Integral Approach to Solution of the Time Dependent Conduction Equation in a Reactor Fuel Rod

The 1-D time dependent conduction equation in cylindrical geometry can be written as

$$(\rho C_p)_f \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r k_f \frac{\partial T}{\partial r} \right\} + q'''(r,t)$$

Assuming no accumulation in the gap or cladding, the boundary condition at the fuel pellet surface is

$$-rk_f \left.\frac{\partial T}{\partial r}\right|_R = U(t)R_o\left[T_s - T_\infty\right]$$

where T_s is the pellet surface temperature and U(t) is the overall heat transfer coefficient between the fuel pellet surface and the coolant given by

$$U(t) = \left\{ \frac{R_o}{R_i H_G} + \frac{R_o}{k_c}\ln\left(\frac{R_o}{R_i}\right) + \frac{1}{h_c(t)} \right\}^{-1} .$$

Multiply the conduction equation by r, and integrate over the pellet radius.

$$(\rho C_p)_f \int_0^R \frac{\partial (rT)}{\partial t} dr = \int_0^R \frac{\partial}{\partial r}\left\{ r k_f \frac{\partial T}{\partial r} \right\} dr + \int_0^R r q'''(r,t) dr$$

$$(\rho C_p)_f \frac{d}{dt}\int_0^R (rT) dr = \left. r k_f \frac{\partial T}{\partial r} \right|_R + \int_0^R r q'''(r,t) dr$$

Define an area averaged property $\bar{\Psi}$ as

$$\bar{\Psi} = \frac{\int_0^R (r\Psi) dr}{\int_0^R r dr} = \frac{\int_0^R (r\Psi) dr}{R^2/2} = \frac{1}{R^2}\int_0^R (2r\Psi) dr$$

Applying the boundary condition at the pellet surface, and averaging the fuel temperature and volumetric heat generation rate gives

$$\frac{R^2}{2}(\rho C_p)_f \frac{d\bar{T}}{dt} = -U(t)R_o\left[T_s - T_\infty\right] + \bar{q}'''(t)\frac{R^2}{2}$$

or

$$R^2(\rho C_p)_f \frac{d\bar{T}}{dt} = -U(t)2R_o\left[T_s - T_\infty\right] + \bar{q}'''(t)R^2$$

The above integrated conduction equation is exact, however solution requires a relationship between the area averaged fuel temperature and the fuel surface temperature. Assume a temperature distribution of the form

$$T(r,t) = T_0(t) + c(t)r^2$$

Subject to 1) $\bar{T}(t) = \frac{1}{R^2}\int_0^R 2rTdr$

2) $-rk_f \left.\frac{\partial T}{\partial r}\right|_R = U(t)R_o\left[T_s - T_\infty\right]$

where $T_0(t)$ and $c(t)$ are unknown and to be determined from Equations 1) and 2) above.

Substitute the assumed temperature distribution into Equation 1.

$$R^2\bar{T}(t) = \int_0^R 2r\left[T_0 + cr^2\right]dr$$

$$= R^2T_0 + 2c\frac{R^4}{4}$$

$$\boxed{T_0 = \bar{T} - \frac{cR^2}{2}}$$

Equation 2 may be written as

$$-r\left.\frac{\partial T}{\partial r}\right|_R = \frac{U(t)R_o}{k_f}\left[T_s - T_\infty\right] = B_i(t)\left[T_s - T_\infty\right]$$

where $B_i(t) = \frac{U(t)R_o}{k_f}$ is the dimensionless Biot Number.

