

Software Requirements Specification for ROC: software estimating the radius of convergence of a power series

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Revision History

Date	Version	Notes
12 October 2020	1.0	First submission

1 Reference Material

This section records information for easy reference.

1.1 Table of Units

A Table of Units is not applicable to ROC.

1.2 Table of Symbols

The table that follows summarizes the mathematical notation used in this document.

symbol	description
$z \in \mathbb{C}$	A member z of the complex numbers \mathbb{C}
$x \in \mathbb{R}$	A member x of the real numbers \mathbb{R}
$\{c_n\} \subset \mathbb{C}$	A sequence of complex numbers whose n^{th} term is $c_n \in \mathbb{C}$
$\{c_n\}_{n=0}^{N-1} \subset \mathbb{C}$	A finite sequence of N complex numbers whose n^{th} term is $c_n \in \mathbb{C}$
$\sum_{n=0}^{\infty} c_n(z - z_0)^n$	A power series. $c_n \in \mathbb{C}$ is the n^{th} coefficient. z^n is the n^{th} power of $z \in \mathbb{C}$. z_0 is the center point.
$\liminf_{n \rightarrow \infty}$	Lower subsequential limit
$\limsup_{n \rightarrow \infty}$	Upper subsequential limit
R_c	Radius of the circle of convergence
\mathbb{R}^d	A d -dimension real vector space
\mathcal{D}	An open subset of \mathbb{R}^d
$[a, b] \subset \mathbb{R}$	Interval of real numbers, $t \in [a, b]$ means $a \leq t \leq b$ and $t \in \mathbb{R}$
$(x_n)_n$	n^{th} Taylor coefficient of $t \in \mathbb{R} \mapsto x \in \mathbb{R}^d$ evaluated at $t_n \in \mathbb{R}$

1.3 Abbreviations and Acronyms

symbol	description
A	Assumption
BM	Bergsma and Mooij Bergsma and Mooij (2016)
CC	Chang and Corliss Chang and Corliss (1982)
C++	The programming language
\mathbb{C}	Complex numbers
DD	Data Definition
FORTRAN 77	The programming language
GD	General Definition
GS	Goal Statement
IM	Instance Model
IVP	Initial Value Problem
JZ	Jorba and Zou Jorba and Zou (2005)
LC	Likely Change
\mathbb{N}	The non-negative integers
ODE	Ordinary Differential Equation
\mathbb{R}	Real numbers
R	Requirement
ROC	Radius of Convergence software developed for this project
SRS	Software Requirements Specification
T	Theoretical Model
TC	Taylor coefficient
TS	Taylor series

2 Introduction

Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{1}$$

is called a *power series*. The number $c_n \in \mathbb{C}$ is the n^{th} coefficient in the power series. The symbol z^n denotes the n^{th} power of the complex number z . This power series is *centered* at $z_0 \in \mathbb{C}$.

In general, a power series will converge or diverge, depending on the magnitude of $z - z_0$. With every power series, there is associated a circle of convergence such that (1) converges if z is in the interior of the circle of convergence or diverges if z is in the exterior of the circle of convergence. The convergence/divergence behavior of (1) on the circle of convergence can not be described so simply. By convention, the entire complex plane is the interior of a circle of infinite radius, and a point is the interior of a circle of zero radius.

This project is concerned with estimating the radius R_c of the circle of convergence.

2.1 Purpose of Document

The purpose of this document is to facilitate communication between the stakeholders and developers during the software development of project ROC by communicating and reflecting its software requirements. The scientific and business problem ROC solves is described in Section 4.1, the “Problem Description”.

2.2 Scope of Requirements

In the late 1800’s, several authors resolved problems concerning the characterization and analysis of singularities for power series. CC discuss this history [Chang and Corliss \(1982\)](#).

The CC approach to series analysis is the topic of the following quote.

Our approach to series analysis was motivated by the observation that series for solutions to ODEs follow a few very definite patterns which are characterized by the location of primary singularities. In general, the coefficients of a power series follow no patterns, so few theorems about truncated series can be proved. However, series which are real-valued on the real axis can have poles, logarithmic branch points, and essential singularities only on the real axis or in conjugate pairs. Further, the effects of all secondary singularities disappear if sufficiently long series are used. ([Chang and Corliss, 1982](#), p. 122)

A primary singularity of (1) is the closest singularity to the series expansion point z_0 in the complex plane. All other singularities are secondary singularities.

