Convex Reparameterizations for Efficient Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Feedback Control

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Optimal Linear Feedback Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Synthesis Is Valuable and Challenging

Renewable resources require performance and robustness for uncertainties

Sunlight Intensity

Wind Speed

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis

- has a long history
- is valuable on applications
- but challenging due to the nonconvexity

Our Objective: Develop Novel Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Design Methods

Methods for Tackling Nonconvexity

Youla[1]

Coprime decomposition

System Level Synthesis[2]

State feedback control

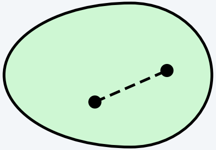
Output feedback control

Input-Output Parameterization[3]

Mixed I[4] **Mixed II**[4]

Output feedback control

Reparameterize in terms of closed-loop system response



Nonconvex

Convex Reparameterization

Convex

Recovering Controller for Original Problem

[1] D. Youla, H. Jabr, and J. Bongiorno, "Modern wiener-hopf design of optimal controllers–part ii: The multivariable case," IEEE Transactions on Automatic Control, vol. 21, no. 3, pp. 319–338, 1976.

[2] J. Anderson, J. C. Doyle, S. H. Low, and N. Matni, "**System level synthesis**," Annual Reviews in Control, vol. 47, pp. 364–393, 2019.

[3] L. Furieri, Y. Zheng, A. Papachristodoulou, and M. Kamgarpour, "An **input–output parametrization** of stabilizing controllers: Amidst youla and system level synthesis," IEEE Control Systems Letters, vol. 3, no. 4, pp. 1014–1019, 2019.

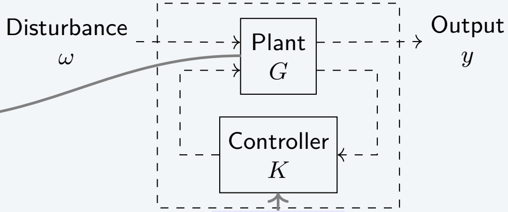
[4] Y. Zheng, L. Furieri, M. Kamgarpour, and N. Li, "System-level, input–output and new parameterizations of stabilizing controllers, and their numerical computation," Automatica, vol. 140, p. 110211, 2022.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ State Feedback in Discrete Time

Problem Formulation after System Level Synthesis

LTI system G:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + \overbrace{\hat{B}w(k)}^{v(k)} \\
y(k) &= Cx(k)
\end{aligned}
$$



$$K = \Phi_u \Phi_x^{-1}$$

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

$$
\begin{aligned}
\min_{K(z)} & \left\| \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} T_{w\to y}(z) - T_{\text{des}}(z) \\ T_{w\to u}(z) \end{bmatrix} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty} \\
\text{s.t.} \quad & T_{v\to x}(z), T_{v\to u}(z) \in \frac{1}{z}\mathcal{RH}_\infty
\end{aligned}
$$

SLS

Convex SLS Problem

$$
\begin{aligned}
\min_{\Phi_x(z),\Phi_u(z)} & \left\| \overbrace{\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C\Phi_x(z)\hat{B} - T_{\text{des}}(z) \\ \Phi_u(z)\hat{B} \end{bmatrix}}^{\tilde{\Phi}(z)} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty} \\
\text{s.t.} \quad & (zI - A)\Phi_x(z) - B\Phi_u(z) = I \\
& \Phi_x(z), \Phi_u(z) \in \frac{1}{z}\mathcal{RH}_\infty
\end{aligned}
$$

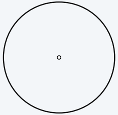
Infinite Dimensionality

- $Q,\ R$ can be chosen by the designer
- T_{des} is the desired system dynamics and can be set as 0
- $\|\bullet\|_{\mathcal{H}_2/\mathcal{H}_\infty} = \|\bullet\|_{\mathcal{H}_2} + \lambda\|\bullet\|_{\mathcal{H}_\infty}$, $\lambda \geq 0$
- $\frac{1}{z}\mathcal{RH}_\infty$ is rational strictly proper Hardy space

Simple Pole Approximation[5] Addresses the Limitations of Finite Impulse Response

Finite Impulse Response (FIR)

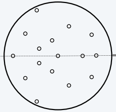
closed-loop poles all lie at the origin



- infeasibility for stabilizable but uncontrollable systems
- high computational cost in systems with large separation of time scales
- unknown to incorporate prior knowledge about optimal closed-loop poles

Simple Pole Approximation (SPA)

any finite selection of stable poles that is closed under complex conjugation



$\mathcal{P}$

- can apply for stabilizable but uncontrollable systems
- low computational cost in practice
- can easily include prior knowledge

The closed-loop system responses are

$$\Phi_x(z) = \sum_{p \in \mathcal{P}} \frac{G_p}{z - p}, \quad \Phi_u(z) = \sum_{p \in \mathcal{P}} \frac{H_p}{z - p},$$

G_p and H_p are complex coefficient matrices

[5] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles—part i: Density and geometric convergence rate in hardy space," IEEE Transactions on Automatic Control, vol. 69, no. 8, pp. 4894–4909, 2024.

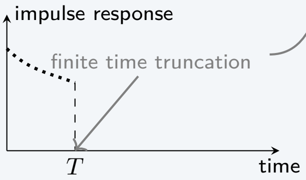
Increased Suboptimality in Prior Work[6] Due to Finite Time Horizon Approximation

Finite Time Horizon Approximation

$$\begin{aligned} T &:= \text{ time horizon} \\ \mathcal{I}_T &:= \text{ impulse response of size } T \\ \mathcal{C}_T &:= \text{ convolution matrix of size } T \end{aligned}$$

$$\|\Phi(z)\|_{\mathcal{H}_2} = \lim_{T\to\infty} \|\mathcal{I}_T\|_F$$

$$\|\Phi(z)\|_{\mathcal{H}_\infty} = \lim_{T\to\infty} \|\mathcal{C}_T\|_2$$



- suboptimality bound is derived under the assumption of solving problem exactly
- suboptimality may not tend to zero as the number of poles diverges

- degraded performance
- higher memory and storage requirements and longer runtime

Goal $\boxed{1}$: eliminate the error of finite time horizon approximation

[6] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles – part ii: System level synthesis beyond finite impulse response," IEEE Transactions on Automatic Control, pp. 1–16, 2024.

KYP Lemma[7] Expresses $\mathcal{H}_2/\mathcal{H}_\infty$ Norms as LMIs

For given transfer function $\tilde{\Phi}(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B}$, if $\tilde{A}$ is stable in the **discrete time** then the following statements hold.

1) $\|\tilde{\Phi}(z)\|_{\mathcal{H}_2} < \gamma_1$ if and only if there exist $K_1 \in \mathbb{S}^{n\times|\mathcal{P}|}$, $Z \in \mathbb{S}^m$, such that

$$\mathrm{Trace}(Z) < \gamma_1, \begin{bmatrix} K_1 & K_1\tilde{A} & K_1\tilde{B} \\ \tilde{A}^\intercal K_1 & K_1 & 0 \\ \tilde{B}^\intercal K_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0, \begin{bmatrix} K_1 & 0 & \tilde{C}^\intercal \\ 0 & I & 0 \\ \tilde{C} & 0 & Z \end{bmatrix} \succ 0.$$

2) $\|\tilde{\Phi}(z)\|_{\mathcal{H}_\infty} < \gamma_2$ if and only if there exists $K_2 \in \mathbb{S}^{n\times|\mathcal{P}|}$,

$$\begin{bmatrix} K_2 & 0 & \tilde{A}^\intercal K_2 & \tilde{C}^\intercal \\ 0 & \gamma_2 I & \tilde{B}^\intercal K_2 & 0 \\ K_2\tilde{A} & K_2\tilde{B} & K_2 & 0 \\ \tilde{C} & 0 & 0 & \gamma_2 I \end{bmatrix} \succ 0$$

[7] C. Scherer and S. Weiland, "Linear matrix inequalities in control," Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, vol. 3, no. 2, 2000.

