

Derive this result from k and $k-1$ recursion relations.

$RC \stackrel{\text{def}}{=} a$. Then

$$\frac{h}{RC} = \frac{h}{a} = k\frac{(v)_{k+1}}{(v)_k} - (k-1)\,\frac{(v)_k}{(v)_{k-1}}$$

For Order s (Use estimated RC from above)

$$\text{order} \stackrel{\text{def}}{=} s = k\frac{(v)_{k+1}}{(v)_k}\frac{RC}{h} - k + 1$$

If the series has one primary pole or logarithmic singularity, this test will detect it

Compute two estimates. Terms $N-2$, $N-1$, N. If the two estimates of h/RC do not agree (Relative backward error), then series does not have one primary real pole or logarithmic singularity

Complex conjugate pair of poles: Six term analysis

Data fitting

Model the given (finite) real series (scale h) by a Complex conjugate pair of poles which is radius a away and has order of singularity s

$b \in \mathbb{R}$ and $\cos\theta \stackrel{\text{def}}{=} b/a$ (Picture would help here)

Model

$$w(z) \stackrel{\text{def}}{=} \left(z^2 - 2bz + a^2\right)^{-s} \qquad s \neq 0, -1, -2, \ldots$$

On circle centered at zero w has Taylor coefficients

$$k(w)_{k+1} = 2(w)_k\,(k+s-1)\,\frac{h}{a}\cos\theta - (w)_{k-1}\,(k+2s-2)\left(\frac{h}{a}\right)^2$$

Reduced derivative formula for u^p with $p = -a$ and $u = z^2 - 2bz + a^2$.

Six term analysis (continued)

Data fitting

Unknowns: $\beta_1 = (h/a)\cos\theta$, $\beta_2 = s\beta_1$, $\beta_3 = (h/a)^2$, $\beta_4 = s\beta_3$

Equations:

$$\begin{aligned} k(w)_{k+1} &= (k-1)\,(w)_k\beta_1 + (w)_k\beta_2 \\ &\quad - (k-2)\,(w)_{k-1}\beta_3 - 2(w)_{k-1}\beta_4 \end{aligned}$$

for $k = N-1, N-2, N-3, N-4$

Linear system. Feasible solution unless $\beta_3 < 0$ or $|\cos\theta| > 1$ or large residual where residual is backward error checked against data from $N-5$ equation

Two values for s may disagree (a lot). Optimization problem minimizing norm of β. Enforce $\beta_3 \geq 0$ and $|\cos\theta| \leq 1$ in box constraints.

Top line analysis: The ratio test

Data fitting

Best linear fit $y(k) = mk + b$ in the 2-norm to the points

$$\left\{(n, \log_{10}|c_n|), (n+1, \log_{10}|c_{n+1}|), \ldots, (N, \log_{10}|c_N|)\right\}$$

for some sufficiently large n so that convergence guaranteed

$\sum_{n=0}^{\infty} c_n$ converges then $c_n \to 0$ as $n \to \infty$ The model parameter m will be negative

Compute the ratio in the ratio test with linear least squares best fit model. Then

$$\log_{10}\left|\frac{y(k+1)}{y(k)}\right| = \log_{10}|y(k+1)| - \log_{10}|y(k)| = m$$

which is independent of k.

Top line analysis (continued)

Data fitting

Have convergence analysis for top-line analysis

To find Order of singularity μ and the best possible *RC*

- Integrate sequence 3 times. Set $\mu = 4$
- Search for the sequence which opens downward.
- Differentiate the result up to 7 times reducing μ by one for each differentiation
- If cannot find such a sequence, conservatively use the final estimate *RC* but μ unknown

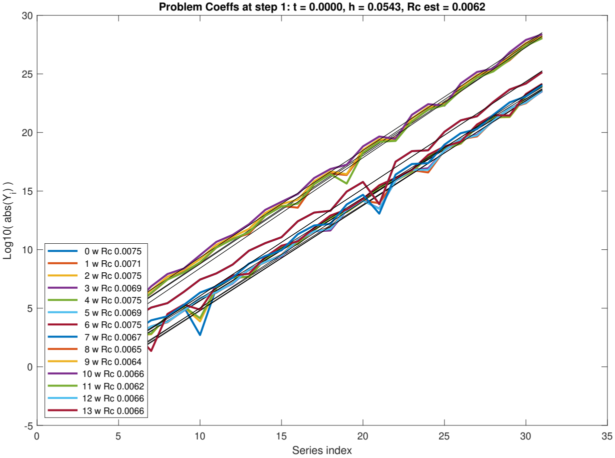

Figure 3: Top-line analysis shows divergent TS and rejected step

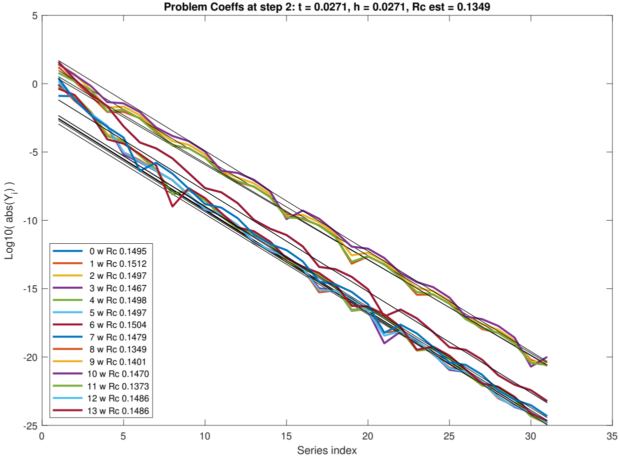

Figure 4: Top-line analysis indicates convergent TS and stepsize