CC proposed a method of four parts to estimate the radius R_c of the circle of convergence as well as the order and location of primary singularities. The top-hump analysis applies

to the power series of entire functions. The 3-term analysis applies to the power series of functions exhibiting a single primary singularity. The 6-term analysis applies to the power series of functions exhibiting a conjugate pair of primary singularities. Whenever these three analysis fail to resolve the R_c , singularity order, and singularity location parameters for the series, the CC method does a top-line analysis. Each CC sub-method works by fitting R_c , singularity order, and singularity location parameters of a known model to the given sequence.

Top-line analysis always applies to power series. It resolves situations where secondary singularities are less distinguishable from primary singularities. However it is less accurate, and it does not have a convergence analysis.

The scope of this ROC project is limited to top-line analysis.

2.3 Characteristics of Intended Reader

This document assumes the intended reader has familiarity with basic real analysis and complex analysis. Sequences and power series as well as the ratio and root tests will be discussed in this document. However, our exposition will only cover the concepts needed for our purposes. For proofs and for a complete exposition of all background materials, the interested reader should consult a beginning level graduate text such as [Rudin \(1976\)](#) for proofs.

2.4 Organization of Document

This document is built on the template recommendations in [Smith and Lai \(2005\)](#); [Smith et al. \(2007\)](#) that seeks to standardize communication tools for software development. The suggested order for reading this SRS document is: Goal Statement (Subsection 4.1.3), Instance Models (Subsection 4.2.5), Requirements (Section 5), Introduction (Section 2), and Specific System Description (Section 4).

3 General System Description

This section provides general information about the system. It identifies the interfaces between the system and its environment, describes the user characteristics, and lists the system constraints.

3.1 System Context

The following equation depicts ROC in symbols:

$$\{c_n\}_{n=0}^{N-1} \rightarrow R_c \tag{2}$$

3.2 User Characteristics

One intended user of ROC is a user of MAPLE or MATLAB. While neither MATLAB nor MAPLE currently implements an estimate for R_c , presumably the companies that develop MATLAB and MAPLE will want to provide such a facility when robust, reliable, and accurate computational tools are available.

After generating a real-valued Taylor series (TS) approximate solution of order p to the ordinary differential equation (ODE) initial-value problem (IVP)

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \in \mathcal{D} \subset \mathbb{R}^d, \quad t \in [t_0, t_{\text{end}}] \subset \mathbb{R}, \quad (3)$$

the TS methods defined in Jorba and Zou [Jorba and Zou \(2005\)](#) (JZ), Bergsma and Mooij [Bergsma and Mooij \(2016\)](#) (BM), and Chang and Corliss [Chang and Corliss \(1982\)](#) (CC) explicitly require an estimate for the TS radius of convergence.

At each (t_n, x_n) , $n \geq 0$, the TS method for the numerical solution of (3) computes Taylor coefficients (TCs) $(x_n)_i$ at t_n to construct the TS approximate solution

$$T(t) = x_n + \sum_{i=1}^p (x_n)_i (t - t_n)^i \quad \text{on} \quad [t_n, t_{n+1}]. \quad (4)$$

Another type of intended user is any developers like BM, JZ, or CC of a TS method seeking the accuracy assurance from knowing the domain $[t_n, t_{n+1}]$ is in the circle of convergence.

3.3 System Constraints

The method developed in this project is expected to be independent of system constraints. However most TS methods are developed in C++ or FORTRAN 77, the goto languages of scientific computing. Certainly a scripting language would not be sufficient for large systems.

4 Specific System Description

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the solution characteristics specification, which presents the assumptions, theories, definitions and finally the instance models.

4.1 Problem Description

ROC is intended to estimate the radius of the circle of convergence of a power series.

4.1.1 Terminology and Definitions

A *sequence* is a function f whose domain is the non-negative integers \mathbb{N} and range is in E , that is, a sequence is the mapping $n \in \mathbb{N} \mapsto f(n) = c_n \in E$. It is customary to denote the sequence f by the symbol $\{c_n\}$ or by c_0, c_1, c_2, \dots . The values of f , that is, the elements c_n

are called the *terms* of the sequence. If A is a subset of E and if $c_n \in A$ for all $n \in \mathbb{N}$, the $\{c_n\}$ is said to be a *sequence in A* . The terms of a sequence need not be distinct. Typically, E is the complex numbers \mathbb{C} or the real numbers \mathbb{R} .