Closed Loop Realizations Preserve Linearity

$$\overbrace{\begin{bmatrix} p_1^r I & & & & \\ & \ddots & & & \\ & & \boxed{p_m^r I} & & \\ & & & \boxed{\begin{matrix}\mathrm{Re}(p_1^c)I & \mathrm{Im}(p_1^c)I \\ -\mathrm{Im}(p_1^c)I & \mathrm{Re}(p_1^c)I\end{matrix}} & & \\ & & & & \ddots & \\ & & & & & \begin{matrix}\mathrm{Re}(p_n^c)I & \mathrm{Im}(p_n^c)I \\ -\mathrm{Im}(p_n^c)I & \mathrm{Re}(p_n^c)I\end{matrix}\end{bmatrix}}^{\bar{A}} \quad \overbrace{\begin{bmatrix} I \\ \vdots \\ \boxed{I} \\ \boxed{\begin{matrix}2I \\ 0\end{matrix}} \\ \vdots \\ 2I \\ 0\end{bmatrix}}^{\bar{B}}$$

contribution of **real** poles

contribution of **complex** poles

$$\bar{C}_x = \begin{bmatrix} G_{p_1^r} \cdots \boxed{G_{p_m^r}} & \boxed{\mathrm{Re}(G_{p_1^c}) \quad \mathrm{Im}(G_{p_1^c})} & \cdots & \mathrm{Re}(G_{p_n^c}) & \mathrm{Im}(G_{p_n^c}) \end{bmatrix}$$

$\bar{C}_x$ is linear w.r.t G_p

$$\bar{C}_u = \begin{bmatrix} H_{p_1^r} \cdots H_{p_m^r} & \mathrm{Re}(H_{p_1^c}) & \mathrm{Im}(H_{p_1^c}) & \cdots & \mathrm{Re}(H_{p_n^c}) & \mathrm{Im}(H_{p_n^c}) \end{bmatrix}$$

$\bar{C}_u$ is linear w.r.t H_p

$(\bar{A}, \bar{B}, \bar{C}_x, 0)$ is a real state space realization of $\Phi_x(z)$
$(\bar{A}, \bar{B}, \bar{C}_u, 0)$ is a real state space realization of $\Phi_u(z)$

$$\tilde{A} = \begin{bmatrix} \bar{A} & 0 \\ 0 & A_{\text{des}} \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} \bar{B}\hat{B} \\ B_{\text{des}} \end{bmatrix}, \; \tilde{C} = \begin{bmatrix} QC\bar{C}_x & -QC_{\text{des}} \\ R\bar{C}_u & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] = \overbrace{\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C\Phi_x(z)\hat{B} - T_{\text{des}}(z) \\ \Phi_u(z)\hat{B} \end{bmatrix}}^{\tilde{\Phi}(z)}$$

Control Design Derivation with Two Parts

Objective

$$\min_{\tilde{\Phi}(z)\in\frac{1}{z}\mathcal{RH}_\infty} \|\tilde{\Phi}(z)\|_{\mathcal{H}_2} + \lambda\|\tilde{\Phi}(z)\|_{\mathcal{H}_\infty}$$

is equivalent to

$$\begin{aligned}\min_{\gamma_1,\gamma_2,\tilde{\Phi}(z)\in\frac{1}{z}\mathcal{RH}_\infty} \quad & \gamma_1 + \lambda\gamma_2 \\ \text{s.t.} \quad & \|\tilde{\Phi}(z)\|_{\mathcal{H}_2} < \gamma_1 \\ & \|\tilde{\Phi}(z)\|_{\mathcal{H}_\infty} < \gamma_2\end{aligned}$$

KYP

SLS Constraint

$$(zI - A)\Phi_x(z) - B\Phi_u(z) = I$$

SPA

$$\sum_{p\in\mathcal{P}} G_p = I$$

$$(pI - A)G_p - BH_p = 0$$

Control Design Framework

Hybrid Domain Control Design Yields an SDP

$$\begin{aligned}
\underset{K_1, K_2, Z, G_p, H_p, \gamma_1, \gamma_2}{\text{minimize}} \quad & \gamma_1 + \lambda \gamma_2 \text{ — linear} \\
\text{subject to} \quad & \mathrm{Tr}(Z) < \gamma_1 \\
& \begin{bmatrix} K_1 & K_1 \tilde{A} & K_1 \tilde{B} \\ \tilde{A}^\intercal K_1 & K_1 & 0 \\ \tilde{B}^\intercal K_1 & 0 & \gamma_1 I \end{bmatrix} \succ 0 \\
& \begin{bmatrix} K_1 & 0 & \tilde{C}(G_p, H_p)^\intercal \\ 0 & I & 0 \\ \tilde{C}(G_p, H_p) & 0 & Z \end{bmatrix} \succ 0 \\
& \begin{bmatrix} K_2 & 0 & \tilde{A}^\intercal K_2 & \tilde{C}(G_p, H_p)^\intercal \\ 0 & \gamma_2 I & \tilde{B}^\intercal K_2 & 0 \\ K_2 \tilde{A} & K_2 \tilde{B} & K_2 & 0 \\ \tilde{C}(G_p, H_p) & 0 & 0 & \gamma_2 I \end{bmatrix} \succ 0
\end{aligned} \quad \text{LMIs}$$

$$\textstyle\sum_{p \in \mathcal{P}} G_p = I \text{ — affine}$$

$$(pI - A) G_p - B H_p = 0 \text{ — linear}$$

Optimization Structure

- objective is a linear combination of new variables
- $\mathcal{H}_2/\mathcal{H}_\infty$ norms are transferred to LMIs in the **state space**
- SLS constraints remain linear/affine in the **frequency domain**
- hybrid domain control design becomes a **semidefinite program (SDP)**

The Control Design Method

- can be solved efficiently
- eliminates the error of finite time horizon approximation

[1] ✓

Suboptimality Bound[8] Works for Our Method

General Suboptimality Bound

$J^* :=$ ground-truth optimal cost

$J(\mathcal{P}) :=$ optimal cost with approximating poles $\mathcal{P}$

$D(\mathcal{P}) :=$ worst-case geometric approx distance

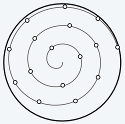
Then under mild assumptions there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P})$$

Spiral Pole Selection

$\mathcal{P}_n :=$ selected n poles on the spiral in the complex conjugate

There exist constants $K, N > 0$ such that $n > N$ implies



$$\frac{J(\mathcal{P}_n) - J^*}{J^*} \leq \frac{K}{\sqrt{n}}$$

- applies directly to our method
- ensures that the suboptimality converges to zero as the number of poles approaches infinity ✓

[1]

[8] M. W. Fisher, G. Hug, and F. Dörfler, "Approximation by simple poles – part ii: System level synthesis beyond finite impulse response," IEEE Transactions on Automatic Control, pp. 1–16, 2024.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ State Feedback in Continuous Time

No Method Exists for Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Design in Continuous Time with SLS

Drawbacks of FIR in Continuous Time

- closed-loop instability
- introduce numerical ill-conditioning into the design problem

Goal

2 develop approximation error bounds of continuous-time SPA theory

3 develop a tractable and efficient mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design method with SLS in continuous time