For $T_s = T_0 + cR^2 = \bar{T} - \frac{cR^2}{2} + cR^2 \Rightarrow T_s = \bar{T} + \frac{cR^2}{2}$

and $\frac{\partial T}{\partial r} = 2cr$

Equation 2 becomes

$$-R\left[2cR\right] = B_i\left\{\bar{T} + \frac{cR^2}{2} - T_\infty\right\}$$

$$-4cR^2 = 2B_i\left[\bar{T} - T_\infty\right] + cR^2Bi$$

$$-cR^2(4 + B_i) = 2B_i\left[\bar{T} - T_\infty\right]$$

$$\boxed{cR^2 = \frac{-2B_i}{(4 + B_i)}\left[\bar{T} - T_\infty\right]}$$

such that

$$T_s = \overline{T} + \frac{cR^2}{2} = \overline{T} - \frac{B_i}{(4+B_i)}\left[\overline{T} - T_\infty\right]$$

and

$$\begin{aligned} T_s - T_\infty &= \overline{T} - T_\infty - \frac{B_i}{(4+B_i)}\left[\overline{T} - T_\infty\right] \\ &= \frac{4(\overline{T}-T_\infty) + B_i(\overline{T}-T_\infty) - B_i(\overline{T}-T_\infty)}{(4+B_i)} \\ &= \frac{4(\overline{T}-T_\infty)}{(4+B_i)} \end{aligned}$$

We can now replace $T_s - T_\infty$ in the integrated conduction equation to give

$$\begin{aligned} R^2(\rho C_p)_f \frac{d\overline{T}}{dt} &= -U(t)2R_o\left[T_s - T_\infty\right] + \overline{q}'''(t)R^2 \\ &= -U(t)R_o\frac{8(\overline{T}-T_\infty)}{(4+B_i)} + \overline{q}'''(t)R^2 \end{aligned}$$

Dividing both sides by the fuel thermal conductivity

$$R^2\frac{(\rho C_p)_f}{k_f}\frac{d\overline{T}}{dt} = -\frac{U(t)R_o}{k_f}\frac{8(\overline{T}-T_\infty)}{(4+B_i)} + \frac{\overline{q}'''(t)R^2}{k_f}$$

$$\frac{R^2}{\alpha_f}\frac{d\overline{T}}{dt} = -\frac{8B_i(t)(\overline{T}-T_\infty)}{(4+B_i(t))} + \frac{\overline{q}'''(t)R^2}{k_f}$$

where $\alpha_f = \dfrac{k_f}{(\rho C_p)_f}$ is the fuel thermal diffusivity. This is a first order ODE which can be solved by a number of means for the new time average fuel temperature. For example, a Crank-Nicholson type approach would be

$$\frac{R^2}{\alpha_f}\left\{\frac{\overline{T}^{t+\Delta t} - \overline{T}^t}{\Delta t}\right\} = \frac{1}{2}\left[-\frac{8B_i(t)(\overline{T}-T_\infty)}{(4+B_i(t))} + \frac{\overline{q}'''(t)R^2}{k_f}\right]^{t+\Delta t} + \frac{1}{2}\left[-\frac{8B_i(t)(\overline{T}-T_\infty)}{(4+B_i(t))} + \frac{\overline{q}'''(t)R^2}{k_f}\right]^{t}$$

which can be solved directly for $\overline{T}^{t+\Delta t}$ given values for the flow rate and T_∞. An alternate approach is to assume that only the average fuel temperature varies significantly over a time step such that the integrated conduction equation is of the form

$$\frac{d\overline{T}}{dt} + \lambda\overline{T} = S$$

where λ and S are assumed constant at either past time, new time or mixed time. This is a first order linear ODE with constant coefficients and has solution

$$\overline{T}(t+\Delta t) = \overline{T}(t)e^{-\lambda\Delta t} + \frac{S}{\lambda}\left\{1 - e^{-\lambda\Delta t}\right\}$$

Given the new time average fuel temperature, the rod surface heat flux is given by

$$q''(t) = U(t)\left(\frac{4}{4+B_i(t)}\right)[\bar{T}(t) - T_\infty] \ .$$

The clad surface temperature can be related to the average fuel temperature through

$$h_c(t)[T_{co}(t) - T_\infty] = U(t)\left(\frac{4}{4+B_i(t)}\right)[\bar{T}(t) - T_\infty] \ .$$