Given a sequence $\{c_n\}$, we use the notation

$$\sum_{n=p}^q c_n \quad \text{with} \quad p \leq q \quad (5)$$

to denote the sum $c_p + c_{p+1} + \cdots + c_q$. With the sequence $\{c_n\}$, we associate a sequence $\{s_n\}$, where

$$s_n \stackrel{\text{def}}{=} \sum_{k=0}^n c_k. \quad (6)$$

For the sequence $\{s_n\}$, we may use the symbolic expression $c_0 + c_1 + c_2 + \dots$ or

$$\sum_{n=0}^{\infty} c_n. \quad (7)$$

The symbol (7) is called an *infinite series* or just *series*. The terms s_n are called *partial sums* of the series, they are just numbers. If $\{s_n\} \rightarrow s$ as $n \rightarrow \infty$, then we say $\{s_n\}$ converges to s , the series converges, and write

$$\sum_{n=0}^{\infty} c_n = s. \quad (8)$$

The number s is the limit of a sequence of sums called the *sum of the series*. If $\{s_n\}$ diverges, then the series is said to diverge.

Given a sequence $\{e_n\}$, consider a sequence $\{n_k\}$ of non-negative integers such that $n_0 < n_1 < n_2 < \cdots$. Then the sequence $\{e_{n_k}\}$ is called a *subsequence* of $\{e_n\}$. If $\{e_{n_k}\}$ converges, its limit is called a *subsequential limit* of $\{e_n\}$.

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x in the extended real number system such that $s_{n_k} \rightarrow x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits plus possibly $+\infty$ and $-\infty$. Define $s^* \stackrel{\text{def}}{=} \sup E$ and $s_* \stackrel{\text{def}}{=} \inf E$. The numbers s^* and s_* are called the *upper limit* and *lower limit* of $\{s_n\}$. We use the notation

$$\liminf_{n \rightarrow \infty} s_n = s_* \quad \text{and} \quad \limsup_{n \rightarrow \infty} s_n = s^*. \quad (9)$$

4.1.2 Physical System Description

This subsection doesn't apply to ROC.

4.1.3 Goal Statements

Given the inputs, the goal statements are:

GS1: Implement CC top line analysis to estimate R_c .

GS2: Use top line analysis to estimate R_c for three problems from real analysis where R_c is known. Report on the accuracy of the result.

GS3: Use top line analysis to estimate the order of a singularity as described in CC (Chang and Corliss, 1982, pp. 127–128).

GS4: Use top line analysis to estimate R_c while solving ODEIVP by the TS method.

4.2 Solution Characteristics Specification

This section characterizes the attributes of an acceptable solution. Both analysts and stakeholders should agree on these attributes so that the solution can be accepted when the project is complete.

4.2.1 Assumptions

Given a tolerance TOL, consider the sequence $\{c_n\}$ and its power series $\sum_{n=0}^{\infty} c_n$ under the following assumptions:

A1: We know an integer N such that, for all $m \geq n \geq N$, $|\sum_{k=n}^m c_k| < \text{TOL}$.

A2: The software ROC will estimate the radius of convergence from a finite number of terms in the power series. It will not compute R_c exactly.

A3: The sequence $\{c_n\}$ is a subset of \mathbb{R} .

4.2.2 Theoretical Models

Applying the terminology and definitions from Subsection 4.1.1, this section records theorems required to identify a convergent/divergent series.

Consider the sequence $\{c_n\}$ and its power series $\sum_{n=0}^{\infty} c_n$. The following TM is used to show that the coefficients in the terms of a series tend to zero as the index of the term tends to infinity.

Number	T1
Label	Cauchy convergence condition
Theorem	A series $\sum_{n=0}^{\infty} c_n$ converges if and only if, for every $\epsilon > 0$, there is an integer N such that $ \sum_{k=n}^m c_k < \epsilon$ whenever $m \geq n \geq N$.
Description	Tools to identify when a series converges.
Source	Theorem 3.22 (Rudin, 1976, p. 59)
Ref. By	Used to choose sequence to approximate and obtain error.