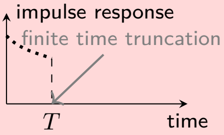
4 apply continuous time approximation error bounds to develop suboptimality bounds for the new method

Limitations of Finite Time Horizon Approximation

$$\begin{aligned} T &:= \text{time horizon} \\ \mathcal{I}_T &:= \text{impulse response at } T \\ \mathcal{C}_T &:= \text{convolution matrix at } T \end{aligned}$$

$$\|\tilde{\Phi}(s)\|_{\mathcal{H}_2} \neq \lim_{T\to\infty} \|\mathcal{I}_T\|_F$$

$$\|\tilde{\Phi}(s)\|_{\mathcal{H}_\infty} \neq \lim_{T\to\infty} \|\mathcal{C}_T\|_2$$



- finite time horizon approximations do **NOT** immediately result in finite dimensional design problems

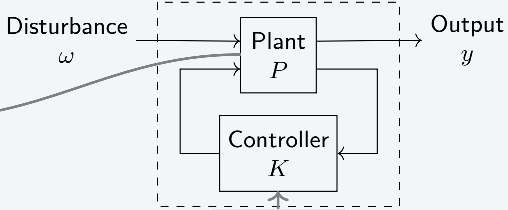
Undeveloped SPA in Continuous Time

- noncompact domain
- DT assumptions do not carry over to CT
- extra $\mathcal{H}_2$ bounds need to be derived
- inapplicable spiral pole selection

Problem Formulation after SLS in Continuous Time

LTI system G:

$$\dot{x}(t) = Ax(t) + Bu(t) + \overbrace{\hat{B}w(t)}^{v(t)}$$
$$y(t) = Cx(t)$$



$K = \Phi_u \Phi_x^{-1}$

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

$$\min_{K(s)} \left\| \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} T_{w\to y}(s) - T_{\text{des}}(s) \\ T_{w\to u}(s) \end{bmatrix} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$
$$\text{s.t.} \quad T_{v\to x}(s), T_{v\to u}(s) \in \frac{1}{s}\mathcal{RH}_\infty$$

SLS

Convex SLS Problem

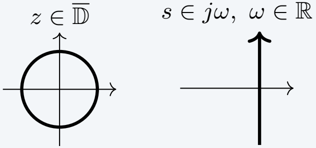
$$\min_{\Phi_x(s),\Phi_u(s)} \left\| \overbrace{\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} C\Phi_x(s)\hat{B} - T_{\text{des}}(s) \\ \Phi_u(s)\hat{B} \end{bmatrix}}^{\tilde{\Phi}(s)} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$
$$\text{s.t.} \quad (sI - A)\Phi_x(s) - B\Phi_u(s) = I$$
$$\Phi_x(s), \Phi_u(s) \in \frac{1}{s}\mathcal{RH}_\infty$$

Infinite Dimensionality

- Q, R can be chosen by the designer
- T_{des} is the desired system dynamics and can be set as 0
- $\| \bullet \|_{\mathcal{H}_2/\mathcal{H}_\infty} = \| \bullet \|_{\mathcal{H}_2} + \lambda \| \bullet \|_{\mathcal{H}_\infty}$, $\lambda \geq 0$
- $\frac{1}{s}\mathcal{RH}_\infty$ is rational strictly proper Hardy space

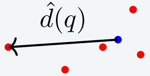
Challenges with Applying the Discrete Time Simple Pole Approximation to Continuous Time

Noncompactness of the domain over which the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are calculated



Discrete-time assumptions are too restrictive

worst case approximation distance upperbound for a single repeated pole $\hat{d}(q) < 1$ doesn't hold in continuous time



$$\hat{d}(q) := \max_{\text{approximating poles } p} |p - q|$$

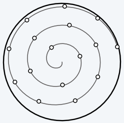
Extra $\mathcal{H}_2$ bounds need to be derived

$$\|T\|_{\mathcal{H}_2} \leq \text{constant} \cdot \|T\|_{\mathcal{H}_\infty}$$

is widely used to establish error bounds and suboptimality bounds in discrete time.

In contrast, it fails to hold in continuous time.

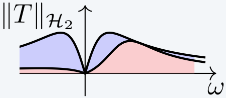
Spiral Pole selection is not applicable

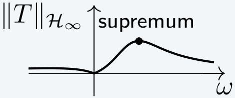


vs ?

Approaches to Address These Challenges

$\mathcal{H}_2$ norm is transformed as an improper integral and $\mathcal{H}_\infty$ norm is upperbounded by Cauchy's bound



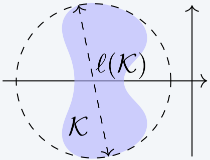


Assumptions changed

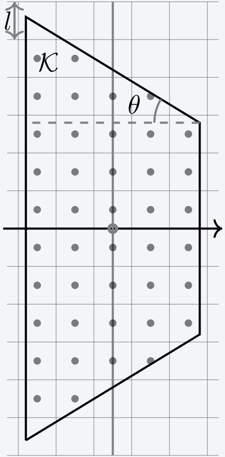
A compact set $\mathcal{K} \subset \mathbb{C}^-$ can be found for arbitrary transfer function in Hardy space

$$\begin{aligned} \hat{d}(q)^k &= \hat{d}(q)^{k-1}\hat{d}(q) \\ &\leq \ell(\mathcal{K})^{m-1}\hat{d}(q) \end{aligned}$$

$k \leq m :=$ multiplicity of q



Grid pole selection for compact set $\mathcal{K}$



$\mathcal{K}$ is specialized as a trapezoide

overlay a grid with cell size l aligning at the center

select center points of cells intersecting with $\mathcal{K}$

Simple Pole Approximation in Continuous Time

Simple Pole Approximation

Let arbitrary transfer function $S \in \frac{1}{s}\mathcal{RH}_\infty$, denote the collection of poles of S as $\mathcal{Q}$, and let $\mathcal{P}$ be a set of poles satisfying Assumptions A1-A3.

- (A1) There exists a compact set $\mathcal{K} \subset \mathbb{C}^-$ that is symmetric with respect to the real axis, such that $\mathcal{P}, \mathcal{Q} \subset \mathcal{K}$
- (A2) $|\mathcal{P}| \geq m_{\max}$, $m_{\max}$ is the largest multiplicity of $\mathcal{Q}$
- (A3) $\mathcal{P}$ is closed under complex conjugation (i.e., $p \in \mathcal{P}$ implies that $\overline{p} \in \mathcal{P}$)

Then there exist constants $c_S = c_S(\mathcal{Q}, G^*_{(q,j)}, \mathcal{K}) > 0$ and $c'_S = c'_S(\mathcal{Q}, G^*_{(q,j)}, \mathcal{K}) > 0$, and constant matrices $\{G_p\}_{p \in \mathcal{P}}$ such that $\sum_{p\in\mathcal{P}} G_p \frac{1}{s-p} \in \frac{1}{s}\mathcal{RH}_\infty$ and

$$\left\| \sum_{p\in\mathcal{P}} G_p \frac{1}{s-p} - S \right\|_{\mathcal{H}_2} \leq c_S \underset{\substack{\uparrow \\ \text{worst-case approximation distance}}}{D(\mathcal{P})} \qquad \left\| \sum_{p\in\mathcal{P}} G_p \frac{1}{s-p} - S \right\|_{\mathcal{H}_\infty} \leq c'_S D(\mathcal{P}).$$

2 ✓

KYP Lemma[9] Expresses $\mathcal{H}_2/\mathcal{H}_\infty$ Norms as LMIs

For given transfer function $\tilde{\Phi}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$, if $\tilde{A}$ is stable in the **continuous time** then the following statements hold.

