Hybrid State Space and Frequency Domain System Level Synthesis for Sparsity-Promoting $\mathcal{H}_2/\mathcal{H}_\infty$ Control Design

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Optimal Linear State Feedback Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Synthesis Is Valuable and Challenging

Renewable resources require performance and robustness for uncertainties

Sunlight Intensity

Wind Speed

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis

- has a long history
- is valuable on applications
- but challenging due to the nonconvexity

Our Objective: Improve Existing Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis

Methods for Tackling Nonconvexity

Youla[1]

Coprime decomposition

System Level Synthesis[2]

State feedback control

Output feedback control

Input-Output Parameterization[3]

Output feedback control

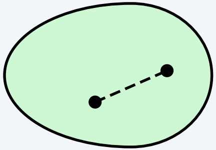
Reparameterize in terms of closed-loop system response

Nonconvex

Convex Reparameterization →

← Recovering Controller for Original Problem

Convex



[1] D. Youla, H. Jabr, and J. Bongiorno, "Modern wiener-hopf design of optimal controllers–part ii: The multivariable case," IEEE Transactions on Automatic Control, vol. 21, no. 3, pp. 319–338, 1976.

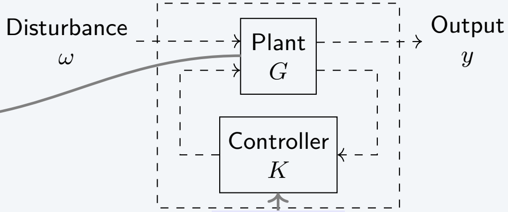
[2] J. Anderson, J. C. Doyle, S. H. Low, and N. Matni, "**System level synthesis**," Annual Reviews in Control, vol. 47, pp. 364–393, 2019.

[3] L. Furieri, Y. Zheng, A. Papachristodoulou, and M. Kamgarpour, "An **input–output parametrization** of stabilizing controllers: Amidst youla and system level synthesis," IEEE Control Systems Letters, vol. 3, no. 4, pp. 1014–1019, 2019.

Problem Formulation after System Level Synthesis

LTI system G:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \overbrace{\hat{B}w(k)}^{v(k)} \\ y(k) &= Cx(k) \end{aligned}$$



$K = \Phi_u \Phi_x^{-1}$

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

$$\min_{K(z)} \left\| \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} T_{w\to y}(z) - T_{\text{des}}(z) \\ T_{w\to u}(z) \end{bmatrix} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{s.t.} \quad T_{v\to x}(z), T_{v\to u}(z) \in \frac{1}{z}\mathcal{RH}_\infty$$

SLS

Convex SLS Problem

$$\min_{\Phi_x(z),\Phi_u(z)} \left\| \overbrace{\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C\Phi_x(z)\hat{B} - T_{\text{des}}(z) \\ \Phi_u(z)\hat{B} \end{bmatrix}}^{\Phi(z)} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{s.t.} \quad (zI - A)\Phi_x(z) - B\Phi_u(z) = I$$

$$\Phi_x(z), \Phi_u(z) \in \frac{1}{z}\mathcal{RH}_\infty$$

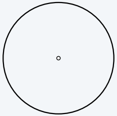
Infinite Dimensionality

- Q, R can be chosen by the designer
- T_{des} is the desired system dynamics and can be set as 0
- $\| \bullet \|_{\mathcal{H}_2/\mathcal{H}_\infty} = \| \bullet \|_{\mathcal{H}_2} + \lambda \| \bullet \|_{\mathcal{H}_\infty}$, $\lambda \geq 0$
- $\frac{1}{z}\mathcal{RH}_\infty$ is rational strictly proper hardy space

Simple Pole Approximation[4] Addresses the Limitations of Finite Impulse Response

Finite Impulse Response (FIR)

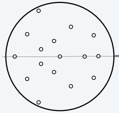
closed-loop poles all lie at the origin



- infeasibility for stabilizable but uncontrollable systems
- high computational cost in systems with large separation of time scales
- unknown to incorporate prior knowledge about optimal closed-loop poles