Number	T2
Label	Convergence of sequence
Theorem	If series $\sum_{n=0}^{\infty} c_n$ converges, then $\lim_{n \rightarrow \infty} c_n = 0$.
Description	If a series converges, then its terms converge to zero.
Source	Theorem 3.28 (Rudin, 1976, p. 60)
Ref. By	Our IM uses this to estimate α .

Number	T3
Label	Root test
Theorem	Given a series $\sum_{n=0}^{\infty} c_n$. Set $\alpha \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \sqrt[n]{ c_n }$. Then <ul style="list-style-type: none"> (a) if $\alpha < 1$, then $\sum_{n=0}^{\infty} c_n$ converges; (b) if $\alpha > 1$, then $\sum_{n=0}^{\infty} c_n$ diverges; (c) if $\alpha = 1$, then this test gives no information.
Description	Tools to identify when a series converges/diverges.
Source	Theorem 3.33 (Rudin, 1976, p. 65)
Ref. By	Used to define R_c .

Number	T4
Label	Ratio test
Theorem	<p>The series $\sum_{n=0}^{\infty} c_n$</p> <p>(a) converges if $\limsup_{n \rightarrow \infty} \left \frac{c_{n+1}}{c_n} \right < 1$,</p> <p>(b) diverges if $\left \frac{c_{n+1}}{c_n} \right \geq 1$ for $n \geq N$, where N is some fixed integer.</p>
Description	Tools to identify when a series converges/diverges.
Source	Theorem 3.34 (Rudin, 1976, p. 66)
Ref. By	Used to define our R_c .

The ratio test is often easier to apply than the root test. However, the root test resolves more application than the ratio test. Both the ratio test and the root test deduce divergence from the statement in Theoretical Model 2, if a series converges, then its terms converge to zero.

Number	T5
Label	Comparing the Ratio test and the Root test
Theorem	<p>For any sequence $\{c_n\}$ of positive (real) numbers,</p> <p>(a) $\liminf_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{c_n}$,</p> <p>(b) $\limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$</p>
Description	This General Definition shows that if the ratio test converges, then the root test converges and if the root test is inconclusive, then the ratio test is inconclusive. Whenever the limit exists and it is unique, then there is equality in (a) and (b) and $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = \lim_{n \rightarrow \infty} \sqrt[n]{c_n}$.
Source	Theorem 3.37 (Rudin, 1976, p. 68)
Ref. By	Our IM uses this for equality between Root test and Ratio test.

4.2.3 General Definitions

The proofs of the theorem in this section apply the terminology and definitions from Subsection 4.1.1 as well as the Theoretical models from Subsection 4.2.2.

The radius of the circle of convergence is defined in the next General Definition, a theorem that enables us to justify and construct the IM so that ROC will estimate R_c .

Number	GD1
Label	Define the radius of the circle of convergence
Theorem	Given any sequence $\{c_n\}$, construct the power series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$. Set $\alpha \stackrel{\text{def}}{=} \limsup_{n \rightarrow \infty} \sqrt[n]{ c_n }$ and $R_c \stackrel{\text{def}}{=} 1/\alpha$. Then $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ converges whenever $ z - z_0 < R_c$.
Description	This General Definition defines R_c , the radius of convergence of the power series. By our convention stated Subsection 4.1.1, $\alpha = 0$ implies $R_c = +\infty$ and $\alpha = +\infty$ implies $R_c = 0$.
Source	Theorem 3.39 (Rudin, 1976, p. 69)
Ref. By	Our IM uses this to determine R_c .

We need to relate the root test to the ratio test to obtain our IM. It is instructive to understand the role of the root test in the proof of GD1.

Inside GD1

Given any sequence $\{c_n\}$, construct the power series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$. Set $a_n = c_n(z - z_0)^n$, and apply the root test TM3 to the series $\sum_{n=0}^{\infty} a_n$.

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = |z - z_0| \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} \stackrel{\text{def}}{=} \frac{|z - z_0|}{R_c}. \quad (10)$$

Obtain from the root test that

- (a) if $|z - z_0| < R_c$, then the power series converges;
- (b) if $|z - z_0| > R_c$, then the power series diverges;
- (c) if $|z - z_0| = R_c$, then this test gives no information.

The next section derives an IM to approximate R_c .

4.2.4 Data Definitions

ROC does not rely on any Data Definitions at this time.

4.2.5 Instance Models

This section transforms the problem defined in Section 4.1 into one which can be translated into software. We will define finite sequences and series to replace the infinite counterparts identified in Sections 4.2.2 and 4.2.3.