1) $\|\tilde{\Phi}(s)\|_{\mathcal{H}_2} < \gamma_1$ if and only if there exist $K_1 \in \mathbb{S}^{n\times|\mathcal{P}|}$, $Z \in \mathbb{S}^m$, such that

$$\mathrm{Trace}(Z) < \gamma_1, \begin{bmatrix} \tilde{A}^\intercal K_1 + K_1\tilde{A} & K_1\tilde{B} \\ \tilde{B}^\intercal K_1 & -\gamma_1 I \end{bmatrix} \succ 0, \begin{bmatrix} K_1 & \tilde{C}^\intercal \\ \tilde{C} & Z \end{bmatrix} \succ 0.$$

2) $\|\tilde{\Phi}(s)\|_{\mathcal{H}_\infty} < \gamma_2$ if and only if there exists $K_2 \in \mathbb{S}^{n\times|\mathcal{P}|}$,

$$\begin{bmatrix} \tilde{A}^\intercal K_2 + K_2\tilde{A} & K_2\tilde{B} & \tilde{C}^\intercal \\ \tilde{B}^\intercal K_2 & -\gamma_2 I & 0 \\ \tilde{C} & 0 & -\gamma_2 I \end{bmatrix} \succ 0$$

[9] C. Scherer and S. Weiland, "Linear matrix inequalities in control," Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, vol. 3, no. 2, 2000.

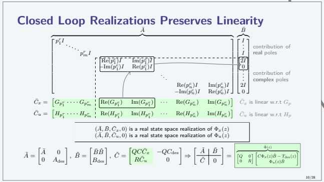
Control Design Derivation with Two Parts

Objective

$$\min_{\tilde{\Phi}(s)\in\frac{1}{s}\mathcal{RH}_\infty} \|\tilde{\Phi}(s)\|_{\mathcal{H}_2} + \lambda\|\tilde{\Phi}(s)\|_{\mathcal{H}_\infty}$$

is equivalent to

$$\begin{aligned}\min_{\gamma_1,\gamma_2,\tilde{\Phi}(s)\in\frac{1}{s}\mathcal{RH}_\infty} \quad & \gamma_1 + \lambda\gamma_2 \\ \text{s.t.} \quad & \|\tilde{\Phi}(s)\|_{\mathcal{H}_2} < \gamma_1 \\ & \|\tilde{\Phi}(s)\|_{\mathcal{H}_\infty} < \gamma_2\end{aligned}$$



+ KYP

SLS Constraint

$$(sI - A)\Phi_x(s) - B\Phi_u(s) = I$$

SPA

$$\sum_{p\in\mathcal{P}} G_p = I$$

$$(pI - A)G_p - BH_p = 0$$

Control Design Framework

Hybrid Domain Control Design Yields an SDP

$$\underset{K_1, K_2, Z, G_p, H_p, \gamma_1, \gamma_2}{\text{minimize}} \quad \gamma_1 + \lambda \gamma_2 \text{ — linear}$$

$$\text{subject to} \quad \text{Tr}(Z) < \gamma_1$$

$$\begin{bmatrix} -\tilde{A}^\intercal K_1 - K_1 \tilde{A}^\intercal & -K_1 \tilde{B} \\ -\tilde{B}^\intercal K_1 & \gamma_1 I \end{bmatrix} \succ 0$$

$$\begin{bmatrix} K_1 & \tilde{C}(G_p, H_p)^\intercal \\ \tilde{C}(G_p, H_p) & Z \end{bmatrix} \succ 0 \quad \text{LMIs}$$

$$\begin{bmatrix} -\tilde{A}^\intercal K_2 - K_2 \tilde{A} & -K_2 \tilde{B} & -\tilde{C}(G_p, H_p)^\intercal \\ -\tilde{B}^\intercal K_2 & \gamma_1 I & 0 \\ -\tilde{C}(G_p, H_p) & 0 & \gamma_2 I \end{bmatrix} \succ 0$$

$$\sum_{p \in \mathcal{P}} G_p = I \text{ — affine}$$

$$(pI - A) G_p - B H_p = 0 \text{ — linear}$$

Optimization Structure

- objective is a linear combination of new variables
- $\mathcal{H}_2/\mathcal{H}_\infty$ norms are transferred to LMIs in the **state space**
- SLS constraints remain linear/affine in the **frequency domain**
- hybrid domain control design becomes a **semidefinite program (SDP)**

The Control Design Method

- can be solved efficiently
- provides a tractable approach for efficiently evaluating the $\mathcal{H}_2/\mathcal{H}_\infty$ norms in continuous time

$\boxed{3}$ ✓

Suboptimality Bounds for SLS in Continuous Time

General Suboptimality Bound

$$
\begin{aligned}
J^* &:= \text{ground-truth optimal cost} \\
J(\mathcal{P}) &:= \text{optimal cost with approximating poles } \mathcal{P} \\
D(\mathcal{P}) &:= \text{worst-case geometric approx distance}
\end{aligned}
$$

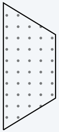
Then under mild assumptions there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P})$$

Grid Pole Selection

$\mathcal{P}_n :=$ selected n poles from the grid in the complex conjugate

There exist constants $K, N > 0$ such that $n > N$ implies



$$\frac{J(\mathcal{P}_n) - J^*}{J^*} \leq \frac{K}{\sqrt{n}}$$

$\boxed{4}$ ✓

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Output Feedback in Discrete Time

Lack of Efficient Output Feedback Control Design

Challenges in Prior Work

Recently developed convex reparameterizations (SLS, IOP, Mixed I, and Mixed II) are combined with finite impulse response approximation

- limited by the drawbacks of finite impulse response method
- no suboptimality bounds ensuring convergence to the infinite dimensional optimal solution as the approximation order increases

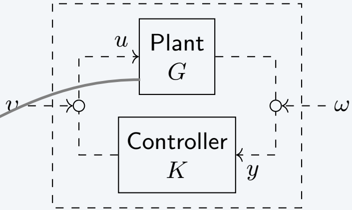
Goal

5. develop tractable and efficient mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback control design methods for each of SLS, IOP, Mixed I, and Mixed II
6. provide suboptimality bounds that guarantee convergence to the ground-truth global optimum for all four new methods using a common theoretical framework

Problem Formulation of Output Feedback Design

The closed-loop system in frequency domain:

$$y(z) = G(z)u(z) + \omega(z)$$
$$u(z) = K(z)y(z) + \upsilon(z)$$



State space representation of plant G:

$$x(k+1) = Ax(k) + Bu(k) + \varsigma(k)$$
$$y(k) = Cx(k) + \omega(k)$$

Closed-loop transfer functions mapping:

$$\begin{bmatrix} x \\ y \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} T_{\varsigma x} & T_{\omega x} & T_{\upsilon x} \\ T_{\varsigma y} & T_{\omega y} & T_{\upsilon y} \\ T_{\varsigma u} & T_{\omega u} & T_{\upsilon u} \end{bmatrix}}_{=:\,T(z)} \begin{bmatrix} \varsigma \\ \omega \\ \upsilon \end{bmatrix}$$

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

$$\min_{K(z)} \left\| \begin{bmatrix} T_{\upsilon y}(z) & T_{\omega y}(z) - I - T_{\omega y}^{\text{des}} \\ T_{\upsilon u}(z) - I & T_{\omega u}(z) \end{bmatrix} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{subject to} \quad T(z) \in \mathcal{RH}_\infty$$

Nonconvex

$T(z) \in \mathcal{RH}_\infty \Leftrightarrow$ one of $\begin{bmatrix} T_{\varsigma x} & T_{\omega x} \\ T_{\varsigma u} & T_{\omega u} \end{bmatrix}, \begin{bmatrix} T_{\omega y} & T_{\upsilon y} \\ T_{\omega u} & T_{\upsilon u} \end{bmatrix}, \begin{bmatrix} T_{\varsigma y} & T_{\omega y} \\ T_{\varsigma u} & T_{\omega u} \end{bmatrix}, \begin{bmatrix} T_{\omega x} & T_{\upsilon x} \\ T_{\omega u} & T_{\upsilon u} \end{bmatrix}$ lies in $\mathcal{RH}_\infty$[10]

[10] Y. Zheng, L. Furieri, M. Kamgarpour, and N. Li, "System-level, input–output and new parameterizations of stabilizing controllers, and their numerical computation," Automatica, vol. 140, p. 110211, 2022.