Simple Pole Approximation (SPA)

any finite selection of stable poles that is closed under complex conjugation



- can apply for stabilizable but uncontrollable systems
- low computational cost in practice
- can easily include prior knowledge

The closed-loop system responses are

$$\Phi_x(z) = \sum_{p \in \mathcal{P}} \frac{G_p}{z - p}, \quad \Phi_u(z) = \sum_{p \in \mathcal{P}} \frac{H_p}{z - p},$$

G_p and H_p are complex coefficient matrices

[4] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles—part i: Density and geometric convergence rate in hardy space," IEEE Transactions on Automatic Control, vol. 69, no. 8, pp. 4894–4909, 2024.

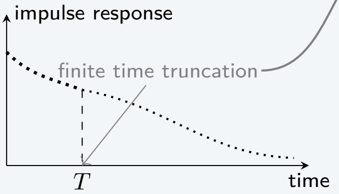
Increased Suboptimality in Prior Work[5] Due to Finite Time Horizon Approximation

Finite Time Horizon Approximation

$$\begin{aligned} T &:= \text{ time horizon} \\ \mathcal{I}_T &:= \text{ impulse response of size } T \\ \mathcal{C}_T &:= \text{ convolution matrix of size } T \end{aligned}$$

$$\|\Phi(z)\|_{\mathcal{H}_2} = \lim_{T\to\infty} \|\mathcal{I}_T\|_F$$

$$\|\Phi(z)\|_{\mathcal{H}_\infty} = \lim_{T\to\infty} \|\mathcal{C}_T\|_2$$



- suboptimality bound is derived under the assumption of solving problem exactly
- suboptimality may not tend to zero as the number of poles diverges

- degraded performance
- higher memory and storage requirements and longer runtime

Goal: eliminate the error of finite time horizon approximation

[5] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles – part ii: System level synthesis beyond finite impulse response," IEEE Transactions on Automatic Control, pp. 1–16, 2024.

KYP Lemma[6] Expresses $\mathcal{H}_2/\mathcal{H}_\infty$ Norms as LMIs

For given transfer function $\Phi(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B}$, if $\tilde{A}$ is stable in the discrete time then the following statements hold.

1) $\|\Phi(z)\|_{\mathcal{H}_2} < \gamma_1$ if and only if there exist $K_1 \in \mathbb{S}^{n\times|\mathcal{P}|}$, $Z \in \mathbb{S}^m$, such that

$$\mathrm{Trace}(Z) < \gamma_1, \begin{bmatrix} K_1 & K_1\tilde{A} & K_1\tilde{B} \\ \tilde{A}^\intercal K_1 & K_1 & 0 \\ \tilde{B}^\intercal K_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0, \quad \begin{bmatrix} K_1 & 0 & \tilde{C}^\intercal \\ 0 & I & 0 \\ \tilde{C} & 0 & Z \end{bmatrix} \succ 0.$$

2) $\|\Phi(z)\|_{\mathcal{H}_\infty} < \gamma_2$ if and only if there exists $K_2 \in \mathbb{S}^{n\times|\mathcal{P}|}$,

$$\begin{bmatrix} K_2 & 0 & \tilde{A}^\intercal K_2 & \tilde{C}^\intercal \\ 0 & \gamma_2 I & \tilde{B}^\intercal K_2 & 0 \\ K_2\tilde{A} & K_2\tilde{B} & K_2 & 0 \\ \tilde{C} & 0 & 0 & \gamma_2 I \end{bmatrix} \succ 0$$

[6] C. Scherer and S. Weiland, "Linear matrix inequalities in control," Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, vol. 3, no. 2, 2000.

