CAS 741: Problem Statement
Radius of Convergence

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Table 1: Revision History

Date	Developer(s)	Change
September 18, 2020	John Ernsthausen	Initial draft
September 25, 2020	John Ernsthausen	Removed connection to DAETS
24 December 2020	John Ernsthausen	Second submission

Construct the power series centered at $z_0 \in \mathbb{R}$

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{1}$$

from a sequence $\left\{c_n\right\}$ of real numbers where the n^{th} term in the sequence corresponds to the n^{th} coefficient in the series. We associate a sequence $\left\{s_n\right\}$ of partial sums

$$s_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} c_k (z - z_0)^k \tag{2}$$

with the power series. If $\left\{s_n\right\} \to s$ as $n \to \infty$, then we say $\left\{s_n\right\}$ converges to s. The number s is the sum of the series, and we write s as (1). If $\left\{s_n\right\}$ diverges, then the series is said to diverge.

We cannot perform an infinite sum on a digital computer, and we cannot identify the set of all z near z_0 such that (1) converges by exhaustive checking. We need to identify the circle of convergence by its radius. A power series always converges inside a circle of convergence of radius R_c.

Chang and Corliss (1) observed that the coefficients of (1) follow a few very definite patterns characterized by the location of primary singularities. Real valued power series can only have poles, logarithmic branch points, and essential singularities. Moreover, these singularities occur on the real axis or in complex conjugate pairs. The effects of secondary singularities disappear whenever sufficiently long power series are used. Recall that a primary singularity of (1) is the closest singularity to the series expansion point z_0 in the complex plane. All other singularities are secondary singularities.

To determine R_c and the order of the singularity μ, Chang and Corliss (1) fit a given finite sequence corresponding to the partial sum s_N for N sufficiently large to a model.

I propose to estimate the R_c of a given power series based on the three term analysis for a real pole, six term analysis for a pair of complex conjugate poles, and top line analysis (1, pp. 127–128). The input is a finite sequence of scaled coefficients and their scaling. The output is R_c and the order of the singularity μ.

References

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