Output Feedback after Convex Reparameterizations

System Level Synthesis (SLS)

$$\underset{\Phi_{\varsigma x}(z),\Phi_{\omega x}(z)\atop \Phi_{\varsigma u}(z),\Phi_{\omega u}(z)}{\text{minimize}} \quad \left\| \underbrace{\begin{bmatrix} C\Phi_{\varsigma x}B & C\Phi_{\omega x} - T_{\omega y}^{\text{des}} \\ \Phi_{\varsigma u}B & \Phi_{\omega u} \end{bmatrix}}_{\tilde{\Phi}_{\text{SLS}}} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{subject to} \quad \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_{\varsigma x} & \Phi_{\omega x} \\ \Phi_{\varsigma u} & \Phi_{\omega u} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{\varsigma x} & \Phi_{\omega x} \\ \Phi_{\varsigma u} & \Phi_{\omega u} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\Phi_{\omega u}, \Phi_{\varsigma x}, \Phi_{\omega x}, \Phi_{\varsigma u} \in \frac{1}{z}\mathcal{RH}_\infty$$

Mixed I (MI)

$$\underset{\Phi_{\varsigma y}(z),\Phi_{\omega y}(z)\atop \Phi_{\varsigma u}(z),\Phi_{\omega u}(z)}{\text{minimize}} \quad \left\| \underbrace{\begin{bmatrix} \Phi_{\varsigma y}B & \Phi_{\omega y} - I - T_{\omega y}^{\text{des}} \\ \Phi_{\varsigma u}B & \Phi_{\omega u} \end{bmatrix}}_{\tilde{\Phi}_{\text{MI}}} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{subject to} \quad \begin{bmatrix} I & -G \end{bmatrix} \begin{bmatrix} \Phi_{\varsigma y} & \Phi_{\omega y} \\ \Phi_{\varsigma u} & \Phi_{\omega u} \end{bmatrix} = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{\varsigma y} & \Phi_{\omega y} \\ \Phi_{\varsigma u} & \Phi_{\omega u} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Phi_{\omega y} \in \mathcal{RH}_\infty, \ \ \Phi_{\varsigma y}, \Phi_{\varsigma u}, \Phi_{\omega u} \in \frac{1}{z}\mathcal{RH}_\infty$$

Infinite Dimensionality

Input Output Parameterization (IOP)

$$\underset{\Phi_{\omega y}(z),\Phi_{\upsilon y}(z)\atop \Phi_{\omega u}(z),\Phi_{\upsilon u}(z)}{\text{minimize}} \quad \left\| \underbrace{\begin{bmatrix} \Phi_{\upsilon y} & \Phi_{\omega y} - I - T_{\omega y}^{\text{des}} \\ \Phi_{\upsilon u} - I & \Phi_{\omega u} \end{bmatrix}}_{\tilde{\Phi}_{\text{IOP}}} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{subject to} \quad \begin{bmatrix} I & -G \end{bmatrix} \begin{bmatrix} \Phi_{\omega y} & \Phi_{\upsilon y} \\ \Phi_{\omega u} & \Phi_{\upsilon u} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{\omega y} & \Phi_{\upsilon y} \\ \Phi_{\omega u} & \Phi_{\upsilon u} \end{bmatrix} \begin{bmatrix} -G \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\Phi_{\omega y}, \Phi_{\upsilon u} \in \mathcal{RH}_\infty, \ \ \Phi_{\upsilon y}, \Phi_{\omega u} \in \frac{1}{z}\mathcal{RH}_\infty$$

Mixed II (MII)

$$\underset{\Phi_{\omega x}(z),\Phi_{\upsilon x}(z)\atop \Phi_{\omega u}(z),\Phi_{\upsilon u}(z)}{\text{minimize}} \quad \left\| \underbrace{\begin{bmatrix} C\Phi_{\upsilon x} & C\Phi_{\omega x} - T_{\omega y}^{\text{des}} \\ \Phi_{\upsilon u} - I & \Phi_{\omega u} \end{bmatrix}}_{\tilde{\Phi}_{\text{MII}}} \right\|_{\mathcal{H}_2/\mathcal{H}_\infty}$$

$$\text{subject to} \quad \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \Phi_{\omega x} & \Phi_{\upsilon x} \\ \Phi_{\omega u} & \Phi_{\upsilon u} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{\omega x} & \Phi_{\upsilon x} \\ \Phi_{\omega u} & \Phi_{\upsilon u} \end{bmatrix} \begin{bmatrix} -G \\ I \end{bmatrix} = \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}$$

$$\Phi_{\upsilon u} \in \mathcal{RH}_\infty, \ \ \Phi_{\omega x}, \Phi_{\upsilon x}, \Phi_{\omega u} \in \frac{1}{z}\mathcal{RH}_\infty$$

Simple Pole Approximation and State Space Representations for Four Objectives

Simple Pole Approximation for 9 closed-loop reparameterized transfer functions

$$\underbrace{\begin{bmatrix} \Phi_{\varsigma x} & \Phi_{\omega x} & \Phi_{\upsilon x} \\ \Phi_{\varsigma y} & \Phi_{\omega y} & \Phi_{\upsilon y} \\ \Phi_{\varsigma u} & \Phi_{\omega u} & \Phi_{\upsilon u} \end{bmatrix}}_{=:\,\Phi(z)}$$

$\sum_{p\in\mathcal{P}} H_p^{\bullet} \frac{1}{z-p}$

$\sum_{p\in\mathcal{P}} H_p^{\bullet} \frac{1}{z-p} + \boxed{I}$ is cancelled in the objective

deliberately chosen realizatioin $(\overline{A}, \overline{B}, \overline{C}_{\bullet}, 0)$

$$\overbrace{\begin{bmatrix} p_1^r I & & & & & & \\ & \ddots & & & & & \\ & & p_m^r I & & & & \\ & & & \mathrm{Re}(p_1^c)I & \mathrm{Im}(p_1^c)I & & \\ & & & -\mathrm{Im}(p_1^c)I & \mathrm{Re}(p_1^c)I & & \\ & & & & & \ddots & & \\ & & & & & & \mathrm{Re}(p_n^c)I & \mathrm{Im}(p_n^c)I \\ & & & & & & -\mathrm{Im}(p_n^c)I & \mathrm{Re}(p_n^c)I \end{bmatrix}}^{\overline{A}} \quad \overbrace{\begin{bmatrix} I \\ \vdots \\ I \\ 2I \\ 0 \\ \vdots \\ 2I \\ 0 \end{bmatrix}}^{\overline{B}}$$

contribution of **real** poles

contribution of **complex** poles

$$\underbrace{\begin{bmatrix} H_{p_1^r}^{\bullet} \cdots H_{p_m^r}^{\bullet} & \mathrm{Re}(H_{p_1^c}^{\bullet}) & \mathrm{Im}(H_{p_1^c}^{\bullet}) & \cdots & \mathrm{Re}(H_{p_n^c}^{\bullet}) & \mathrm{Im}(H_{p_n^c}^{\bullet}) \end{bmatrix}}_{\overline{C}_{\bullet}}$$