Closed Loop Realizations Preserve Linearity

$$\overbrace{\begin{bmatrix} p_1^r I & & & & & & \\ & \ddots & & & & & \\ & & p_m^r I & & & & \\ & & & \mathrm{Re}(p_1^c)I & \mathrm{Im}(p_1^c)I & & \\ & & & -\mathrm{Im}(p_1^c)I & \mathrm{Re}(p_1^c)I & & \\ & & & & & \ddots & & \\ & & & & & & \mathrm{Re}(p_n^c)I & \mathrm{Im}(p_n^c)I \\ & & & & & & -\mathrm{Im}(p_n^c)I & \mathrm{Re}(p_n^c)I \end{bmatrix}}^{\bar{A}} \quad \overbrace{\begin{bmatrix} I \\ \vdots \\ I \\ 2I \\ 0 \\ \vdots \\ 2I \\ 0 \end{bmatrix}}^{\bar{B}}$$

contribution of **real** poles

contribution of **complex** poles

$$\bar{C}_x = \begin{bmatrix} G_{p_1^r} & \cdots & G_{p_m^r} & \mathrm{Re}(G_{p_1^c}) & \mathrm{Im}(G_{p_1^c}) & \cdots & \mathrm{Re}(G_{p_n^c}) & \mathrm{Im}(G_{p_n^c}) \end{bmatrix}$$

$\bar{C}_x$ is linear w.r.t G_p

$$\bar{C}_u = \begin{bmatrix} H_{p_1^r} & \cdots & H_{p_m^r} & \mathrm{Re}(H_{p_1^c}) & \mathrm{Im}(H_{p_1^c}) & \cdots & \mathrm{Re}(H_{p_n^c}) & \mathrm{Im}(H_{p_n^c}) \end{bmatrix}$$

$\bar{C}_u$ is linear w.r.t H_p

$(\bar{A}, \bar{B}, \bar{C}_x, 0)$ is a real state space realization of $\Phi_x(z)$
$(\bar{A}, \bar{B}, \bar{C}_u, 0)$ is a real state space realization of $\Phi_u(z)$

$$\tilde{A} = \begin{bmatrix} \bar{A} & 0 \\ 0 & A_{\mathrm{des}} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \bar{B}\hat{B} \\ B_{\mathrm{des}} \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} QC\bar{C}_x & -QC_{\mathrm{des}} \\ R\bar{C}_u & 0 \end{bmatrix}, \qquad \Phi(z) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]$$

Control Design Derivation with Two Parts

Objective

$$\min_{\Phi(z)\in\frac{1}{z}\mathcal{RH}_\infty} \|\Phi(z)\|_{\mathcal{H}_2} + \lambda\|\Phi(z)\|_{\mathcal{H}_\infty}$$

is equivalent to

$$\begin{aligned}\min_{\gamma_1,\gamma_2,\Phi(z)\in\frac{1}{z}\mathcal{RH}_\infty} \quad & \gamma_1 + \lambda\gamma_2 \\ \text{s.t.} \quad & \|\Phi(z)\|_{\mathcal{H}_2} < \gamma_1 \\ & \|\Phi(z)\|_{\mathcal{H}_\infty} < \gamma_2\end{aligned}$$

KYP

SLS Constraint

$$(zI - A)\Phi_x(z) - B\Phi_u(z) = I$$

SPA

$$\sum_{p\in\mathcal{P}} G_p = I$$

$$(pI - A)G_p - BH_p = 0$$

Control Design Framework

Hybrid Domain Control Design Yields a SDP

$$
\begin{aligned}
\min_{K_1, K_2, Z, G_p, H_p, \gamma_1, \gamma_2} \quad & \gamma_1 + \lambda \gamma_2 \text{ — linear} \\
\text{subject to} \quad & \left.\begin{aligned}
& \mathrm{Tr}(Z) < \gamma_1 \\
& \begin{bmatrix} K_1 & K_1 \tilde{A} & K_1 \tilde{B} \\ \tilde{A}^\intercal K_1 & K_1 & 0 \\ \tilde{B}^\intercal K_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0 \\
& \begin{bmatrix} K_1 & 0 & \tilde{C}(G_p, H_p)^\intercal \\ 0 & I & 0 \\ \tilde{C}(G_p, H_p) & 0 & Z \end{bmatrix} \succ 0 \\
& \begin{bmatrix} K_2 & 0 & \tilde{A}^\intercal K_2 & \tilde{C}(G_p, H_p)^\intercal \\ 0 & \gamma_2 I & \tilde{B}^\intercal K_2 & 0 \\ K_2 \tilde{A} & K_2 \tilde{B} & K_2 & 0 \\ \tilde{C}(G_p, H_p) & 0 & 0 & \gamma_2 I \end{bmatrix} \succ 0
\end{aligned}\right\} \text{LMIs} \\
& \textstyle\sum_{p \in \mathcal{P}} G_p = I \text{ — affine} \\
& (pI - A)\, G_p - B\, H_p = 0 \text{ — linear}
\end{aligned}
$$