Given $\text{TOL} > 0$ and any sequence $\{c_n\}$, construct the power series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$. Use Assumption 1 to obtain an integer N such that, for all $m \geq n \geq N$, $|\sum_{k=n}^m c_k| < \text{TOL}$. It is no loss in generality to assume $N > 30$, else replace N with 30. TM1 says that $\sum_{n=0}^{\infty} c_n$ converges.

We now invoke Assumption 2 and extract 15 elements $\{c_{N-14}, c_{N-13}, \dots, c_N\}$ from the sequence. With $k = i + N - 14$, obtain the best linear fit $y(k) = mk + b$ in the 2-norm to the points

$$\{(N - 14, \log_{10} |c_{N-14}|), (N - 13, \log_{10} |c_{N-13}|), \dots, (N, \log_{10} |c_N|)\}, \quad (11)$$

that is, find m and b such that $\sum_{i=0}^{14} |\log_{10} |c_{i+N-14}| - y(i + N - 14)|^2$ is minimized. Because $\sum_{n=0}^{\infty} c_n$ converges, TM2 says that $c_n \rightarrow 0$ as $n \rightarrow \infty$. The model parameter m will be negative.

Compute the ratio in the ratio test TM4 with our model. Then

$$\log_{10} \left| \frac{y(k+1)}{y(k)} \right| = \log_{10} |y(k+1)| - \log_{10} |y(k)| = m, \quad (12)$$

which is independent of k . By continuity of \log_{10} and the best linear fit function y as well as $\log_{10} \left| \frac{y(k+1)}{y(k)} \right|$ approximates $\log_{10} \left| \frac{c_{k+1}}{c_k} \right|$, we observe that $\log_{10} \left| \frac{y(k+1)}{y(k)} \right| \rightarrow m$ is approximately $\log_{10} \left| \frac{c_{k+1}}{c_k} \right| \rightarrow m$. By TM5, the ratio test TM4 converging implies the root test TM3 converges, because they converge to a single limit and the same limit. See description section of TM5. It follows from GD1 that the power series converges whenever $m < 0$ and diverges whenever $m > 0$. Moreover GD1 says $R_c = 1/10^m$.

Number	IM1
Label	Approximating the radius of convergence
Inputs	N and approximation points (11)
Output	R_c or confirmation of divergence.
Description	Approximate the radius of the circle of convergence, $R_c \approx 1/10^m$.
Sources	The full details are my own contribution. However the ideas are inspired by conversations with G. Corliss and N. Nedialkov.
Ref. By	Final product.

Chang and Corliss [Chang and Corliss \(1982\)](#) and ASrefreal. The sequence $\{c_n\}$ is a subset of \mathbb{R} .

4.2.6 Input Data Constraints

Properly scaled sequence.

4.2.7 Properties of a Correct Solution

Error tail.

A correct solution must exhibit .

Table 1: Output Variables

Var	Physical Constraints
T_W	$T_{\text{init}} \leq T_W \leq T_C$ (by A??)

5 Requirements

This section provides the functional requirements, the business tasks that the software is expected to complete, and the nonfunctional requirements, the qualities that the software is expected to exhibit.

5.1 Functional Requirements

Positive.

Chang and Corliss. Concave down. Envelope.

Reflect known cases.

Use top line analysis to estimate R_c for three problems from real analysis where R_c is known. Report on the accuracy of the result.

Use top line analysis to estimate the order of a singularity as described in CC (Chang and Corliss, 1982, pp. 127–128).

Use top line analysis to estimate R_c while solving ODEIVP by the TS method.

The method developed in this project is expected to be independent of system constraints. However most TS methods are developed in C++ or FORTRAN 77, the goto languages of scientific computing. Certainly a scripting language would not be sufficient for large systems.

R1:

R2:

R3:

R4:

R5:

5.2 Nonfunctional Requirements

Section 3.2 This section summarizes the knowledge/skills expected of the user. Measuring usability, which is often a required non-function requirement, requires knowledge of a typical user.

6 Likely Changes

[Likely change] Error estimate. Sum the tail using linear model as truth.

LC1:

7 Unlikely Changes

LC2:

8 Traceability Matrices and Graphs

9 Values of Auxiliary Constants

ROC does not have symbolic parameters at this time.

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