$\overline{C}_{\bullet}$ is linear w.r.t $H_p^{\bullet}$

$\bullet \in \{\varsigma x, \varsigma y, \varsigma u, \omega x, \omega y, \omega u, \upsilon x, \upsilon y, \upsilon u\}$

Closed-loop state space realizations

$$\tilde{A} = \begin{bmatrix} \overline{A} & & \\ & \overline{A} & \\ & & A_{\text{des}}^{\omega y} \end{bmatrix}, \quad \underbrace{\begin{bmatrix} \overline{B} & & \\ & & \overline{B} \\ & & B_{\text{des}}^{\omega y} \end{bmatrix}}_{B_1}, \quad \underbrace{\begin{bmatrix} \overline{B} & B & \\ & & \overline{B} \\ & & B_{\text{des}}^{\omega y} \end{bmatrix}}_{B_2}$$

$$\tilde{C}_{\text{SLS}} = \begin{bmatrix} C\overline{C}_{\varsigma x} & C\overline{C}_{\omega x} & -C_{\text{des}}^{\omega y} \\ \overline{C}_{\varsigma u} & \overline{C}_{\omega u} & \end{bmatrix}, \quad \tilde{B}_{\text{SLS}} = B_2$$

$$\tilde{C}_{\text{IOP}} = \begin{bmatrix} \overline{C}_{\upsilon y} & \overline{C}_{\omega y} & -C_{\text{des}}^{\omega y} \\ \overline{C}_{\upsilon u} & \overline{C}_{\omega u} & \end{bmatrix}, \quad \tilde{B}_{\text{IOP}} = B_1$$

$$\tilde{C}_{\text{MI}} = \begin{bmatrix} \overline{C}_{\varsigma y} & \overline{C}_{\omega y} & -C_{\text{des}}^{\omega y} \\ \overline{C}_{\varsigma u} & \overline{C}_{\omega u} & \end{bmatrix}, \quad \tilde{B}_{\text{MI}} = B_2$$

$$\tilde{C}_{\text{MII}} = \begin{bmatrix} C\overline{C}_{\upsilon x} & C\overline{C}_{\omega x} & -C_{\text{des}}^{\omega y} \\ \overline{C}_{\upsilon u} & \overline{C}_{\omega u} & \end{bmatrix}, \quad \tilde{B}_{\text{MII}} = B_1$$

$$\tilde{C}_{\circ}(zI - \tilde{A})^{-1}\tilde{B}_{\circ} = \tilde{\Phi}_{\circ}(z)$$

$\circ \in$ {SLS, IOP, MI, MII}

Control Design Derivation–IOP as an Example

Objective

$$\min_{\tilde{\Phi}_{\mathsf{IOP}}(z)\in\frac{1}{z}\mathcal{RH}_\infty} \|\tilde{\Phi}_{\mathsf{IOP}}(z)\|_{\mathcal{H}_2} + \lambda\|\tilde{\Phi}_{\mathsf{IOP}}(z)\|_{\mathcal{H}_\infty}$$

is equivalent to

$$\min_{\gamma_1,\gamma_2,\tilde{\Phi}_{\mathsf{IOP}}(z)\in\frac{1}{z}\mathcal{RH}_\infty} \gamma_1 + \lambda\gamma_2$$

$$\text{subject to} \quad \begin{aligned} \|\tilde{\Phi}_{\mathsf{IOP}}(z)\|_{\mathcal{H}_2} &< \gamma_1 \\ \|\tilde{\Phi}_{\mathsf{IOP}}(z)\|_{\mathcal{H}_\infty} &< \gamma_2 \end{aligned}$$

closed-loop realization $(\tilde{A}, \tilde{B}, \tilde{C}_{\mathsf{IOP}}, 0)$

KYP lemma

LMIs in state space

IOP affine constraints after SPA

$\forall\lambda \in \sigma \setminus \mathcal{P}$ and $k \in I_{m_\lambda}$

$$\sum_{p\in\mathcal{P}}\sum_{j=0}^{m_\lambda-k}\frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}}H_p^{\omega u} = 0$$

$$\sum_{p\in\mathcal{P}}\sum_{j=0}^{m_\lambda-k}\frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}}H_p^{\upsilon u} + G_\lambda^k = 0$$

$$\sum_{p\in\mathcal{P}}H_p^{\omega y}\sum_{j=0}^{m_\lambda-k}\frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} + G_\lambda^k = 0$$

$$\sum_{p\in\mathcal{P}}H_p^{\omega u}\sum_{j=0}^{m_\lambda-k}\frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} = 0$$

$\forall p \in \mathcal{P} \setminus \sigma$

$$H_p^{\omega y} - \sum_{\lambda\in\sigma}\sum_{k=1}^{m_\lambda}\frac{G_\lambda^k}{(p-\lambda)^k}H_p^{\omega u} = 0$$

$$H_p^{\upsilon y} - \sum_{\lambda\in\sigma}\sum_{k=1}^{m_\lambda}G_\lambda^k\frac{1}{(p-\lambda)^k}H_p^{\upsilon u} = 0$$

$$H_p^{\upsilon y} - H_p^{\omega y}\sum_{\lambda\in\sigma}\sum_{k=1}^{m_\lambda}G_\lambda^k\frac{1}{(p-\lambda)^k} = 0$$

$$H_p^{\upsilon u} - H_p^{\omega u}\sum_{\lambda\in\sigma}\sum_{k=1}^{m_\lambda}G_\lambda^k\frac{1}{(p-\lambda)^k} = 0$$

$$I_{m_\lambda} = \{1, \dots m_\lambda\}$$
$$I_2^{m_q} = \{2, \dots m_q\}$$

$\forall q \in \sigma \cap \mathcal{P}$

$$\sum_{\substack{\lambda\in\sigma\\ \lambda\neq q}}\sum_{k=1}^{m_\lambda}\frac{G_\lambda^k}{(q-\lambda)^k}H_q^{\omega u} + \sum_{\substack{p\in\mathcal{P}\\ p\neq q}}\sum_{j=0}^{m_q-1}\frac{-G_q^{1+j}}{(p-q)^{1+j}}H_p^{\omega u} = H_q^{\omega y}$$

$$\sum_{\substack{\lambda\in\sigma\\ \lambda\neq q}}\sum_{k=1}^{m_\lambda}\frac{G_\lambda^k H_q^{\upsilon u}}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P}\\ p\neq q}}\sum_{j=0}^{m_q-1}\frac{-G_q^{1+j}H_p^{\upsilon u}}{(p-q)^{1+j}} + G_q^1 = H_q^{\upsilon y}$$

$$\sum_{\substack{\lambda\in\sigma\\ \lambda\neq q}}\sum_{k=1}^{m_\lambda}\frac{H_q^{\omega y}G_\lambda^k}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P}\\ p\neq q}}H_p^{\omega y}\sum_{j=0}^{m_q-1}\frac{-G_q^{1+j}}{(p-q)^{1+j}} + G_q^1 = H_q^{\upsilon y}$$

$$H_q^{\omega u}\sum_{\substack{\lambda\in\sigma\\ \lambda\neq q}}\sum_{k=1}^{m_\lambda}\frac{G_\lambda^k}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P}\\ p\neq q}}H_p^{\omega u}\sum_{j=0}^{m_q-1}\frac{-G_q^{1+j}}{(p-q)^{1+j}} = H_q^{\upsilon u}$$