Optimization Structure

- objective is a linear combination of new variables
- $\mathcal{H}_2/\mathcal{H}_\infty$ norms are transferred to LMIs in the **state space**
- SLS constraints remain linear/affine in the **frequency domain**
- hybrid domain control design becomes a **semidefinite program (SDP)**

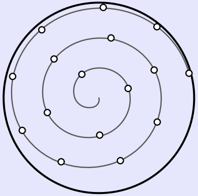
The Control Design Method

- can be solved efficiently
- eliminates the error of finite time horizon approximation

Suboptimality Bound[7] Works for Our Method

Spiral Pole Selection

$\mathcal{P}_n :=$ selected n poles on the spiral in the complex conjugate



Suboptimality Bound

$$\begin{aligned} J^* &:= \text{ground-truth optimal cost} \\ J(\mathcal{P}_n) &:= \text{optimal cost with approximating } \mathcal{P}_n \end{aligned}$$

Then under mild assumptions there exists a constant $K > 0$ and $N > 0$ such that $n \geq N$ implies

$$\frac{J(\mathcal{P}_n) - J^*}{J^*} \leq \frac{K}{\sqrt{n}}$$

- applies directly to our method
- ensures that the suboptimality converges to zero as the number of poles approaches infinity

[7] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles – part ii: System level synthesis beyond finite impulse response," IEEE Transactions on Automatic Control, pp. 1–16, 2024.

Sparsity-Promoting Method Enhances Robustness and Fixed-Order Performance

The Number of Poles $\uparrow$

- increases the computational cost of the design problem
- reduces the robustness of the resulting controller

Sparsity constraints:

$$\sum_{p\in\mathcal{P}} \mathbb{1}(G_p) \leq l, \quad \sum_{p\in\mathcal{P}} \mathbb{1}(H_p) \leq l$$

nonconvex $\Downarrow$ group lasso[8]

Sparsity penalty function:

$$\gamma_1 + \lambda\gamma_2 + \sigma_x \sum_{p\in\mathcal{P}} \|G_p\|_F^2 + \sigma_u \sum_{p\in\mathcal{P}} \|H_p\|_F^2$$

Optimal Sparse Selection

$$\boxed{\begin{array}{ll} \min\limits_{K_1,K_2,Z,G_p,H_p,\gamma_1,\gamma_2} & \gamma_1 + \lambda\gamma_2 + \sigma_x \sum\limits_{p\in\mathcal{P}} \|G_p\|_F^2 + \sigma_u \sum\limits_{p\in\mathcal{P}} \|H_p\|_F^2 \\ \text{s.t.} & \text{Same Constraints} \end{array}}$$

conic program

[8] M. Yin, A. Iannelli, M. Khosravi, A. Parsi, and R. S. Smith, "Linear time-periodic system identification with grouped atomic norm regularization," IFAC-PapersOnLine, vol. 53, no. 2, pp. 1237–1242, 2020.