$$\sum_{\substack{p\in\mathcal{P}\\ p\neq q}}\sum_{j=0}^{m_q-k}\frac{-G_q^{k+j}}{(p-q)^{k+j}}H_p^{\omega u} + G_q^{k-1}H_q^{\omega u} = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P}\\ p\neq q}}\sum_{j=0}^{m_q-k}\frac{-G_q^{k+j}H_p^{\upsilon u}}{(p-q)^{k+j}} + G_q^{k-1}H_q^{\upsilon u} + G_q^k = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P}\\ p\neq q}}\sum_{j=0}^{m_q-k}\frac{-H_p^{\omega y}G_q^{k+j}}{(p-q)^{k+j}} + H_q^{\omega y}G_q^{k-1} + G_q^k = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P}\\ p\neq q}}H_p^{\omega u}\sum_{j=0}^{m_q-k}\frac{-G_q^{k+j}}{(p-q)^{k+j}} + H_q^{\omega u}G_q^{k-1} = 0, \qquad \forall k \in I_2^{m_q}$$

$$G_q^{m_q}H_q^{\omega u} = 0,\; G_q^{m_q}H_q^{\upsilon u} = 0,\; H_q^{\omega y}G_q^{m_q} = 0,\; H_q^{\omega u}G_q^{m_q} = 0$$

$\mathcal{P}$	σ	G_λ^k
SPA pole selection	plant G poles	partial fraction decomposition coefficients of G

Hybrid Domain Method Yields an SDP

$$\underset{K_1^\circ,\, K_2^\circ,\, Z^\circ,\, H_p^\bullet,\, \gamma_1^\circ,\, \gamma_2^\circ}{\text{minimize}} \quad \gamma_1^\circ + \lambda \gamma_2^\circ$$

$$\text{subject to} \quad \begin{bmatrix} K_1^\circ & \star & \star \\ \tilde{A}^\intercal K_1^\circ & K_1^\circ & \star \\ \tilde{B}_\circ^\intercal K_1^\circ & 0 & \gamma_1^\circ I \end{bmatrix} \succ 0, \quad \begin{bmatrix} K_1^\circ & \star & \star \\ 0 & I & \star \\ \tilde{C}_\circ(H_p^\bullet) & 0 & Z^\circ \end{bmatrix} \succ 0$$

$$\begin{bmatrix} K_2^\circ & \star & \star & \star \\ 0 & \gamma_2^\circ I & \star & \star \\ K_2^\circ \tilde{A} & K_2^\circ \tilde{B}_\circ & K_2^\circ & \star \\ \tilde{C}_\circ(H_p^\bullet) & 0 & 0 & \gamma_2^\circ I \end{bmatrix} \succ 0, \quad \mathrm{Tr}(Z^\circ) < \gamma_1^\circ$$

Optimization Structure

- objective is a linear combination of new variables
- $\mathcal{H}_2/\mathcal{H}_\infty$ norms are transferred to LMIs in the **state space**
- IOP constraints remain linear/affine in the **frequency domain**
- hybrid domain control design becomes a **semidefinite program (SDP)**

The Control Design Method

- can be solved efficiently
- provides a unified tractable approach for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback control

[5] ✓

IOP affine constraints after SPA

$\forall \lambda \in \sigma \setminus \mathcal{P}$ and $k \in I_{m_\lambda}$

$$\sum_{p\in\mathcal{P}} \sum_{j=0}^{m_\lambda - k} \frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} H_p^{\omega u} = 0$$

$$\sum_{p\in\mathcal{P}} \sum_{j=0}^{m_\lambda - k} \frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} H_p^{\upsilon u} + G_\lambda^k = 0$$

$$\sum_{p\in\mathcal{P}} H_p^{\omega y} \sum_{j=0}^{m_\lambda - k} \frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} + G_\lambda^k = 0$$

$$\sum_{p\in\mathcal{P}} H_p^{\omega u} \sum_{j=0}^{m_\lambda - k} \frac{-G_\lambda^{k+j}}{(p-\lambda)^{1+j}} = 0$$

$\forall p \in \mathcal{P} \setminus \sigma$

$$H_p^{\omega y} - \sum_{\lambda\in\sigma} \sum_{k=1}^{m_\lambda} \frac{G_\lambda^k}{(p-\lambda)^k} H_p^{\omega u} = 0$$

$$H_p^{\upsilon y} - \sum_{\lambda\in\sigma} \sum_{k=1}^{m_\lambda} G_\lambda^k \frac{1}{(p-\lambda)^k} H_p^{\upsilon u} = 0$$

$$H_p^{\upsilon y} - H_p^{\omega y} \sum_{\lambda\in\sigma} \sum_{k=1}^{m_\lambda} G_\lambda^k \frac{1}{(p-\lambda)^k} = 0$$

$$H_p^{\upsilon u} - H_p^{\omega u} \sum_{\lambda\in\sigma} \sum_{k=1}^{m_\lambda} G_\lambda^k \frac{1}{(p-\lambda)^k} = 0$$

$$I_{m_\lambda} = \{1, \dots m_\lambda\}$$
$$I_2^{m_q} = \{2, \dots m_q\}$$

$\forall q \in \sigma \cap \mathcal{P}$

$$\sum_{\substack{\lambda\in\sigma \\ \lambda\neq q}} \sum_{k=1}^{m_\lambda} \frac{G_\lambda^k}{(q-\lambda)^k} H_q^{\omega u} + \sum_{\substack{p\in\mathcal{P} \\ p\neq q}} \sum_{j=0}^{m_q-1} \frac{-G_q^{1+j}}{(p-q)^{1+j}} H_p^{\omega u} = H_q^{\omega y}$$

$$\sum_{\substack{\lambda\in\sigma \\ \lambda\neq q}} \sum_{k=1}^{m_\lambda} \frac{G_\lambda^k H_q^{\upsilon u}}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P} \\ p\neq q}} \sum_{j=0}^{m_q-1} \frac{-G_q^{1+j} H_p^{\upsilon u}}{(p-q)^{1+j}} + G_q^1 = H_q^{\upsilon y}$$

$$\sum_{\substack{\lambda\in\sigma \\ \lambda\neq q}} \sum_{k=1}^{m_\lambda} \frac{H_q^{\omega y} G_\lambda^k}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P} \\ p\neq q}} H_p^{\omega y} \sum_{j=0}^{m_q-1} \frac{-G_q^{1+j}}{(p-q)^{1+j}} + G_q^1 = H_q^{\upsilon y}$$

$$H_q^{\omega u} \sum_{\substack{\lambda\in\sigma \\ \lambda\neq q}} \sum_{k=1}^{m_\lambda} \frac{G_\lambda^k}{(q-\lambda)^k} + \sum_{\substack{p\in\mathcal{P} \\ p\neq q}} H_p^{\omega u} \sum_{j=0}^{m_q-1} \frac{-G_q^{1+j}}{(p-q)^{1+j}} = H_q^{\upsilon u}$$

$$\sum_{\substack{p\in\mathcal{P} \\ p\neq q}} \sum_{j=0}^{m_q-k} \frac{-G_q^{k+j}}{(p-q)^{k+j}} H_p^{\omega u} + G_q^{k-1} H_q^{\omega u} = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P} \\ p\neq q}} \sum_{j=0}^{m_q-k} \frac{-G_q^{k+j} H_p^{\upsilon u}}{(p-q)^{k+j}} + G_q^{k-1} H_q^{\upsilon u} + G_q^k = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P} \\ p\neq q}} \sum_{j=0}^{m_q-k} \frac{-H_p^{\omega y} G_q^{k+j}}{(p-q)^{k+j}} + H_q^{\omega y} G_q^{k-1} + G_q^k = 0, \qquad \forall k \in I_2^{m_q}$$

$$\sum_{\substack{p\in\mathcal{P} \\ p\neq q}} H_p^{\omega u} \sum_{j=0}^{m_q-k} \frac{-G_q^{k+j}}{(p-q)^{k+j}} + H_q^{\omega u} G_q^{k-1} = 0, \qquad \forall k \in I_2^{m_q}$$

$$G_q^{m_q} H_q^{\omega u} = 0, \; G_q^{m_q} H_q^{\upsilon u} = 0, \; H_q^{\omega y} G_q^{m_q} = 0, \; H_q^{\omega u} G_q^{m_q} = 0$$

$\mathcal{P}$	σ	G_λ^k
SPA pole selection	plant G poles	partial fraction decomposition coefficients of G

Suboptimality Bounds for Four Convex Reparameterization Methods

General SLS suboptimality bound

Let J^* denote the optimal cost of original SLS problem, and let $J(\mathcal{P})$ denote the optimal cost applying SPA for any choice of $\mathcal{P}$. Under mild assumptions, then there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P}).$$

General MI suboptimality bound

Let J^* denote the optimal cost of original MI problem, and let $J(\mathcal{P})$ denote the optimal cost applying SPA for any choice of $\mathcal{P}$. Under mild assumptions, then there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P}).$$

$\boxed{6}$ ✓

General IOP suboptimality bound

Let J^* denote the optimal cost of original IOP problem, and let $J(\mathcal{P})$ denote the optimal cost applying SPA for any choice of $\mathcal{P}$. Under mild assumptions, then there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P}).$$

General MII suboptimality bound

Let J^* denote the optimal cost of original MII problem, and let $J(\mathcal{P})$ denote the optimal cost applying SPA for any choice of $\mathcal{P}$. Under mild assumptions, then there exists a constant $\hat{K} > 0$ such that

$$\frac{J(\mathcal{P}) - J^*}{J^*} \leq \hat{K} D(\mathcal{P}).$$

Numerical Example: Wind Turbine Interfaced to the Power Grid

$$
\begin{aligned}
\dot{x}_{pll} &= v^q \\
\dot{\theta}_{pll} &= k_{p,pll} v^q + k_{i,pll} x_{pll} \\
C\dot{v} &= -C\omega^* J_2 v + i_s - i \\
L\dot{i}_s &= -\left(L\omega^* J_2 + R I_2\right) i_s + v_s - v \\
M\dot{\omega}_r &= p - p_0 \omega_r^m
\end{aligned}
$$

$$[p\ q]' = T_{\text{des}}(z)[\omega\ v]'$$

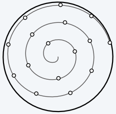
$$T_{\text{des}}(z) = \begin{bmatrix} \frac{-1.1052}{z-0.944} & \\ & \frac{-1.1389}{z-0.944} \end{bmatrix}$$

A wind turbine interfaced to the power grid via a power converter

Pole Selection: first incorporate the plant poles and the poles of the desired transfer function

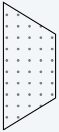
Spiral Method (Discrete Time)

select the remaining 10 poles along an Archimedes spiral

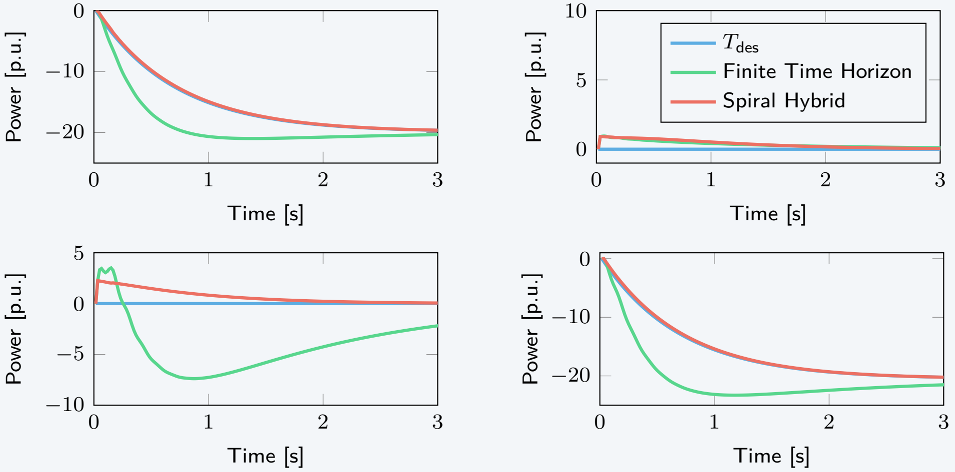


Grid Method (Continuous Time)

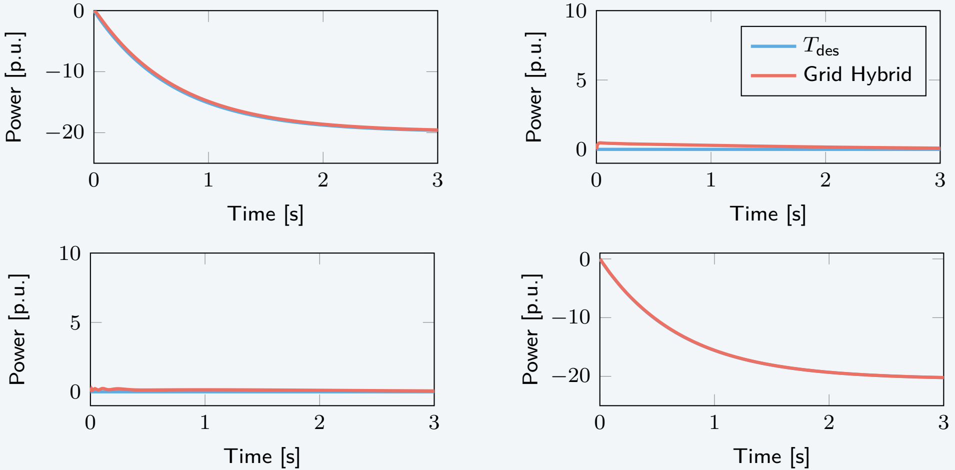
select the remaining 10 poles from grid



Hybrid Domain Method Outperforms Finite Time Horizon Method in Discrete Time



Hybrid Domain Method Maintains Excellent Performance in Continuous Time



Conclusion

State Feedback (DT)	State Feedback (CT)	Output Feedback (DT)
◦ hybrid state space and frequency domain formulation ◦ eliminates finite time horizon approximations	◦ continuous-time SPA theory ◦ 1^{st} tractable control design method for $\mathcal{H}_2/\mathcal{H}_\infty$ control with SLS in continuous time ◦ 1^{st} suboptimality bounds for SLS in continuous time	◦ 1^{st} tractable $\mathcal{H}_2/\mathcal{H}_\infty$ control design framework does not rely on FIR with only four known output feedback convex reparameterizations without relying on coprime decomposition ◦ 1^{st} suboptimality bounds for output feedback $\mathcal{H}_2/\mathcal{H}_\infty$ control with these four convex reparameterizations (SLS, IOP, MI, MII)

All optimizations are convex SDPs and converge to the global optimum of the infinite dimensional problem as the number of poles diverges

Finally, we demonstrated on the test case of control design for a wind turbine with power converter interface, and showed superior performance compared to the prior methods.

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