Numerical Example: Wind Turbine Interfaced to the Power Grid

$$A = \begin{bmatrix} 0.8046 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1177 & -0.0112 & -0.1332 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9889 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.0111 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0.9066 & -0.0364 & 1.0218 & 0 \\ 0 & 0.364 & 0.9066 & -0.2406 & -1.0201 \end{bmatrix}, \hat{B} = \begin{bmatrix} -0.4885 & 0 \\ 0 & 1.0069 \\ 2.5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

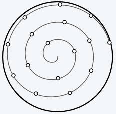
$$A_{\text{des}} = \begin{bmatrix} 0.944 & \\ & 0.944 \end{bmatrix}, B_{\text{des}} = \begin{bmatrix} -1.1052 & \\ & -1.1389 \end{bmatrix}, C_{\text{des}} = I_2.$$

A wind turbine interfaced to the power grid via a power converter

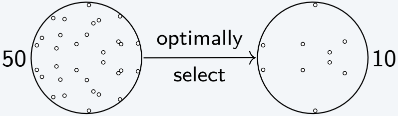
Pole Selection: first incorporate the plant poles and the poles of the desired transfer function

Spiral Method

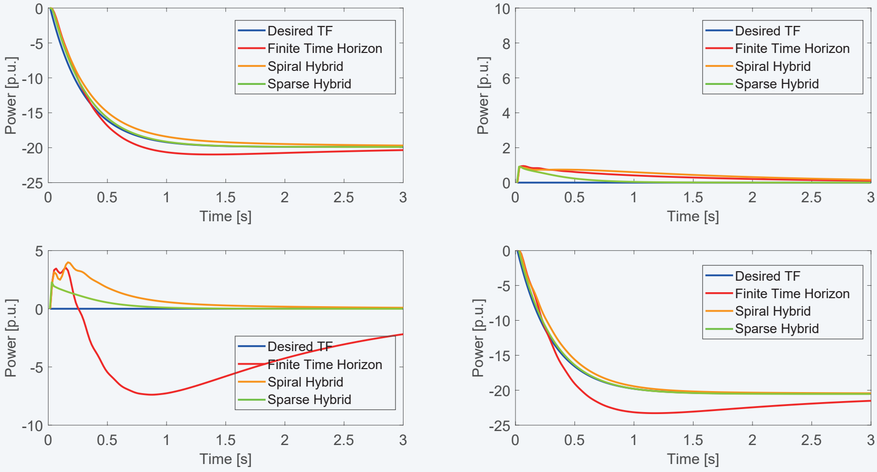
select the remaining 10 poles along an Archimedes spiral



Sparsity-Promoting Method



Sparse Hybrid Method Outperforms Finite Time Horizon Method



Conclusion

We developed a novel hybrid state space and frequency domain method for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design which

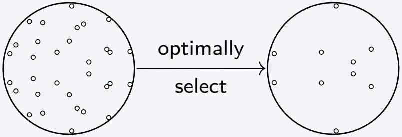
- reduced suboptimality
- improved performance
- less computational cost

Hybrid Domain Method

State Space	**Frequency Domain**
◦ mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm	◦ SLS constraint
	◦ objective
LMIs	**Linear/Affine**

Semidefinite Program

Sparsity-Promoting Method



Conic Program

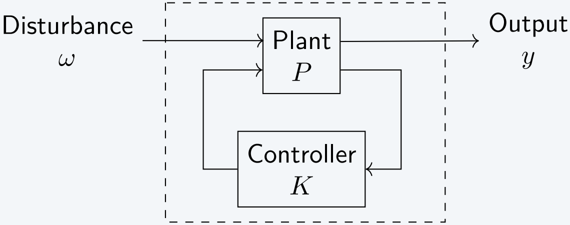
Both Can Be Solved Efficiently

We demonstrated on the test case of control design for a wind turbine with power converter interface, and showed superior performance compared to the prior methods.

Ongoing Work

Ongoing Work

A recent work on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis combining SLS and SPA in continuous time[9]



while we focus on

- Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis in continuous time
- Suboptimality bound for SPA combined with SLS is nontrivial and has been certified

An upcoming manuscript is coming soon!

[9] Y. Du and J. S. Li, "State feedback system level synthesis in continuous time," arXiv preprint arXiv:2410.08135, 2024.

Thank you for your attention

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